Sectoral Shifts, Production Networks, and the Term Structure of Equity

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Abstract

In this paper, I argue that the term structure of equity can serve as a diagnostic to evaluate the relationship between business cycle variation and long-run growth generated in given macroeconomic model. As an application, I explore the asset pricing implications of a multi-sector production network model and use this to shed light on relative importance of idiosyncratic and aggregate shocks in sectoral total factor productivity (TFP). Though aggregate TFP in the U.S. over the last 60 years has grown approximately 1.4 percent annually, these gains have been dispersed across individual sectors, with some sectors even seeing substantial declines. This dispersion is either the result of idiosyncratic sectoral trends or aggregate shocks that shift the composition of the economy without necessarily increasing long-run aggregate output. I show that while as much as 40% of the total variation in TFP growth across sectors can be accounted for by aggregate shifts, the short-term aggregate effects of these shocks implied by the model are too small to account for the stylized fact that the term structure of equity is downward sloping, suggesting a need for other sources of business cycle variation.

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1 Introduction

One of the main objectives of this paper is to demonstrate how asset pricing data, specifically that associated with the term structure of equity, can be used to inform a macroeconomic model. An asset with cash flows or dividend payments payed over time can be viewed as a collection of claims to the individual payments in each period, often called "dividend strips." That is, when the price of a claim to the cash flows $\{D_t\}$ given a stochastic discount factor process $\{S_t\}$ is

$$P_t = \mathbb{E}_t \left[\sum_{\tau=1}^{\infty} \frac{S_{t+\tau}}{S_t} D_{t+\tau} \right], \tag{1}$$

the price of the τ-horizon dividend strip is

$$P_t^\tau = \mathbb{E}_t \left[\frac{S_{t+\tau}}{S_t} D_{t+\tau} \right] \tag{2}$$

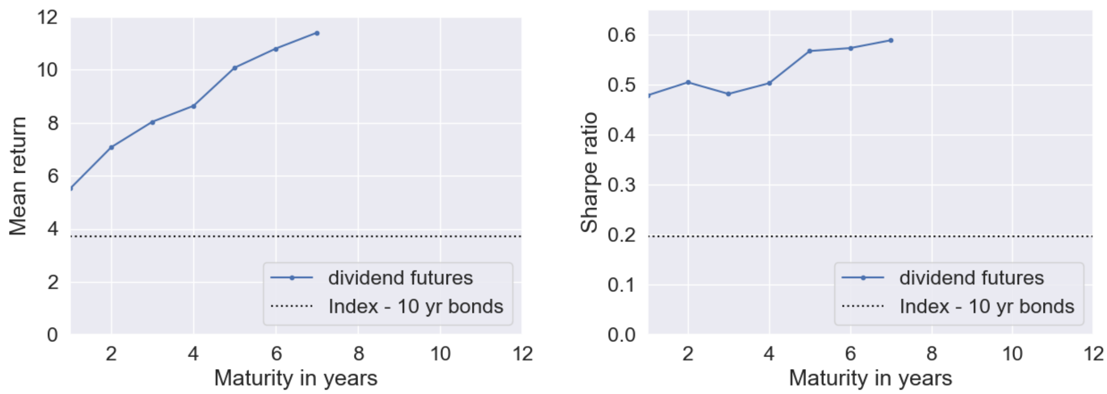
and $P_t = \sum_{\tau=1}^{\infty} P_t^\tau$. The return on these dividend strips,

$$R_{t+1}^\tau = \frac{P_{t+1}^{\tau-1}}{P_t^\tau}, \tag{3}$$

constitute the *term structure of equity.*[1] There is a growing literature in finance seeks to decompose broad market indexes, such as the S&P 500, into these strips and to measure the associated returns. Within this literature, there is growing evidence that this term structure is downward sloping, in the sense that the average returns on longer horizon strips is lower than shorter horizon strips (see, e.g., van Binsbergen, Brandt, and Koijen 2012; van Binsbergen et al. 2013; Gormsen 2018). In this paper, I argue that this information can be used to inform the dynamics of a macroeconomic model. Though most macroeconomic models of consumption and investment are not used to examine this term structure, they nonetheless contain theoretical predictions for it (Borovička and Hansen, 2014). As can be seen in (2), because the prices of dividend strips P_t^τ contain richer information about future dividends $D_{t+\tau}$ and discount rates $S_{t+\tau}$ than the price of the index P_t, the information contained in these prices and their associated returns represent a rich avenue of opportunities to explore the ability of asset prices to inform macroeconomic models.

[1]There are several different definitions of the term structure of equity. Alternatives include "equity yields" (dividend-price ratios associated with equity securities) as well as returns on assets held over different horizons. Here I focus on the term structure as equity as defined by holding period returns on dividend strips.

Figure 1: Term structure of equity



Average returns and Sharpe ratios for dividend futures across maturities of one to seven years. These are dividend futures of dividends paid out by the S&P 500, over a sample of daily returns spanning Nov. 2002 - Jul. 2014, as reported in van Binsbergen and Koijen (2017). Figures reported are annualized. The dotted line represents the figures associated with the index, in excess of returns on 10 year Treasury bonds.

In Figure 1 I plot summary statistics for the average returns and Sharpe ratios on dividend futures based on the dividends paid by the S&P 500 index, as reported in van Binsbergen and Koijen (2017). A *dividend future* is a forward claim on a dividend strip. It is defined as a security where an investor enters into an agreement at time t in which she agrees to pay P_t at time $t + \tau$ in exchange for a risky cash flow $D_{t+\tau}$. These assets are closely related to dividend strips since, by no-arbitrage, the return on this future must be equal to the holding period return on the dividend strip in excess of the risk-free bond with the same maturity. The dotted line in the figure reports the average return on the S&P 500 index, in excess of the average holding period returns on 10 year Treasury bonds. Since the return on the index can be expressed (up to a first-order approximation) as a weighted average of the returns on the dividend strips associated with the index, the return on the index in excess of the 10 year bonds approximates a weighted average of the dividend futures over all horizons $\tau > 0$.[2] Since these dividend futures are only traded for maturities of one to seven years, the right tail of the plot must be inferred by the returns on the index. The key stylized fact to be seen in Figure 1 is that, while returns and Sharpe

[2] See van Binsbergen and Koijen (2017) for a more complete discussion.

ratios may appear to be rising over the maturities 1–7, the lower average returns on the underlying index suggest that the term structure must be downward sloping at higher, unobserved maturities. Loosely speaking, this means that aggregate cash flows at shorter horizons must be in some sense riskier than cash flows at longer horizons. As is common within the macro-finance literature, if we take the S&P 500 as a proxy for the market portfolio and assume that dividends are proportional to aggregate consumption, then the properties associated with the returns on dividend futures should tell us something about the dynamics of aggregate consumption and output. One of the main results of this paper is to show that the risk exposures associated with these dividend futures are exactly equal to the impulse response functions of aggregate consumption. Thus, the risk premia plotted in Figure 1 are equal to a weighted sum of these impulse response functions, weighted by the model implied risk prices. Furthermore, I show that if we impose the restriction that the term structure of equity is downward sloping, this amounts to putting restrictions on the relationship between the permanent and transitory components of cash flow growth.

As an application of using the term structure of equity to evaluate a macroeconomic model, I examine a simple multi-sector production networks model through the lens of asset pricing. In this model, the source of uncertainty that I consider are shocks to total factor productivity (TFP). Because of the way that the shocks propagate through the production networks, the shape of these networks has important implications for the dynamics of the model. The dynamics of aggregate output and investment depend on the shape of the production network and its interaction with the covariance structure of TFP growth, and as such, help determine the shape of the term structure of equity. Given that at the heart of this model is series of decisions about investment and consumption, comparing the model's implicit asset pricing predictions against those observed in the data is a reasonable exercise. Given that the slope of the term structure of equity depends on the relationship between business cycle fluctuations and long-run growth, analyzing the term structure implied by the model serves as a convenient and readily interpretable way to evaluate the model-implied relationship between these objects.

One of the costs of modeling asset prices in a macroeconomic model is that, to get realistic asset price behavior, is that the economist must model the sources of non-stationarity. To keep things simple, I assume that sectoral TFP follows a random walk. Thus, any transitory variation must arise from the mechanisms within the model. However, although I assume sectoral TFP individually follows a random walk, I allow TFP growth across sectors to be correlated via latent common factors. I will show that if any of these common factors induce some amount of negative correlation among sectoral TFP growth, these shocks can generate transitory variation in aggregate output. I will consider a kind of shock that I interpret as a sectoral shift—a shock that alters the composition of the economy without necessarily increasing or decreasing long-run output. Using the structural factor analysis procedure outlined in Foerster, Sarte, and Watson (2011), I will estimate the contribution of these latent factors to sectoral TFP growth and estimate the importance of such shifts. After taking this model to the data, I then quantify the contribution of each shock to the risk premia observed at various horizons within the term structure of equity and compare them to those induces by the other shocks in the model. This information can then be used to evaluate which sources of risk are important for producing realistic asset prices. As an extension, I add transitory variation to TFP to measure the amount of such variation must be added to TFP growth to match the observed asset pricing facts.

After qualitatively demonstrating the ability of a production networks to help determine the shape of the term structure of equity, I will proceed with a more full quantitative exploration. I use industry input-output data from the Bureau of Economic Analysis (BEA) to inform the structure of the intermediate goods network and capital use tables to inform the structure of the investment network. I use the model's implied equilibrium dynamics to infer TFP shocks from sectoral output data. I then explore the inferred distribution and dynamics of these TFP shocks in an unrestricted setting as well as under the restrictions imposed by the moment conditions implied by the term structure of equity. I then explore various calibrations and assess the ability the shape of the intermediate goods network and investment networks to meaningfully contribute to the slope of the term structure.

For a short preview of the relevant data, see Table 1. This table shows annual growth rates for TFP and value added across 30 sectors comprising the US economy over 1960-2013. While aggregate TFP growth over this time period has grown about 1.4 percent per year, this growth has not been shared equally among sectors. While TFP in sectors such as Communications and Services have grown more than 1.4 percent per year over this period, TFP in sectors such as Metals, Non-electric Machinery, or Apparel have seen substantial declines. This dispersion may represent, for example, differences in technological trends across industries or losses in efficiency due to decreases in economies of scale (say, resulting from shifts due to globalization). While I don't take a stand on the economic origins of this dispersion, I do measure the degree to which TFP growth across industries can be explained latent common factors relative to idiosyncratic, sector-specific shocks. I show that, in this model, idiosyncratic, sector-specific shocks do not help to produce a downward sloping term structure of equity. The only shocks that help in this regard are

Table 1: Sectoral Growth in the US (1960-2013)

Industry Name	TFP	Value Added	Industry Name	TFP	Value Added
Agricult.	0.46	1.53	Metals	-1.89	0.65
Mining	0.16	1.39	Fabricated Metals	0.21	1.50
Oil/gas extract.	-0.35	1.00	NE Machinery	-1.53	4.53
Construction	1.50	0.72	E Machinery	2.62	4.30
Food etc.	0.07	1.50	Vehicles	0.97	2.75
Textile	-0.77	0.26	Other Trans. Eq.	1.55	1.94
Apparel	-1.44	-1.29	Instruments	1.69	4.40
Lumber	0.45	1.41	Misc. Manuf.	0.72	2.25
Furniture	0.14	1.66	Warehousing	1.15	2.66
Paper	-0.04	1.37	Communications	3.57	5.35
Printing	0.31	1.94	Utilities	0.39	1.54
Chemical	1.12	2.39	Whole. And Retail	1.85	3.32
Petroleum Refin.	-0.04	1.41	FIRE	1.38	3.81
Plastics	1.69	3.31	Services	2.11	3.57
NM Minerals	0.31	0.85	Govt.	1.69	2.32
			Aggregate	1.40	2.92

Here I show real annual growth rates for a decomposition of the US economy into 30 different sectors, given in percentage points. The row labeled "Aggregate" is a share-weighted average of the 30 sectors. The data comes from Dale Jorgensen's KLEMS data set and the BEA Industry Accounts data. TFP is estimated using the procedure of Foerster, Sarte, and Watson (2011), as discussed in Section 4.

the latent common shocks that shift TFP between sectors, since these shocks cause short-term fluctuations in output without necessarily changing long-run output. We will see that, though these shift shocks can explain up to 40% of the variation in TFP growth, the aggregate effects of these shocks are too small to explain the downward sloping term structure of equity.

1.1 Related Literature

A recent literature has developed that documents that the term structure of equity is downward sloping. The study of holding-period returns on cash flows payoffs (e.g., dividend strips) over alternative horizons is one way to characterize this term structure. van Binsbergen, Brandt, and Koijen (2012) and van Binsbergen et al. (2013) develop and explore empirical counterparts to these holding-period returns and this term structure. They document that the term structure is downward sloping on average, meaning that the expected holding-period return on short-maturity equity is higher than the return on long-maturity equity. This downward slope is inconsistent with many traditional asset pricing models, such as the long-run risk model of Bansal and Yaron (2004) the external habits model of Campbell and Cochrane (1999). Consequently, addressing this challenge represents an important and active area of research within macroeconomics and finance.

Lettau and Wachter (2007) and Hansen, Heaton, and Li (2008) were among the first to emphasize the importance of the term structure of risk prices and risk exposures for asset pricing. The model of Lettau and Wachter (2007) exhibits a downward-sloping term structure, as it is designed to. However, the stochastic discount factor analyzed is specified exogenously and does not represent a fully-fledged model of equilibrium. A logical next step would be to explore micro foundations that could give risk to such a model. What is more, though standard consumption-based asset pricing models already consider the joint modeling of asset prices and aggregate consumption, an equally important endeavor is to is to consider the joint restrictions between these series and other components within general equilibrium, such as aggregate output and investment. The literature that studies asset prices in full general equilibrium models with production does exactly this. As emphasized by Borovička and Hansen (2014), a fully specified dynamic stochastic general equilibrium model will have predictions for asset prices, including the term structure of risk premia. Thus, recent evidence regarding the term structure of risk prices and risk exposures represents a rich opportunity to "examine macroeconomic models through the lens of asset pricing."[3]

Several models have had some success in explaining the average downward sloping term structure of risk premia. A few examples of models that capture this feature are Ai et al. (2012), Andries, Eisenbach, and Schmalz (2014), and Nakamura et al. (2013). Some do so by modifying the preferences of the representative agent, altering their beliefs, or considering alternative technology specifications. van Binsbergen and Koijen (2017) provide a nice overview of such models. In this paper, I focus on the effect of network-induced shock propagation within a standard, frictionless production economy.

With regard to the role of the production network of intermediate goods on asset pricing, Herskovic (2018) explores a related question. In that paper, he decomposes consumption growth into three factors, where one is an aggregation of TFP shocks and the other two are related to innovations in the shape of the production networks. TFP growth, along with changes in the network structure over time, proxy for consumption growth and shocks to any of these factors should be priced in equilibrium. In contrast, the shape of the network in my model is constant over time. Furthermore, I distinguish between the network for intermediate goods and the network for investment goods and emphasize the ability of the investment network to propagate shocks over time. It is this gradual shock propagation that helps to inform the shape of the term structure of equity.

In a similar vein, Richmond (2019) explores centrality in global trade networks and shows a strong relationship between centrality and both interest rates and currency risk premia. The main mechanism in this paper is that central countries' consumption growth is more exposed to global consumption growth shocks via the trade network. This exposure is a contemporaneous exposure and it explains differences in the currency risk premia. In contrast, cross-sectional differences in risk-premia in my paper may arise due to differences in exposures across time. Though I also explore the effects of centrality on risk, the mechanism that I explore leads to a different measure of centrality depending on the horizon analyzed. Risk exposures in the short-run depend more on the intermediate goods network, while risk exposures in the long run tilt more towards the investment network.

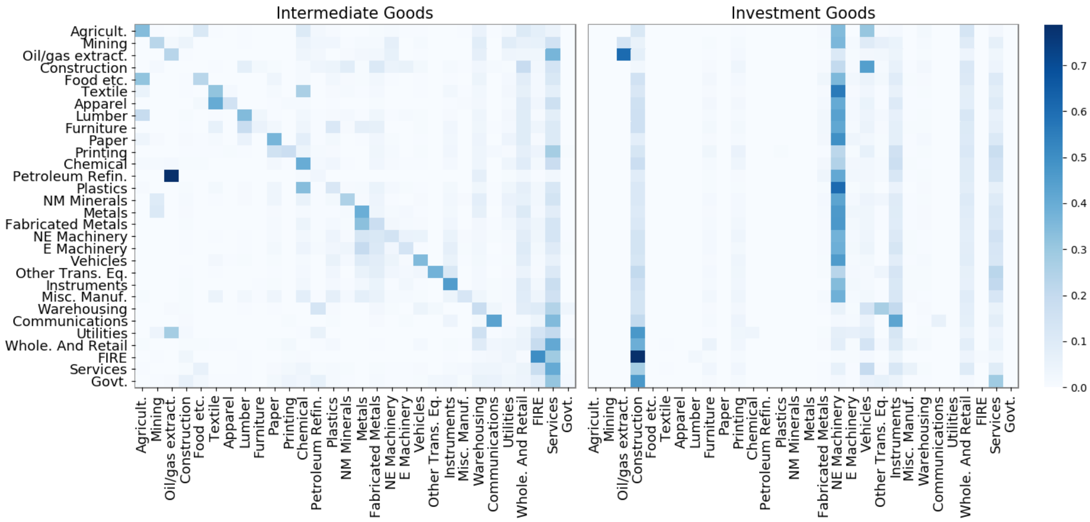
Also, the mechanism I propose relies on the hypothesis that shocks propagate through the network over time. Several recent papers provide evidence that these effects exist and are strong. Barrot and Sauvagnat (2016) and Carvalho et al. (2016) use natural disasters as a source of exogenous variation to identify firm- or industry-

[3] Hansen, Heaton, and Li (2008), Hansen and Scheinkman (2012), Borovicka et al. (2011), Hansen (2012), and Borovička and Hansen (2014) are examples of papers that develop tools and methods to examine macroeconomic models in this way.

level idiosyncratic shocks. Acemoglu, Akcigit, and Kerr (2015) explore a variety of instruments and show that supply shocks transmit from supplier to customer and that shocks resembling demand shocks transmit from customer to supplier. An important paper related to the mechanism explored in my model is vom Lehn and Winberry (2019). While other papers have also explored the consequences of the shape of production networks, this paper emphasizes that the implications of the shape of the intermediate goods network are different from the investment network. They document that the investment network is dominated by a few "investment hubs" and that this structure is important for understanding the business cycle and the nature of sectoral comovement. Shocks to investment hubs have larger and more persistent effects on aggregate GDP and employment and these shocks lead many of these effects. In this paper, I adopt their framework and study the asset pricing implications of the model. However, it's important to note that I consider a different source of variation. While they de-trend the data to study how the model propagates transitory shocks, I consider the variation stemming from the stochastic trends in sectoral TFP growth. I do this because, in order to produce a realistic model of asset prices, I must model the sources of non-stationarity in the model. While the transitory variation in TFP growth is certainly very important, modeling it adds additional complexity to the model that I avoid for now. As an extension, I will later consider the case in which TFP is modeled as a unit root process.

As a note, within the context of the production networks model, I follow vom Lehn and Winberry (2019) and distinguish between the network of intermediate goods and the network of investment goods—goods that are used in the production of new capital. These networks can be represented conveniently as a matrix of cost shares, as illustrated in Figure 2. Since each entry represents a share of expenditures that the row industry spends on the output of the column industry, darker columns represent industries that play an important role producing goods used by many other industries. In this sense, these industries are more central within their respective networks. Many papers within the macroeconomics network literature emphasize how properties such as the centrality of an industry increase an industry's importance in determining aggregate outcomes. One concept that I introduce in this paper is that the appropriate centrality measure depends on the horizon in question. I show that shocks to industries central within the intermediate goods network propagate quickly, while shocks to industries central within the investment network tend to play out more slowly over time. This leads to a measure I call "short-run centrality" and another I call "long-run centrality," which are largely determined by the intermediate goods network and the investment network, respectively. If there are significant differences between the intermediate goods network and the investment

Figure 2: Empirical Production Networks



Heatmaps of empirical production network for intermediate goods and for investment goods. In the intermediate goods plot, each entry depicts the total expenditures of the row industry on the intermediate goods produced by the column industry, divided by the column industry's total expenditures on intermediate goods. The investment network plot is similar, but for expenditures on investment goods. Calculated from 1997 BEA Input-Output tables and Capital Flows table.

network, as there appear to be in Figure 2, then the covariance structure of shocks across industries will play an important role in determining the riskiness of claims to aggregate output over various horizons and, thus, the term structure of equity.

2 A framework for the term structure equity

I begin by introducing the several asset pricing objects of interest in a generic model, including risk prices, risk exposures, and the term structure of equity. I will reference these definitions and results later when analyzing the multisector production network model. Consider a macroeconomic model in which the vector of state variables x_t can be written as a linear state-space model,

$$x_{t+1} = Gx_t + Hw_{t+1}, \tag{4}$$

where G is an $N \times N$ matrix with spectral radius less than one, H is a $N \times M$ constant matrix, and $w_{t+1} \sim \mathcal{N}(0, I)$ is an i.i.d. random vector representing the underlying structure shocks of the model. Suppose further that the model-implied stochastic discount factor (SDF) and a given cash flow process (e.g., aggregate dividends) can be written as

$$\log S_{t+1} - \log S_t = \mu_s + U_s' x_t + \lambda_s' w_{t+1} \tag{5}$$

$$\log D_{t+1} - \log D_t = \mu_d + U_d' x_t + \lambda_d' w_{t+1}, \tag{6}$$

where μ_s and μ_d are constants, and U_s, U_d, λ_s, and λ_d are conforming vectors.

By definition of the SDF, the gross returns over the period t to $t+1$, R_{t+1}, satisfy

$$\mathbb{E}\left[\frac{S_{t+1}}{S_t} R_{t,t+1} \mid x_t\right] = 1.$$

Supposing that we can express the log one-period returns of a given asset as

$$\log R_{t+1} = \mu_r + U_r' x_t + \lambda_r' w_{t+1},$$

for some fixed μ_r, U_r, and λ_r, the one-period risk-premium takes on a simple form, given in Lemma 1.

Lemma 1. *The risk premium, the expected returns associated with R_{t+1} in excess of the short-term risk-free rate, is equal to a product of risk prices and risk exposures:*

$$\log E[R_{t+1}] - \log[R^f_{t+1}] = -\underbrace{\lambda_s}_{\text{risk-prices}} \cdot \underbrace{\lambda_r,}_{\text{risk-exposures}} \tag{7}$$

where R^f_{t+1} is the one-period risk-free rate implied by the SDF.[4] *The vector λ_s thus represents a vector of* risk prices *associated with exposure to each source of risk in w_{t+1}. The vector λ_r represents the vector of* risk exposures*, defining the exposure of the returns R_{t+1} to each source of risk.*

The risk prices represent the marginal risk-premium associated with an additional unit of risk while the risk exposures represent the quantities of each source of risk that the asset with return R_{t+1} is exposed to.

Similarly, for any given cash flow process $\{D_t\}$ (e.g. aggregate dividends), the price of an asset that pays such cash flows is

$$P_t = \mathbb{E}_t\left[\sum_{k=}^{\infty} \frac{S_{t+k}}{S_t} D_{t+k}\right],$$

[4] See the Appendix, equations (77) and (83) for the proof.

where the conditional expectation at time t is evaluated conditional on the state x_t. A *dividend strip* is a claim to one of the individual dividend payments of this asset. Denote the price of the τ-horizon dividend strip as

$$P_t^\tau = \mathbb{E}_t\left[\frac{S_{t+\tau}}{S_t} D_{t+\tau}\right].$$

The return associated with holding this asset for one period (the one-period holding period return) is

$$R_{t+1}^\tau = \frac{P_{t+1}^{\tau-1}}{P_t^\tau}. \tag{8}$$

These are analogous to the holding period returns on zero-coupon bonds, in which the payment $D_{t+\tau}$ is fixed. However, the price of dividend strip also contains information associated with the risky cash flow and its interaction with the evolution of discount factor over the given horizon. Studying dividend strips can therefore improve our understanding of investor preferences and the dynamics of the endowments or technologies that drive the risky cash flows.

In this setting, the risk premium associated with holding period returns has a simple characterization, given in Proposition 3. However, I preface this with a definition of impulse response functions.

Definition 2. Let $\psi_s(\tau)$ be the impulse response function of the SDF (in levels) at horizon τ. That is,

$$\Delta\mathbb{E}_{t+1}\left[\log S_{t+\tau}\right] = \psi_s(\tau) \cdot w_{t+1}.$$

Define ψ_d similarly as the impulse response function for dividends.

Most of the derivations that I will present are written in terms of the impulse response functions, provided a convenient way to link asset pricing results with commonly studied objects in macroeconomics. Now, define $R_{t+1}^{\tau,f}$ as the holding period return associated with holding the zero-coupon risk-free bond with maturity τ and the short-term risk-free rate as $R_{t+1}^f = R_{t+1}^{f,1}$. This gives us the following characterization of the returns on dividend strips.

Proposition 3. *The risk premium associated with the holding-period return on the τ-horizon dividend strip is*

$$\log\mathbb{E}[R_{t+1}^\tau] - \log\mathbb{E}[R_{t+1}^f] = -\underbrace{\lambda_s}_{\textit{risk-prices}} \cdot \underbrace{(\psi_s(\tau) - \psi_s(1) + \psi(\tau))}_{\textit{risk-exposures}}, \tag{9}$$

The proof of this claim is given in the appendix, in Section A.1. To better understand this formula, first note that the impulse response functions measure the sensitivity of each process to the underlying shocks over alternative horizons. λ_s describes the prices associated with a unit of risk from each source while $\psi_d(\tau)$ describes the quantities of risk that the dividend process is exposed to at horizon τ.[5] An interpretation of (9) is that the risk exposures associated with the holding-period return are comprised of two components: a dividend-risk channel, embodied in the term $\psi_d(\tau)$, and a valuation channel, embodied in the term $\psi_s(\tau) - \psi_s(1)$. The dividend-risk channel captures the risk associated with fluctuations in the cash-flow process and the valuation channel captures the risk associated with changing prices of the claim.

With this result in hand, the derivation of the expected returns associated with dividend futures follows immediately. This, along with the expected holding period returns on τ-maturity risk-free bonds, is given in Corollary 4.

Corollary 4. *Since the zero-coupon bond with maturity τ has fixed a fixed cash flow, the risk premium associated with holding this bond is*

$$\log \mathbb{E}\left[R^{f,\tau}_{t+1}\right] - \log \mathbb{E}\left[R^{f}_{t+1}\right] = -\underbrace{\lambda_s}_{\textit{risk prices}} \cdot \underbrace{(\psi_s(\tau) - \psi_s(1)).}_{\textit{risk exposures}} \tag{10}$$

Thus, expected return on dividend strip in excess of the risk-free bond with the same maturity is

$$\log \mathbb{E}\left[R^{\tau}_{t+1}\right] - \log \mathbb{E}\left[R^{f,\tau}_{t+1}\right] = -\underbrace{\lambda_s}_{\textit{risk prices}} \cdot \underbrace{\psi_d(\tau).}_{\textit{risk exposures}} \quad . \tag{11}$$

This derivation can be interpreted as the expected return on a dividend future with maturity τ.

