Quantifiers

Jon Beattie

February 14, 2018

1 Mark each statement as True or False. Justify each answer.

(a) The symbol "$\forall$" means "for every".

True; This is the definition of "$\forall$"

(b) The negation of a universal statement is another universal statement.

False; The negation of a universal statement is an existential statement.

(c) The symbol "$\ni$" is read "such that".

True; This is the definition of "$\ni$"

2 Mark each statement True or False. Justify each answer.

(a) The symbol "$\exists$" means "there exist several".

False; "$\exists$" means "there exists" which may only refer to a single entity.

(b) If a variable is used in the antecedent of an implication without being quantified, then the universal quantifier is assumed to apply.

True; This is a general rule.

(c) The order in which quantifiers are used affects the truth value.

True; Suppose I were to say for every number x there exists some number y such that $x - y = 0$. This is true for $x = y$. However, If I were to say there exists some y such that for all x, $x - y = 0$, this would be false. If $x \neq y$, then this assertion breaks.

3 Write the negation of each statement.

(a) All the roads in Yellowstone are open.

Some road in Yellowstone is closed.

(b) Some fish are green.

All fish are not green.

(c) No even integer is prime.

There exists an even integer that is prime.

(d) $\exists x < 3 \ni x^2 \geq 10$.

$\forall x > 3, x^2 > 10$

(e) $\forall x$ in $A, \exists y < k \ni 0 < f(y) < f(x)$

$\exists x$ in $A \ni \forall y < k, 0 \geq f(y) \geq f(x)$

(f) If $n > N$, then $\forall x$ in S, $\mid f_n(x) - f(x) \mid < \epsilon$.

$n > N$ and $\exists x$ in $S \ni \mid f_n(x) - f(x) \mid \geq \epsilon$.

4 Write the negation of each statement.

(a) Some basketball players at Central High are short.

All basketball players at Central High are tall.

(b) All of the lights are on.

Some of the lights are off.

(c) No bounded interval contains infinitely many integers.

Some bounded interval doesn't contain infinitely many integers.

(d) $\exists x$ in $S \ni x \geq 5$.

$\forall x$ in $S \ni x < 5$.

(e) $\forall x \ni 0 < x < 1, f(x) < 2$ or $f(x) > 5$.

$\exists x \ni 0 < x < 1, 2 < f(x) < 5$.

(f) If $x > 5$, then $\exists y > 0 \ni x^2 > 25 + y$.

$x > 5$ and $\forall y > 0, x^2 < 25 + y$.

5 Determine the truth value of each statement, assuming that x, y, and z are real numbers.

(a) $\exists x \ni \forall y \exists z \ni x + y = z$.

True

(b) $\exists x \ni \forall y$ and $\forall z, x + y = z$.

False

(c) $\forall x$ and $\forall y, \exists z \ni y - z = x$.

True

(d) $\forall x$ and $\forall y, \exists z \ni xz = y$.

False

(e) $\exists x \ni \forall y$ and $\forall z, z > y$ implies that $z > x + y$.

True

(f) $\forall x, \exists y$ and $\exists z \ni z > y$ implies that $z > x + y$.

True

6 Determine the truth value of each statement, assuming that x, y and z are real numbers.

(a) $\forall x$ and $\forall y, \exists z \ni x + y = z$.

True

(b) $\forall x \exists y \ni \forall z, x + y = z$.

False

(c) $\exists x \ni \forall y, \exists z \ni xz = y$.

True

(d) $\forall x$ and $\forall y, \exists z \ni yz = x$.

False

(e) $\forall x \exists y \ni \forall z, z > y$ implies that $z > x + y$.

False

(f) $\forall x$ and $\forall y, \exists z \ni z > y$ implies that $z > x + y$.

True

7 Below are two strategies for determining the truth value of a statement involving a positive number x and another statement $P(x)$. Find some $x > 0$ such that $P(x)$ is true. Let x be the name for any number greater than 0 and show $P(x)$ is true. For each statement below, indicate which strategy is more appropriate.

