Applied Linear Algebra Homework One

Jesse Bannon

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1 Matrices Excersize

Theorem 1 *Let A be an adjacency matrix for a graph G. $A^p_{i,j}$ represents the number of walks of length p from vertex i to j, $i, j \in G$.*

Proof We prove that an adjecency matrix A for a graph G is nilpotent if and only if G is a directed acylic graph using a case proof that considers all possible graphs.

Case 1 $A = 0$

We can consider G to be a DAG since G has no cycles, and $A^p = 0$ for $p > 0$. Thus A is nilpotent and G is a DAG.

Case 2 G contains a cycle

Using **Theorom 1**, we know that $A^p \neq 0$ for $p > 0$.

Let C be the transitive closure from vertex i to i, where $i \in G$ has a cycle. There exists a walk of p steps from vertex i to some vertex $j \in C$ for $p > 0$.

Thus, $\exists A^p_{i,j} \in A^p \rightarrow A^p_{i,j} > 0$ for $p > 0$.
Thus A is not nilpotent and G is not a DAG.

Case 3 G contains no cycles

Using **Theorom 1**, we know that $A^p = 0$ for some $p > 0$.
Let $n = |V(G)|$

Since there are no cycles, G is considered a DAG. The largest path from any vertex $i \in G$ can only contain every other vertex $\in G$. Thus, the largest possible number of walks for any path $\in G$ can only be $n - 1$.

Thus, $A^n = 0$, making A nilpotent. A is always nilpotent if G is a DAG.

Thus, A is nilpotent if and only if G is a DAG. We can also conclude that the smallest value of p for which $A^p = 0$ indicates that the largest path $\in G$ contains $p - 1$ vertices.