The expression (11) is particularly useful because, it demonstrates that in this setting we can control for the valuation channel by simply netting out the returns associated with holding a risk free bond with the same maturity as the dividend strip.Thus, the risk exposures in this expression are simply equal to the impulse responses of dividends ψ_d. Furthermore, this expression can conveniently be interpreted as the return on a dividend future. A *dividend future* is a forward claim on a dividend strip that is defined as a security where an investor enters into an agreement

[5] In the more general framework of Borovička and Hansen (2014), these are shock-value and shock-exposure elasticities, respectively, and their product $\psi_s(j) \cdot \psi_d(j)$, represents the shock-price elasticities at each horizon j. See Section A.4.4 in the appendix for a short primer.

at time t in which she agrees to pay P_t at time $t + \tau$ in exchange for the cash flow $D_{t+\tau}$. By no-arbitrage, the return on this future must be equal to the holding period return on the dividend strip in excess of the risk-free bond with the same maturity. I will use the returns to these claims over all horizons $\tau = 1, 2, 3, \ldots$ to represent the *term structure of equity.* I argue that observations of these average returns constitute moment conditions that put restrictions on the dynamics of aggregate dividends, as characterized by the impulse responses $\psi_d(\tau)$.

As a corollary, also note that this framework admits a simple framework for the Sharpe ratios.

Corollary 5. *From the definition of the impulse response functions and the derivation of the dividend future returns in Proposition 3,*

$$Cov(\Delta \log S_{t+1}, \Delta \mathbb{E}_{t+1} [\log D_{t+\tau}]) = \psi_s(1) \cdot \psi_d(\tau)$$

and

$$Var(\log R^{\tau}_{t+1} - \log R^{f,\tau}_{t+1}) = \|\psi_d(\tau)\|^2.$$

It then follows that the ratio of this risk premium to the standard deviation of the excess returns is

$$SR_\tau := \frac{\log \mathbb{E}[R^{\tau}_{t+1}] - \log \mathbb{E}[R^{f,\tau}_{t+1}]}{\sqrt{Var(\log R^{\tau}_{t+1} - \log R^{f,\tau}_{t+1})}} = -\psi_s(1) \cdot \frac{\psi_d(\tau)}{\|\psi_d(\tau)\|}. \tag{12}$$

Note that the definition of SR_τ I give above deviates slightly from the common definition of a Sharpe ratio. I examine the ratio SR_τ for ease of discourse. However, the proper definition of a Sharpe ratio could be substituted relatively easily.

2.1 The information content of the long-run slope of the term structure

Here I discuss the information contained by the extremes of the term structure. As demonstrated in (11), the risk exposures associated with the dividend futures are equal to the impulse response functions of dividends. These impulse responses have a useful interpretation in the limit, $\psi_d(\infty) = \lim_{\tau\to\infty} \psi_d(\tau)$, to help us understand the information content of a downward sloping term structure of equity.

To see this, consider the following decomposition.

Lemma 6. *Given an arbitrary log linear process of the form* $\log Y_{t+1} - \log Y_t = \mu_y + U_y' x_t + \lambda_y' w_{t+1}$ *with* $x_{t+1} = Gx_t + Hw_{t+1}$*, the process can be decomposed into a deterministic trend, permanent, and transitory component. That is,*

$$\log Y_t = \underbrace{t\,\mu_y}_{\textit{det. trend}} + \underbrace{\sum_{k=1}^{t} M_y w_k}_{\textit{permanent component}} + \underbrace{F_y x_t}_{\textit{stationary component}} + \underbrace{\log Y_0 - F_y x_0}_{\textit{initial conds.}},$$

where

$$F_y \coloneqq -U_y'(I-G)^{-1}$$
$$M_y \coloneqq \lambda_y' + U_y'(I-G)^{-1}H.$$

Furthermore, given the impulse response function of Y*,* ψ_y*, note that*

$$\psi_y'(\infty) \coloneqq \lim_{\tau\to\infty} \psi_y'(\tau) = M_y \tag{13}$$

and

$$\psi_y'(1) - \psi_y'(\infty) = F_y H. \tag{14}$$

Since the term structure compares the returns associated with short-term assets with long-term assets, consider a comparison of

$$\log \mathbb{E}\left[R_{t+1}^{\tau}\right] - \log \mathbb{E}\left[R_{t+1}^{f,\tau}\right]\Bigg|_{\tau=1}$$

with

$$\log \mathbb{E}\left[R_{t+1}^{\tau}\right] - \log \mathbb{E}\left[R_{t+1}^{f,\tau}\right]\Bigg|_{\tau\to\infty}.$$

According to (11) in Corollary 4, the risk exposures associated with the short-term asset depend on $\psi_d(1)$ and those associated with the long-term asset depend on $\psi_d(\infty)$. Applying Lemma 6, we see that the risk exposures in the long-run depend exclusively on the permanent component of dividend growth while those associated with the short-term asset depend on a combination of the permanent and transitory components.

Interestingly, from (10) we can see that the risk exposures associated with the holding period returns on long-term bonds, $\psi_s(\infty) - \psi_s(1)$, depends exclusively on the transitory component of the SDF, S_t, mirroring the result of Alvarez and Jermann (2005) (who show this in more generality).

2.2 The macroeconomic implications of the slope of the term structure of equity

Up until this point, I have only considered a reduced form representation of the stochastic discount factor, S_t, and computed prices and expected returns associated with a generic cash flow process D_t. If we now put some structure on the economy, we can derive the restrictions that the term structure imposes on the economy.

Begin by supposing that the cash flow process D_t represents aggregate consumption, or at least is proportional to consumption. As argued by (Lucas, 1978), a portfolio consisting of all available assets in the economy to all future consumption (and leisure, if the model includes labor and human capital). This portfolio is called the wealth portfolio. Similarly, when the cash flow process equals the total dividends payed by the aggregate stock market, the claim to these dividends is the market portfolio (Gordon, 1962). To that extent that the market portfolio represents a sufficient proxy for the wealth portfolio,[6], we can then explore the restrictions that the term structure of equity associated with, say, the S&P 500 puts on the macroeconomy.

Now, suppose also that consumption $\mathcal{C}_t$ is defined exogenously by

$$\log \mathcal{C}_{t+1} - \log \mathcal{C}_t = \mu_{\mathcal{C}} + U_{\mathcal{C}}' x_t + \lambda_{\mathcal{C}}' w_{t+1} \tag{15}$$

and that a representative household has Epstein-Zin preferences given by the recursion

$$V_t = \{(1-\beta)(\mathcal{C}_t)^{1-\rho} + \beta[\mathcal{R}_t(V_{t+1})]^{1-\rho}\}^{1/(1-\rho)} \tag{16}$$

where $\mathcal{R}_t$ is the certainty equivalent operator defined by

$$\mathcal{R}_t(V_{t+1}) \equiv \mathbb{E}_t[(V_{t+1})^{1-\gamma}]^{1/(1-\gamma)},$$

ρ^{-1} is the elasticity of intertemporal substitution, and γ is the risk-aversion parameter.

- Suppose that aggregate consumption can be written as

  $$\log \mathcal{C}_{t+1} - \log \mathcal{C}_t = \mu_{\mathcal{C}} + U_{\mathcal{C}}' x_t + \lambda_{\mathcal{C}}' w_{t+1}.$$

  Under these assumptions, the SDF is

  $$\frac{S_{t+1}}{S_t} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left[\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t[V_{t+1}^{1-\gamma}]}\right]^{\frac{\rho-\gamma}{1-\gamma}}.$$

[6] See the critique of Roll 1977

- When $\rho = 1$ with $\gamma > 1$, the agent still exhibits a concern for long-run risk. However, the form of the SDF simplifies and can be written in the previously described form,

$$\log S_{t+1} - \log S_t = \mu_s + U_s' x_t + \lambda_s' w_{t+1},$$

  where μ_s, U_s, and λ_s are functions of the model parameters G, H, $\mu_{\mathcal{C}}$, $U_{\mathcal{C}}$, $\lambda_{\mathcal{C}}$.

- Note that $\psi_s(1) = \lambda_s$ defines risk-prices.

- Convenient to look at the short horizon compared to the limiting long-horizon case: Slope is negative when $0 > -\psi_s(1) \cdot (\psi_d(\infty) - \psi_d(1))$.

$$\begin{aligned}
&- \psi_s(1) \cdot (\psi_d(\infty) - \psi_d(1)) = \\
&\quad = \gamma\eta \underbrace{\lambda_{\mathcal{C}} \cdot (U_{\mathcal{C}}'(I - G)^{-1} H)}_{\text{Cov. of contemp } \mathcal{C}_t \text{ and transitory}} + (\gamma - 1)\eta \underbrace{\beta U_{\mathcal{C}}'(I - \beta G)^{-1} H H'(I - G')^{-1} U_{\mathcal{C}}.}_{\approx \text{ Var. of transitory}} \\
&\quad \approx \gamma\eta \operatorname{Cov}_t(\log \mathcal{C}_{t+1} - \log \mathcal{C}_t, -F_{\mathcal{C}} x_{t+1}) + (\gamma - 1)\eta \operatorname{Var}_t(-F_{\mathcal{C}} x_{t+1}) \qquad (17)
\end{aligned}$$

- This approximation is exact when $\beta \to 1$.

- As we see, a downward slope here requires that contemporaneous consumption growth shocks are positively correlated with the transitory component of consumption growth. That is, a positive shock to consumption today must be associated with a degree of mean reversion in the future.

- Recall that $\gamma = 1$ is the log utility case.

3 A multi-sector production network model

I use a standard multi-sector production model, amended so that households have recursive preferences of the Epstein-Zin variety (Epstein and Zin, 1989). The production and investment technology specification is otherwise standard, as in Foerster, Sarte, and Watson (2011).

3.1 Model Description

Consider an economy with n distinct industries, indexed $i = 1, \ldots, n$. Each industry produces a quantity Q_{it} of a distinct good. Industries have Cobb-Douglas production technologies with constant returns to scale, transforming intermediate goods, capital, and labor into a new product. The gross output of good i is

$$Q_{i,t} = \exp(\xi_{i,t}) K_{it}^{a_i^k} L_{it}^{a_i^\ell} M_{it}^{a_i^m}, \quad i = 1, \ldots, n, \tag{18}$$

where $\xi_t = (\xi_{1,t}, \ldots, \xi_{n,t})'$ is the vector of log total factor productivity associated with each sector, with capital K_{it}, labor L_{it}, and M_{it} a bundle of intermediate goods used in the production of good i at time t. a_i^k, a_i^ℓ, and a_i^m are fixed parameters and, since the production function features constant returns to scale, $a_i^k + a_i^\ell + a_i^m = 1$.

In each sector i, the capital stock follows the law of motion,

$$K_{i,t+1} = I_{it} + (1-\delta) K_{it},$$

where I_{it} is a bundle of investment goods used in sector i and δ is the depreciation rate common to all sectors.

The bundle of intermediates goods used by i is a an aggregation of goods produced by other industries,

$$M_{it} = \prod_{j=1}^{n} M_{ijt}^{a_{ij}}.$$

When a_{ij} is higher, it means that good j is more important in producing good i. With respect to intermediate goods, a_{ij} characterizes the input-output linkages between sectors. With respect to intermediate goods, I summarize the input-output linkages between sectors with the matrix $\boldsymbol{A} = [a_i^m a_{ij}]$, which I refer to as the intermediate goods network input-output matrix, or just the *input-output matrix*.

The bundle of investment goods used in sector i is formed according to the constant returns to scale technology

$$I_{it} = \prod_{j=1}^{n} I_{ijt}^{\theta_{ij}},$$

with $\sum_{j=1}^{n} \theta_{ij} = 1$ and I_{ijt} as the quantity of good j used to produce investment in sector i at time t. I summarize these linkages between sectors with the matrix $\boldsymbol{\Theta} = [a_i^k \theta_{ij}]$, which is referred to as the *investment network matrix*.

The goods produced in each sector can be used as intermediate goods applied to the production to other goods, can be used towards the capital investments in a particular sector, or can be consumed. Thus, each sector is subject to the resource constraint,

$$C_{jt} + \sum_{i=1}^{n} M_{ijt} + \sum_{i=1}^{n} I_{ijt} = Q_{jt},$$

where C_{jt} denotes the quantity of good j that is consumed at time t by a representative household.

I assume that the economy has a representative household. This household divides time between labor allocated to the various industries, L_{it} for $i = 1, \ldots, n$, and leisure $\mathcal{L}_t$. The household consumes the n different goods C_{it}, which it aggregates with a Cobb-Douglas aggregator,

$$C_t = \prod_{i=1}^{n} C_{i,t}^{\alpha_i}, \tag{19}$$

with $\alpha = (\alpha_1, \ldots, \alpha_n)'$ and $1 = \sum_i \alpha_i$. This household has Epstein-Zin utility, given by the recursion

$$V_t = \{(1-\beta)(\mathcal{C}_t)^{1-\rho} + \beta[\mathcal{R}_t(V_{t+1})]^{1-\rho}\}^{1/(1-\rho)} \tag{20}$$

where $\mathcal{C}_t = \mathcal{L}_t^{1-s_c} C_t^{s_c}$ is a measure of per-period utility, $s_c \in [0,1]$ controls preferences for consumption relative to leisure, $\mathcal{R}_t$ is the certainty equivalent operator defined by

$$\mathcal{R}_t(V_{t+1}) \equiv \mathbb{E}[(V_{t+1})^{1-\gamma} \mid \mathcal{F}_t]^{1/(1-\gamma)},$$

ρ^{-1} is the elasticity of intertemporal substitution, and γ is the risk-aversion parameter. The households is endowed with H units of labor/leisure, so that is

$$H = \mathcal{L}_t + \sum_{i=1}^{n} L_{it}.$$

Finally, let TFP growth be distributed as a dynamic factor model,

$$\begin{aligned} \Delta\xi_{i,t+1} &= \mu_{\xi,i} + \beta_{ai} z_t + \beta_{ai}\varepsilon_{a,t+1} + \beta_{bi}\varepsilon_{b,t+1} + \varepsilon_{i,t+1} \\ z_{t+1} &= \phi_z z_t + \varepsilon_{z,t+1} \end{aligned} \tag{21}$$

where $\varepsilon_t = (\varepsilon_{1,t}, \ldots, \varepsilon_{n,t}, \varepsilon_{a,t}, \varepsilon_{b,t}, \varepsilon_{z,t})' \overset{\text{iid}}{\sim} \mathcal{N}(0, \Sigma)$ and $\mu_{\xi,i}$ is a set of constants. Furthermore, as a factor model, let $\varepsilon_{a,t}$, $\varepsilon_{b,t}$, and $\varepsilon_{z,t}$ be aggregate (common) shocks, whereas $\varepsilon_{i,t}$ are idiosyncratic, industry-specific shocks. That is, assume $\varepsilon_{i,t}$ is uncorrelated with $\varepsilon_{j,t}$ for all $i \neq j$ for $i, j \in 1, 2, \ldots, n$ and that and $\varepsilon_{i,t}$ is uncorrelated with $\varepsilon_{x,t}$, for $x = a, b, z$. $\varepsilon_{z,t}$ drives a slow-moving, transitory component to growth. While the derivations presented in this paper hold with a more general model for TFP, I assume this form in order illustrate several properties of asset prices in a multisector model in a simpler setting.

The competitive equilibrium of this economy is defined in the usual way. That is, a competitive equilibrium is a set of prices and quantities such that the representative household maximizes her utility while taking prices and the wage as given, the representative firm in each sector maximizes its profits while taking prices and the wage as given, and all markets clear. Since I assume no frictions, I solve this model via the social planner's problem.

In the following section, I define the risk prices and risk premia associated with aggregate vs idiosyncratic shocks. In Section 3.2, I examine the competitive equilibrium dynamics and risk prices via a first-order approximation around the non-stochastic balanced growth path. This approximated solution will be especially useful as its simple form will make econometric evaluation simpler. In Section 3.3, I will explore some qualitative features of the equilibrium in a simplified versions of the model. This more stylized environment will help to better understand the model's underlying mechanisms.

3.2 Equilibrium solution via log-linearization

To simplify the analysis of the model, assume that the elasticity of substitution $\rho^{-1} = 1$ and that $\gamma > 1$. Under this assumption, utility as characterized in equation (20) simplifies to

$$\log V_t = (1 - \beta) \log \mathcal{C}_t + \beta \log \mathcal{R}_t(V_{t+1}).$$

Though preferences under these assumptions simplify greatly, note that the assumption that $\gamma \neq \rho$ ensures that households still exhibit a concern for long-run risk.[7] I then examine equilibrium dynamics and asset prices by analyzing a first-order approximation around a balanced growth path. Though certainty equivalence applies to quantity dynamics under this approximation, assets still exhibit positive risk premia and equilibrium still imposes joint restrictions on quantity dynamics and the term structure of risk premia. Furthermore, since my analysis revolves primarily around output dynamics and consumption, I will abstract away from labor supply decisions by letting labor be supplied inelastically $L^*_{it} = L_i$.

Under these assumptions, the non-stochastic balanced growth path of the model is analytically tractable and a linear approximation of the first-order conditions and the resource constraints around this path yields a vector ARMA(1,1) model for sectoral

[7] Note that when $\rho = \gamma$, preferences collapse to CRRA utility and that when $\rho = \gamma = 1$, preferences become log-utility. However, here I assume that $\rho = 1$ and I will typically assume that $\gamma > 1$. Under this assumption, the result is still a non-time-additive von Neumann-Morgenstern utility function.

output growth,

$$\Delta q_{t+1} = \Phi \Delta q_t + \Pi_0 \Delta \xi_{t+1} + \Pi_1 \Delta \xi_t, \tag{22}$$

where $q_{it} = \log Q_{it}$ is log output in sector i, $q_t = [q_{it}]$ is the vector of output in each sector, and $\Delta q_{t+1} = q_{t+1} - q_t$ is the vector of log output growth. Φ, Π_0, and Π_1 are $N \times N$ matrices that depends only on the model parameters a^k, a^ℓ, a^m, A, Θ, δ, α, β, s_c, and γ. Furthermore,

$$\Delta \log \mathcal{C}_{t+1} = s_c \alpha' \Delta q_{t+1}, \tag{23}$$

where $\alpha = [\alpha_i]$.

The balanced growth path and approximation derivation is given in the appendix, in Section A.2. Note that the shocks w_{t+1} in (4) are orthogonal, with $w_{t+1} \sim \mathcal{N}(0, I)$. Since the shocks in $\varepsilon_{t+1} \sim \mathcal{N}(0, \Sigma)$ are not, the final step here is to choose an orthogonalization P, where

$$\varepsilon_{t+1} = P w_{t+1}. \tag{24}$$

Thus, by stacking and applying this orthogonalization, we can write the equilibrium solution as a linear state space model of the form (4).

The stochastic discount factor I now discuss the derivation of the stochastic discount factor (SDF). In equilibrium with $\rho = 1$, the SDF is

$$\frac{S_{t+1}}{S_t} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-1} \left[\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t[V_{t+1}^{1-\gamma}]} \right]$$

and can be written as $\log S_{t+1} - \log S_t = \mu_s + U_s' x_t + \lambda_s' w_{t+1}$, where μ_s, U_s, and λ_s are constants that depend on the model parameters. The proof of this is given in the appendix, in Section A.5. Also note that risk prices λ_s can be decomposed into two parts—a part capturing investor concern for long-run risk $\lambda_{s,LR}$ and myopic risk-prices $\lambda_{s,SR}$,

$$\lambda_s = \lambda_{s,SR} + (\gamma - 1)\lambda_{s,LR}. \tag{25}$$

This interpretation comes from the fact that when $\gamma = \rho = 1$, the Epstein-Zin utility functions collapses into log-utility and investors no longer exhibit a concern for long-run risk.

Thus, have shown that SDF takes on the form in (5) and that the state variables follow dynamics of the form in (4).

3.2.1 Term Structure of Equity in Equilibrium

I now turn to analyzing equilibrium asset prices. To derive the term structure of equity as outlined in Section 2, I need to define aggregate dividends within the model. As discussed previously, the price of the aggregate stock market, the *market portfolio*, equals the sum of discounted future dividend payments (Gordon, 1962). Similarly, a portfolio comprised of ownership of all capital within the economy—the *wealth portfolio*—amounts to a claim to all future consumption (and leisure value) (Lucas, 1978). Thus, using the market portfolio as a proxy for the wealth portfolio amounts to connecting the asset prices associated with a broad-based market index with the implied asset prices of claims to consumption in equilibrium. A common assumption in the macro-finance literature is to assume that aggregated dividends are proportional to consumption, taking into account the aggregate leverage of traded firms. Thus, I assume that aggregate dividends are equal to a levered consumption process,

$$\log D_t = \eta \log \mathcal{C}_t, \tag{26}$$

where $\eta \geq 1$ is the leverage factor.

We have now specified a cash flow process that satisfies (6). We have previously shown that SDF takes on the form in (5) and that the state variables follow dynamics of the form in (4). Given a dividends process that takes the form (6), the formulas for the term structure of holding period returns described in Proposition 3 apply to this solution of the model. Having established this, I will now derive the risk exposures associated with dividends and equilibrium risk prices.

Risk Exposures of Dividend Futures Given the simple expression for the risk premium associated with dividend futures, given in (11), as well as the corresponding data given in Figure 1, I will focus on these. Recall that the returns on the dividend futures have the interpretation as the return on a dividend strip in excess of the risk-free bond with the same maturity. I'll refer to this as the risk exposures associated with the returns on the dividend futures.

Recalling that the risk-exposures associated the dividend futures are equal to impulse response function of the dividend process, risk exposures in the very short term is

$$\psi_d'(1) = \eta \, s_c \alpha' \Pi_0 \Psi_\xi(1), \tag{27}$$

where $\Psi_\xi(1)$ is a matrix of impulse responses sectoral TFP at a one-period horizon, defined so as to satisfy

$$\Delta \mathbb{E}_{t+1}[\xi_{t+\tau}] = \Psi_\xi(\tau)\, w_{t+1}. \tag{28}$$

Under this definition, entry (i, j) of $\Psi_\xi(\tau)$ represents the impulse response of TFP of sector i to a unit impulse to the j'th shock at horizon τ. Note that $\alpha'\Pi_0$ is a vector and that η and s_c are scalars. From this we can see that the one-period risk exposures are a set of weighted sums, where the risk exposure associated with each shock is a weighted sum of the impulse response functions of TFP, weighted by the vector $\alpha'\Pi_0$. Substituting so as to see this more explicitly, we get

$$
\begin{aligned}
\psi_d'(1) &= \eta\, s_c \underbrace{\alpha'\Pi_0}_{\substack{\text{sector}\\\text{weights}}} \underbrace{\Psi_\xi(1)}_{\substack{\text{TFP}\\\text{IRFs}}} \\
&= \eta\, s_c \alpha'\Pi_0 \left[\underbrace{\operatorname{diag}\begin{pmatrix}\sigma_1\\ \vdots \\ \sigma_n\end{pmatrix}}_{\varepsilon_{it} \text{ for all } i} \quad \underbrace{\sigma_a \begin{bmatrix}\beta_{a1}\\ \vdots \\ \beta_{an}\end{bmatrix}}_{\varepsilon_{at}} \quad \underbrace{\sigma_b \begin{bmatrix}\beta_{b1}\\ \vdots \\ \beta_{bn}\end{bmatrix}}_{\varepsilon_{bt}} \quad \underbrace{\begin{bmatrix}0\\ \vdots \\ 0\end{bmatrix}}_{\varepsilon_{zt}} \right],
\end{aligned}
\tag{29}
$$

where σ_i for $i = 1, \ldots, n$ is the volatility of the idiosyncratic shocks, ε_{it}, σ_a is the volatility of the first common factor ε_{at}, and σ_b is the volatility of the second, ε_{bt}, defined in the TFP growth process in (21). Thus, the risk exposure associated with the idiosyncratic shocks is just σ_i multiplied by the sector's weight given in the vector $\eta\, s_c\alpha'\Pi_0$. The risk exposures associated with the common, aggregate shocks is a weighted sum of the factor loadings and the weights $\alpha'\Pi_0$ and scaled by scalars η and s_c. For the shock ε_{at}, this is a weighted sum of the β_{at}, multiplied by the volatility σ_a. Since the slow moving component governed by z_t has no instantaneous effect, it's risk exposure is zero at the short horizon.

The long-term risk exposures, defined by

$$
\psi_d(\infty) \coloneqq \lim_{\tau\to\infty} \psi_d(\tau),
$$

take on a slightly more complicated form. This expression is developed in the appendix. However, the expression is simplified if we assume $\Delta\xi_{it}$ to be i.i.d. This amounts to supposing that the slow-moving component z_t in (21) is shut-off, setting $\varepsilon_{z,t} = 0$ for all t. Under this assumption,

$$
\psi_d'(\infty)\Big|_{\Delta\xi_t \text{ iid}} = \eta\, s_c\alpha'(I-\Phi)^{-1}\Pi_0\Psi_\xi(\infty)\Big|_{\Delta\xi_t \text{ iid}} \tag{30}
$$

$$
= \eta\, s_c\alpha'(I-\Phi)^{-1}\Pi_0 \left[\operatorname{diag}\begin{pmatrix}\sigma_1\\ \vdots \\ \sigma_n\end{pmatrix} \quad \sigma_a \begin{bmatrix}\beta_{a1}\\ \vdots \\ \beta_{an}\end{bmatrix} \quad \sigma_b \begin{bmatrix}\beta_{b1}\\ \vdots \\ \beta_{bn}\end{bmatrix} \quad \begin{bmatrix}0\\ \vdots \\ 0\end{bmatrix}\right], \tag{31}
$$

where $\Psi_\xi(\infty) = \lim_{\tau\to\infty} \Psi_\xi(\tau)$. The interpretation here is similar to the short-term exposures, except here the impulse responses of sectoral TFP are weighted by the vector $\alpha'(I-\Phi)^{-1}\Pi_0$ rather than $\alpha'\Pi_0$. I will argue later that these weights can be interpreted as a measure of sectoral centrality within the production networks and that the short-term weights are governed largely by centrality within the intermediate goods network and that the long-term weights are tilted towards centrality in the investment network. Note that these weights do not sum to one. Though I will later discuss the interpretation of these weights as a measure of centrality, I define them now as such.

Definition 7. I call the weights vector $\alpha'\Pi_0$ in the short-term case the *short-term centrality* vector and the weights vector $\alpha'(I-\Phi)^{-1}\Pi_0$ in the long-term case the *long-term centrality* vector.

The weights in the intermediate term can be interpreted as interpolating these two extremes. This difference in the weights over the time leads to different risk exposures and, thus, different risk premia on dividend futures at each horizon.

