Schemes

Algebraic Geometry

Thursday 18 December 2025

Schemes

Have seen:

$$\left\{ \begin{array}{c} \text{affine varieties } X \text{ over } k \\ k \text{ algebraically closed} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{finitely generated reduced} \\ k\text{-algebras} \end{array} \right\}$$

Main problem:
Solutions of equations over more general rings?

$$\mathbb{Z}, \quad \mathbb{F}_p, \quad \mathbb{R}, \quad k[x]/(x^2), \quad \mathbb{C}[[x]]$$

Plan of action:

▶ Define geometric object associated to any ring R

$\Rightarrow$ affine schemes

1) Affine schemes

Definition (Affine schemes)

- R ring. Then set

$$\mathrm{Spec}(R) = \{\mathfrak{p} \trianglelefteq R \mid \mathfrak{p} \text{ prime}\}$$

is called the **spectrum / affine scheme associated to** R.

Example

Let $R = A[X]$ and X be an affine variety over k alg. closed field:

$$\left\{ \textbf{non-empty irreducible affine subvarieties of } Y \right\} \iff \left\{ \textbf{prime ideals in } A(Y) \right\},$$

Y affine variety.

$$\mathrm{Spec}(R) \cong \left\{ Y \subset X \mid Y \text{ irreducible subvariety of } X \right\}.$$

Special cases.

▶ $R = k[x] = A(\mathbb{A}^1_k)$, k algebraically closed:

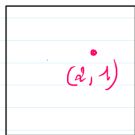$$\mathrm{Spec}(R) = \{ (x - a) \mid a \in k \} \cup \{(0)\}.$$

▶ $R = \mathbb{R}[x]$:

$$\mathrm{Spec}(R) \text{ contains the prime ideal } (x^2 + 1).$$
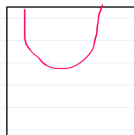
▶ $R = \mathbb{Z}$:

$$\mathrm{Spec}(\mathbb{Z}) = \{ (p) \mid p \text{ prime} \} \cup \{(0)\}.$$

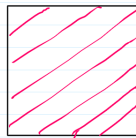# Example: points in $\mathrm{Spec}(k[x_1, x_2])$

$\mathrm{Spec}\left(K[x_1, x_2]\right)$ $(x_1 - 2, x_2 - 1)$ $(x_2 - x_1^2)$ $(\mathcal{O})$
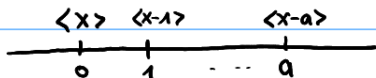
Definition (Nilradical)

Let R ring. The *nilradical* of R is

$$\sqrt{(0)} \;=\; \{\, f \in R \mid \exists\, n \geq 1,\ f^n = 0 \,\}.$$

We denote it by

$$\mathrm{Nil}(R) := \sqrt{(0)}.$$

Nilradical

Remark: R ring, $\quad J = \sqrt{(0)}$ nilradical.

$$f \in J, \ \mathfrak{p} \in \operatorname{Spec}(R) \ \Rightarrow \ \exists\, m \in \mathbb{N} : f^m = 0 \in \mathfrak{p} \ \Rightarrow \ f \in \mathfrak{p}.$$

$\varphi : R \longrightarrow R/J$ quotient map induces:

$$\operatorname{Spec}(R/J) \ \xrightarrow{\ 1:1\ } \operatorname{Spec}(R) \quad \text{(bijection)}$$

$$\mathfrak{q} \longmapsto \varphi^{-1}(\mathfrak{q}).$$

SpecR is sometimes…blind

Ex: $\mathrm{Spec}(K[x]/\langle x^2 \rangle) \xrightarrow{1:1} \mathrm{Spec}(K[x]/\langle x \rangle) = \mathrm{Spec}\, K = \{\langle 0 \rangle\}.$

$$J = \langle x \rangle$$

$\Rightarrow$ as set $\mathrm{Spec}(R)$ cannot detect nilpotent elements.