(a) $\forall x > 0, P(x)$.

Use the second method.

(b) $\exists x > 0 \ni P(x)$.

Use the first method.

(c) $\exists x > 0 \ni \neg P(x)$.

Use the first method.

(d) $\forall x > 0, \neg P(x)$.

Use the second method.

8 Which of the following best identifies f as a constant function, where x and y are real numbers

(a) $\exists x \ni \forall y, f(x) = y$.

This isn't a function.

(b) $\forall x \exists y \ni f(x) = y$.

This isn't necessarily constant.

(c) $\exists y \ni \forall x, f(x) = y$.

f is constant.

(d) $\forall y \exists x \ni f(x) = y$.

This isn't constant.

9 Determine the truth value of each statement, assuming x is a real number.

(a) $\exists x$ in $[2, 4] \ni x < 7$.

True

(b) $\forall x$ in $[2, 4], x < 7$.

True

(c) $\exists x \ni x^2 = 5$.

True

(d) $\forall x, x^2 = 5$.

False

(e) $\exists x \ni x^2 \neq -3$.

True

(f) $\forall x, x^2 \neq -3$.

True

(g) $\exists x \ni x \div x = 1$.

True

(h) $\forall x, x \div x = 1$.

False

10 Determine the truth value of each statement, assuming x is a real number.

(a) $\exists x$ in $[3, 5] \ni x \geq 4$.

True

(b) $\forall x$ in $[3, 5]$, $x \geq 4$.

False

(c) $\exists x \ni x^2 \neq 3$.

True

(d) $\forall x, x^2 \neq 3$.

False

(e) $\exists x \ni x^2 = -5$.

False

(f) $\forall x, x^2 = -5$.

False

(g) $\exists x \ni x - x = 0$.

True

(h) $\forall x, x - x = 0$.

True

Exercises 2.11 to 2.19 give certain properties of functions that we shall encounter later in the text. You are to do two things: (a) rewrite the defining conditions in logical symbolism using $\forall$, $\exists$, $\ni$, and $\implies$ as appropriate; and (b) write the negation of part (a) using the same symbolism. It is not necessary that you understand precisely what each term means.

Example: A function f is odd iff for every x, $f(-x) = -f(x)$.

(a) defining condition: $\forall x, f(-x) = -f(x)$.
(b) negation: $\exists x \ni f(-x) \neq -f(x)$.

11 A function f is *even* if for every x, $f(-x) = f(x)$.

12 A function f is *periodic* iff there exists a $k > 0$ such that for every x, $f(x+k) = f(x)$.

13 A function f is *increasing* iff for every x and for every y, if $x \leq y$, then $f(x) \leq f(y)$.

14 A function f is *strictly decreasing* iff for every x and for every y, if $x < y$, then $f(x) > f(y)$.

15 A function f is *injective* iff for every x and y in A, if $f(x) = f(y)$, then $x = y$.

16 A function $f : A \to B$ is *surjective* iff for every y in B there exists an x in A such that $f(x) = y$.

17 A function $f : D \to R$ is *continuous* at $c \in D$ iff for every $\epsilon > 0$ there is a $\delta > 0$ such that $\mid f(x) - f(y) \mid < \epsilon$ whenever $\mid x - c \mid < \delta$ and $x \in D$.

18 A function f is *uniformly continuous on a set* S iff for every $\epsilon > 0$ there is a $\delta > 0$ such that $\mid f(x) - f(y) \mid < \epsilon$ whenever x and y are in S and $\mid x - y \mid < \delta$.

19 The real number L is the *limit* of the function $f : D \to R$ at the point c iff for each $\epsilon > 0$ there exists a $\delta > 0$ such that $\mid f(x) - L \mid < \epsilon$ whenever $x \in D$ and $0 < \mid x - c \mid < \delta$.