Before continuing, I also provide another convenient derivation. Suppose we restore the slow-moving component z_t in (21). Then, if we assume that $\Pi_1 = \mathbf{0}$, then the long-term risk exposures also take on a convenient form:

$$\psi_d'(\infty)\Big|_{\Pi_1=\mathbf{0}} = \eta\, s_c \alpha'(I-\Phi)^{-1}\Pi_0 \Psi_\xi(\infty)\Big|_{\Pi_1=\mathbf{0}} \tag{32}$$

and

$$\Psi_\xi(\infty) = \left[\underbrace{\operatorname{diag}\begin{pmatrix}\sigma_1\\ \vdots \\ \sigma_n\end{pmatrix}}_{\varepsilon_{it}\text{ for all } i} \quad \underbrace{(\sigma_a + \rho_{za}\sigma_z(1-\phi_z)^{-1})\begin{bmatrix}\beta_{a1}\\ \vdots \\ \beta_{an}\end{bmatrix}}_{\varepsilon_{at}} \quad \underbrace{\sigma_b\begin{bmatrix}\beta_{b1}\\ \vdots \\ \beta_{bn}\end{bmatrix}}_{\varepsilon_{bt}} \quad \underbrace{\left(\sigma_z(1-\phi_z)^{-1}\sqrt{1-\rho_{za}^2}\right)\begin{bmatrix}\beta_{a1}\\ \vdots \\ \beta_{an}\end{bmatrix}}_{\varepsilon_{zt}} \right], \tag{33}$$

where I have orthogonalized the shocks, ordering ε_{at} first and ρ_{za} is the correlation between ε_{at} and ε_{zt}. From this we can make the following observation about the term structure. If we want to impose a downward sloping term structure of equity to work through lower risk exposures in the long term relative to the short term, then we can do so in two ways. From (29) and (31), we see that we can achieve lower risk exposures via the changing sector weights, changing from $\alpha'\Pi_0$ to $\alpha'(I-\Phi)^{-1}\Pi_0$. If the long-term more heavily weights sectors with lower factor loadings, then these exposures will decrease. From (33) we see that we can also get decreasing risk exposures by imposing a negative correlation between the permanent shocks ε_{at} and the slow-moving component z_t. By setting $\rho_{za} < 0$, the effects of ε_{at} will shrink over time.

Note that, though the equilibrium solution given in equation (22) admits a VARMA(1,1), the parameters of the model are not easily interpretable in terms of the model primitives. In the Section 3.3, I consider a simpler case of the model that produces analytical expressions Φ, Π_0, and Π_1 that will shed additional light on the nature of the sectoral weights discussed here. However, before moving to this simpler case, I proceed with a discussion of the risk prices.

Risk prices I now turn my attention to deriving risk prices in equilibrium. I will derive the two components discussed in (25). The expression for the myopic risk prices, $\lambda_{s,SR}$, share proportionally the same form as the short-term risk exposures,

$$\lambda'_{s,SR} = -s_c \alpha' \Pi_0 \Psi_\xi(1) = -\frac{1}{\eta}\psi'_d(1). \tag{34}$$

The risk prices arising from a concern for long-run risk are approximately equal to the long-run risk exposures. If we suppose $\Delta\xi_t$ is i.i.d. as we did before, then

$$\lambda'_{s,LR} = -s_c \alpha'(I - \beta\Phi)^{-1}\Psi_(\infty)\Big|_{\Delta\xi_t \text{ iid}}. \tag{35}$$

Otherwise, we can simply state that

$$\lambda_{s,LR} \approx -\frac{1}{\eta}\psi_d(\infty), \tag{36}$$

where the approximation is exact in the limit where the subjective discount factor goes to one, $\beta \to 1$.

Note that in calculating the risk premium associated with dividend futures, the risk prices do not change with the horizon. Rather, the risk prices λ_s as expression in (25) are a linear combination of $\lambda_{s,SR}$ and $\lambda_{s,LR}$. When γ is larger, the investor puts more weight into the long-term risk prices. As we see in (12), a changing Sharpe ratio over the term structure is due to risk exposures changing, tilting towards sources of risk with a higher or lower risk price.

I now turn to a special case of equilibrium that will help to interpret these expressions.

3.3 The special case of full depreciation

In this section, I explore the special case that admits a closed form solution that will allow for a clearer characterization of equilibrium prices and quantities. In addition to assuming that $\rho = 1$, as we did in the previous section, I will assume that capital depreciates fully after one period, $\delta = 1$. Note that in this case, I do not need to assume that labor is inelastically supplied. The absence of capital ensures that labor supply is constant. I will discuss the dynamics of sectoral output in this case as well as the resulting asset prices, including the term structure of equity. This setting will allow me to cleanly characterize short-run and long-run centrality in terms of the intermediate goods and investment networks.

Equilibrium in this special case is described in Proposition 8, the proof of which is given in the appendix, Section A.2.3.

Proposition 8. *Suppose $\rho = 1$ and $\delta = 1$. Let q_t be the vector of log-output at time t such that $q_{it} = \log Q_{it}$ and let ξ_t be the vector of log TFP shocks, $\xi_{it} = \log \Xi_{it}$. Let Δ be the difference operator, so that $\Delta q_{t+1} = q_{t+1} - q_t$. Then, output growth in equilibrium must satisfy*

$$\Delta q_{t+1} = (I - A)^{-1}\Theta\Delta q_t + (I - A)^{-1}\Delta\xi_{t+1}, \tag{37}$$

where $A = [a_i^m a_{ij}]$ and $\Theta = [a_i^k \theta_{ij}]$. Equilibrium leisure and labor is constant and, furthermore,

$$\begin{aligned} \Delta \log \mathcal{C}_{t+1} &= s_c \alpha' \Delta q_{t+1} \\ &= s_c \alpha' (I - A)^{-1}\Theta\Delta q_t + s_c \alpha' (I - A)^{-1}\Delta\xi_{t+1}, \end{aligned} \tag{38}$$

where $\alpha = [\alpha_i]$.

As can be seen, the resulting dynamics can be considered a special case of form of the dynamics seen previously in equation (22), with $\Phi = (I - A)^{-1}\Theta$, $\Pi_0 = (I - A)^{-1}$, and $\Pi_1 = 0$. In this special case, we can see that the investment network's role in determining the relationship between output growth today and output growth tomorrow. When capital plays no role in production, $a_i^k = 0$, this intertemporal relationship disappears, with $\Theta = \mathbf{0}$. Under that assumption, output growth in previous periods would not predict future output growth and propagation through the intermediate goods network happens instantaneously,

$$\Delta q_{t+1} = (I - A)^{-1}\Delta\xi_{t+1}. \tag{39}$$

Evidence of this instantaneous propagation can be seen in the multiplier on log TFP growth, as

$$(I - A)^{-1} = I + A + A^2 + A^3 + \dots,$$

where the term A represents the effect after one degree of separation, A^2 represents the effect after two degrees of separation, etc. All of these effects occur simultaneously, leading to (39). In contrast, shocks propagating through the investment network propagate gradually over time. Similarly, if we to omit the intermediate goods network with $A = \mathbf{0}$, then we can simplify (37) so that gradual propagation through the investment network would manifest as

$$\Delta q_{t+1} = (I - \Theta L)^{-1} \Delta \xi_{t+1}, \tag{40}$$

where L here is the lag operator. Evident from,

$$(I - \Theta L)^{-1} = I + \Theta L + \Theta^2 L^2 + \Theta^3 L^3 + \dots,$$

the term representing the effects of after one degree of separation, Θ, occurs only after one period has passed and the term representing the effects after two degrees, Θ^2, occurs occurs after two periods has passed, etc. This difference in the role of each network in the propagation of shocks results in different implications for asset prices and the term structure of risk premia.

3.3.1 Risk Prices and Risk Exposures

Given that this simplified version of the model is a special case, I can leverage the derivations from Section 3.2.1. I present the risk exposures in this simplified case as Corollary 9. The proof of this is given in the appendix, in Section 3.3.

Corollary 9. *When $\delta = 1$, the*

- *risk exposures for the short-horizon dividend futures are given by*

$$\psi_d'(1) = \eta \, s_c \alpha' (I - A)^{-1} \Psi_\xi(1), \tag{41}$$

  where the sector weights $\alpha'(I - A)^{-1}$ depend on centrality in the intermediate goods network. This is the short-run centrality *vector.*

- *The risk exposures for the long-horizon risk exposures are*

$$\psi_d'(\infty) = \eta \, s_c \alpha' (I - (A + \Theta))^{-1} \Psi_\xi(\infty),$$

  where the sector weights $\alpha'(I - (A + \Theta))^{-1}$ depend on the sum of the intermediate goods network and investment network matrices. This is the long-run centrality *vector.*

Centrality Interpretation Within the network theory literature, *alpha centrality* is a measure of a node's centrality within a network. It determines the centrality of a node by computing a weighted sum of the centrality of its neighbors, weighted by the strength of the connection between its neighbors, and then adding some baseline level of "centrality" to each node. For example, let $c(\alpha, A)$ be the alpha centrality of the intermediate goods network as represented by the matrix A where α is the vector of baseline centrality given to each industry. Then, by definition, the alpha centrality is characterized as the vector c that solves

$$\alpha' + c'A = c'.$$

Since

$$c(\alpha, A) = \alpha'(I - A)^{-1}$$

solves this equation, $\alpha'(I-A)^{-1}$ is the alpha centrality of the network A. Recall that α is the vector of Cobb-Douglas shares in the consumption aggregator, so this expression measures the centrality of industries within the intermediate goods network, weighted by the importance of each industry within the consumption aggregator. With this in mind, we also see that long-run centrality in this case

$$c(\alpha, A + \Theta) = \alpha'(I - (A + \Theta))^{-1}$$

measures centrality within a network formed by summing the shares of the intermediate goods network A and the investment network Θ. Thus, the centrality measure in the long-run is tilted towards giving more weight to industries that are central within the investment network.

Risk exposures in the intermediate term Given the cyclicality built into TFP growth $\Delta\xi_t$ and its interaction with the autoregressive form for output relative to TFP, the risk exposures of the dividend process (the impulse response function) take on a somewhat complicated form. However, when $\Delta\xi_t$ is i.i.d., this is greatly simplified. In this case, the risk exposures in the intermediate term are

$$\psi_d'(\tau)\Big|_{\Delta\xi_t \text{ iid}} = \eta\, s_c \alpha'(I - \Phi)^{-1}(I - \Phi^\tau)\Pi_0 \Psi_\xi(\tau)\Big|_{\Delta\xi_t \text{ iid}} \tag{42}$$

where $\Phi = (I - A)^{-1}\Theta$ and $\Pi_0 = (I - A)^{-1}$. This also has an interpretation similar to alpha centrality. Alpha centrality has a recursive definition but can be interpreted as counted walks that are discounted by distance. Since

$$(I - \Phi)^{-1}(I - \Phi^\tau) = I + A^2 + \ldots + A^{\tau-1},$$

the intermediate term risk exposures can be though of as a truncated form of alpha centrality which counts walks with of length $\tau - 1$ or shorter.

	a_{ij}		θ_{ij}	
Sector i, j	1	2	1	2
1	.9	.1	.1	.9
2	.9	.1	.1	.9

Table 2: Intermediate goods network and investment network for two-sector example. Let the cost shares in the intermediate goods network, a_{ij}, be defined so that industry 1 is an intermediate goods hub and industry 2 is an investment hub. Let $a_i^m = .4$, $a_i^k = .4$, $a_i^\ell = .2$.

Risk Prices As demonstrated in equations (34) and (36), the short-run and long-run risk exposures can also be used to describe equilibrium risk prices. Substituting the expressions derived in Corollary 9 gives these risk prices a convenient characterization in the full depreciation model. Again, the depend on the impulse responses of TFP, weighted by short-run and long-run centrality. When $\gamma = 1$ and the utility function collapses into log utility, the risk prices depend only on short-run centrality. As γ becomes larger, risk prices depend more on long-run centrality.

The composition of these risk prices and risk exposures determine the risk premia that make up the term structure of equity.

3.3.2 Imposing a downward sloping term structure

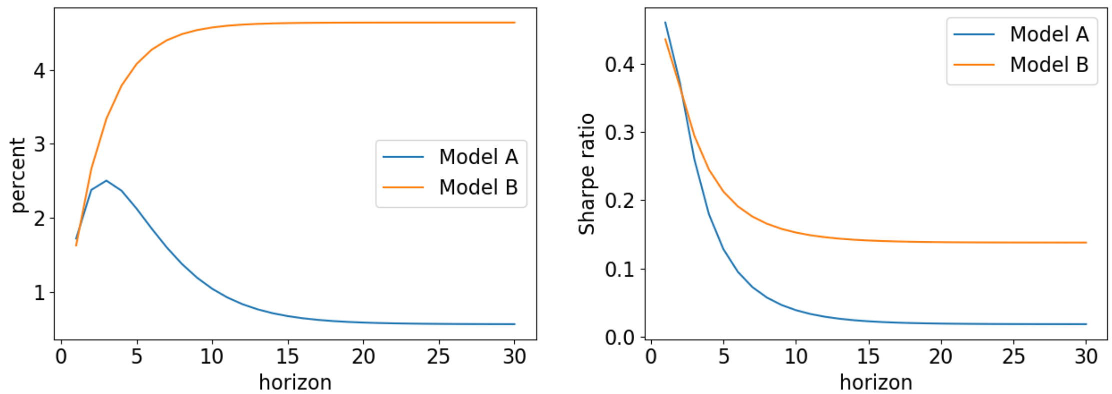
I now consider the conditions we must impose on the distribution of TFP in (21) in order to obtain a downward sloping term structure over the long-run. I will show that in a single-sector model, the only way to obtain this is to imposes mean-reversion in the level of TFP. In (21), an example of this would be to imposing a negative correlation between ε_{zt} and ε_{at}. In a multisector model such as this one, a downward sloping term structure can also be achieved by imposing a negative correlation between shocks to intermediate goods hubs and investment hubs. I will illustrate this in a simple, two-sector model.

Consider a two-sector example where 1 sector is a clear intermediates goods hub and the other is an investment hub, with the intermediate goods network and investment network defined in Table 2. In the following examples, let the model have the parameters defined in Table 3, with $\gamma = 10$.

Mean-reverting Component Delivers Downward Sloping Term Structure In this example, suppose that we reinsert the autoregressive common factor, keeping all other parameters the same as before. Let $\beta_{ai} = 0.2$, $\text{Std}(\varepsilon_{za}) = 0.02$, and $\phi_z = 0.7$.

Sector i	$\mu_{\xi,i}$	β_{zi}	β_{bi}	$\mathrm{std}(\varepsilon_{i,t})$	α_i
1	0.005	1	0	0.02	0.5
2	0.005	1	0	0.02	0.5

Table 3: Model parameters of two-sector example.



(a) Expected annual returns on dividends futures.

(b) Sharpe ratio on dividends futures

Figure 3: A model with mean reversion in TFP delivers a downward sloping term structure of equity.

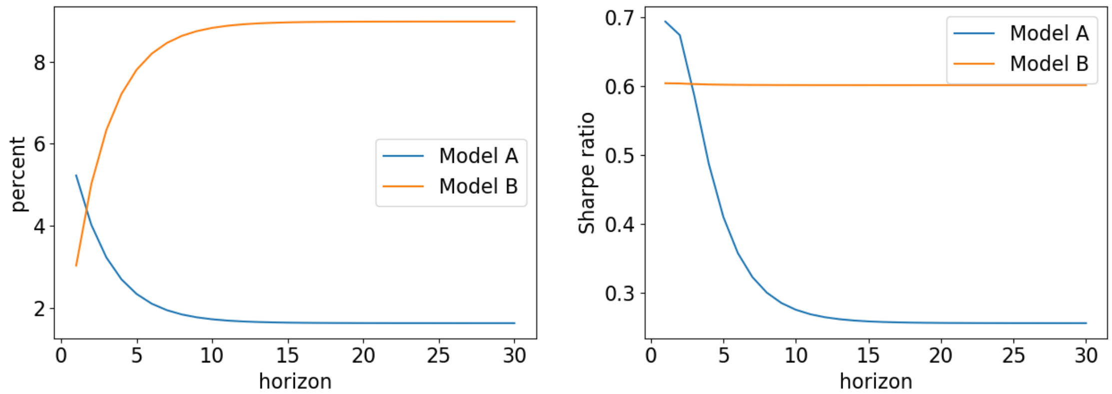
Then, consider alternatives, Model A and Model B.

- Model A: Negative correlation with transitory component, $\mathrm{Corr}(\varepsilon_{at}, \varepsilon_{zt}) = -0.2$
- Model B: Positive correlation with transitory component, $\mathrm{Corr}(\varepsilon_{at}, \varepsilon_{zt}) = 0.2$

As illustrated in Figure 3, mean reversion as determined by $\mathrm{Corr}(\varepsilon_{at}, \varepsilon_{zt}) = -0.2$ delivers a downward sloping term structure of equity.

Negative Correlation of Intermediate Goods Hubs and Investment Hubs Delivers Downward Sloping Term Structure In this example, remove the autoregressive common factor so that all shocks are permanent, i.i.d. shocks. Let $\mathrm{Std}(\varepsilon_{at}) = 0.15$, and $\mathrm{Std}(\varepsilon_{it}) = 0.02$. Consider again two alternatives, Model A and Model B:

- Model A: Negatively correlated sectors, $\beta_{a1} = 1$, $\beta_{a2} = -1$,



(a) Expected annual returns on dividends futures.

(b) Sharpe ratio on dividends futures

Figure 4: A model with negative correlation between intermediate goods hubs and investment hubs delivers a downward sloping term structure of equity.

- Model B: Positively correlated sectors, $\beta_{a1} = 0.2$, $\beta_{a2} = 0.2$,

Model A features a negative correlation between the investment hub and the intermediate goods hub and, thus, exhibits a downward sloping term structure, illustrated in Figure 4.

4 Empirical Evidence

In this section, I take the simplified, full depreciation model of Section 3.3 and the benchmark model of Section 3.2 to the data. In each case, the empirical exercise takes on the following steps:

1. **Model Filter:** Estimate sectoral TFP growth from the implied model filter, following the procedure developed by Foerster, Sarte, and Watson (2011). For example, in the full depreciation case this involves solving $\Delta\xi_t$ in terms of lags of Δq_t in (37).

2. **Fit Factor Model:**With sectoral TFP growth $\Delta\xi_t$ recovered, estimate the panel of sectoral TFP as a linear factor model with latent common factors. This estimates the degree to which comovement in TFP is driven by common, "aggregate" shocks, relative to idiosyncratic movements.

3. **Factor Loadings Rotation:** The factor loadings recovered in the previous step are only identified up to an orthogonal rotation. That is, we may choose an alternative rotation of the factor loadings to help us to interpret the fit of the model. Conveniently, a two-factor model appears to provide the best fit in both cases. I therefore choose a rotation so that one factor has no long-run impact on aggregate output. This factor will be called the "shift shock." Almost surely, there are two such rotations to satisfy this restriction. I therefore choose the rotation that sets the other factor has a positive long-run impact on aggregate output.

4. **Implied Term Structure (decomposed by shock):** After choosing a particular rotation of the factor loadings, I can then measure the risk exposures and, thus, the risk premia associated with each shock at each point in time. This allows me to express how much each shock contributes to the term structure of equity at each horizon in terms of financial returns. We will see that the shift shock contributes a downward sloping term structure. All other shocks tend to contribute to an upward sloping term structure. The size of the shift-shock is no large enough to imply that the aggregate term structure is downward sloping.

Data Description In my main analysis, I use data from the BEA IO Tables and BEA Capital Flow tables, the BEA Industry Accounts, and Dale Jorgenson's KLEMS data set. Following the same or similar procedures as used elsewhere in the production networks literature, such as in Atalay (2017) and vom Lehn and Winberry (2019), I use these to measure the empirical intermediate goods network and investment network; labor, investment, and intermediate goods shares; and consumption shares. The BEA IO Tables and the Capital Flows tables are used to construct the networks and the factor shares in production. I use Dale Jorgenson's 35-sector KLEMS data set and the BEA Industry Accounts, following the procedure outlined in Atalay (2017), to produce a measure of sectoral output growth over the years 1960–2013. I describe this data in more detail in the Appendix, in section B.

Model Filter In order to estimate sectoral TFP, I follow the procedure of Foerster, Sarte, and Watson (2011). In the case of the full-depreciation model, I solve (37) for TFP growth, giving

$$\Delta \xi_{t+1} = (I - A)\Delta q_{t+1} - \Theta \Delta q_t.$$

Since output growth is observed and the intermediate goods network and investment good network can be estimated from the BEA Input-Output and Capital Flows

Factor num.	1	2	3	4	5	6	total
$\Delta\xi_{it}$	0.31	0.23	0.04	0.04	0.03	0.03	0.67
Δq_{it}	0.58	0.06	0.04	0.03	0.02	0.02	0.76

Table 4: The proportion of total sample variance explained by the k-th factor, R^2, for the series $\Delta\xi_{it}$ (TFP growth) and Δq_{it} (output). This uses output data over the sample of 1960–2013 for the benchmark model.

tables, we have everything we need to solve for $\Delta\xi_t$. The process is similar for the benchmark model, except that the coefficients on output growth are a function of other model parameters.

This step is important because, as emphasized by Foerster, Sarte, and Watson (2011), a factor analysis of Δq_{t+1} would overestimate the importance of common factors because of the way that shocks are propagated through the production networks.

Factor analysis of TFP growth I now estimate a statistical factor model of the panel of TFP growth. Throughout, I will estimate this factor model using maximum likelihood. In Table 4 I report the proportion of sample variance explained by each factor when a 6-factor model is used. I do this for the sample of 1960–2013 using TFP measured in the benchmark case. The results using the full-depreciation case are similar. As we can see, a great majority of the variation is explained by the first two factors. For this reason, I will proceed using only a two-factor model, as in

$$\Delta\xi_{i,t+1} = \mu_{\xi,i} + \beta_{ai}\varepsilon_{a,t+1} + \beta_{bi}\varepsilon_{b,t+1} + \varepsilon_{i,t+1}, \tag{43}$$

with mean zero, normally distributed shocks $\varepsilon_{i,t}$ and $\varepsilon_{x,t}$ that are all mutually uncorrelated for $x = a, b$ and all $i = 1, \ldots, n$.

Factor Loading Rotation I now want to consider a rotation of the factor loadings such that the shock ε_{at} has a positive long-run effect of aggregate output while ε_{bt} has a zero long-run effect. Solving for this rotation is easy since the long-run impact can be determined by the weighted sum of the factor loadings, weighted by the long-run centrality scores described in Section 3.2.1.

After solving for this particular rotation, I summarize new configuration in Table 5. Each shock is assumed to have unit standard deviation. For each shock, I report the the proportion of total TFP variation that can be explained by the shock and the

Table 5: Summary of Common Shocks

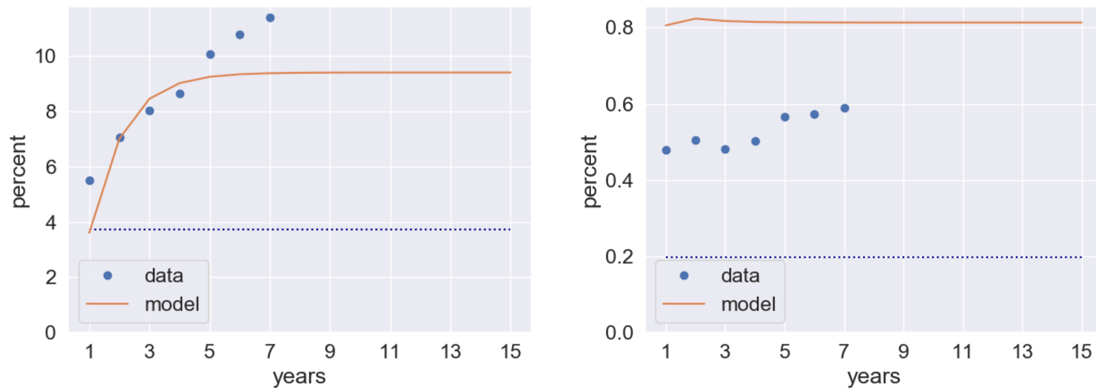
Model	Common Factor	Proportion of total variance (R^2)	Short-run Impact	Long-run Impact
benchmark	$\varepsilon_{a,t}$	0.15	-0.02	0.09
	$\varepsilon_{b,t}$	0.39	-0.02	0.00
full depreciation	$\varepsilon_{a,t}$	0.30	0.02	0.05
	$\varepsilon_{b,t}$	0.08	0.00	0.00

Here I summarize the analysis of the factor loadings.

short-run (one-period) and long-run impact on consumption. In the full-depreciation case, the shift shocks, ε_{bt}, account for 8% of the variation in TFP but has almost no short-run or long-run impact. In the benchmark case, the shift shocks appear to explain nearly 40% of the total variance of TFP growth. A shift has a small short-run impact of consumption. The common growth shock, ε_{at}, accounts for about 15% of TFP growth and has a large long-run impact on consumption.

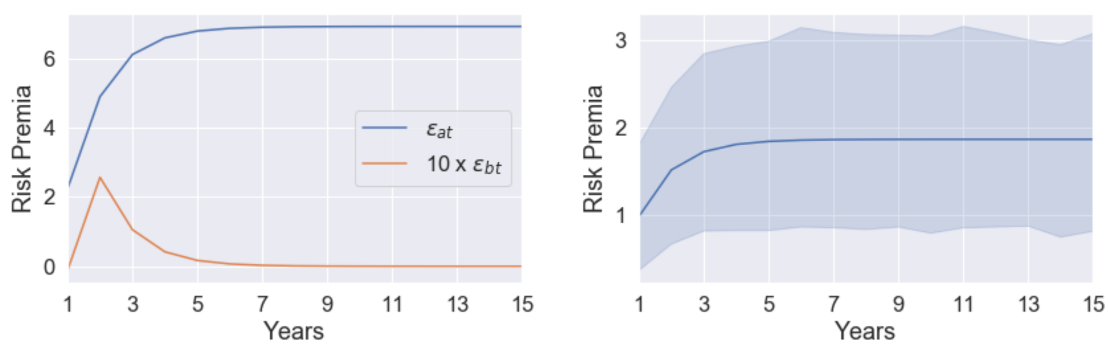
Implied Term Structure of Equity Here I report the model's implied term structure of equity. Table 5 reports the model's (full-depreciation version) predicted returns and Sharpe ratios for $\gamma = 25$, $\eta = 2$, $s_c = 0.9$, and $\beta = 0.95$.

In Table 6, I decompose the model-implied expected returns into the contributions from each shock. In Panel (a), the blue line is the contribution of the shock ε_{at}. The orange line is the contribution of the shift shock, ε_{bt}. However, I have multiplied the effect by 10 so that it is more visible. Otherwise, as we can see, the effect is small. In Panel (b), I report the average contributions of the 30 sectoral shocks. I've multiplied the effect by 30, so that the blue solid line reflects the total contribution of the idiosyncratic sectoral shocks. As we can see, the sectro



(a) Expected annual returns on dividends futures. (b) Sharpe ratio on dividends futures

Figure 5: This plots the model's implied term structure of equity for $\gamma = 25$, $\eta = 2$, $s_c = 0.9$, and $\beta = 0.95$. For this parameterization, the model appears to match the rising expected returns in the short term. However, The term structure does not bend back down as we would hope. Sharpe ratios, also, are mostly flat.