2) Functions and vanishing loci on affine schemes

Before:

- X affine variety $\Rightarrow A(X)$ coordinate ring
- $f \in A(X)$, $x \in X \Rightarrow$ evaluation $f(x) \in k$
- $I \trianglelefteq A(X) \Rightarrow$ vanishing locus $V(I) \subset X$

Question: How to generalize with $\mathrm{Spec}(R)$?

Idea:

- Functions on $\mathrm{Spec}(R) \Rightarrow$ elements f of R
- How to evaluate at $\mathfrak{p} \in \mathrm{Spec}(R)$?

Residue values

Let R be a ring and let $\mathfrak{p} \in \mathrm{Spec}(R)$.

Definition (Residue values and values of f at a prime)

(a) **Residue field at $\mathfrak{p}$:**

$$K(\mathfrak{p}) = \mathrm{Frac}(R/\mathfrak{p}).$$

(b) **Value of $f \in R$ at $\mathfrak{p}$:** denoted $f(\mathfrak{p})$, is defined as the image of f under the composite ring homomorphism

$$R \longrightarrow R/\mathfrak{p} \longrightarrow K(\mathfrak{p}).$$

where

$$f(\mathfrak{p}) = \frac{[f]}{1} \in K(\mathfrak{p}).$$

Key fact

$$f(\mathfrak{p}) = 0 \iff f \in \mathfrak{p}.$$

Indeed :

The image of f under: $R \longrightarrow R/\mathfrak{p} \longrightarrow K(\mathfrak{p})$ is injective !

If f evaluates to 0 at $\mathfrak{p} \Rightarrow$ by injectivity of the second map it must evaluate to 0 at the second map and thus we must have value 0 on the first map $\Leftrightarrow f \in \mathfrak{p}$.

$$\Rightarrow \quad f(\mathfrak{p}) = 0 \iff f \in \mathfrak{p}$$

Example

(a) Points

$R = A(X)$, X affine variety, $k = \overline{k}$.

$a \in X$, $\quad \mathfrak{p} = I_X(a) \in \mathrm{Spec}(R)$ maximal ideal

$$\Rightarrow \quad R/\mathfrak{p} \simeq k$$

So

$$f : A(X) \longrightarrow A(X)/\mathfrak{p} = K(\mathfrak{p}) = k$$
$$\text{(field)}$$

(b) Link with "rational map":

$\mathfrak{p} = I_X(Y)$, $Y \subset X$ irreducible subvariety,

so

$$R/\mathfrak{p} \simeq A(Y) \qquad \text{and} \qquad K(\mathfrak{p}) = \mathrm{Frac}(A(Y)) = K(Y)$$

(where $K(Y)$ set of rational functions $Y \dashrightarrow \mathbb{A}^1_k$)

Zero loci & ideals

Definition.

R ring.

(a) $S \subset R \Rightarrow$ the zero locus of S is:

$$V(S) = \big\{ \mathfrak{p} \in \mathrm{Spec}(R) \ \big| \ f(\mathfrak{p}) = 0 \ \forall\, f \in S \big\} \subset \mathrm{Spec}(R).$$

(b) $X \subset \mathrm{Spec}(R) \Rightarrow$ the ideal of X:

$$I(X) = \big\{ f \in R \ \big| \ f(\mathfrak{p}) = 0 \ \forall\, \mathfrak{p} \in X \big\} \trianglelefteq R.$$

Question : Is the ideal of X really an ideal ?

Have seen: $f \in \mathfrak{p} \iff f(\mathfrak{p}) = 0.$ Thus:

▶
$$V(S) = \big\{ \mathfrak{p} \in \mathrm{Spec}(R) \mid f \in \mathfrak{p}, \ \forall f \in S \big\}$$
$$= \big\{ \mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \supset S \big\}$$

▶
$$I(X) = \big\{ f \in R \mid f \in \mathfrak{p}, \ \forall \mathfrak{p} \in X \big\}$$
$$= \bigcap_{\mathfrak{p} \in X} \mathfrak{p}$$

Nothing new !

Properties of $V(-)$ and $I(-)$

- $V(S) = V(\langle S \rangle)$

- $V(S_1) \cup V(S_2) = V(S_1 S_2)$

- $\displaystyle\bigcap_{i \in I} V(S_i) = V\left(\bigcup_{i \in I} S_i\right)$

- $V(1) = \varnothing$ Do you see why?