(a) Contribution of common shocks ε_{at} and ε_{bt} to the term structure of equity returns. (b) Contribution of the idiosyncratic sectoral shocks to the term structure of equity returns.

Figure 6: These figures decompose the expected returns on dividend futures at each horizon into the contributions coming from each source of uncertainty. As we can see, only the shift shock, ε_{bt}, appears to contribute to a negatively sloped term structure. However, it is not nearly large enough to overcome the effects of the other shocks.

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6 Figures and Tables

Table 6: Summary Statistics for Industry Output, including Customer and Supplier Industries

Pooled panel data	count	mean	std	min	25%	50%	75%	max
y_t (Real output in 1997 dollars)	19438	8604.977	97676.108	10.809	1048.119	2464.364	5157.131	5099306.008
$\Delta \log y_t$	19076	0.016	0.123	-1.095	-0.038	0.021	0.077	1.669
Ave. Supplier $\Delta \log y_t$	19076	0.012	0.049	-0.393	-0.007	0.012	0.039	0.458
Ave. Customer $\Delta \log y_t$	19076	0.011	0.054	-0.498	-0.003	0.006	0.030	0.552
Within industry medians	count	mean	std	min	25%	50%	75%	max
y_t (Real output in 1997 dollars)	362	5171.372	11082.287	61.218	1211.018	2649.236	4983.316	127060.745
$\Delta \log y_t$	362	0.022	0.025	-0.048	0.008	0.021	0.034	0.210
Ave. Supplier $\Delta \log y_t$	362	0.015	0.012	-0.005	0.009	0.015	0.020	0.096
Ave. Customer $\Delta \log y_t$	362	0.014	0.015	-0.012	0.004	0.011	0.022	0.126
Within industry standard deviations	count	mean	std	min	25%	50%	75%	max
y_t (Real output in 1997 dollars)	362	8381.943	86413.213	61.973	394.279	907.158	1956.114	1260155.979
$\Delta \log y_t$	362	0.113	0.044	0.032	0.084	0.105	0.136	0.289
Ave. Supplier $\Delta \log y_t$	362	0.044	0.019	0.001	0.032	0.046	0.057	0.105
Ave. Customer $\Delta \log y_t$	362	0.041	0.034	0.000	0.012	0.035	0.061	0.183

Summary statistics for manufacturing industries within the NBER-CES Manufacturing Industry Database merged with network data from the Bureau of Economic Analysis 1987 Benchmark Input-Output tables. In the first panel, I compute summary statistics across the pooled observations within the panel. In the second and third, I compute the median and standard deviation over time within industries, and then compute summary statistics across industries. Supplier averages are weighted by the expenditures shares (fraction of expenditures going to supplier) and customer averages are weighted by revenue shares (fraction of revenue accounted for by customer). Output here is the real value of total shipments of each industry.

Table 7: Industry Output Dynamics

	$y_{it} = \Delta \log(\text{Output}_{it})$			
	(1)	(2)	(3)	(4)
$(A\, y_{t-1})_i$ (suppliers, 1-step)	0.298**	0.117	0.213**	0.289***
	(0.146)	(0.196)	(0.090)	(0.100)
$(\widehat{A}'\, y_{t-1})_i$ (customers, 1-step)	0.109	0.030	0.067	0.002
	(0.151)	(0.126)	(0.123)	(0.105)
$(A^2\, y_{t-1})_i$ (suppliers, 2-steps)		0.523		−0.285
		(0.578)		(0.233)
$((\widehat{A}^2)'\, y_{t-1})_i$ (customers, 2-steps)		0.173		0.166
		(0.157)		(0.129)
Time FE	No	No	Yes	Yes
Firm FE	Yes	Yes	Yes	Yes
Observations	8,932	8,932	8,932	8,932
R^2 (within)	0.0096	0.0103	0.0026	0.0029

Note: *p<0.1; **p<0.05; ***p<0.01

Estimation using manufacturing output data from the NBER-CES Manufacturing Industry Database over the years 1987 to 2011 and the BEA Input-Output Benchmark Table from 1987. Total output here is the real value of total shipments of each industry. To account for serial and spatial dependence, standard errors are constructed following Driscoll and Kraay (1998). $A = [a_{ij}]$ represents the fraction of total expenditures that industry i spends on industry j. Given the vector of total output of each industry, $A\, q_{t-1}$ represents the vector of weighted sums of the change in output of the industry's suppliers, weighted by the industry's expenditure shares. $(A\, q_{t-1})_i$ is the i'th element of the resulting vector. $A^2\, q_{t-1}$ is similar, but is weighted by the expenditure shares that the industry implicitly spends on its suppliers' suppliers. The "customer share" matrix $\widehat{A} = [\widehat{a}_{ij}]$ is defined such that customer industry i purchases the fraction $\widehat{a}_{ij}$ of the total industry output of supplying industry j. Thus, $\widehat{A}'\, q_{t-1}$ represents the sum change in the output of the industry's customers, weighted by the industry's "customer shares". The final regressor analogously measures a sum weighted by the customers' customer shares.

Table 8: Industry Output Dynamics Benchmark

	$y_{it} = \Delta \log(\text{Output}_{it})$			
	(1)	(2)	(3)	(4)
(ave. $y_{j,t-1}$ suppliers, 1-step)	0.173*	−0.013	0.159**	0.036
	(0.099)	(0.125)	(0.062)	(0.061)
(ave. $y_{j,t-1}$ customers, 1-step)	0.134	−0.061	0.087	0.029
	(0.127)	(0.150)	(0.094)	(0.110)
(ave. $y_{j,t-1}$ suppliers, 2-steps)		0.327		0.363**
		(0.266)		(0.141)
(ave. $y_{j,t-1}$ customers, 2-steps)		0.294		0.095
		(0.254)		(0.131)
Time FE	No	No	Yes	Yes
Firm FE	Yes	Yes	Yes	Yes
Observations	8,932	8,932	8,932	8,932
R^2 (within)	0.0118	0.0144	0.0042	0.005

Note: *p<0.1; **p<0.05; ***p<0.01

Panel regression of an output growth on the weighted average output growth of it customers and suppliers. Supplier averages are weighted by the expenditures shares (fraction of expenditures going to supplier) and customer averages are weighted by revenue shares (fraction of revenue accounted for by customer). Also includes controls for average output growth of suppliers' suppliers and customers' customers (labeled "two-step"), using implicit expenditure and revenue shares derived from input-output tables. Estimation uses manufacturing output data from the NBER-CES Manufacturing Industry Database over the years 1987 to 2011 and the BEA Input-Output Benchmark Table from 1987. Output here is the real value of total shipments of each industry. To account for serial and spatial dependence, standard errors are constructed following Driscoll and Kraay (1998).

Table 9: Industry Output Dynamics, Decomposed Effects

	$y_{it} = \Delta \log(\text{Output}_{it})$					
	(1)	(2)	(3)	(4)	(5)	(6)
$a_{ii} y_{i,t-1}$	-0.098	-0.134	-0.023	-0.084	-0.152	-0.146
	(0.228)	(0.242)	(0.214)	(0.229)	(0.232)	(0.235)
$\sum_{j=1}^{N} \mathbb{1}_{i \neq j} a_{ij} y_{j,t-1}$ (suppliers)	0.440	0.323**			0.393**	0.319***
	(0.269)	(0.137)			(0.172)	(0.119)
$\sum_{j=1}^{N} \mathbb{1}_{i \neq j} \widehat{a}_{ji} y_{j,t-1}$ (customers)			0.220	0.045	0.088	0.022
			(0.237)	(0.155)	(0.194)	(0.150)
Time FE	No	Yes	No	Yes	No	Yes
Firm FE	Yes	Yes	Yes	Yes	Yes	Yes
Observations	8,932	8,932	8,932	8,932	8,932	8,932
R^2 (within)	0.0097	0.003	0.004	2e-04	0.0102	0.0031

Note: *p<0.1; **p<0.05; ***p<0.01

Estimation using manufacturing output data from the NBER-CES Manufacturing Industry Database over the years 1987 to 2011 and the BEA Input-Output Benchmark Table from 1987. Total output here is the real value of total shipments of each industry. To account for serial and spatial dependence, standard errors are constructed following Driscoll and Kraay (1998). $A = [a_{ij}]$ represents the fraction of total expenditures that industry i spends on industry j. The "customer share" matrix $\widehat{A} = [\widehat{a}_{ij}]$ is defined such that customer industry i purchases the fraction $\widehat{a}_{ij}$ of the total industry output of supplying industry j.

Appendix to
"Sectoral Shifts, Production Networks, and the Term Structure of Equity"

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A Proofs and Derivations

A.1 Derivations from Section 2

A.1.1 Proof of Proposition 3

- The price of a dividend strip, using the conditional log-normal framework from before, is

$$\begin{aligned} P_t^\tau &= D_t \mathbb{E}_t \left[\frac{S_{t+\tau}}{S_t} \frac{D_{t+\tau}}{D_t} \right] \\ &= D_t \exp \left\{ \tau(\mu_s + \mu_d) + (U_s + U_d)'(I - G)^{-1}(I - G^\tau)x_t + \frac{1}{2} \sum_{k=1}^{\tau} \|\psi_s(k) + \psi_d(k)\|^2 \right\} \end{aligned}$$

- The k-period holding period return is

$$\begin{aligned} \log R_{t,t+k}^\tau = \log \left(\frac{P^{\tau-k}}{P_t^\tau} \right) \\ &= -k\mu_s - \frac{1}{2} \sum_{j=\tau-k+1}^{\tau} \|\psi_s(j) + \psi_d(j)\|^2 \\ &\quad - U_s'(I - G)^{-1}(I - G^k)x_t \\ &\quad + \sum_{j=1}^{k} (\psi_d'(j) + (U_s + U_d)'(I - G)^{-1}(I - G^{\tau-k})G^{j-1}H)w_{t+k+1-j}. \end{aligned} \tag{44}$$

Proof.

$$
\begin{aligned}
\log R^{\tau}_{t,t+k} &= \log\left(\frac{P^{\tau-k}}{P^{\tau}_{t}}\right) \\
&= \log\left(\frac{D_{t+k}}{D_t}\frac{P^{\tau-k}_{t+k}/D_{t+k}}{P^{\tau}_{t}/D_t}\right) \\
&= -k\mu_s + U_d'(I-G)^{-1}(I-G^k)x_t + \sum_{j=1}^{k}\psi_d(j)\cdot w_{t+1+k-j} - \frac{1}{2}\sum_{j=\tau-k+1}^{\tau}\|\psi_s(j)+\psi_d(j)\|^2 \\
&\quad + (U_s+U_d)'(I-G)^{-1}\left[(I-G^{\tau-k})G^k - (I-G^{\tau})\right]x_t \\
&\quad + (U_s+U_d)'(I-G)^{-1}(I-G^{\tau-k})\sum_{j=1}^{k}G^{j-1}Hw_{t+k+1-j} \\
&= -k\mu_s + \sum_{j=1}^{k}\psi_d(j)\cdot w_{t+1+k-j} - \frac{1}{2}\sum_{j=\tau-k+1}^{\tau}\|\psi_s(j)+\psi_d(j)\|^2 \\
&\quad - U_s'(I-G)^{-1}(I-G^k)x_t \\
&\quad + (U_s+U_d)'(I-G)^{-1}(I-G^{\tau-k})\sum_{j=1}^{k}G^{j-1}Hw_{t+k+1-j} \\
&= -k\mu_s - \frac{1}{2}\sum_{j=\tau-k+1}^{\tau}\|\psi_s(j)+\psi_d(j)\|^2 \\
&\quad - U_s'(I-G)^{-1}(I-G^k)x_t \\
&\quad + \sum_{j=1}^{k}(\psi_d'(j) + (U_s+U_d)'(I-G)^{-1}(I-G^{\tau-k})G^{j-1}H)w_{t+k+1-j}.
\end{aligned}
$$

□

- When $k = 1$, the holding period return is

$$
\begin{aligned}
\log R^{\tau}_{t,t+1} &= -\mu_s - \frac{1}{2}\|\psi_s(\tau)+\psi_d(\tau)\|^2 - U_s'x_t \\
&\quad + (\psi_d'(1) + (U_s+U_d)'(I-G)^{-1}(I-G^{\tau-1})H)w_{t+1} \\
&= \mu_\tau + U_\tau' x_t + \lambda_\tau' w_{t+1},
\end{aligned}
\tag{45}
$$

where

$$\begin{aligned}
\mu_\tau &= -\mu_s - \frac{1}{2}\|\psi_s(\tau) + \psi_d(\tau)\|^2 \\
U_\tau &= -U_s \\
\lambda_\tau &= \psi_d(1) + (\psi_s(\tau) - \psi_s(1) + \psi_d(\tau) - \psi_d(1)) \\
&= \psi_s(\tau) - \psi_s(1) + \psi_d(\tau).
\end{aligned}$$

It follows that

$$\begin{aligned}
\log R^{\tau}_{t,t+1} - \log R^{f}_{t,t+1} = &-\frac{1}{2}\|\psi_s(\tau) - \psi_s(1) + \psi_d(\tau)\|^2 - (\psi_s(\tau) - \psi_s(1) + \psi_d(\tau)) \cdot \psi_s(1) \\
&+ (\psi_s(\tau) - \psi_s(1) + \psi_d(\tau)) \cdot w_{t+1}.
\end{aligned}$$

- The result in (9) then follows from (77). The rest of the proposition follows from this.
- For a more complete discussion, see Section A.4.

A.2 Derivations from Section 3

A.2.1 Full Model, First-Order Conditions

Since the welfare theorems apply, competitive equilibrium can be obtained by solving the social planner's problem. A series of monotonic transformations of the utility recursion simplifies the expression of the objective function so that the social planner solves

$$\begin{aligned}
V^{1-\rho}(\{K_{it}, \Xi_{it}\}) = &\max_{\{I_{ijt}\},\{M_{ijt}\},\{K_{i,t+1}\},\{L_{it}\}} \quad (1-\beta)\mathcal{C}^{1-\rho} + \beta\left(\mathcal{R}_t(V_{t+1})\right)^{1-\rho} \\
&\text{subject to} \quad K_{i,t+1} = \prod_{j=1}^{n} I_{ijt}^{\theta_{ij}} + (1-\delta_j)K_{it}
\end{aligned} \tag{46}$$

and, substituting where applicable,

$$\begin{aligned}
\mathcal{C}_t &= \mathcal{L}_t^{1-s_c} C_t^{s_c} \\
C_t &= \prod_{i=1}^n C_{it}^{\alpha_i} \\
I_{it} &= \prod_{j=1}^n I_{ijt}^{\theta_{ij}} \\
M_{it} &= \prod_{j=1}^n M_{ijt}^{a_{ij}} \\
Q_{it} &= \Xi_{it} K_{it}^{a_i^k} L_{it}^{a_t^\ell} \left(\prod_{j=1}^n M_{ijt}^{a_{ij}} \right)^{a_i^m} \\
C_{jt} &= Q_{jt} - \sum_{i=1}^n M_{ijt} - \sum_{i=1}^n I_{ijt} \\
\mathcal{L}_t &= H - \sum_{i=1}^n L_{it} \\
\mathcal{R}_t(V_{t+1}) &= \mathbb{E}_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}} .
\end{aligned}$$

First-order conditions Let λ_{it} be the Lagrange multiplier associated with the constraints on Capital dynamics. Then, the first order conditions are as follows.

- Those associated with I_{ijt} are

$$(1-\beta)(1-\rho) s_c \frac{\mathcal{C}^{1-\rho}}{C_t} \alpha \frac{C_t}{C_{jt}} (-1) + \lambda_{it} \left(-\theta_{ij} \frac{I_t}{I_{ijt}} \right) = 0 \tag{47}$$

- and with $K_{i,t+1}$ are

$$\begin{aligned}
\beta(1-\rho)\mathcal{R}_t(V_{t+1})^{-\rho} \frac{\partial R_t}{\partial K_{i,t+1}} + \lambda_{it} = 0 \\
\beta(1-\rho)\mathcal{R}_t(V_{t+1})^{-\rho} \mathcal{R}_t^{\gamma} \mathbb{E}_t \left[V_{t+1}^{-\gamma} \frac{\partial V_{t+1}}{\partial K_{i,t+1}} \right] = -\lambda_{it},
\end{aligned} \tag{48}$$

with $\mathcal{R}_t = \mathcal{R}_t(V_{t+1})$. Combining these two gives

$$(1-\beta) s_c \frac{\mathcal{C}_t^{1-\rho}}{C_{jt}} \alpha_j = \beta \mathcal{R}_t^{\gamma-\rho} \mathbb{E}_t \left[V_{t+1}^{-\gamma} \frac{\partial V_{t+1}}{\partial K_{i,t+1}} \right] \theta_{ij} \frac{I_{it}}{I_{ijt}}. \tag{49}$$

- The envelope theorem implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}K_{it}}V_t^{1-\rho} &= (1-\rho)V_t^{-\rho}\frac{\mathrm{d}V_t}{\mathrm{d}K_{it}} \\ &= (1-\beta)(1-\rho)s_c\alpha_i\frac{\mathcal{C}_t^{1-\rho}}{C_{it}}a_i^k\frac{Q_{it}}{K_{it}} - \lambda_{it}(1-\delta).\end{aligned}$$

Combining this with the first-order conditions for investment and capital gives

$$\alpha_j\frac{\mathcal{C}_t^{1-\rho}}{C_{jt}} = \mathbb{E}_t\left[\beta\left(\frac{V_{t+1}}{\mathcal{R}_t}\right)^{\rho-\gamma}\theta_{ij}\frac{I_{it}}{I_{ijt}}\left(\alpha_i\frac{\mathcal{C}_{t+1}^{1-\rho}}{C_{i,t+1}}a_i^k\frac{Q_{i,t+1}}{K_{i,t+1}} + \alpha_j\frac{\mathcal{C}_{t+1}^{1-\rho}}{C_{j,t+1}}\frac{I_{ij,t+1}}{I_{i,t+1}}\frac{1}{\theta_{ijt}}(1-\delta)\right)\right]. \tag{50}$$

The pieces of this equation can be interpreted as follows. The term,

$$\underbrace{\alpha_j\frac{\mathcal{C}_t^{1-\rho}}{C_{jt}}}_{\substack{\sim\text{marginal utility}\\ \text{of good } j}},$$

is proportional to the marginal utility of good j. The conditional expectations contains a risk adjustment, the marginal increase in the investment in good i with respect to the contribution of the investment good j, and the marginal utility associated with an increase in the capital of type i,

$$\mathbb{E}_t\left[\beta\underbrace{\left(\frac{V_{t+1}}{\mathcal{R}_t}\right)^{\rho-\gamma}}_{\substack{\text{risk adjustment of}\\ \text{next period utility}}}\underbrace{\theta_{ij}\frac{I_{it}}{I_{ijt}}}_{\substack{\text{marginal transform. of}\\ \text{good } j \text{ into inv. } i.}}\underbrace{(\ldots)}_{\substack{\text{marginal utility per}\\ \text{unit of capital type } i}}\right].$$

This final marginal utility term can be broken down as follows. It is the marginal utility associated with an increase in capital of type i, with one part coming from the marginal product of that capital and the part coming from the increase in the capital stock remaining after depreciation in the following period:

$$\left(\underbrace{\alpha_i\frac{\mathcal{C}_{t+1}^{1-\rho}}{C_{i,t+1}}}_{\substack{\sim\text{marginal utility}\\ \text{of good } i}}\underbrace{a_i^k\frac{Q_{i,t+1}}{K_{i,t+1}}}_{\substack{\text{marginal product of}\\ \text{capital in sector } i}} + \underbrace{\alpha_j\frac{\mathcal{C}_{t+1}^{1-\rho}}{C_{j,t+1}}}_{\substack{\sim\text{marginal utility}\\ \text{of good } j}}\underbrace{\frac{I_{ij,t+1}}{I_{i,t+1}}\frac{1}{\theta_{ijt}}}_{\substack{\text{marginal transformation}\\ \text{into good } j\\ \text{per unit of capital } i}}\underbrace{(1-\delta)}_{\substack{\text{capital remaining}\\ \text{after depreciation}}}\right).$$

Note that the risk adjustment term $\beta \left(\frac{V_{t+1}}{\mathcal{R}_t}\right)^{\rho-\gamma}$ arising from the assumption of non-expected utility, via Epstein-Zin recursive utility. A expected, when $\gamma = \rho$, this term becomes unity.

- Also, note that when we assume that the consumption bundle $\mathcal{C}_t$ is the numeraire with price normalized to one, $P_t = 1$. The price index is then

$$P_t = \prod_{i=1}^{n} \left(\frac{P_{it}}{\alpha_i}\right)^{\alpha_i},$$

so that the price of a unit of good i satisfies

$$P_{it} = \alpha_i \frac{\mathcal{C}_t}{C_{it}}.$$

The stochastic discount factor (SDF) is then

$$\frac{S_{t+1}}{S_t} = \beta \left(\frac{V_{t+1}}{\mathcal{R}_t}\right)^{\gamma-\rho} \left(\frac{\mathcal{C}_{t+1}}{\mathcal{C}_t}\right)^{-\rho}. \tag{51}$$

We can then write the first-order conditions for investment and capital as

$$1 = \mathbb{E}_t \left[\frac{S_{t+1}}{S_t} R_{ij,t+1}\right], \tag{52}$$

where $R_{ij,t+1}$ is the return associated with investing a unit of the consumption good j into capital of type i,

$$R_{ij,t+1} = \theta_{ij} \frac{I_{it}}{I_{ijt}} \frac{1}{P_{jt}} \left(P_{i,t+1} a_i^k \frac{Q_{i,t+1}}{K_{i,t+1}} + P_{j,t+1} \frac{I_{ij,t+1}}{I_{i,t+1}} \frac{1}{\theta_{ij}} (1-\delta) \right). \tag{53}$$

- The first-order conditions with respect to the intermediate goods M_{ijt} are

$$\frac{\alpha_j}{C_{jt}} = \frac{\alpha_i}{C_{it}} a_i^m Q_{it} M_{ijt} a_{ij}. \tag{54}$$

- The first-order conditions of the household's problem with respect to L_{it} are

$$\frac{1-s_c}{\mathcal{L}_t} = s_c \frac{\alpha_i}{C_{it}} a_i^\ell Q_{it} L_{it}. \tag{55}$$

A.2.2 Non-stochastic balanced growth path and log-linearization

These derivations still need to be typeset. Coming soon.

A.2.3 Proposition 8: Case with Full Depreciation

Here I use the first-order conditions derived from the full model, as derived in the previous section, Section A.2.1. When $\delta = 1$ and as ρ approaches 1 in the limit, the program (46) becomes

$$
\begin{aligned}
\log V(\{K_{it}, \Xi_{it}\}) = \max_{\{I_{ijt}\},\{M_{ijt}\},\{K_{i,t+1}\},\{L_{it}\}} \quad & (1-\beta)\log \mathcal{C}_t + \beta \log\left(\mathcal{R}_t(V_{t+1})\right) \\
\text{subject to} \quad & K_{i,t+1} = \prod_{j=1}^{n} I_{ijt}^{\theta_{ij}}.
\end{aligned}
\tag{56}
$$

The first-order conditions derived in the previous section will hold, after substituting $\rho = 1$ and $\delta = 1$.

To solve for the optimal policy functions and the value function, I proceed by using the method of undetermined coefficients. I make a direct guess as to the functional form of V_t. From this, I derive the optimal policy functions and determine the value of V_{t+1}. The first-order conditions along with market clearing would then give us the restrictions needed to determine these unknown coefficients. I then verify that this guess indeed solves the functional equation that characterizes equilibrium.

Guess Value Function and Evaluate First-Order Conditions

- Guess that

$$
V(\{K_{it}\}, \{\xi_{it}\}) = \prod_{i}^{n} K_{it}^{s_c(1-\beta)a_i^k \nu_i} \exp\left\{J(\xi_t)\right\} V_0, \tag{57}
$$

  for some unknown constants ν_i, for $i = 1, \ldots, n$, an unknown $\mathcal{F}_t$-measurable function J, and a constant V_0.

- Under this assumption,

$$
\frac{\partial V_t}{\partial K_{it}} = s_c(1-\beta)a_i^k \nu_i \frac{V_t}{K_{it}}.
$$

  Substituting this into the first-order conditions for investment and capital, (49), implies that

$$
\begin{aligned}
\frac{\alpha_i}{C_{jt}} &= \beta a_i^k \nu_i \theta_{ij} \frac{I_{it}}{I_{ijt}} \mathbb{E}_t\left[\left(\frac{V_{t+1}}{\mathcal{R}_t}\right)^{1-\gamma} \frac{1}{K_{i,t+1}}\right] \\
\frac{\alpha_j}{C_{jt}} &= \beta a_i^k \nu_i \theta_{ij} \frac{1}{I_{ijt}},
\end{aligned}
\tag{58}
$$

where the second line follows from

$$K_{i,t+1} = I_{it}$$

and

$$\mathbb{E}_t\left[\left(\frac{V_{t+1}}{\mathcal{R}_t}\right)^{1-\gamma}\right] = \mathbb{E}_t\left[\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t\left[V_{t+1}^{1-\gamma}\right]}\right] = 1.$$

- For convenience, define $\widetilde{\Theta} = [\theta_{ij}]$ and $\widetilde{A} = [a_{ij}]$. Since these are the Cobb-Douglas aggregator shares that sum to one, $\widetilde{\Theta}\mathbb{1} = \widetilde{A}\mathbb{1} = \mathbb{1}$, where $\mathbb{1}$ is a vector of ones.

- Let $d_{jt} = \alpha_j \frac{\sum_{i=1}^n I_{ijt}}{C_{jt}}$, with the vector $d_t = [d_{jt}]$. Then, continuing with equation (58), this implies that

$$d' \equiv d_t' = \beta\nu'\Theta,$$

where $\Theta = \text{diag}(a^k)\widetilde{\Theta} = [a_i^k\theta_{ij}]$ and, since d_t is constant, I've defined $d = d_t$.

- From the first-order conditions for M_{ijt}, (54) combined with the market clearing conditions for goods,

$$\begin{aligned}Q_{jt} - \sum_{i=1}^n I_{ijt} - C_{jt} = \sum_{i=1}^n M_{ijt} &= \sum_{i=1}^n \frac{\alpha_i}{\alpha_j}\frac{C_{jt}}{C_{it}}a_i^m Q_{it}a_{ij} \\ \frac{Q_{jt}}{C_{jt}} - \frac{\sum_{i=1}^n I_{ijt}}{C_{jt}} &= 1 + \sum_{i=1}^n \frac{\alpha_i}{\alpha_j}a_i^m a_{ij}\frac{Q_{it}}{C_{it}}.\end{aligned} \tag{59}$$

Let $b_{jt} = \alpha_j \frac{Q_{jt}}{C_{jt}}$, with the vector $b_t = [b_{jt}]$. With $A = \text{diag}(a^m)\widetilde{A} = [a_i^m a_{ij}]$, (59) implies

$$b_t' - d' = \alpha' + b_t' A$$

with the solution

$$\begin{aligned}b' \equiv b_t' &= (\alpha + d)'(I - A)^{-1} \\ b' &= (\alpha' + \beta\nu'\Theta)(I - A)^{-1},\end{aligned} \tag{60}$$

where I've defined $b = b_t$, since b_t is constant. Also, since we assume $0 \leq a_i^m < 1$, $(I - A)$ is invertible.

- From the first-order conditions for L_{it}, we have

$$L_{it} = \frac{s_c}{1-s_c} b_i a_i^\ell \mathcal{L}_t.$$

Market clearing for hours worked implies that

$$\mathcal{L}_t = H \left(1 + \frac{s_c}{1-s_c} \sum_{i=1}^n a_i^\ell b_i \right)^{-1}.$$

Evidently, $L_{it} = L_i$ and $\mathcal{L}_t = \mathcal{L}$ are constant.

- By definition of b,

$$C_{it} = \frac{\alpha_i}{b_i} Q_{it}.$$

From eq. (54),

$$M_{ijt} = \frac{b_i}{b_j} a_i^m a_{ij} Q_{jt}.$$

From eq. (58),

$$I_{ijt} = \beta \frac{\nu_i}{b_j} a_i^k \theta_{ij} Q_{jt}.$$

- To compute the dynamics of output, take logs of the production function and substitute the optimal values of M_{ijt} and L_{it}. This gives

$$\log Q_{it} = \log \Xi_{it} + a_i^k \log K_{it} + a_i^m \sum_{j=1}^n a_{ij} \left(\log Q_{jt} + \left(\frac{b_i}{b_j} a_i^m a_{ij} \right) \right) + a_i^\ell \log L_i.$$

In matrix-vector form, this is

$$q_t = \xi_t + \mathrm{diag}(a^k) k_t + \mathrm{diag}(a^m) \widetilde{A} q_t + g^y,$$

where k_t is a vector of log capital $k_{it} = \log K_{it}$, diag is the operator that creates a matrix with the argument vector on the diagonal and g^y is a vector of constants based on the model parameters. This implies

$$(I - A) q_t = \xi_t + \mathrm{diag}(a^k) k_t + g^y. \tag{61}$$

Since $K_{i,t+1} = I_{it} = \prod_{j=1}^n I_{ijt}^{\theta_{ij}}$, and substituting the first-order conditions for I_{ijt} and $K_{i,t+1}$, we have

$$\log K_{i,t+1} = \sum_{j=1}^n \theta_{ij} \log \left(\log Q_{jt} + \left(\beta \nu_i \theta_{ij} \frac{1}{b_j} \right) \right).$$

Thus,

$$k_{t+1} = \widetilde{\Theta} q_t + \widetilde{\Theta} g^k, \tag{62}$$

where g^k is a vector of constants based on the model parameters. Thus,

$$q_{t+1} = (I - A)^{-1} \Theta q_t + (I - A)^{-1} \xi_{t+1} + (I - A)^{-1}(g^y + \Theta g^k).$$

and

$$\Delta q_{t+1} = (I - A)^{-1} \Theta \Delta q_t + (I - A)^{-1} \Delta \xi_{t+1}$$

- The dynamics of capital can also be computed. Using equations (61) and (62),

$$\begin{aligned} k_{t+1} &= \widetilde{\Theta}(I - A)^{-1} \mathrm{diag}(a^k) k_t + \widetilde{\Theta}(I - A)^{-1} \xi_t + g^{kd} \qquad (63) \\ \mathrm{diag}(a^k) k_{t+1} &= \Theta (I - A)^{-1} \mathrm{diag}(a^k) k_t + \Theta (I - A)^{-1} \xi_t + \mathrm{diag}(a^k) g^{kd}, \end{aligned}$$

where

$$g^{kd} = \widetilde{\Theta}(I - A)^{-1}(g^y + (I - A) g^k).$$

An interpretation of the effects here is as follows:

$$k_{t+1} = \underbrace{\widetilde{\Theta}}_{\substack{\text{investment network,} \\ \text{bundle investments}}} \underbrace{(I - A)^{-1}}_{\substack{\text{instantaneous effect of} \\ \text{intermediate goods network}}} \underbrace{\mathrm{diag}(a^k)}_{\substack{\text{relative importance of} \\ \text{capital in production}}} k_t + \widetilde{\Theta}(I - A)^{-1} \xi_t + g^{kd} \tag{64}$$

Verify Guess of Value Function I will now verify that the guess of the value function is a solution to the value function recursion.

- Since $\rho = 1$,

$$\log V_t = (1 - \beta) \log \mathcal{C}_t + \beta \log \mathcal{R}_t(V_{t+1}), \tag{65}$$

where the optimal policy functions have been substituted in.

- Begin by evaluating $\mathcal{R}_t(V_{t+1})$.

$$
\begin{aligned}
&\mathbb{E}_t\Big[\exp\Big\{(1-\gamma)\log V_{t+1}\Big\}\Big] = \\
&= \mathbb{E}_t\Big[\exp\Big\{(1-\gamma)s_c(1-\beta)\sum_{i=1}^{n} a_i^k \nu_i \log K_{i,t+1} + (1-\gamma)\log V_0 + (1-\gamma)J(W_{t+1})\Big\}\Big] \\
&= \mathbb{E}_t\Big[\exp\Big\{(1-\gamma)s_c(1-\beta)\nu'\Big[\Theta(I-A)^{-1}\xi_{t+1} + \Theta(I-A)^{-1}\mathrm{diag}(a^k)k_t + \mathrm{diag}(a^k)g^{kd}\Big] \\
&\qquad + (1-\gamma)\log V_0 + (1-\gamma)J(W_{t+1})\Big\}\Big] \\
&= \exp\Big\{(1-\gamma)s_c(1-\beta)\nu'\Big[\Theta(I-A)^{-1}\mathrm{diag}(a^k)k_t + \mathrm{diag}(a^k)g^{kd}\Big] + (1-\gamma)\log V_0\Big\} \\
&\qquad \times \mathbb{E}_t\Big[\exp\Big\{(1-\gamma)s_c(1-\beta)\nu'\Theta(I-A)^{-1}\xi_{t+1} + (1-\gamma)J(W_{t+1})\Big\}\Big]
\end{aligned}
$$

Then,

$$
\begin{aligned}
\log\mathcal{R}_t(V_{t+1}) = s_c(1-\beta)\nu'\Big(\Theta(I-A)^{-1}\mathrm{diag}(a^k)k_t + \mathrm{diag}(a^k)g^{kd}\Big) + (1-\gamma)\log V_0 + \\
+ \frac{1}{1-\gamma}\log\mathbb{E}_t\Big[\exp\Big\{(1-\gamma)s_c(1-\beta)\nu'\Theta(I-A)^{-1}\xi_{t+1} + \\
+ (1-\gamma)J(\xi_{t+1})\Big\}\Big]
\end{aligned}
$$

.

- Now, evaluating $\mathcal{C}_t$,

$$
\begin{aligned}
\log\mathcal{C}_t &= (1-s_c)\log\mathcal{L} + s_c(g^c + \alpha'(I-A)^{-1}g^y \\
&\qquad + \alpha'(I-A)^{-1}\xi_t + \alpha'(I-A)^{-1}\mathrm{diag}(a^k)k_t) \\
&= s_c\alpha'(I-A)^{-1}\mathrm{diag}(a^k)k_t + s_c\alpha'(I-A)^{-1}\xi_t + g^{c*},
\end{aligned}
$$

where g^{c*} is a constant that is a function of the model parameters.

- Now, we can use these to substitute into the value function recursion (65). Since (65) must hold for all values of the states K_{it}. The left-hand side of this recursion, in terms of the state variables, is given by the guess of the value function in (57):

$$
\log V(\{Q_{it}\},\{W_{it}\}) = \sum_{i}^{n} s_c(1-\beta)a_i^k\nu_i\log K_{it} + J(\xi_t) + \log V_0.
$$

The right-hand side is given by the derivations of $\log \mathcal{C}_t$ and $\log \mathcal{R}_t(V_{t+1})$. Analyzing the coefficients associated with k_t on the left-hand side and right-hand side, we see that ν must satisfy

$$s_c(1-\beta)\nu' \mathrm{diag}(a^k) = (1-\beta)s_c\alpha'(I-A)^{-1}\mathrm{diag}(a^k) + \beta s_c(1-\beta)\nu'\theta,$$

which implies that

$$\nu' = \alpha'(I-A)^{-1}(I-\beta\Theta(I-A)^{-1})^{-1}. \tag{66}$$

Note, as an aside, that when $A = 0$, then $b' = \alpha'(I-\beta\Theta) = \nu'$.

- Furthermore, isolating the terms on each side that depend on the shocks, we have that the function J is characterized by

$$J(\xi_t) = \beta\frac{1}{1-\gamma}\log \mathbb{E}_t\Big[\exp\Big\{(1-\gamma)s_c(1-\beta)\nu'\Theta(I-A)^{-1}\xi_{t+1} + (1-\gamma)J(\xi_{t+1})\Big\}\Big] \tag{67}$$

and V_0 is defined by the remaining constants. Note that when ξ_{t+1} has a joint normal conditional distribution, conditional on information at time t, this is

$$\begin{aligned} J(\xi_t) = \beta\mathbb{E}_t&\left[s_c(1-\beta)\nu'\Theta(I-A)^{-1}\xi_{t+1} + J(\xi_{t+1})\right] \\ &+ \beta\frac{1}{2}(1-\gamma)\mathrm{Var}_t\left(s_c(1-\beta)\nu'\Theta(I-A)^{-1}\xi_{t+1} + J(\xi_{t+1})\right). \end{aligned}$$

- This concludes the verification that our guessed value function solves the recursion and satisfies the first-order conditions.

□

A.2.4 Risk prices and risk exposures in the full depreciation case

Here I give the proof of Proposition 9.

This still needs to be typeset. Coming soon.

A.3 Derivations from Section 4

Here I formalize the argument laid out in Section A.7 regarding the procedure for disentangling common, aggregate shocks from idiosyncratic shocks propagated through the input-output network. This argument was first made by Foerster, Sarte, and Watson (2011) and, for convenience, I reproduce the crux of it here.

Suppose that output growth follows the equilibrium process in equation (22). Suppose again that innovations to productivity have a common, aggregate component, and an idiosyncratic component. That is, let

$$\varepsilon_t = \Lambda_a \nu_t^a + \nu_t^s,$$

where ν_t^a is a $K \times 1$ vector and common to all industries and ν_t^s is an $n \times 1$ vector of idiosyncratic shocks. Λ_a is an $n \times K$ matrix of coefficients reflecting each industry's exposure to the K common shocks. Assume that (ν_t^a, ν_t^s) are serially uncorrelated, and that ν_t^a and ν_t^s are mutually uncorrelated. Assume further that the idiosyncratic shocks are uncorrelated, so that $\Sigma_{\nu\nu} = \mathbb{E}[\nu_t^s \nu_t^s]$ is a diagonal matrix.

Under these assumptions, industry output growth can be written as a dynamic factor model

$$\Delta q_t = \Lambda(L) F_t + u_t, \tag{68}$$

where

$$\Lambda(L) = (I - \Phi L)^{-1}(\Pi_0 + \Pi_1 L)\Lambda_a,$$

$F_t = \nu_t^a$, and

$$u_t = (I - \Phi L)^{-1}(\Pi_0 + \Pi_1 L)\nu_t^s.$$

Importantly, the elements of u_t are a linear combination of of ν_t^s. While the elements of ν_t^s are uncorrelated, the elements of u_t need not be. The matrix $(I - \Phi L)^{-1}(\Pi_0 + \Pi_1 L)$ embodies the effects of network connections in the model and a reduced form factor analysis of industry growth rates may overestimate the importance of aggregate shocks.

To fix this, we can construct a filter based on a calibration of the structural general equilibrium model. Using (22) we see that

$$\varepsilon_t = (\Pi_0 + \Pi_1 L)^{-1}(I - \Phi L)\Delta q_t. \tag{69}$$

Since we have estimates of Π_0, Π_1, and Φ from the calibration, we can construct the right-hand side. We can then use factor analytic methods on the filtered industry growth data to estimate the contributions of the aggregate shocks ν_t^a and ν_t^s.

A.4 Notation and Framework for Risk Prices and Risk Premia

In this section I lay out the notation and describe a framework that will serve as a convenient setting to discuss risk prices and risk premia. I assume a state-space representation of the economy in which the state vector, x_t, follows a VAR(1) process. Such a framework encompasses a broad class of economic models, including several parameterizations of the model considered here.[8]

Let x_t be a $N \times 1$ state vector capturing the state of the economy. Assume that it follows a VAR(1) process,

$$x_{t+1} = Gx_t + Hw_{t+1}, \tag{70}$$

with w_{t+1} as a $M \times 1$ vector of i.i.d. shocks, $w_{t+1} \sim \mathcal{N}(0, \mathcal{I})$, and G and H as conforming matrices with the spectral radius of G less than one. Suppose that the returns on a given asset can be expressed as

$$\log R_{t,t+1} = \mu_r + U_r' x_t + \lambda_r' w_{t+1},$$

and the stochastic discount factor as

$$\log S_{t+1} - \log S_t = \mu_s + U_s' x_t + \lambda_s \cdot w_{t+1},$$

where μ_r and μ_s are constants, U_r and U_s are $N \times 1$ constant vectors, and λ_r and λ_s are $M \times 1$ constant vectors. Defining these in this way allows, for example, the conditional expected returns of each to vary with the state. As a stochastic discount factor, for any such asset, the no arbitrage condition holds

$$\mathbb{E}_t \left[\frac{S_{t+1}}{S_t} R_{t,t+1} \right] = 1.$$

In the remainder of this section, I will define and derive several economic objects and quantities of interest.

A.4.1 Impulse response functions

Before computing risk prices and risk-premia, I begin with some preliminary calculations. Continuing with the economy and assumptions from the previous section, I compute impulse response function of series of interest. Computing these impulse response functions will ease the computations of risk prices later one.

- A useful way to think about impulse responses is in terms of changes in conditional expectations. For example, let Y_t be a log-linear stochastic process, with

$$\log Y_{t+1} - \log Y = \mu_y + U_y' x_t + \lambda_y' w_{t+1}.$$

[8] See, for example, Hansen and Sargent (2014) for a treatment on the broad array of economies that can be modeled in such a framework.

Then, for $\tau \geq 1$, define $\psi_y(\tau)$ to be the impulse response function of Y_t in levels at horizon τ. That is,

$$\Delta \mathbb{E}_{t+1} \left[\log Y_{t+\tau}\right] = \psi_y(\tau) \cdot w_{t+1},$$

where Δ is the difference operator, defined such that $\Delta \mathbb{E}_{t+1}[\log Y_{t+\tau}] = \mathbb{E}_{t+1}[\log Y_{t+\tau}] - \mathbb{E}_t[\log Y_{t+\tau}]$.

- Before proceeding, note that

$$x_t = Gx_{t-1} + Hw_t = G^t x_0 + \sum_{k=0}^{t} G^k H w_{t-k} = \sum_{k=0}^{\infty} G^k H w_{t-k}. \tag{71}$$

- Suppose that we can express the log one-period returns of a given asset as

$$\log R_{t,t+1} = \mu_r + U_r' x_t + \lambda_r' w_{t+1},$$

for some μ_r, U_r, and λ_r. The constant vector U_r controls the dependence of the conditional expectation of returns on the state vector and λ_r controls the exposures of returns to the shocks.

- Now consider the log cumulative returns over τ periods, $\log R_{t,t+\tau} = \sum_{k=1}^{\tau} \log R_{t+k-1,t+k}$. Define the impulse response function over the first τ periods,

$$\psi_r'(k) = \lambda_r' + U_r'(I - G)^{-1}(I - G^{k-\tau})H$$

for $k = 1, \ldots, \tau$. This is defined such that

$$\begin{aligned}
\log R_{t,t+\tau} &= \sum_{k=1}^{\tau} \log R_{t+k-1,t+k} \\
&= \tau\mu_r + U_r'(I - G)^{-1}(I - G^{\tau})x_t + \sum_{k=1}^{\tau} \psi_{r,\tau}'(k) w_{t+1+\tau-k}.
\end{aligned}$$

Proof. Using $x_{t+k} = Gx_{t+k-1} + Hw_{t+k} = G^k x_t + \sum_{i=0}^{k-1} G^i H w_{t+k-i}$,

$$\begin{aligned}
\log R_{t,t+\tau} &= \sum_{k=0}^{\tau-1} \log R_{t+k,t+k+1} \\
&= \sum_{k=0}^{\tau-1} \mu_r + U_r' x_{t+k} + \lambda_r' w_{t+k+1} \\
&= \tau\mu_r + U_r' \sum_{k=0}^{\tau-1} G^k x_t + U_r' \sum_{k=0}^{\tau-1} \sum_{i=0}^{k-1} G^i H w_{t+k-i} + \sum_{k=0}^{\tau-1} \lambda_r' w_{t+k+1}.
\end{aligned}$$

Gathering terms and simplifying,

$$\begin{aligned}\sum_{k=0}^{\tau-1}\sum_{i=0}^{k-1} G^i H w_{t+k-i} &= \sum_{k=1}^{\tau-1}\sum_{i=0}^{k-1} G^i H w_{t+k-i} \\ &= \sum_{k=1}^{\tau-1}\sum_{i=0}^{\tau-k-1} G^i H w_{t+k} \\ &= \sum_{k=1}^{\tau-1}(I-G)^{-1}(I-G^{\tau-k})H w_{t+k},\end{aligned}$$

where the last equality can be applied when $I-G$ is invertible. Then,

$$\begin{aligned}\log R_{t,t+\tau} &= \tau\mu_r + U_r'\sum_{k=0}^{\tau-1} G^k x_t + \sum_{k=1}^{\tau-1}\left(\lambda_r' + U_r'\sum_{i=0}^{\tau-k-1} G^i H\right) w_{t+k} + \lambda_r' w_{t+\tau} \\ &= \tau\mu_r + U_r'(I-G)^{-1}(I-G^{\tau})x_t + \sum_{k=1}^{\tau}\psi_r(\tau-k)\cdot w_{t+k}.\end{aligned}$$

Note that when the spectral radius of G is less than one, which we assume here throughout, we can write

$$\left(\sum_{i=0}^{\tau-1} G^i\right) = (I-G)^{-1}(I-G^{\tau}) = (I-G^{\tau})(I-G)^{-1}.$$

□

- For future calculations, I'll also need to compute the impulse response function of $\log S_t$. The process is defined in log differences,

$$\log S_{t+1} - \log S_t = \mu_s + U_s' x_t + \lambda_s \cdot w_{t+1},$$

but I will need to impulse response of the log level. Let ψ_s be the impulse response function of $\log S_t$ over the first τ periods so that

$$\log S_{t+\tau} = \log S_t + \tau\mu_s + U_s'(I-G)^{-1}(I-G^{\tau})x_t + \sum_{k=1}^{\tau}\psi_s(k)\cdot w_{t+1+\tau-k} \quad (72)$$

with

$$\psi_s'(k) = \lambda_s' + U_s'(I-G)^{-1}(I-G^{k-1})H \qquad \text{for } k = 1,..,\tau.$$

Since the derivation follows similarly from the derivation for the τ-period cumulative returns, I omit the proof.

A.4.2 Risk prices, risk exposures, and risk premia

Using the definitions and derivations from the previous sections, I define risk prices and risk-premia and use the no-arbitrage restrictions to derive a simple characterization. I first calculate the risk-free rate and then analyze the returns processes on risky assets.

Risk-free rate and risk-free bonds

- Using the notation in (105), the one-period conditional risk-free rate is given by
$$\log R^f_{t,t+1} = -\mu_s - U_s' x_t - \frac{1}{2}\|\lambda_s\|^2.$$
This is derived by
$$\begin{aligned}\log R^f_{t,t+1} &= -\log \mathbb{E}_t\left[\frac{S_{t+1}}{S_t}\right] \\ &= -\log \mathbb{E}_t\left[\exp\left\{\mu_s + U_s' x_t + \lambda_s \cdot w_{t+1}\right\}\right].\end{aligned}$$

- Let B_t^τ be the price of the time t price of a risk-free zero coupon bond that pays out a unit amount τ periods in the future. Then,
$$B_t^\tau = \mathbb{E}_t\left[\frac{S_{t+\tau}}{S_t}\right].$$
In terms of the underlying system parameters, this is
$$\log B_t^\tau = \tau\mu_s + U_s'(I-G)^{-1}(I-G^\tau)x_t + \frac{1}{2}\sum_{k=1}^{\tau}\|\psi_s(k)\|^2 \tag{73}$$
To derive this,
$$\begin{aligned}\log B_t^\tau &= \log \mathbb{E}_t\left[\frac{S_{t+\tau}}{S_t}\right] \\ &= \log \mathbb{E}_t\left[\exp\left\{\tau\mu_s + U_s'\sum_{k=1}^{\tau} G^{k-1}x_t + \sum_{k=1}^{\tau}\psi_s(k)\cdot w_{t+\tau+1-k}\right\}\right] \\ &= \tau\mu_s + U_s'(I-G)^{-1}(I-G^\tau)x_t + \frac{1}{2}\sum_{k=1}^{\tau}\|\psi_s(k)\|^2.\end{aligned}$$

- The log gross *yield-to-maturity* of the zero coupon bond maturing in τ periods is
$$\log Y_t^\tau = -\frac{1}{\tau}\log B_t^\tau. \tag{74}$$
In this case, since the bond is a zero coupon bond, the yield-to-maturity is equal to the yield.

- Let $R_{t,t+\tau}^{f,\tau}$ be the gross return on the risk-free bond that matures in τ periods, held to maturity. That is,
$$R_{t,t+\tau}^{f,\tau} = \frac{1}{B_t^\tau}. \tag{75}$$
In terms of the yield and in terms of model parameters, this is
$$R_{t,t+\tau}^{f,\tau} = \tau \log Y_t^\tau = -\tau\mu_s - U_s'(I-G)^{-1}(I-G^\tau)x_t - \frac{1}{2}\sum_{k=1}^{\tau}\|\psi_s(k)\|^2. \tag{76}$$
This follows from (73).

- Note that $R_{t,t+1}^f = R_{t,t+1}^{f,1} = \log Y_t^1$.

Risk prices and conditional risk-premia

- Using the notation and framework of the previous subsections, I call $\psi_r(k)$ as the *risk exposures* and $\psi_s(k)$ as the *risk prices* at the k-period horizon. These are vectors where the components describe the exposures and risk prices associated with the components of the shock vector w_{t+k}, respectively. At the short-term, one-period horizon, $\psi_r(1) = \lambda_r$ and $\psi_s(1) = \lambda_s$. The product of these risk exposures and risk prices comprise the *risk premia*.

- Suppose that we can express the log one-period returns of a given asset as
$$\log R_{t,t+1} = \mu_r + U_r' x_t + \lambda_r' w_{t+1},$$
for some μ_r, U_r, and λ_r. Then, the one-period conditional risk-premium is given by
$$\log E_t[R_{t,t+1}] - \log R_{t,t+1}^f = -\lambda_s \cdot \lambda_r. \tag{77}$$
Note that the expression $\lambda_s \cdot \lambda_r$ is the conditional covariance between the log stochastic discount factor and the given asset's log returns, $\mathrm{Cov}_t\left(\log\frac{S_{t+1}}{S_t}, \log R_{t,t+1}\right) = \lambda_s \cdot \lambda_r$.

Proof. This can be computed as follows. From the pricing equation, we have

$$\begin{aligned}
1 &= \mathbb{E}_t\left[\frac{S_{t+1}}{S_t}R_{t,t+1}\right] \\
&= \mathbb{E}_t[\exp\{s_{t+1} - s_t + \log R_{t,t+1}\}] \\
0 &= \mu_s + \mu_r + (U_s + U_r)'x_t + \frac{1}{2}\|\lambda_s + \lambda_r\|^2 \\
0 &= \mu_s + \mu_r + (U_s + U_r)'x_t + \frac{1}{2}\|\lambda_s\|^2 + \frac{1}{2}\|\lambda_r\|^2 + \lambda_s \cdot \lambda_r.
\end{aligned}$$

This implies that

$$\begin{aligned}
\log \mathbb{E}_t[R_{t,t+1}] &= \mu_r + U_t'x_t + \frac{1}{2}\|\lambda_r\|^2 \\
&= -\mu_s - U_s'x_t - \frac{1}{2}\|\lambda_s\|^2 - \lambda_s \cdot \lambda_r
\end{aligned}$$

Subtracting the expression for the one-period risk-free rate, the one-period conditional risk-premium is

$$\log E_t[R_{t,t+1}] - \log R^f_{t,t+1} = -\lambda_s \cdot \lambda_r.$$

□

- Now consider the returns over τ periods, $\log R_{t,t+\tau} = \sum_{k=0}^{\tau-1} \log R_{t+k,t+k+1}$. The $\tau-$period conditional risk-premium is

$$\log E_t[R_{t,t+\tau}] - \log R^{f,\tau}_{t,t+\tau} = -\sum_{k=1}^{\tau} \psi_r(k) \cdot \psi_s(k), \qquad (78)$$

where $\psi_r'(k) = \lambda_r' + U_r'(I - G)^{-1}(I - G^{k-1})H$ when is the impulse response function of the cumulative returns of the asset over the periods $1 \leq k \leq \tau$ so that

$$\begin{aligned}
\log R_{t,t+\tau} &= \sum_{k=1}^{\tau} \log R_{t+k-1,t+k} \\
&= \tau\mu_r + U_r'(I - G)^{-1}(I - G^{\tau})x_t + \sum_{k=1}^{\tau} \psi_r(k) \cdot w_{t+\tau+1-k}.
\end{aligned}$$

Proof. The derivation of the $\tau-$period case proceeds in a manner similar to the one-period case, using the no-arbitrage condition

$$1 = \mathbb{E}_t\left[\frac{S_{t+\tau}}{S_t}R_{t,t+\tau}\right].$$

This puts restrictions on the excess returns,

$$\begin{aligned}
1 &= \mathbb{E}_t\left[\exp\left\{\tau(\mu_s+\mu_r)+(U_s+U_r)'(I-G)^{-1}(I-G^\tau)x_t+\sum_{k=1}^{\tau}(\psi_s(k)+\psi_r(k))\cdot w_{t+\tau+1-k}\right\}\right] \\
&= \exp\left\{\tau(\mu_s+\mu_r)+(U_s+U_r)'(I-G)^{-1}(I-G^\tau)x_t+\frac{1}{2}\sum_{k=1}^{\tau}\|\psi_s(k)+\psi_r(k)\|^2\right\} \\
0 &= \tau(\mu_s+\mu_r)+(U_s+U_r)'(I-G)^{-1}(I-G^\tau)x_t \\
&\qquad +\frac{1}{2}\sum_{k=1}^{\tau}(\|\psi_s(k)\|^2+\|\psi_r(k)\|^2+2\psi_s(k)\cdot\psi_r(k)).
\end{aligned} \tag{79}$$

This implies that

$$\begin{aligned}
\mathbb{E}_t[R_{t,t+\tau}] &= \exp\left\{\tau\mu_r+U_r'(I-G)^{-1}(I-G^\tau)x_t+\frac{1}{2}\sum_{k=1}^{\tau}\|\psi_r(k)\|^2\right\} \\
&= \exp\left\{-\tau\mu_s-U_s'(I-G)^{-1}(I-G^\tau)x_t-\frac{1}{2}\sum_{k=1}^{\tau}(\|\psi_s(k)\|^2+2\psi_s(k)\cdot\psi_r(k))\right\}
\end{aligned} \tag{80}$$

and, thus, that

$$\log\mathbb{E}_t[R_{t,t+\tau}]-\log\mathbb{E}_t[R^{f,\tau}_{t,t+\tau}] = -\sum_{k=1}^{\tau}\psi_s(k)\cdot\psi_r(k).$$

As an aside, note that since x_t takes values in $\mathbb{R}^N$ and is unbounded, then we must have $U_r=-U_s$.