- $V(0) = \mathrm{Spec}(R)$ Do you see why?

Nothing new !

Properties of $V(-)$ and $I(-)$

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- $V(S_1) \cup V(S_2) = V(S_1 S_2)$
- $\displaystyle \bigcap_{i \in I} V(S_i) = V\left(\bigcup_{i \in I} S_i \right)$
- $V(1) = \varnothing$ (any ideal containing 1 is the full ring)
- $V(0) = \operatorname{Spec}(R)$ (0 is contained in any prime ideal)

3) The Zariski topology

Definition (Zariski topology):

Endow the affine scheme $\mathrm{Spec}(R)$ with topology

$$\mathcal{Z} = \big\{ V(S) \subset \mathrm{Spec}(R) \,\big|\, S \subset R \big\}$$

Remark:

(a) Connectedness, irreducibility, .. ✓

(b) Zariski topology is not T_1: points not necessarily closed!

For $\mathfrak{p} \in \mathrm{Spec}(R)$:

$$\overline{\{\mathfrak{p}\}} = \bigcap_{\mathfrak{p} \in V(S)} V(S) = V\left(\bigcup_{\mathfrak{p} \supset S} S \right) = V(\mathfrak{p}) = \big\{ \mathfrak{q} \in \mathrm{Spec}(R) \,\big|\, \mathfrak{q} \supset \mathfrak{p} \big\}$$
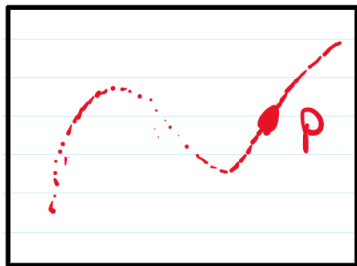
Hence $\{\mathfrak{p}\}$ is closed $\iff$ $\mathfrak{p}$ maximal ideal.
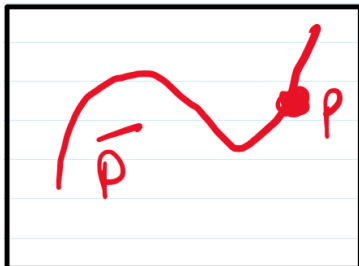
Example

$$R = k[x_1, x_2], \qquad \mathfrak{p} = \langle x_2 - x_1^3 + x_1 \rangle \quad \implies \quad Y = V(\mathfrak{p}) \subset \mathbb{A}_k^2$$

$$\overline{\{\mathfrak{p}\}} = \{\mathfrak{p}\} \cup \langle x_1 - a_1, \ x_2 - a_2 \rangle, \qquad (a_1, a_2) \in Y$$

Example — geometric intuition

$Y = V(\mathfrak{p})$ and a closed point

$\overline{\{\mathfrak{p}\}}$

Proposition (Scheme-theoretic Nullstellensatz)

Let R be a ring.

(a) For $X \subset (R)$ closed, we have

$$V(I(X)) = X.$$

(b) For $J \trianglelefteq R$ an ideal, we have

$$I(V(J)) = \sqrt{J}.$$

In particular, we have a 1:1 correspondance :

$$\left\{ \begin{array}{c} \text{closed subsets} \\ \text{of } Spec R \end{array} \right\} \quad \underset{V(\cdot)}{\overset{I(\cdot)}{\longleftrightarrow}} \quad \left\{ \begin{array}{c} \text{radical ideals} \\ \text{in } R \end{array} \right\}$$

Proof

We prove the $\subset$ in (b) :

$$I(V(J)) = I\big(\{\mathfrak{p} \in Spec(R) \mid \mathfrak{p} \supset J\}\big)$$
$$= \bigcap_{\substack{\mathfrak{p} \in Spec(R) \\ \mathfrak{p} \supset J}} \mathfrak{p}.$$