□

Now, I consider some alternate expressions.

- Consider the expression where we examine the conditional expectation of the log differences rather than the log differences in the conditional expectations. We see that

$$\begin{aligned}
\mathbb{E}_t[\log R_{t,t+1}-\log R^f_{t,t+1}] &= \mathbb{E}_t\left[\mu_r+\mu_s+(U_s+U_r)'x_t+\frac{1}{2}\|\lambda_s\|^2+\lambda_r\cdot w_{t+1}\right] \\
&= -\frac{1}{2}\|\lambda_r\|^2-\lambda_s\cdot\lambda_r.
\end{aligned}$$

  The expression $\frac{1}{2}\|\lambda_r\|^2$ is the conditional variance of the log returns, the Jensen's inequality term.

- Applied to longer horizons, this is

$$\mathbb{E}_t[\log R_{t,t+\tau} - \log R^{f,\tau}_{t,t+\tau}] = -\frac{1}{2}\sum_{k=1}^{\tau}\|\psi_r(k)\|^2 - \sum_{k=1}^{\tau}\psi_r(k)\cdot\psi_s(k).$$

Proof. Using (79), we can derive this from

$$\log R_{t,t+\tau} - \log R^{f,\tau}_{t,t+\tau} = \sum_{k=1}^{\tau}\psi_r(k)\cdot w_{t+\tau+1-k} - \frac{1}{2}\sum_{k=1}^{\tau}\left(\|\psi_r(k)\|^2 + 2\psi_s(k)\cdot\psi_r(k)\right). \tag{81}$$

□

Unconditional risk premia Here I derive the unconditional counterparts to the previous expressions.

- The unconditional expected value of the τ-period horizon risk-free rate is

$$\begin{aligned}\mathbb{E}\left[R^f_{t,t+\tau}\right] &= \mathbb{E}\exp\left\{-\tau\mu_s - U_s'(I-G)^{-1}(I-G^\tau)\sum_{j=1}^{\infty}G^{j-1}Hw_{t+1-j} - \frac{1}{2}\sum_{k=1}^{\tau}\|\psi_s(k)\|^2\right\}\\ &= \exp\left\{-\tau\mu_s - \frac{1}{2}\sum_{j=1}^{\infty}\|U_s'(I-G)^{-1}(I-G^\tau)G^{j-1}H\|^2 - \frac{1}{2}\sum_{k=1}^{\tau}\|\psi_s(k)\|^2\right\}.\end{aligned} \tag{82}$$

To derive, substitute (71) into (76) and compute the unconditional expectation.

- The unconditional risk premium over a τ-period horizon:

$$\log\mathbb{E}[R_{t,t+\tau}] - \log\mathbb{E}[R^{f,\tau}_{t,t+\tau}] = -\sum_{k=1}^{\tau}\psi_s(k)\cdot\psi_r(k). \tag{83}$$

Proof. Using (80) and (71),

$$\begin{aligned}\mathbb{E}[R_{t,t+\tau}] = \mathbb{E}\left[\mathbb{E}_t\left[R_{t,t+\tau}\right]\right] = \exp\Bigg\{ &-\tau\mu_s - \frac{1}{2}\sum_{j=1}^{\infty}\|U_s'(I-G)^{-1}(I-G^\tau)G^{j-1}H\|^2\\ &- \frac{1}{2}\sum_{k=1}^{\tau}\left(\|\psi_s(k)\|^2 + 2\psi_s(k)\cdot\psi_r(k)\right)\Bigg\}.\end{aligned}$$

Then, cancel terms from (82).

$\square$

- Alternatively, the unconditional risk premium where we take expectations after differencing is simple:

$$\mathbb{E}[\log R_{t,t+1} - \log R^f_{t,t+1}] = \mathbb{E}[\mathbb{E}_t[\log R_{t,t+1} - \log R^f_{t,t+1}]] = -\frac{1}{2}\|\lambda_r\|^2 - \lambda_s \cdot \lambda_r.$$

- Over a τ-period horizon, this is

$$\mathbb{E}[\log R_{t,t+\tau} - \log R^{f,\tau}_{t,t+\tau}] = -\frac{1}{2}\sum_{k=1}^{\tau}\|\psi_r(k)\|^2 - \frac{1}{2}\sum_{k=1}^{\tau}\psi_s(k)\cdot\psi_r(k). \tag{84}$$

This is computed using (81).

- Unconditional covariance of excess returns of one portfolio with the excess returns of another. Suppose the log one-period returns of portfolio 1 are

$$\log R^1_{t,t+1} = \mu_{r^1} + U'_{r^1}x_t + \lambda_{r^1}w_{t+1}$$

and the log one-period returns of portfolio 2 are

$$\log R^2_{t,t+1} = \mu_{r^2} + U'_{r^2}x_t + \lambda_{r^2}w_{t+1}.$$

Then,

$$\mathrm{Cov}(\log R^1_{t,t+1} - \log R^f_{t,t+1}, \log R^2_{t,t+1} - \log R^f_{t,t+1}) = \lambda_{r^1}\cdot\lambda_{r^2}.$$

To see this, note that

$$\log R_{t,t+1} - \log R^f_{t,t+1} = \mu_r + \mu_s + (U_s + U_r)'x_t + \frac{1}{2}\|\lambda_s\|^2 + \lambda_r \cdot w_{t+1} \tag{85}$$

$$= -\frac{1}{2}\|\lambda_r\|^2 - \lambda_s\cdot\lambda_r + \lambda_r\cdot w_{t+1}. \tag{86}$$

This implies that the unconditional variance of excess returns:

$$\mathrm{Var}(\log R_{t,t+1} - \log R^f_{t,t+1}) = \lambda_r\cdot\lambda_r = \|\lambda_r\|^2.$$

- Similarly, over τ-periods,

$$\mathrm{Cov}(\log R^1_{t,t+\tau} - \log R^{f,\tau}_{t,t+\tau}, \log R^2_{t,t+\tau} - \log R^{f,\tau}_{t,t+\tau}) = \sum_{k=1}^{\tau}\psi_{r^1}(k)\cdot\psi_{r^2}(k).$$

This follows from (81). Note that this work because of the homoskedasticity of the model. Because volatility is not time varying, the conditional expectation of the excess returns doesn't vary with the state.

- From (86), we see that the excess returns only depend on the shock at time $t+1$. Since w_{t+1} is i.i.d., w_{t+1} is uncorrelated with x_t. Thus,

$$\mathrm{Cov}(\log R_{t,t+1} - \log R^f_{t,t+1}, \log S_{t+1} - \log S_t) = \lambda_r \cdot \lambda_s$$

  and, similarly,

$$\mathrm{Cov}(\log R_{t,t+\tau} - \log R^{f,\tau}_{t,t+\tau}, \log S_{t+\tau} - \log S_t) = \sum_{k=1}^{\tau} \psi_r(k) \cdot \psi_s(k). \tag{87}$$

- This allows us to express the risk premium over a τ-period horizon in terms of conditional moments:

$$\log \mathbb{E}[R_{t,t+\tau}] - \log \mathbb{E}[R^{f,\tau}_{t,t+\tau}] = -\mathrm{Cov}(\log R_{t,t+\tau} - \log R^{f,\tau}_{t,t+\tau}, \log S_{t+\tau} - \log S_t). \tag{88}$$

  This follows from (81).

Miscellaneous Calculations

- The j-th autocovariance for the state VAR(1) process is

$$\mathrm{Cov}(x_t, x_{t-j}) = \sum_{i=0}^{\infty} G^{j+i} H H' (G^i)'.$$

  The unconditional variance can also be expressed as follows,

$$\begin{aligned} \Sigma_{xx} &= G \Sigma_{xx} G' + HH' \\ \mathrm{vec}\ \Sigma_{xx} &= (G \otimes G)\ \mathrm{vec}\ \Sigma_{xx} + \ \mathrm{vec}\ HH' \\ \mathrm{vec}\ \Sigma_{xx} &= (I - G \otimes G)^{-1}\ \mathrm{vec}\ HH', \end{aligned}$$

  where $\Sigma_{xx} = \mathrm{Cov}(x_t, x_t)$.

A.4.3 Term structure of equity

Before discussing the term structure of equity, I briefly discuss term structure of interest rates on riskless bonds and the holding period returns associated with long term bonds.

Yields and Bond Returns

- Recall the definition and derivation of the yields, Y_t^τ, in (74) and (76). The one-period holding period return on such a bond is defined as

$$R_{t,t+1}^{f,\tau} \coloneqq \frac{B_{t+1}^{\tau-1}}{B_t^\tau}. \tag{89}$$

In accord with (74),

$$R_{t,t+\tau}^{f,\tau} = \frac{1}{B_t^\tau},$$

since $B_t^\tau\Big|_{\tau=0} = 1$ for any t.

- The one-period holding period return on a τ-horizon bond is

$$\log R_{t,t+1}^{f,\tau} = -\mu_s - \frac{1}{2}\|\psi_s(\tau)\|^2 - U_s' x_t + (\psi_s(\tau) - \psi(1)) \cdot w_{t+1}. \tag{90}$$

Expected excess return over the short-term interest rate is

$$\log \mathbb{E}\left[R_{t,t+1}^{f,\tau}\right] - \log \mathbb{E}\left[R_{t,t+1}^{f}\right] = -(\psi_s(\tau) - \psi_s(1)) \cdot \psi_s(1). \tag{91}$$

This is called the *term premium*.

Proof. Using (73),

$$\begin{aligned}
\log R_{t,t+1}^{f,\tau} &= (\tau-1)\mu_s + U_s'(I-G)^{-1}(I-G^{\tau-1})(Gx_t + Hw_{t+1}) + \frac{1}{2}\sum_{j=1}^{\tau-1}\|\psi_s(j)\|^2 \\
&\quad - \tau\mu_s - U_s'(I-G)^{-1}(I-G^\tau)x_t - \frac{1}{2}\sum_{j=1}^{\tau}\|\psi_s(j)\|^2 \\
&= -\mu_s - U_s'(I-G)^{-1}\left[(I-G^\tau) - (I-G^{\tau-1})G\right]x_t \\
&\quad U_s'(I-G)^{-1}(I-G^{\tau-1})Hw_{t+1} - \frac{1}{2}\|\psi_s(\tau)\|^2 \\
&= -\mu_s - U_s'x_t + U_s'(I-G)^{-1}(I-G^{\tau-1})Hw_{t+1} - \frac{1}{2}\|\psi_s(\tau)\|^2 \\
&= -\mu_s - \frac{1}{2}\|\psi_s(\tau)\|^2 - U_s'x_t + (\psi_s(\tau) - \psi_s(1)) \cdot w_{t+1}
\end{aligned}$$

Calculating the risk-premium associated with this return, i.e. the term premium, follows from (83).

$\square$

Term structure of equity

- Consider a dividend process

$$\log D_{t+1} - \log D_t = \mu_d + U_d' x_t + \lambda_d w_{t+1}.$$

Let the price of a claim to the dividend payout τ periods in the future be

$$P_t^\tau = \mathbb{E}_t\left[\frac{S_{t+\tau}}{S_t} D_{t+\tau}\right].$$

The return to holding this claim to maturity is

$$R_{t,t+\tau}^\tau = \frac{D_{t+\tau}}{P_t^\tau}$$

and the holding period return, holding the claim for $k \leq \tau$ periods is

$$R_{t,t+k}^\tau = \frac{P_{t+k}^{\tau-k}}{P_t^\tau}.$$

- van Binsbergen et al. (2013) define the *equity yield* as as

$$e_{t,\tau} = \frac{1}{\tau}\log\left(\frac{1}{(P_t^\tau/D_t)}\right) = \frac{1}{\tau}\log\left(\frac{D_t}{P_t^\tau}\right).$$

and the *forward equity yield* as

$$e_{t,\tau}^f = \frac{1}{\tau}\log\left(\frac{D_t}{F_t^\tau}\right) = \frac{1}{\tau}\log\left(\frac{D_t}{P_t^\tau}\right) - \log Y_t^\tau,$$

where F_t^τ is the futures (forward) price of the dividend strip. The second equality holds by no-arbitrage, with $F_t^\tau = P_t^\tau \cdot (Y_t^\tau)^\tau$.

- The price of a dividend strip, using the conditional log-normal framework from before, is

$$\begin{aligned} P_t^\tau &= D_t \mathbb{E}_t\left[\frac{S_{t+\tau}}{S_t}\frac{D_{t+\tau}}{D_t}\right] \\ &= D_t \exp\left\{\tau(\mu_s + \mu_d) + (U_s + U_d)'(I-G)^{-1}(I-G^\tau)x_t + \frac{1}{2}\sum_{k=1}^{\tau}\|\psi_s(k) + \psi_d(k)\|^2\right\} \end{aligned}$$

- The k-period holding period return is

$$
\begin{aligned}
\log R^{\tau}_{t,t+k} &= \log\left(\frac{P^{\tau-k}}{P^{\tau}_t}\right) \\
&= -k\mu_s - \frac{1}{2}\sum_{j=\tau-k+1}^{\tau} \|\psi_s(j)+\psi_d(j)\|^2 \\
&\quad - U_s'(I-G)^{-1}(I-G^k)x_t \\
&\quad + \sum_{j=1}^{k}(\psi_d'(j) + (U_s+U_d)'(I-G)^{-1}(I-G^{\tau-k})G^{j-1}H)w_{t+k+1-j}.
\end{aligned}
\tag{92}
$$

Proof.

$$
\begin{aligned}
\log R^{\tau}_{t,t+k} &= \log\left(\frac{P^{\tau-k}}{P^{\tau}_t}\right) \\
&= \log\left(\frac{D_{t+k}}{D_t}\frac{P^{\tau-k}_{t+k}/D_{t+k}}{P^{\tau}_t/D_t}\right) \\
&= -k\mu_s + U_d'(I-G)^{-1}(I-G^k)x_t + \sum_{j=1}^{k}\psi_d(j)\cdot w_{t+1+k-j} - \frac{1}{2}\sum_{j=\tau-k+1}^{\tau}\|\psi_s(j)+\psi_d(j)\|^2 \\
&\quad + (U_s+U_d)'(I-G)^{-1}\left[(I-G^{\tau-k})G^k - (I-G^{\tau})\right]x_t \\
&\quad + (U_s+U_d)'(I-G)^{-1}(I-G^{\tau-k})\sum_{j=1}^{k}G^{j-1}Hw_{t+k+1-j} \\
&= -k\mu_s + \sum_{j=1}^{k}\psi_d(j)\cdot w_{t+1+k-j} - \frac{1}{2}\sum_{j=\tau-k+1}^{\tau}\|\psi_s(j)+\psi_d(j)\|^2 \\
&\quad - U_s'(I-G)^{-1}(I-G^k)x_t \\
&\quad + (U_s+U_d)'(I-G)^{-1}(I-G^{\tau-k})\sum_{j=1}^{k}G^{j-1}Hw_{t+k+1-j} \\
&= -k\mu_s - \frac{1}{2}\sum_{j=\tau-k+1}^{\tau}\|\psi_s(j)+\psi_d(j)\|^2 \\
&\quad - U_s'(I-G)^{-1}(I-G^k)x_t \\
&\quad + \sum_{j=1}^{k}(\psi_d'(j) + (U_s+U_d)'(I-G)^{-1}(I-G^{\tau-k})G^{j-1}H)w_{t+k+1-j}.
\end{aligned}
$$

□

- When $k = 1$, the holding period return is

$$\begin{aligned}\log R^{\tau}_{t,t+1} &= -\mu_s - \frac{1}{2}\|\psi_s(\tau) + \psi_d(\tau)\|^2 - U_s' x_t \\ &\quad + (\psi_d'(1) + (U_s + U_d)'(I - G)^{-1}(I - G^{\tau-1})H)w_{t+1} \\ &= \mu_\tau + U_\tau' x_t + \lambda_\tau' w_{t+1},\end{aligned} \tag{93}$$

where

$$\begin{aligned}\mu_\tau &= -\mu_s - \frac{1}{2}\|\psi_s(\tau) + \psi_d(\tau)\|^2 \\ U_\tau &= -U_s \\ \lambda_\tau &= \psi_d(1) + (\psi_s(\tau) - \psi_s(1) + \psi_d(\tau) - \psi_d(1)) \\ &= \psi_s(\tau) - \psi_s(1) + \psi_d(\tau).\end{aligned}$$

Note that no arbitrage in this context requires that the conditional expectation of the holding period returns move one-for-one with the SDF. If ψ_d is zero at all horizons, then the claim is the return on holding a risk-free bond and should match the holding period returns of the risk-free bonds derived earlier.

- It follows that

$$\begin{aligned}\log R^{\tau}_{t,t+1} - \log R^{f}_{t,t+1} = &-\frac{1}{2}\|\psi_s(\tau) - \psi_s(1) + \psi_d(\tau)\|^2 - (\psi_s(\tau) - \psi_s(1) + \psi_d(\tau)) \cdot \psi_s(1) \\ &+ (\psi_s(\tau) - \psi_s(1) + \psi_d(\tau)) \cdot w_{t+1}.\end{aligned}$$

- When holding this claim to maturity, the return is

$$R^{\tau}_{t,t+\tau} = \frac{D_{t+\tau}}{P^{\tau}_t} = \exp\left\{\sum_{k=1}^{\tau} \mu_d + U_d' x_{t+k-1} + \lambda_d' w_{t+k}\right\} \frac{D_t}{P^{\tau}_t}.$$

The risk premium associated with this return is

$$\log \mathbb{E}\left[R^{\tau}_{t,t+\tau}\right] - \log \mathbb{E}\left[R^{f}_{t,t+\tau}\right] = -\sum_{k=1}^{\tau} \psi_s(k) \cdot \psi_d(k).$$

The derivation is a simple application of (83).

- The risk premium associated with the k-period holding period returns is

$$\log \mathbb{E}\left[R^{\tau}_{t,t+k}\right] - \log \mathbb{E}\left[R^{f}_{t,t+k}\right] = -\sum_{j=1}^{k} \psi_s(j) \cdot \Big(\psi_d(j) + (U_s + U_d)'(I-G)^{-1}(I - G^{\tau-k})G^{j-1}H\Big)$$
$$= -\sum_{j=1}^{k} \psi_s(j) \cdot \psi_d(j)$$
$$-\sum_{j=1}^{k} \psi_s(j) \cdot (U_s + U_d)'(I-G)^{-1}(I - G^{\tau-k})G^{j-1}H. \quad (94)$$

Proof. Use (92) and apply the formula in (83).

□

- Combining this result with the term structure of interest rates results, we see that we must have

$$\log \mathbb{E}\left[R^{\tau}_{t,t+1}\right] - \log \mathbb{E}\left[R^{f}_{t,t+1}\right] = \log \mathbb{E}\left[R^{f,\tau}_{t,t+1}\right] - \log \mathbb{E}\left[R^{f}_{t,t+1}\right] - \psi_s(1) \cdot \psi_d(\tau) \quad (95)$$

or, alternatively,

$$\log \mathbb{E}\left[R^{\tau}_{t,t+1}\right] - \log \mathbb{E}\left[R^{f,\tau}_{t,t+1}\right] = -\psi_s(1) \cdot \psi_d(\tau). \quad (96)$$

Proof. Use (93) and apply (83). Then, apply equation (91), which says

$$\log \mathbb{E}\left[R^{f,\tau}_{t,t+1}\right] - \log \mathbb{E}\left[R^{f}_{t,t+1}\right] = -(\psi_s(\tau) - \psi_s(1)) \cdot \psi_s(1) = -\psi_s(\tau) \cdot \psi_s(1) + \psi_s(1) \cdot \psi_s(1).$$

□

- In terms of unconditional moments,

$$\mathrm{Cov}(\log R^{f,\tau}_{t,t+1} - \log R^{f}_{t,t+1}, \log S_{t+1} - \log S_t) = (\psi_s(\tau) - \psi_s(1)) \cdot \psi_s(1), \quad (97)$$

and

$$\mathrm{Cov}(\log R^{\tau}_{t,t+1} - \log R^{f}_{t,t+1}, \log S_{t+1} - \log S_t) = (\psi_s(\tau) - \psi_s(1) + \psi_d(\tau)) \cdot \psi_s(1). \quad (98)$$

Also,

$$\mathrm{Cov}(\log R^{\tau}_{t,t+1} - \log R^{f,\tau}_{t,t+1}, \log S_{t+1} - \log S_t) = \psi_d(\tau) \cdot \psi_s(1). \quad (99)$$

Proof. The first holds because the conditional expectation of the excess returns on the bond don't depend on the state,

$$\log R_{t,t+1}^{f,\tau} - \log R_{t,t+1}^{f} = -\frac{1}{2}\|\psi_s(\tau)\|^2 + \frac{1}{2}\|\psi_s(1)\|^2. + (\psi_s(\tau) - \psi_s(1)) \cdot w_{t+1}.$$

Same with the second, as seen in equation (93). The third, by extension.

□

- The holding period return, held for k periods, on a dividend strip paying of at horizon $\tau \geq k$ is related to dividend growth and equity yields as follows:

$$\begin{aligned} R_{t,t+k}^{\tau} &= \frac{D_{t+k}}{D_t} \cdot \frac{P^{\tau-k}/D_{t+k}}{P_t^{\tau}/D_t} \\ \frac{1}{k}\log R_{t,t+k}^{\tau} &= \frac{1}{k}\log\frac{D_{t+k}}{D_t} + \frac{\tau}{k}e_{t,\tau} - \frac{\tau-k}{k}e_{t+k,\tau-k}. \end{aligned}$$

  The risk associated with the holding period return can be thought of as coming from two sources: risk associated with dividend growth and risk associated with fluctuations in the future equity yield $e_{t+k,\tau-k}$. This can be seen in the previous derivation. In equation (94), the first summation in the premium is a result of exposure to dividend growth risk and the second summation is a result of exposure to changes in the equity yields (fluctuations in the price dividend ratio of the maturing dividend strip).

- Note that in a frictionless environment, when a representative households has Epstein-Zin preferences and the elasticity of intertemporal substitution (EIS) is equal to one,
  - the price dividend ratio of the wealth portfolio is constant.
  - Also, relatedly, in the case our linear state space model, with Epstein-Zin preferences and EIS of one, we will have $U_s + U_d = \vec{0}$. Thus, the holding period return will only depend on the periods held k and not on the horizon τ.
  - The equity yields will be constant (or at least $e_{t+k,\tau-k}$ will be known at time t).

- As an additional exercise to understand (94), consider the case where $\tau = 2$ and $k = 1$. In this case,

$$\log \mathbb{E}\left[R_{t,t+1}^{2}\right] - \log \mathbb{E}\left[R_{t,t+1}^{f}\right] = -\psi_s(1)\cdot\psi_d(1) - \psi_s(1)\cdot(U_s + U_d)'H.$$

Define $\Delta E_{t+1}[X_{t+2}] = E_{t+1}[X_{t+2}] - E_t[X_{t+2}]$. Then,

$$\Delta \mathbb{E}_{t+1}\left[\log S_{t+2} - \log S_{t+1}\right] = U_s' H w_{t+1}.$$

Thus,

$$\psi_s(1) \cdot U_d' H = \mathrm{Cov}_t(\log S_{t+1} - \log S_t, \Delta \mathbb{E}_{t+1}\left[\log D_{t+2} - \log D_{t+1}\right])$$

and

$$\begin{aligned}
\log \mathbb{E}_t\left[R^2_{t,t+1}\right] - \log \mathbb{E}\left[R^f_{t,t+1}\right] = &-\mathrm{Cov}_t(\log D_{t+1} - \log D_t, \log S_{t+1} - \log S_t) \\
&- \mathrm{Cov}_t(\Delta \mathbb{E}_{t+1}\left[\log S_{t+2} - \log S_{t+1}\right], \log S_{t+1} - \log S_t) \\
&- \mathrm{Cov}_t(\Delta \mathbb{E}_{t+1}\left[\log D_{t+2} - \log D_{t+1}\right], \log S_{t+1} - \log S_t).
\end{aligned}$$

With this, we see that the risk associated with the holding-period returns on this τ-horizon dividend strip comes from exposure to fluctuations in dividend growth, fluctuations in expected future discount rates, and fluctuations in expected future dividend growth.

A.4.4 Dynamic Value Decomposition

Some of the derivations rely on the application of Epstein-Zin preferences with the elasticity of intertemporal substitution $\rho^{-1} = 1$. With these assumption, we obtain expressions for risk-prices are analytically tractable and take on a simple log-normal form. In cases where such tractability is impossible, such as when $\rho \neq 1$, a useful set of tools is the Dynamic Value Decomposition (DVD) developed and described in Hansen and Scheinkman (2012), Borovicka et al. (2011), Hansen (2012), and Borovička and Hansen (2014). This dynamic value decomposition defines a set of elasticities that can be interpreted as pricing counterparts to impulse response functions. These elasticities are a set of measures used to measure the contribution of exposure to individual shocks to risk prices, asset prices, and expected payoffs at alternative horizons. They are designed to accommodate nonlinearities in the stochastic evolution of the model's underlying Markov process. It does so by tracing out changes in conditional expectations of future quantities in response to a marginal change in exposures to shocks. In this regard, they are conceptually similar to to some nonlinear versions of impulse response functions. These elasticities can be viewed as a generalization of the risk-prices and risk-exposures analyzed in the preceding sections in the sense that the elasticities are exactly the impulse response functions arising from the underlying VAR system. In this section, I'll present a short primer on the decomposition and show how it applies to the log-normal case analyzed before. Borovička and Hansen (2014) show how these can be applied to second-order perturbations solutions of DSGE models featuring Epstein-Zin preferences and that the elasticities in such cases take on a quasi-analytic form. For this reason, these are especially useful in this paper.

These will be useful when I solve the model using higher order methods and when $\rho \neq 1$.