By a classical result of commutative algebra,

$$\bigcap_{\mathfrak{p} \supset J} \mathfrak{p} = \sqrt{J}.$$

This concludes the proof. $\qquad\qquad\square$

Again : Same properties of $V(-)$ & $I(-)$

(a) $J_1, J_2 \trianglelefteq R \quad \Rightarrow$

- $$V(J_1) \cup V(J_2) = V(J_1 J_2) = V(J_1 \cap J_2)$$

- $$V(J_1) \cap V(J_2) = V(J_1 + J_2)$$

(b) $X_1, X_2 \subset \operatorname{Spec}(R)$ closed

- $$I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$$

- $$I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$$

Distinguished opens in affine schemes

Definition (Distinguished open subsets):

R ring, $f \in R$, the set

$$D(f) = \operatorname{Spec}(R) \setminus V(f) = \big\{ \mathfrak{p} \in \operatorname{Spec}(R) \mid f \notin \mathfrak{p} \big\},$$

is called the *distinguished open subset* of f in $\operatorname{Spec}(R)$.

▶ $U \subset \operatorname{Spec}(R)$ open $\Rightarrow U = \operatorname{Spec}(R) \setminus V(S)$ for some $S \subset R$.

▶
$$U = \bigcup_{f \in S} D(f).$$

i.e. As for affine varieties: distinguished open subsets form a basis of the topology of an affine scheme $\operatorname{Spec}(R)$.

i.e. Any $U \subset \operatorname{Spec}(R)$ is a union of distinguished opens (non ness. finite).

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$$U = \mathrm{Spec}(R) \setminus \bigcap_{f \in S} V(f) = \bigcup_{f \in S} \big(\mathrm{Spec}(R) \setminus V(f)\big) = \bigcup_{f \in S} D(f).$$

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i.e. Any $U \subset \mathrm{Spec}(R)$ is a union of distinguished opens (non ness. finite).

Regular functions and the structure sheaf

Question. For an open subset $U \subset Spec(R)$, what are the regular functions on U?

Definition 12.16 (Regular functions)

Let R be a ring and $U \subset Spec(R)$ an open subset.
A *regular function* on U is a family

$$\varphi = (\varphi_{\mathfrak{p}})_{\mathfrak{p} \in U}, \qquad \varphi_{\mathfrak{p}} \in R_{\mathfrak{p}} \text{ for all } \mathfrak{p} \in U,$$

such that

$$\begin{cases} \forall \, \mathfrak{p} \in U, \ \exists \, f, g \in R \text{ and an open } U_{\mathfrak{p}} \text{ with } \mathfrak{p} \in U_{\mathfrak{p}} \subset U, \\ f \notin \mathfrak{q} \text{ and } \varphi_{\mathfrak{q}} = \dfrac{g}{f} \in R_{\mathfrak{q}} \quad \forall \, \mathfrak{q} \in U_{\mathfrak{p}}, \end{cases} \tag{$*$}$$

$$\mathcal{O}_{Spec(R)}(U) := \{\text{regular functions on } U\}.$$

Ring structure:

$$f + g = (f_{\mathfrak{p}} + g_{\mathfrak{p}})_{\mathfrak{p} \in U}, \qquad f \cdot g = (f_{\mathfrak{p}} g_{\mathfrak{p}})_{\mathfrak{p} \in U}.$$

Remark:

Condition $(*)$ is local $\Rightarrow \mathcal{O}_{\mathrm{Spec}(R)}$ is a **sheaf of rings** (the *structure sheaf*).

Let $\mathfrak{p} \trianglelefteq R$ be a prime ideal. Then $R_{\mathfrak{p}}$ is a local ring whose maximal ideal $P_{\mathfrak{p}}$ has residue field $K(\mathfrak{p})$.

Hence:

If $\varphi \in \mathcal{O}_{\mathrm{Spec}(R)}(U)$, then there is a well-defined value

$$\varphi(\mathfrak{p}) \in K(\mathfrak{p}), \qquad \forall \, \mathfrak{p} \in U,$$

with

$$\varphi_{\mathfrak{p}} \in K(\mathfrak{p}).$$

Stalks of regular functions

Lemma 1

Let R be a ring. For any point $\mathfrak{p} \in \mathrm{Spec}(R)$, the stalk $\mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}}$ of the structure sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ at $\mathfrak{p}$ is isomorphic to the localization $R_{\mathfrak{p}}$.

Proof Sketch.

Define

$$\Phi : \mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}} \longrightarrow R_{\mathfrak{p}}$$

by

$$[(U, \varphi)] \longmapsto \varphi_{\mathfrak{p}}.$$

This map sends the class of a family

$$\varphi = (\varphi_{\mathfrak{q}})_{\mathfrak{q} \in U} \in \mathcal{O}_{\mathrm{Spec}(R)}(U)$$

in the stalk $\mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}}$ to its value at $\mathfrak{q} = \mathfrak{p}$.

The inverse map is given by

$$\Psi : R_{\mathfrak{p}} \longrightarrow \mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}}, \qquad \frac{g}{f} \longmapsto \left[\left(D(f), \ \varphi = (\tfrac{g}{f})_{\mathfrak{q} \in D(f)} \right) \right],$$

with $\mathfrak{p} \in D(f)$.

One checks that

$$\Phi \circ \Psi = \mathrm{id} \qquad \text{and} \qquad \Psi \circ \Phi = \mathrm{id}.$$

Hence $\mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}} \simeq R_{\mathfrak{p}}$.

Let $[(U, \varphi)] \in \mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}}$. By definition of regular functions, there exists $\mathfrak{p} \in U_{\mathfrak{p}} \subset U$ and $g, f \in R$ such that

$$\varphi_{\mathfrak{q}} = \frac{g}{f} \quad \forall \mathfrak{q} \in U_{\mathfrak{p}}.$$

Hence

$$[(U, \varphi)] = \left[\left(U_{\mathfrak{p}}, \ (\tfrac{g}{f})_{\mathfrak{q} \in U_{\mathfrak{p}}} \right) \right] = \Psi\left(\frac{g}{f} \right) = \Psi\big(\Phi([(U, \varphi)]) \big).$$

Therefore

$$\Psi \circ \Phi = \mathrm{id}.$$

Regular functions on distinguished open sets

Proposition 1

Let R be a ring and let $f \in R$. Then

$$\mathcal{O}_{\mathrm{Spec}(R)}\big(D(f)\big) \simeq R_f.$$

In particular

Taking $f = 1$, the global regular functions on $\mathrm{Spec}(R)$ are

$$\mathcal{O}_{\mathrm{Spec}(R)}\big(\mathrm{Spec}(R)\big) \simeq R.$$

Example

Let

$$R = K[x]/(x^2), \qquad K \text{ a field.}$$

There are nilpotent elements in R, hence

$$\mathcal{O}_{\mathrm{Spec}(R)}(\mathrm{Spec}(R)) = R,$$

by proposition 1 ; i.e. the ring of global regular functions has nilpotent elements.

Comparison with affine varieties

This cannot happen for an affine variety X, since $A(X)$ is always a quotient of a polynomial ring by a radical ideal $I(X)$.

Let

$$R = K[x]/(x^2), \qquad K \text{ a field.}$$

Let

$$\varphi = a + bx \in R, \qquad a, b \in K.$$

Then

$$\varphi^n = a^n + na^{n-1}bx,$$

so φ is nilpotent

$$\iff a = 0 \iff \varphi \in \langle x \rangle.$$

φ not uniquely determined by its values at points !

Conversely

$$\mathfrak{p} = \langle x \rangle$$

is the maximal ideal, and

$$R/(\mathfrak{p}) \simeq K \quad \text{(a field)}.$$

Therefore

$$\mathrm{Spec}(R) = \{\mathfrak{p}\} \quad \text{and} \quad K(\mathfrak{p}) \simeq K.$$

$\varphi(\mathfrak{p}) = a \in K(\mathfrak{p})$, i.e. φ is *not uniquely determined* by its values at points of $\mathrm{Spec}(R)$.

Locally ringed spaces and their morphisms

Have given

$$X = \left(\operatorname{Spec}(R),\ \mathcal{O}_{\operatorname{Spec}(R)} \right)$$

a structure of ringed space.