Definitions Given the stochastic discount factor S_t, a payoff D_t (e.g., a dividend strip), and a parameterized perturbation $\mathcal{E}_1(r)$ (here, a random variable that converges in probability to one when r converges to 0), we can compute elasticities that measure the sensitivity of the logarithm of expected returns, prices, or expected cash flows to a change in exposure to normalized shocks underlying the model. Let $\mathcal{E}_1(r)$ be a Radon-Nykodym derivative or perturbation parameterized by the radius r for a given vector function α_0,

$$\log \mathcal{E}_1(r) = r\alpha_0(x_0) \cdot W_1 - \frac{r^2}{2}|\alpha_0(x_0)|^2.$$

As defined in Borovička and Hansen (2014), we have

- *Shock-exposure elasticity* at horizon t (effect on expected cash flow):

$$\rho_d(t,x) = \frac{\mathrm{d}}{\mathrm{d}r} \log \mathbb{E}[D_t \mathcal{E}_1(r) \mid x_0 = x]\Big|_{r=0}$$

- *Shock-value elasticity* at horizon t (effect on present discounted value of cash flow, assuming $S_0 = 1$):

$$\rho_c(t,x) = \frac{\mathrm{d}}{\mathrm{d}r} \log \mathbb{E}[S_t D_t \mathcal{E}_1(r) \mid x_0 = x]\Big|_{r=0}$$

- *Shock-price elasticity* at horizon t (effect on log expected return on claim):

$$\rho_p(t,x) = \rho_d(t,x) - \rho_c(t,x).$$

Example: the log-normal case Here I calculate the three defined shock elasticities in a log-normal environment. I'll demonstrate that the shock elasticities are exact and that the results have an intuitive interpretation.

Suppose now that this cash flow is a log-normal process defined by

$$\log D_{t+1} = \mu_d + U_d \cdot x_t + \lambda_d \cdot w_{t+1}$$

where x_t is a state variable defined as

$$x_{t+1} = Gx_t + Hw_{t+1}$$

and w_{t+1} is an i.i.d. normally distributed random vector. Furthermore, let S_t be a stochastic discount factor with

$$\log S_{t+1} - \log S_t = \mu_s + U_s' x_t + \lambda_s \cdot w_{t+1}.$$

Compute the Shock-Exposure Elasticity The shock exposure elasticity is defined as

$$\begin{aligned} \rho_d(t,x) &= \left.\frac{\mathrm{d}}{\mathrm{d}r} \ln \mathbb{E}[D_t \mathcal{E}_1(r) \mid x_0 = x]\right|_{r=0} \\ &= \alpha_0 \cdot \frac{\mathbb{E}[D_t w_1 \mid x_0 = x]}{\mathbb{E}[D_t \mid x_0 = x]}. \end{aligned}$$

Expanding, $D_t = D_0 \exp\left(\sum_{i=1}^t \ln D_i - \ln D_{i-1}\right)$, we can calculate

$$\begin{aligned} \mathbb{E}[D_t w_1 \mid x_0 = x] &= \mathbb{E}_t\left[D_0 \exp\left(\sum_{i=1}^t \ln D_i - \ln D_{i-1}\right) w_1 \mid x_0 = x\right] \\ &= \mathrm{Cov}\left(D_0 \exp\left(\sum_{i=1}^t \ln D_i - \ln D_{i-1}\right), w_1 \mid x_0 = x\right). \end{aligned}$$

Since $\sum_{i=1}^t \ln D_i - \ln D_{i-1}$ and w_1, conditional on $x_0 = x$, are jointly normally distributed, we can apply Stein's lemma to get

$$\begin{aligned} &\mathrm{Cov}\left(D_0 \exp\left(\sum_{i=1}^t \ln D_i - \ln D_{i-1}\right), w_1 \mid x_0 = x\right) \\ &\quad = \mathbb{E}[D_t \mid x_0 = x]\ \mathrm{Cov}\left(\sum_{i=1}^t \ln D_i - \ln D_{i-1}, w_1 \mid x_0 = x\right). \end{aligned}$$

Thus, the shock exposure elasticity in this log-normal case is simply the impulse response function of $\log D_t$. Using the notation analogous to (72), the shock exposure elasticity of the cash flow D_t is

$$\rho_d(t) = \alpha_0 \cdot \psi_d(t) = \alpha_0 \cdot \left(\lambda_d' + U_d'(I-G)^{-1}(I-G^{t-1})H\right).$$

Compute the Shock-Value Elasticity Given that

$$\log(D_{t+1}S_{t+1}) - \log(D_t S_t) = (\log D_{t+1} - \log D_t) - (\log S_{t+1} - \log S_t),$$

the shock value elasticity calculation amounts to summing the impulse responses of S_t and D_t. That is,

$$\rho_c(t) = \alpha_0 \cdot (\psi_d(t) + \psi_s(t)).$$

Compute the Shock-Price Elasticity

$$\rho_p(t) = \rho_d(t) - \rho_v(t) = \alpha_0 \cdot \psi_s(t)$$

A.5 Risk prices in an economy with Epstein-Zin preferences

In this section, I derive risk prices in an economy with Epstein-Zin preferences that is governed by a VAR(1) state space. I derive the solution where the risk aversion parameter is different from the reciprocal of the elasticity of intertemporal substitution, $\gamma \neq \rho$ and the elasticity of intertemporal substitution is one, $1/\rho = 1$. Under such an assumption, the household still has concern for long-run risk, but the solution admits an analytic solution. In Section A.5.1, I solve the utility recursion to obtain an expression for the stochastic discount factor. In Section

Many other results from this paper follow as a corollary to the derivations presented within this section. Note that the derivations here follow those within Hansen, Heaton, and Li (2008).

A.5.1 Solving for stochastic discount factor

In this section, I solve the utility recursion resulting from the assumed Epstein-Zin preferences and the given stochastic process governing the dynamics of the state of the economy. This will give us a characterization of the stochastic discount factor in terms of the underlying state dynamics and will allow a clean expression for risk-prices and risk-premia over arbitrary horizons.[9]

Let a representative agent have Epstein-Zin preferences, defined by the recursion

$$V_t = \{(1-\beta)(C_t)^{1-\rho} + \beta[\mathcal{R}_t(V_{t+1})]^{1-\rho}\}^{1/(1-\rho)},$$

[9] When dealing with Epstein-Zin preferences, there are generally two approaches in the literature to taking the SDF to the data. On approach is to rewrite the continuation value term in terms of the return on the wealth portfolio,

$$R^W_{t+1} = \frac{W_{t+1}}{W_t - C_t}. \tag{100}$$

where

$$\mathcal{R}_t(V_{t+1}) \equiv \mathbb{E}[(V_{t+1})^{1-\gamma} \mid \mathcal{F}_t]^{1/(1-\gamma)}.$$

Using the framework laid out in the previous section, again let x_t be a $N \times 1$ state vector capturing the state of the economy. Assume that it follows the VAR(1) process defined in equation (70), with w_{t+1} as a $M \times 1$ vector of i.i.d. shocks, $w_{t+1} \sim \mathcal{N}(0, \mathcal{I})$ and G and H as conforming matrices. Let log consumption growth be given by

$$c_{t+1} - c_t = \mu_c + U_c \cdot x_t + \lambda_c \cdot w_{t+1}. \tag{102}$$

The assumed preferences imply the stochastic discount factor (SDF),

$$\begin{aligned}
\frac{S_{t+1}}{S_t} &= (MV_{t+1}) \frac{MC_{t+1}}{MC_t} \\
&= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{\mathcal{R}_t(V_{t+1})} \right)^{\rho-\gamma} \\
&= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left[\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t[V_{t+1}^{1-\gamma}]} \right]^{\frac{\rho-\gamma}{1-\gamma}}.
\end{aligned}$$

To characterize this SDF in terms of the assumed dynamics for the state (70) and consumption growth (102), we need to solve for the continuation values V_t in terms of the same. The utility recursion that characterizes Epstein-Zin preferences, paired with the assumed state and consumption growth dynamics, results in a functional equation that we may solve. Rearranging terms,

$$\begin{aligned}
V_t &= \{(1-\beta)(C_t)^{1-\rho} + \beta[\mathcal{R}_t(V_{t+1})]^{1-\rho}\}^{1/(1-\rho)} \\
\frac{V_t}{C_t} &= \left\{ (1-\beta) + \beta \left[\mathcal{R}_t \left(\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right) \right]^{1-\rho} \right\}^{1/(1-\rho)},
\end{aligned}$$

This involves an application of Euler's theorem and results in

$$\frac{S_{t+1}}{S_t} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\theta\rho} (R^w_{t+1})^{\theta-1}, \tag{101}$$

where $\theta = \frac{1-\gamma}{1-\rho}$. While some papers use the returns on broad market indices (e.g., the S&P 500 index) as a proxy for R^W_t, in general this return in unobservable. The Roll critique applies here. The aggregate value of stock market indices make up only a small portion of total wealth in the economy. Importantly, it is missing the value of human capital. Furthermore, the returns on these proxies may not adequately capture the long-run effects of certain shocks to the macroeconomy. In models such as those of Bansal and Yaron (2004) and Hansen, Heaton, and Li (2008), the risk associated with these long-run shocks is crucial. To alleviate the shortcomings of using such a proxy, the second approach is to solve for the continuation value directly in terms of stochastic process governing the state of the economy, including consumption growth.

so we need only solve $v_t - c_t \equiv \log \frac{V_t}{C_t}$ as a function of the state x_t.

Adding the assumption that the elasticity of intertemporal substitution is one Here I derive the solution for risk prices in terms of the underlying state dynamics, in the case that $\rho = 1$. This requires us to solve for the SDF, and thus the continuation values of utility. If we assume that $\rho = 1$, then we may obtain an analytic solution. Otherwise, we must approximate the solution. Note that when $\rho = \gamma$, the assumed preferences collapse into CRRA preferences. Further, when $\rho = \gamma = 1$, they become log-utility. When $\rho = 1$ and $\gamma > 1$, the agent's preferences are not of the time-additive von Neumann–Morgenstern expected utility variety and the agent still exhibits a concern for long-run risk.

Solving for the continuation value. Calculating the limit of the utility recursion as $\rho \to 1$ (applying l'Hopital's rule), we get

$$\begin{aligned}
v_t - c_t &= \frac{\beta}{1-\gamma} \ln \left(\mathbb{E} \left[\left(\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right)^{1-\gamma} \middle| \mathcal{F}_t \right] \right) \\
&= \frac{\beta}{1-\gamma} \ln \left(\mathbb{E}_t \left[\exp \left\{ (1-\gamma)(v_{t+1} - c_{t+1} + c_{t+1} - c_t) \right\} \right] \right) \\
&= \frac{\beta}{1-\gamma} \ln \left(\mathbb{E}_t \left[\exp \left\{ (1-\gamma)(v_{t+1} - c_{t+1} + \mu_c + U_c' x_t + \lambda_c \cdot w_{t+1}) \right\} \right] \right).
\end{aligned}$$

The end result here is a difference equation in $v_t - c_t$. In order to solve this difference equation, I proceed by the method of undetermined coefficients. Let us postulate that $v_t - c_t = \phi(x_t) = \mu_v + U_v \cdot x_t$ for some presently unknown coefficients μ_v and U_v. Substituting,

$$\begin{aligned}
\mu_v + U_v' x_t &= \frac{\beta}{1-\gamma} \ln \left(\mathbb{E}_t \left[\exp \left\{ (1-\gamma)(\mu_v + U_v'(G x_t + H w_{t+1}) + \mu_c + U_c' x_t + \lambda_c \cdot w_{t+1}) \right\} \right] \right) \\
&= \beta \left(\mu_v + \mu_c + (U_v' G + U_c') x_t + \frac{1}{2}(1-\gamma)^2 \| U_v' H + \lambda_c' \|^2 \right).
\end{aligned}$$

Then, matching terms, we see that we must have

$$\begin{aligned}
U_v' &= \beta U_c' (I - \beta G)^{-1} \\
\mu_v &= \frac{\beta}{1-\beta} \left(\mu_c + (1-\gamma)^2 \frac{1}{2} \| U_v' H + \lambda_c' \|^2 \right).
\end{aligned}$$

Note that these all conform, since if x_t is $N \times 1$ and w_{t+1} is $M \times 1$, then U_v and U_c are both $1 \times N$, G is $N \times N$, H is $N \times M$, and λ_c is $1 \times M$.

In summary, we get

$$v_t - c_t = \mu_v + U_v' x_t,$$

with

$$\begin{aligned}
\mu_v &= \frac{\beta}{1-\beta}\left(\mu_c + (1-\gamma)\frac{1}{2}\|U_v' H + \lambda_c\|^2\right) \\
U_v' &= \beta U_c'(I - \beta G)^{-1}.
\end{aligned}$$

Characterizing the SDF Now that we have solved for the continuation value in terms of the state dynamics, we can solve for the stochastic discount factor in terms of the state dynamics. In the limit as $\rho \to 1$,

$$\begin{aligned}
\frac{S_{t+1}}{S_t} &= \beta\left(\frac{C_{t+1}}{C_t}\right)^{-1}\left[\frac{V_{t+1}^{1-\gamma}}{\mathbb{E}_t[V_{t+1}^{1-\gamma}]}\right] \\
&= \beta\left(\frac{C_{t+1}}{C_t}\right)^{-1}\frac{\left(\frac{V_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{1-\gamma}}{\mathbb{E}_t\left[\left(\frac{V_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]}.
\end{aligned}$$

Computing,

$$\begin{aligned}
\mathbb{E}_t\left[\left(\frac{V_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right] &= \mathbb{E}_t[\exp\{(1-\gamma)((v_{t+1} - c_{t+1}) + (c_{t+1} - c_t))\}] \\
&= \exp\left[(1-\gamma)(\mu_v + \mu_c + (U_v' G + U_c')x_t) + \frac{1}{2}(1-\gamma)^2\|U_v' H + \lambda_c'\|^2\right]
\end{aligned}$$

Then, defining $s_{t+1} \equiv \ln S_{t+1}$, substituting in the definition for consumption dynamics, and substituting in our solution for the continuation value, we calculate

$$\begin{aligned}
s_{t+1} - s_t &= \ln\beta - (c_{t+1} - c_t) + (1-\gamma)((v_{t+1} - c_{t+1}) + (c_{t+1} - c_t)) \\
&\quad - \left[(1-\gamma)(\mu_v + \mu_c + (U_v' G + U_c')x_t) + \frac{1}{2}(1-\gamma)^2\|U_v' H + \lambda_c'\|^2\right] \\
&= \ln\beta - \gamma(\mu_c + U_c' x_t + \lambda_c \cdot w_{t+1}) + (1-\gamma)U_v' H w_{t+1} \\
&\quad - \left[(1-\gamma)(\mu_c + U_c' x_t) + \frac{1}{2}(1-\gamma)^2\|U_v' H + \lambda_c'\|^2\right] \\
s_{t+1} - s_t &= \mu_s + U_s' x_t + \lambda_s \cdot w_{t+1},
\end{aligned}$$

where

$$\begin{aligned}
\mu_s &= \ln\beta - \mu_c - \frac{1}{2}(1-\gamma)^2\|U_v' H + \lambda_c'\|^2 \\
U_s &= -U_c \\
\lambda_s' &= -(\gamma - 1)(\lambda_c' + U_v' H) - \lambda_c'.
\end{aligned}$$

Summary Here I summarize the assumptions and results. To summarize, x_t is an $N \times 1$ state vector capturing the state of the economy,

$$x_{t+1} = Gx_t + Hw_{t+1},$$

with w_{t+1} as a $M \times 1$ vector of i.i.d. shocks, $w_{t+1} \sim \mathcal{N}(0, \mathcal{I})$ and G and H as conforming matrices. Log consumption growth is

$$c_{t+1} - c_t = \mu_c + U_c \cdot x_t + \lambda_c \cdot w_{t+1}.$$

Solving the utility recursion for the case when $\rho = 1$, we get

$$v_t - c_t = \mu_v + U_v' x_t, \tag{103}$$

with

$$\begin{aligned} \mu_v &= \frac{\beta}{1-\beta}\left(\mu_c + (1-\gamma)\frac{1}{2}\|U_v' H + \lambda_c\|^2\right) \\ U_v' &= \beta U_c'(I - \beta G)^{-1}. \end{aligned} \tag{104}$$

The log SDF is

$$s_{t+1} - s_t = \mu_s + U_s' x_t + \lambda_s \cdot w_{t+1} \tag{105}$$

with

$$\begin{aligned} \mu_s &= \ln\beta - \mu_c - \frac{1}{2}(1-\gamma)^2\|U_v' H + \lambda_c'\|^2 \\ U_s &= -U_c \\ \lambda_s' &= -(\gamma - 1)(\lambda_c' + U_v' H) - \lambda_c'. \end{aligned} \tag{106}$$

□

Note that when $\gamma = 1$, preferences collapse to the log-normal case. Thus, let

$$\begin{aligned} \lambda_{s,SR} &= -\lambda_c \\ \lambda_{s,LR} &= -(\lambda_c' + U_v' H)', \end{aligned} \tag{107}$$

so that

$$\lambda_s = \lambda_{s,SR} + (\gamma - 1)\lambda_{s,LR}.$$

These are defined with the interpretation that $\lambda_{s,LR}$ is the portion of risk prices that are due to concern for long-run risk. When $\gamma = 1$, we have $\gamma = \rho$, and preferences thus collapse into the expected utility case—the log-utility case, in particular—and $\lambda_{s,LR} = 0$. Therefore, we can interpret $\lambda_{s,LR}$ as the component of risk prices due to a concern for long-run risk, under the assumption that $\rho = 1$.

A.5.2 Pricing a claim to aggregate consumption, the wealth portfolio

Since utility V_t is homogeneous of degree one in C_t and realizations of V_{t+1}, we can apply an infinite dimensional version of Euler's theorem to get

$$\frac{W_t}{C_t} = \frac{1}{1-\beta}\left(\frac{V_t}{C_t}\right)^{1-\rho},$$

where aggregate wealth is defined as the present discounted value of current and future consumption,

$$W_t = \mathbb{E}_t\left[\sum_{\tau=0}^{\infty}\frac{S_{t+\tau}}{S_t}C_{t+\tau}\right].$$

Define the return to the wealth portfolio as $R^W_{t+1} = \frac{W_{t+1}}{W_t - C_t}$,

$$\begin{aligned}
R^w_{t+1} &= \frac{W_{t+1}}{W_t - C_t} \\
&= \frac{W_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\frac{1}{\frac{W_t}{C_t}-1} \\
&= \frac{1}{1-\beta}\left(\frac{V_{t+1}}{C_{t+1}}\right)^{1-\rho}\frac{1}{\frac{1}{1-\beta}\left(\frac{V_t}{C_t}\right)^{1-\rho}-1}\frac{C_{t+1}}{C_t}.
\end{aligned}$$

When $\rho = 1$, this simplifies to

$$R^w_{t+1} = \frac{1}{\beta}\frac{C_{t+1}}{C_t}$$

and in terms of the state dynamics as written in (70) and (102),

$$\log R^w_{t+1} = -\log\beta + \mu_c + U_c' x_t + \lambda_c \cdot w_{t+1}.$$

From this, we can use (77) to easily calculate the one-period condition risk premium on a claim to aggregate consumption,

$$\begin{aligned}
\log\mathbb{E}_t\left[R^w_{t+1}\right] - \log R^f_t &= -\lambda_c\cdot\lambda_s = -\lambda_c\cdot\lambda_{s,SR} - \lambda_c\cdot\lambda_{s,LR} \\
&= \|\lambda_c\|^2 + (\gamma-1)\,\lambda_c\cdot(\lambda_c + U_v H).
\end{aligned} \tag{108}$$

Using (78), the normalized $\tau-$period conditional risk premium is

$$\begin{aligned}
\frac{1}{\tau}\left(\log \mathbb{E}_t\left[R^w_{t,t+\tau}\right] - \log R^{f,\tau}_{t,t+\tau}\right) &= -\frac{1}{\tau}\sum_{k=1}^{\tau}\psi_c(k)\cdot\psi_s(k) \\
&= -\frac{1}{\tau}\sum_{k=1}^{\tau}\left(\lambda_c' + U_c'\left(\sum_{i=1}^{k}G^{i-1}\right)H\right)\cdot\left(\lambda_s' + U_s'\left(\sum_{i=1}^{k}G^{i-1}\right)H\right) \\
&= -\frac{1}{\tau}\sum_{k=0}^{\tau-1}\left[\lambda_c\cdot\lambda_s + U_c'\left(\sum_{i=1}^{k}G^{i-1}\right)H\lambda_s + U_s'\left(\sum_{i=1}^{k}G^{i-1}\right)H\lambda_c + \right. \\
&\qquad \left. + U_c'\left(\sum_{i=1}^{k}G^{i-1}\right)HH'\left(\sum_{i=1}^{k}G^{i-1}\right)'U_s\right], \qquad (109)
\end{aligned}$$

where $\sum_{i=1}^{k}G^{i-1} = (I-G)^{-1}(I-G^k)$, where $\psi_c'(\tau) = \lambda_c' + U_c'(I-G)^{-1}(I-G^\tau)H$ is the impulse response function for C_t and $\psi_s'(\tau) = \lambda_s' + U_s'(I-G)^{-1}(I-G^\tau)H$ is the impulse response function for S_t.

- Now, consider a claim to consumption in the period $t+\tau$. Priced in time t, this is

  $$P_t^\tau = \mathbb{E}_t\left[\frac{S_{t+\tau}}{S_t}C_{t+\tau}\right].$$

  The holding period return on this claim, held for k periods is

  $$R^{\tau,\tau-k}_{t,t+k} = \frac{P^{\tau-k}_{t+k}}{P^\tau_t}.$$

  The risk premium associated with this return is

  $$\log\mathbb{E}\left[R^\tau_{t,t+k}\right] - \log\mathbb{E}\left[R^f_{t,t+k}\right] = -\sum_{j=1}^{k}\psi_c(j)\cdot\psi_s(j).$$

  Proof. Using the derivation from Section A.4.2, notice that $U_s = -U_c$.

  □

A.6 Network theory and measures of centrality

In graph theory (network theory), a graph (network) is made up of nodes (vertices) and edges (the connections between vertices). In an unweighted graph, edges between nodes either exist or they don't and are indicated with one or zero. A weighted graph is a generalization in which each edge has a numerical weight associated with it. A directed graph is a graph in which the connections between nodes have a direction associated with them.

Define a graph as an ordered pair $(\mathcal{N}, A)$, where $\mathcal{N} = \{1, 2, ..., n\}$ is a set of nodes and $A = [A_{ij}]$ is a matrix representing a possibly weighted and directed graph. This matrix A is called an adjacency matrix. In an unweighted, undirected graph, $A_{ij} = 1$ indicates an undirected connection between nodes i and j. Otherwise, the elements are zero. When the graph is undirected, A is symmetric. When the graph is directed, $A_{ij} = 1$ indicates a connection *from* node j to node i. Otherwise, the element is equal to zero. In the case of a weighted graph, the elements of A are real numbers and, in most applications, are non-negative.[10]

One exercise of interest in the study of networks is to measure the importance or influence of each node within a graph based on the node's positioning within the graph. Such measures are called measures of centrality. One of the simplest such measures is called *Degree centrality* . Degree centrality of a node i measures the number of edges that are connected to node i. In the case of a weighted graph, this measures the sum of the weights of the connected edges. This is expressed,

$$\vec{C}_{\text{degree}} \coloneqq \sum_{j=1}^{n} A_{ji} = A'\mathbb{1},$$

where $\mathbb{1}$ is a vector of ones.

Another measure of centrality is *Eigenvector centrality*, sometimes referred to as Bonacich centrality.[11] This measure is defined recursively. It captures the idea that a node has higher eigenvector centrality if it connected to another node that itself has high eigenvector centrality. $\vec{C}_{EV} = x$ such that x solves

$$x_i = \frac{1}{\lambda} \sum_{j=1}^{n} A_{ij} x_j,$$

or in matrix form,

$$Ax = \lambda x,$$

where λ is the principal eigenvalue of the adjacency matrix A and x is the associated normalized eigenvector.[12]

[10] For more details, good references include Jackson (2010) and Newman (2018).

[11] See Bonacich (2002).

[12] As a note, one drawback of Eigenvector centrality is that, when the graph is acyclic, all nodes will have centrality zero. There are variants of Eigenvector centrality that address this problem. Katz centrality can be thought of as one of these.

A third measure is *Katz centrality*, which measures the number of connections between other nodes, including connections that run through other nodes in the network. Each such path is weighted by an attenuation factor $\beta \in (0,1)$ so that links to distant nodes are given less weight. This measure can be expressed as[13]

$$\vec{C}_{\text{Katz}} \coloneqq \sum_{k=0}^{\infty} \beta^k (A^k)' \mathbb{1} = \sum_{k=0}^{\infty} \sum_{j=1}^{n} \beta^k (A^k)_{ji}.$$

This definition of Katz centrality can be interpreted by noting that if A represents a non-weighted, undirected network, then the element at location (i,j) of A^k reflects the total number of k degree connections between nodes i and j. Assuming that $\beta < 1/\lambda$, where λ is the principal eigenvalue of A, Katz centrality can be expressed as

$$\vec{C}_{\text{Katz}} = (I - \beta A')^{-1} \mathbb{1},$$

where I is a conforming identity matrix and $\mathbb{1}$ is a vector of ones. Katz centrality, like Eigenvector centrality, can be interpreted as a recursive measure of influence. In Eigenvector centrality, the centrality of a node i is a linear function of the centrality of the nodes that node i is connected to. In Katz centrality, the centrality of node i is an affine function of the centrality of the nodes that it is connected to:

$$x_i = \beta \sum_{j=1}^{n} A_{ij} x_j + 1. \tag{110}$$

That is, each node is endowed with some small amount of centrality "for free." In matrix terms, this is

$$x = \beta A x + \mathbb{1}$$

and thus $x = (I - \beta A)^{-1} \mathbb{1}$. Katz centrality can be thought of as a generalization of degree centrality and Eigenvector centrality because when β is small, most weight is placed on first-order connections. In light of eq. (110), when β is larger, more weight is placed on the recursive term rather than the constant term. When the attenuation factor approaches $1/\lambda$ from below, where λ is the principal eigenvalue of A, Katz centrality converges to Eigenvector centrality,

$$\lim_{\beta \nearrow \frac{1}{\lambda}} \vec{C}_{\text{Katz}} = \vec{C}_{EV}.$$

For a proof, see section A.6.1 below.

[13] There are different conventions for Katz centrality. In some cases, the infinite sum starts at $k = 1$, thus not including the initial term I. In Newman (2018), the sum starts at $k = 0$, as it does here.

Weighted Katz Centrality For our purposes, it will be useful to consider a slight generalization of Katz centrality called *weighted Katz centrality.* This allows us to endow each node with a prior sense of centrality, apart from the centrality dictated by the network connections. Recall that Katz centrality introduces a forcing term that ensures that nodes have a positive level of centrality so that the centrality of each node is an affine function of the centrality of the nodes that its connected to. In a sense, each term is given a unit amount of centrality "for free." Weighted Katz centrality allows this forcing terms (or starting amount) to vary across nodes. Let α by an $n \times 1$ vector of positive real numbers. Then weighted Katz centrality solves the equation

$$x = \beta A x + \alpha,$$

so that

$$\vec{C}_{\text{WKatz}} \coloneqq (I - \beta A)^{-1} \alpha.$$

This measure is sometimes called *Alpha centrality.*[14]

A.6.1 Proof: Eigenvector centrality as the limiting case of Katz centrality

Eigenvector centrality (sometimes called Bonacich centrality) is defined as the vector x that solves

$$Ax = \kappa x,$$

where κ is the principal (largest, most positive) eigenvalue of the adjacency matrix A. Katz centrality is defined as the vector y that solves

$$y = \alpha A y + \mathbf{1},$$

where $0 \leq \alpha < \kappa^{-1}$ and $\mathbf{1}$ is a conforming vector of ones. That is, $y = (\mathbf{I} - \alpha A)^{-1}\mathbf{1}$.