We would like to have: morphisms $f : X \to Y$ of affine schemes

$\iff$ morphisms $f : X \to Y$ of ringed spaces

$\iff$ f pulls back $\varphi \in \mathcal{O}_Y(U)$ to $f^*\varphi \in \mathcal{O}_X\left(f^{-1}(U)\right)$.

Problem:

Definition of $f^*\varphi = \varphi \circ f$ assumes $\varphi : V \to K$.

But here regular functions are no longer defined by values at points.

Recall / Convention:

Assume for any ringed space $(X, \mathcal{O}_X)$, then for $U \subset X$ open we have

$$\mathcal{O}_X(U) \subset \{\varphi : U \to K \mid \varphi \text{ function}\}.$$

Solution

Need to include the data

$$f^* : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X\big(f^{-1}(V)\big).$$

These arbitrary homomorphisms need to satisfy a certain compatibility condition with the continuous map $f : X \to Y$.

Motivation:

Let $f : X \to Y$ be a morphism of (pre-)varieties.

- $f^* : \mathcal{O}_Y(V) \to \mathcal{O}_X\big(f^{-1}(V)\big)$ is compatible with restrictions;
- $f_a^* : \mathcal{O}_{Y,f(a)} \to \mathcal{O}_{X,a}$ for all $a \in X$.

$$I_{f(a)} = \{\varphi \in \mathcal{O}_{Y,f(a)} \mid \varphi(f(a)) = 0\}, \qquad I_a = \{\psi \in \mathcal{O}_{X,a} \mid \psi(a) = 0\}.$$

Remark

Have:

$$(f_a^*)^{-1}(I_a) = \left\{ \varphi \in \mathcal{O}_{Y,f(a)} \mid \varphi(f(a)) = 0 \right\} = I_{f(a)}.$$

i.e. inverse image of a maximal ideal in $\mathcal{O}_{X,a}$ is a maximal ideal in $\mathcal{O}_{Y,f(a)}$.

As the stalks of the structure sheaf of an affine scheme are local rings, by Lemma 1 we can use this as a compatibility condition between the map f and the pull-back f^*

Restriction and Localization

Definition 3.16 (Restriction of a (pre-)sheaf)

Let $\mathcal{F}$ be a presheaf on a topological space X, and let $U \subset X$ be open. The *restriction* of $\mathcal{F}$ to U is the presheaf $\mathcal{F}|_U$ on U defined by

$$\mathcal{F}|_U(V) := \mathcal{F}(V) \qquad \text{for every open set } V \subset U,$$

with the same restriction maps as $\mathcal{F}$. If $\mathcal{F}$ is a sheaf on X, then $\mathcal{F}|_U$ is a sheaf on U.

Lemma 3.19 (Stalks as localizations)

Let X be an affine variety and $a \in X$. Then the stalk of the structure sheaf at a is isomorphic (as a K-algebra) to the localization of the coordinate ring at the maximal ideal $I(a) \lhd A(X)$:

$$\mathcal{O}_{X,a} \cong A(X)_{I(a)} \cong \left\{ \frac{g}{f} \ : \ f, g \in A(X), \ f(a) \neq 0 \right\}.$$

In particular, $\mathcal{O}_{X,a}$ is a local ring with unique maximal ideal

Locally ringed spaces

Definition (Locally ringed space)

A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that, for every point $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring.

Lemma 1

Let R be a ring. For any point $\mathfrak{p} \in \mathrm{Spec}(R)$, the stalk $\mathcal{O}_{\mathrm{Spec}(R),\mathfrak{p}}$ of the structure sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ at $\mathfrak{p}$ is isomorphic to the localization $R_{\mathfrak{p}}$.

Examples

(a) By Lemma 3.19, every (pre-)variety is a locally ringed space.

(b) Every open subset of a locally ringed space, endowed with the restricted structure sheaf (Definition 3.16), is again a locally ringed space.

(c) By Lemma 1, every affine scheme (and hence, by (b), every open subset of an affine scheme) is a locally ringed space.