As $\alpha \nearrow \kappa^{-1}$, Katz centrality converges to eigenvector centrality.

Proof. Consider the definition of Eigenvector centrality given above where $\mathcal{C}^e = \frac{x}{\|x\|}$ and the definition of Katz centrality, where $\mathcal{C}^k(\alpha) = \frac{(\mathbf{I}-\alpha A)^{-1}\mathbf{1}}{\|(\mathbf{I}-\alpha A)^{-1}\mathbf{1}\|}$. When $\alpha < \kappa^{-1}$, recall that we can write $y = (\mathbf{I} - \alpha A)^{-1}\mathbf{1} = \mathbf{1} + \alpha A \mathbf{1} + \alpha^2 A^2 \mathbf{1} + \ldots$.

Define $a_n(\alpha) = \mathbf{1} + \alpha A \mathbf{1} + \alpha^2 A \mathbf{1} + \ldots + \alpha^n A^n \mathbf{1}$, let $a_0 = \mathbf{1}$. Define $b_n(\alpha) = \|a_n(\alpha)\|$.

Given these definitions, we would like to calculate the limit

$$\lim_{\alpha \nearrow \kappa^{-1}} \mathcal{C}^k(\alpha) = \lim_{\alpha \nearrow \kappa^{-1}} \lim_{n \to \infty} \frac{a_n(\alpha)}{b_n(\alpha)},$$

[14] https://en.wikipedia.org/wiki/Alpha_centrality

where κ is the principal eigenvalue of A.

Note that $\lim_{n\to\infty} a_n/b_n$ is defined for all values $\alpha \in [0, \kappa^{-1})$. Also, a_n and b_n are finite for $n < \infty$. We can thus switch the order of the limits so that

$$\lim_{\alpha \nearrow \kappa^{-1}} \mathcal{C}^k(\alpha) = \lim_{n\to\infty} \lim_{\alpha \nearrow \kappa^{-1}} \frac{a_n(\alpha)}{b_n(\alpha)} = \lim_{n\to\infty} \frac{a_n(\kappa^{-1})}{b_n(\kappa^{-1})}$$

Define

$$x_{n+1} = \frac{A^{n+1}x_0}{\|A^{n+1}x_0\|},$$

with $x_0 = \mathbf{1}$. Assuming the needed conditions for the power iteration algorithm, $x_{n+1} \to x$, where x is an eigenvector associated with principal eigenvalue of A.

Since $\det\left(\mathbf{I} - \frac{1}{\kappa}A\right) = 0$, we know that a_n and b_n diverge when $\alpha = \kappa^{-1}$. Assume that A is nonnegative (network edge weights are nonnegative). Then b_n is also strictly monotonic. This allows us to use the Stolz–Cesàro theorem

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}.$$

Calculating,

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{\alpha^{n+1}A^{n+1}\mathbf{1}}{\|a_{n+1}\| - \|a_n\|} = \frac{\kappa^{-n-1}\|A^{n+1}\mathbf{1}\|\, x_{n+1}}{\|a_{n+1}\| - \|a_n\|} \geq \frac{\kappa^{-n-1}\|A^{n+1}\mathbf{1}\|\, x_{n+1}}{\|\kappa^{-n-1}A^{n+1}\mathbf{1}\|} = x_{n+1},$$

where the inequality follows from the reverse triangle inequality. Since $\|a_n(\kappa^{-1})\|$ diverges and is monotonically increasing, the inequality holds with equality in the limit. Thus,

$$\lim_{\alpha \nearrow \kappa^{-1}} \mathcal{C}^k(\alpha) = \lim_{n\to\infty} \frac{a_n(\kappa^{-1})}{b_n(\kappa^{-1})} = \lim_{n\to\infty} x_n = x = \mathcal{C}^e.$$

□

A.7 Decomposing shocks into aggregate and idiosyncratic components

As specified, this model is closely related to the model analyzed in Foerster, Sarte, and Watson (2011). In that paper, the authors propose a procedure for decomposing industry productivity shocks. The key challenge in identifying the idiosyncratic component of sectoral shocks is disentangling the confounding effect of the network connections. That is, even if the productivity shocks in each industry were completely independent of each other, because production in each industry depends on the output of each other industry, output among the sectors will be correlated. A simple reduced-form factor analysis of the panel of industry output growth in this case would be biased, mistaking comovement due to network connectivity for the presence of common variation in industrys' productivity shocks. To correct for this, the authors construct a filter based on the calibrated model's equilibrium process. I apply their procedure here. For a full presentation of this challenge and its resolution, see the appendix, Section A.3 Here, I simply produce the filter to be used to disentangle the common variation from the effects of the network connections and discuss how I will use the results.

Suppose that sectoral shocks exhibit i.i.d. growth, so that

$$\log \Xi_{t+1} = \log \Xi_t + \varepsilon_{t+1}, \tag{111}$$

where $\varepsilon_t = (\varepsilon_{1,t}, \ldots, \varepsilon_{n,t})'$ is a vector-valued martingale difference process with covariance matrix $\Sigma_{\varepsilon,\varepsilon}$. Then, a first-order approximation around the balanced growth path results in model dynamics given by

$$(I - \Phi L)\Delta q_{t+1} = (\Pi_0 + \Pi_1 L)\varepsilon_t. \tag{112}$$

Supposing again that shocks to sectoral productivity have a common component and idiosyncratic components as described by equation (**??**), we see that

$$\varepsilon_t = (\Pi_0 + \Pi_1 L)^{-1}(I - \Phi L)\Delta q_t. \tag{113}$$

Since we observe Δq_t and we can estimate Π_0, Π_1, and Φ, (e.g., via calibration) we can construct the filtered series to which we can apply standard factor analytic methods. Such an analysis on the filtered industry growth data will allow we to recover ν_t^a and ν_t^s and to estimate the contributions of each to observed risk-premia. With these series, I can estimate the various exposures of each sector to the aggregate shocks as well as the volatilities of each sector's idiosyncratic shocks. These shocks will then determine the risk prices in the economy, which I can compare to observed risk prices.

I still need to decide how important this is and I still need to give this procedure some more thought. In particular, the measurement of transitory vs permanent TFP shocks will make a crucial difference for asset prices. I need to decide how I will deal with this. The examples presented in Section 3.3 did not have TFP shocks with permanent shocks.

B Data

B.1 Input-Output Accounts Data

I begin by described the procedure I use to measure the network of intersectoral trade. To construct a measure of the flow of dollars between producers and purchasers within the U.S. economy, I use the Input-Output accounts from the Bureau of Economic Analysis (BEA). These data cover all industrial sectors as well as household production and government entities. The core of the Input-Output accounts consists of two basic national-accounting tables: the "make" table, which records the production of commodities by industries, and the "supply" table, which records the uses of commodities by intermediate and final users. Also, the BEA publishes these tables at various levels of granularity. The most detailed tables are the "Benchmark Input-Output Data", which are coded at a 6-digit level and comprise between 405-544 industries, depending on the year. These tables are published every five years, each edition published with a five-year lag, starting with the 1982 tables up until the most recently published 2012 tables. I use these tables to construct a measure of cash flows between industries, with which I estimate the production technologies featured in my model. The high level of granularity available in the benchmark tables allows me to explore the heterogeneity in the network properties of various industries with the greatest precision possible.

Constructing the Inter-Industry Cash Flow Matrix As noted, I use the "make" and "use" tables from the BEA's Input-Output accounts to construct a measure of cash flows between industries within the U.S. economy. In general, the make table is an $I \times C$ matrix where the entry MAKE_{ic} records the amount of commodity c in dollars that is produced by industry i and the subset of the use table that records the purchases of intermediate users is a $C \times I$ matrix where USE_{ci} is the dollar amount of commodity c used by industry i. Following the procedure outlined in Ahern and Harford (2014), I construct a matrix of cash flows by first constructing the "share" matrix,

$$\text{SHARE}_{ic} = \frac{\text{MAKE}_{ic}}{\sum_{c'=1}^{C} \text{MAKE}_{ic'}},$$

that records the percentage of commodity c produced by industry i and then the cash flow matrix,

$$\text{FLOW}_{ij} = \sum_{c=1}^{C} \text{USE}_{ci} \cdot \text{SHARE}_{jc},$$

that records the dollar value of the products flowing from industry j to industry i.

Note that the BEA also publishes Input-Output requirements tables, such as industry-by-industry or commodity-by-commodity total requirements tables. The industry-by-industry total requirements table, for example, shows the production required, both directly and indirectly, from each industry j per dollar of delivery to final use of each industry i. The use of these tables are inappropriate for the purposes of this paper, however, since these measures include indirect requirements. That is, these measures of the requirements for the output of industry i include the inputs from its direct suppliers (those industries directly supplying industry i) as well as its indirect suppliers (the suppliers of the suppliers to industry i, etc.).

Give a description of the SUPP matrix below. Describe how SUPP is used to match production technology. (The A matrix.) It's the cost share matrix of production expenditures.

The SUPP matrix normalizes by summing across suppliers s:

$$\text{SUPP}_{ij} = \frac{\text{FLOW}_{ij}}{\sum_{s=1}^{I} \text{FLOW}_{is}}$$

Needs addition and discussion of CUST matrix.

Redefinitions The BEA IO make and use tables are published in two varieties: the standard tables and the supplementary tables. The standard tables are constructed before "redefinitions" of selected secondary products and the supplementary tables are constructed after. These redefinitions are one of several methods for handling the accounting of secondary products. In the tables after redefinition, the make and use tables are modified so as to better conform to a "commodity-technology" assumption. Under this assumption, it is assumed that the production of a given commodity requires a unique set of inputs, regardless of which industry produces that commodity. Under these redefinitions, the secondary products and their associated inputs are excluded from the industry that produced them and are included in the industry in which they are primary.[15] These reallocations are only made in cases where the production process for the secondary product is very dissimilar to that for the industry's primary product. [16] The result of these redefinitions is a set of tables that

[15] Note that redefinitions do not affect the definition of the commodity or the measurement of the total output of the commodity. However, redefinitions do affect the measure of industry output.

[16] Horowitz and Planting (2009) give the following example. "The production production process for restaurant services provided in hotels is very different from that of lodging services. Therefore, for the supplementary tables, the output and inputs for these restaurant services are moved or redefined from the hotel industry to the restaurant industry."

represent a more homogeneous relationship between input structure and products and, as such, comprise a more useful tool for analyzing the relationships between industries (Horowitz and Planting, 2009). For this reason, I use the supplementary tables (those constructed after redefinition) in my analysis.

C Robustness Checks

Here I outline additional robustness checks of my model. I begin with a test of the presumed dynamics of output in equilibrium.

C.1 Tests of Equilibrium Quantity Dynamics

The key feature present in the model is that a shock to a particular sector propagates downstream through the production network, from suppliers to customers, and do so gradually over time. As described earlier, shocks propagate throughout the intermediate goods network quickly and through the investment network gradually. Gradual propagation means that a an increase in output growth to one sector should predict an increase in the output of its customers in the following period. I present a set of regressions that test whether this is the case. In the following regressions, I do not distinguish between the investment network and the intermediate goods network. Rather, I simply take the BEA IO tables as a singular production network and test whether shocks to a given sector output predict increasing in downstream sectors. In this sense, the specification that I test matches the specification of Long and Plosser (1983). As a robustness check, I use a different data set than the KLEMS data set used in the main analysis. To analyze the dynamics of sectoral output, I use industry-level output and productivity data for manufacturing from the NBER-CES Manufacturing Industry Database (Becker, Gray, and Marvakov, 2013). This data set includes annual data from U.S. manufacturing sector for the period from 1958 to 2011. It includes manufacturing data, including output and productivity data. While this data set only covers manufacturing sectors, it allows for an analysis of intersectoral dynamics at a finer granularity than the KLEMS data set. This NBER-CES data set includes 473 industries, where the analysis with the KLEMS is aggregated to include at most 30 sectors.

I've recently switched to focusing on the investment network rather than intermediate goods network. For this reason, I currently only have regressions based on the intermediate goods network. These are below. I will add the regressions using the investment network soon.

To construct the production network, I use the same BEA-IO tables as in the main analysis. However, I only use the BEA-IO tables and not the capital flow tables. The description of these is given in Section 4

C.1.1 Testing simplified production dynamics

In the specification of Section 3.3, the dynamics of sectoral output take on a simple form, described in Proposition 8:

$$\Delta q_{t+1} = (I - A)^{-1}\Theta\Delta q_t + (I - A)^{-1}\Delta\xi_{t+1}. \tag{37}$$

If we eliminate the effects of the intermediate-goods network by setting $A = I$, for example, output follows equation (40), which I repeat here:

$$\Delta q_{t+1} = \Theta\Delta q_t + \Delta\xi_{t+1}. \tag{40}$$

This form has a simple interpretation. Since Δq_t is the vector of log output growth and $\Theta = [a_i^k\theta_{ij}]$ is a matrix of expenditure shares on investment goods, next period's output growth in any sector is a weighted average of the output growth in that sector's suppliers' output growth. Recall that $a_i^k\theta_{ij}$ represents the fraction of total expenditures that industry i spends on the investment goods produced by industry j. Thus, a simple diagnostic test of the these dynamics would be to explore whether output in supplying industries predicts future output in customer industries.

To this end, consider constructing a panel regression of the form (40), where we attempt to estimate an unknown matrix Θ. Due to the high dimensionality of Θ, such a regression would be infeasible. To get around this problem, I instead take Θ to be the cost shares from the BEA's IO tables. I then estimate a panel regression of the form

$$\Delta q_{it} = k_i + \delta_t + b\, x_{i,t-1} + \epsilon_{it}, \tag{114}$$

where $x_{it} = (\Theta\,\Delta q_t)_i$ is the i'th element of the vector $\Theta\,\Delta q_t$, and k_i, δ_t and b are parameters to be estimated. k_i and δ_t are industry and time fixed effects, respectively. If the model defined by (40) holds exactly, then we should observe $b = 1$.

As discussed in Section 3.3, the main difference between the intermediate-goods network and the investment network is that shocks propagate through the investment network gradually, whereas shocks propagate through the intermediate-goods network instantaneously. This is expressed in the autoregressive coefficient $(I-A)^{-1}$ in (37). Since $(I - A)^{-1} = I + A + A + A^2 + \ldots$, the coefficient $(I - A)^{-1}$ represents the full, cumulative effect of propagation through the network. Ignoring the effect of the investment network, the effect over one step through the intermediate-goods network would manifest as a $(I-A)^{-1}\Delta q_t$. In practice, it's reasonable to assume that shocks would, to some degree, propagate gradually through the intermediate-goods network as well. To explore this, I estimate the panel regression model (114) again, this time using the the intermediate-goods network only $x_{it} = (A\,\Delta q_{t-1})_i$. I measure A by calculating cost shares from the BEA's Benchmark Input-Output tables. Data on industry output comes from the NBER-CES Manufacturing Industry Database and is thus restricted to only manufacturing industries. The manufacturing industries are identified by their 1987 Standard Industry Classification (SIC) four-digit codes.[17]

I will also do this with Dale Jorgensen's 35-sector KLEMS dataset. I use the NBER manufacturing industry database because it allows for greater granularity (more industries).

At this point, I should note that the form of the regression model in (114) is somewhat restrictive. To make this test more meaningful, I add some additional flexibility to the empirical model of output growth. (37) implies that shocks propagate downstream, from supplier industries to customer industries. A meaningful alternative hypothesis would be that shocks may also propagate upstream, from customers to suppliers. A concern would be that a n estimate of $b > 0$ in (114) may be the result this potential upstream propagation rather than the downward propagation implied in equilibrium. Therefore, I estimate a model that allows for shocks to propagate both upstream, from customers to suppliers, and downstream, from suppliers to customers. To capture this upstream effect, I compute the "customer share" matrix $\widehat{A} = [\widehat{a}_{ij}]$. This is defined such that customer industry i purchases the fraction $\widehat{a}_{ij}$ of the total industry output of supplying industry j. Accordingly, we must have $\sum_{i=1}^{n} \widehat{a}_{ij} = 1$. As an example, if $\widehat{a}_{12} = 1$, industry 1 is the only customer of industry 2. Thus, $\widehat{A}'\,\Delta q_{t-1}$ represents the weighted average output growth of each industry's customers, weighted by the industry's "customer shares". If I include this term as a regressor, this allow the regression to distinguish between upstream and downstream effects. This leads to a panel regression of the form

$$\Delta q_{i,t} = k_i + \delta_t + b_d\, x_{i,t-1}^{\text{suppliers}} + b_u\, x_{i,t-1}^{\text{customers}} + \epsilon_{it}, \tag{115}$$

[17] To facilitate the merging of data, I use the 1987 BEA tables and industry output data from 1987-2011. The NBER-CES Manufacturing Industry Database covers the years from 1958 to 2011. Results appear to be robust to the choice of year of the IO tables.

where

$$x_{it}^{\text{suppliers}} = (A\,\Delta q_t)_i = a_i^m \sum_{j=1}^{n} a_{ij} \Delta q_{jt}$$

$$x_{it}^{\text{customers}} = (\widehat{A}\,\Delta q_t)_i = \sum_{j=1}^{n} \widehat{a}_{ji} \Delta q_{jt}.$$

$x_{it}^{\text{suppliers}}$ is the weighted average output growth of industry i's suppliers, scaled by a_i^m, the fraction of expenditures that industry i spends on intermediate goods (compared to, e.g., labor or capital). The weights, a_{ij} are the fraction of intermediate goods expenditures that industry i spends on the output of industry j. The second term, $x_{it}^{\text{customers}}$, is the weighted average of customer output growth, weighted by the customer shares $\widehat{a}_{ji}$. Since equilibrium dynamics in (40) feature propagation in the downstream direction only, the upstream effect should be null ($b_u = 0$) and the downstream effect should be positive ($b_d > 0$). If (40) holds exactly, we should observe $b_d = 1$. Columns (1) and (3) of Table 7 reports the results of these regressions. I use the manufacturing output data from the NBER-CES Manufacturing Industry Database over the years 1987 to 2011 and the BEA Input-Output Benchmark Table from 1987. Total output here is the real value of total shipments of each industry. To account for serial and spatial dependence, standard errors are constructed following Driscoll and Kraay (1998).

Table 8 estimates a simplified form of these same regressions. In (115), instead of using the model implied coefficients $A = [a_i^k a_{ij}]$, I simply compute the weighted average of customer and supplier average output growth as the upstream and downstream regressors. Specifically, I let

$$x_{it}^{\text{suppliers}} = \sum_{j=1}^{n} a_{ij} \Delta q_{jt}$$

$$x_{it}^{\text{customers}} = \sum_{j=1}^{n} \widehat{a}_{ji} \Delta q_{jt}.$$

The customer regressor stayed the same. Now, the downstream regressor can also be interpreted exactly as a weighted average, since $\sum_{j=1}^{n} a_{ij} = 1$. Columns (1) and (3) report the results of these regressions.

Columns (2) and (4) of this same table, Table 8, present the weighted averages of upstream and downstream industries, using instead the implied customer shares and cost shares of the industries 2-steps away in the supply chain. That is, these are expenditure shares that the industry implicitly spends on its suppliers' suppliers and, similarly, customer share weights based on customers' customer shares. This is motivated in the following sub-section.

Sampling frequency and the speed of shock propagation A potential problem with the estimation procedure described above is that the data generating process may describe a pattern of shock propagation faster than the frequency at which the data is sampled. To demonstrate the issue arising from such a mismatch, suppose that industry output follows (40). However, suppose that we only observe output in every other period (e.g., Δq_t, Δq_{t+2}, $\Delta q_{t+4}, \ldots$). Then, the relationship between Δq_{t+2} and Δq_t depends on these second-degree connections encoded in Θ^2,

$$\begin{aligned}\Delta q_{t+2} &= \Theta \Delta q_{t+1} + \Delta \xi_{t+2} \\ &= \Theta^2 \Delta q_t + \Theta \Delta \xi_{t+1} + \Delta \xi_{t+2}.\end{aligned} \tag{116}$$

In this case, it becomes appropriate to regress the components of the vector Δq_{t+2} on the the components of the vector $\Theta^2 \, \Delta q_t$. Note that rows in the matrix Θ represent the shares of expenditures that row industry i spends on column industry j. Analogously, the rows of the matrix Θ^2 represent the shares of expenditures that row industry i implicitly spends on column industry j, taking into account the expenditure shares of industry i's suppliers' suppliers. Thus, Θ represents expenditure shares on suppliers 1-step up the supply chain, while Θ^2 represents implicit expenditure shares on suppliers 2-steps up the supply chain. Thus, the appropriate weights on supplier output in our regression depends on the sampling frequency of our data as well as the speed with which shocks are propagated through supply chains as dictated by the underlying data generating process. The industry output data in both tables are sampled at an annual frequency. Since it's possible that shocks propagate at a faster rate than this, I also estimate a model of the form

$$\begin{aligned}\Delta q_{i,t} = k_i + \delta_t + b_{d1} \, x_{i,t-1}^{\text{downstream,1-step}} + b_{u1} \, x_{i,t-1}^{\text{upstream,1-step}} \\ + b_{d2} \, x_{i,t-1}^{\text{downstream,2-step}} + b_{u2} \, x_{i,t-1}^{\text{upstream,2-step}} + \epsilon_{it},\end{aligned} \tag{117}$$

where

$$\begin{aligned} x_{it}^{\text{downstream,1-step}} &= (A \, \Delta q_t)_i \\ x_{it}^{\text{upstream,1-step}} &= (\widehat{A}' \, \Delta q_t)_i \\ x_{it}^{\text{downstream,2-step}} &= (A^2 \, \Delta q_t)_i \\ x_{it}^{\text{upstream,2-step}} &= ((\widehat{A}^2)' \, \Delta q_t)_i, \end{aligned}$$

where regressor $(A^2 \, q_t)_i$ measures the dependency of industry i's output on supplier output two steps up the supply chain and the regressor $(\widehat{A}^2)' \, q_t$ measures the dependency on customer output two steps down the supply chain.

Additionally, it should be noted that equation (116) indicates that any mismatch in frequency could result in serious cross-sectional dependence in the error terms. This dependence would otherwise lead to underestimation of the size of the standard errors. To account for this, as well as potential serial correlation, I construct standard errors using the nonparametric, robust covariance matrix estimator proposed by Driscoll and Kraay (1998). Columns (2) and (4) of Table 7 reports the results of these regressions.

Decomposing network- and own-effects After testing the models proposed in regression equations (115) and (117), a useful follow-up question is to ask how much of the effects can be attributed to variation in the output of connected industries and how much can be attributed autocorrelation in the components of q_{it}. For example, in equation (115), we can explore this question by decomposing the terms $x_{i,t-1}^{\text{suppliers}}$ and $x_{i,t-1}^{\text{customers}}$. Consider decomposing the terms as follows,

$$\begin{aligned} x_{it}^{\text{suppliers}} &= \sum_{j=1}^{n} a_{ij} \Delta q_{jt} = a_{ij} \Delta q_{jt} + \sum_{j \neq i}^{n} a_{ij} \Delta q_{jt} \\ x_{it}^{\text{customers}} &= \sum_{j=1}^{n} \widehat{a}_{ji} \Delta q_{jt} = \widehat{a}_{ii} \Delta q_{it} + \sum_{j \neq i}^{n} \widehat{a}_{ji} \Delta q_{jt}. \end{aligned}$$

This motivates a regression model of the form

$$\Delta q_{i,t} = k_i + \delta_t + b_{\text{self}} \Delta q_{i,t-1} + b_{d,\text{others}} x_{i,t-1}^{\text{downstream, other}} + b_{u,\text{others}} x_{i,t-1}^{\text{upstream, other}} + \epsilon_{it}, \tag{118}$$

with

$$\begin{aligned} x_{it}^{\text{downstream, other}} &= \sum_{j \neq i}^{n} a_{ij} \Delta q_{jt} \\ x_{it}^{\text{upstream, other}} &= \sum_{j \neq i}^{n} \widehat{a}_{ji} \Delta q_{jt}. \end{aligned}$$

The parameter b_{self} controls the size of the own-effect and the parameters $b_{d,\text{others}}$ and $b_{u,\text{others}}$ control the network effects from supplier and customer output, respectively. The results of this regression are presented in Table 9.

Regression Results Overview I begin by presenting a related set of benchmark regressions in Table 8. These use proper weighted averages and are thus more easily interpreted. The output growth of an industry is regressed on the weighted average output growth of it customers and suppliers. Supplier averages are weighted by the expenditures shares (fraction of expenditures going to supplier) and customer averages are weighted by revenue shares (fraction of revenue accounted for by customer). This also includes controls for average output growth of suppliers' suppliers and customers' customers (labeled "two-step"), using implicit expenditure and revenue shares derived from the input-output tables. This table show large, significant effects in the downstream direction, from suppliers to customers, and no meaningful effects in the upstream direction. A 1% increase in the output growth of supplying industries is associated with a output growth between 0.16-0.36% in the customer industry the following year. To get an idea of the economic significance of this relationship, we see from Table 6 that within this sample,[18] the standard deviation of supplier output growth is about 5%. Regressions (2) and (4) include terms for supplier and customer variation two steps up or down the supply chain. In regression (4), we see that the downstream effect shows up in the term accounting for output growth two steps up the supply chain. As described in the discussion regarding equation (117), this may be due to the low sampling frequency data used. The data give output on an annual basis. The regressions (3) and (4) in Table 8 feature time fixed effects to absorb variation due to an aggregate trend. Regardless, individual industries may also feature trend components distinct from the aggregate trend. For this reason, regressions using first differences may be preferred.

Next, I compute the results of the regression described in (115). These are provided in regressions (1) and (3) in Table 7. I again use the first differences of output to account for non-stationarity in the data. The regressions (3) and (4) in Table 7 feature time fixed effects to absorb variation due to an aggregate trend. The results, as we can see, appear to support the proposed dynamics in equation (**??**). The downstream effects are larger than the upstream effects in both Tables 7 and 8. We can reject the hypothesis that the coefficient on the downstream effect is zero at the 5% or 1% level, depending on particular regression specification.

As described previously, we want to ensure that the previous results are not simply driven by autocorrelation in industry output growth. This is a concern since most industries appear to purchase significant amounts of intermediate goods from firms within their own industry. Table 9 presents the results of regression equation (118) in which the network effect of other suppliers is separated from the own-effect (potential autocorrelation in q_{it}) of supplying to one-self. The results in this table demonstrate that the majority of the downstream effect is due to fluctuations in

[18] Whether we calculate this by pooling observations or by, say, computing the median within-industry standard deviation, the numbers happen to be approximately the same.

other suppliers. That is, the network effect dominates.