Morphisms of locally ringed spaces

Definition
A *morphism of locally ringed spaces*

$$(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

consists of the following data:

- a continuous map

$$f : X \longrightarrow Y;$$

- for every open set $V \subset Y$, a ring homomorphism

$$f_V^* : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X\big(f^{-1}(V)\big),$$

called the *pull-back*.

Morphisms of locally ringed spaces

Compatibility conditions

The pull-back maps must satisfy the following conditions:

(i) **Compatibility with restrictions.** For $W \subset V \subset Y$ open and $\varphi \in \mathcal{O}_Y(V)$,

$$f_W^*(\varphi|_W) = \left(f_V^*(\varphi) \right)\big|_{f^{-1}(W)}.$$

(ii) **Locality on stalks.** For every point $p \in X$, the induced homomorphism on stalks

$$f_p^* : \mathcal{O}_{Y,f(p)} \longrightarrow \mathcal{O}_{X,p}$$

is local, i.e.

$$(f_p^*)^{-1}(I_p) = I_{f(p)},$$

where I_p and $I_{f(p)}$ denote the maximal ideals of $\mathcal{O}_{X,p}$ and $\mathcal{O}_{Y,f(p)}$, respectively.

Any morphism of (pre-)varieties is a morphism of locally ringed spaces.

Let R and S be rings. There is a bijection:

$$\Big\{\text{morphisms } \mathrm{Spec}(R) \to \mathrm{Spec}(S)\Big\} \ 1{:}1 \ \Big\{\text{ring homomorphisms } S \to R\Big\},$$

given by

$$f \longmapsto f^*.$$

In particular, this yields:

$$\Big\{\text{affine schemes}\Big\}/\text{isomorphisms} \ 1{:}1 \ \Big\{\text{rings}\Big\}/\text{isomorphisms}.$$

From affine schemes to schemes

As in the case of (pre-)varieties in Chapter 5, the transition from affine schemes to arbitrary schemes is now simply obtained by gluing.

The only difference is that we now allow gluing infinitely many affine schemes.

Definition (Scheme).
A *scheme* is a locally ringed space that has an open cover by affine schemes.

Morphisms of schemes are just morphisms of locally ringed spaces.

Remark.
There are the equivalents of pre-varieties and these could be more natural to call *pre-schemes*.

However, schemes having a separation property will be called *separated schemes*.

From prevarieties to schemes

As the transition from affine schemes to arbitrary schemes is given by the same gluing construction as for prevarieties, we can associate not only an affine scheme to an affine variety, but also a scheme to any prevariety.

Let's focus here on the key statements and ideas.

(Prevarieties as schemes)

Proposition 3

Let X be a prevariety over an algebraically closed field K.

(a) The set X_{sch} of irreducible closed subsets of X carries a natural scheme structure.

(b) Open subsets of X correspond exactly to open subsets of X_{sch}, and under this identification

$$\mathcal{O}_X(U) = \mathcal{O}_{X_{\mathrm{sch}}}(U).$$

(c) Every morphism of prevarieties $f : X \to Y$ induces a morphism of schemes

$$f_{\mathrm{sch}} : X_{\mathrm{sch}} \to Y_{\mathrm{sch}}.$$

Sketch a)

Choose a finite affine open cover

$$X = \bigcup_{i \in I} U_i, \qquad U_i \text{ affine varieties.}$$

For each $i \in I$, set

$$U_{i,\mathrm{sch}} := \mathit{SpecA}(U_i),$$

an affine scheme.

For $i, j \in I$, set the overlap

$$U_{ij} := U_i \cap U_j.$$

induce isomorphisms on coordinate rings:

$$A(U_i)|_{U_{ij}} \cong A(U_j)|_{U_{ij}}.$$

By Proposition 2, these ring isomorphisms induce isomorphisms of affine schemes :

$$\phi_{ij} : \; U_{j,\mathrm{sch}}|_{U_{ij}} \xrightarrow{\sim} U_{i,\mathrm{sch}}|_{U_{ij}}.$$

Glue $Spec(A(U_i))$ together along the same isomorphisms ——¿ X_{sch}