Time-varying Mixed Graphical Models

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Data Science Amsterdam Meetup

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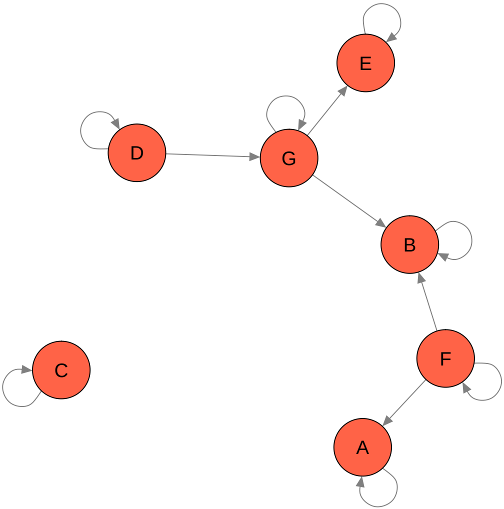
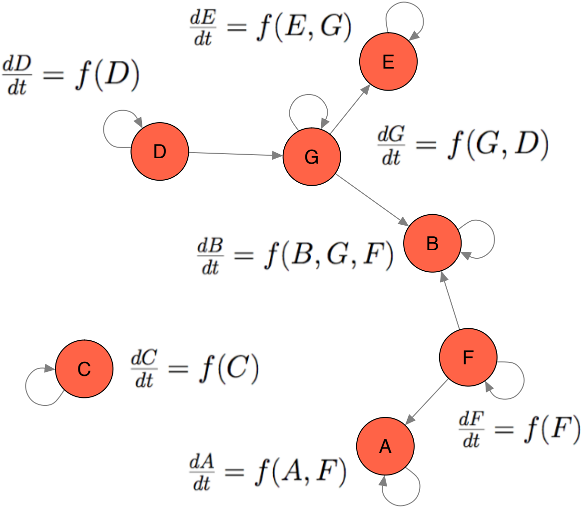
Multivariate System



Multivariate System



$$\frac{dE}{dt} = f(E, G)$$

$$\frac{dD}{dt} = f(D)$$

$$\frac{dG}{dt} = f(G, D)$$

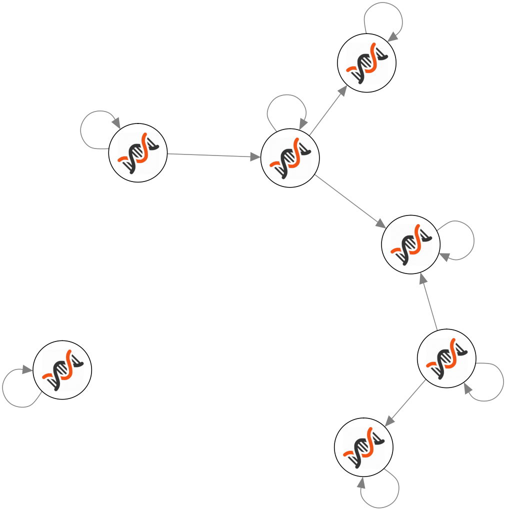
$$\frac{dB}{dt} = f(B, G, F)$$

$$\frac{dC}{dt} = f(C)$$

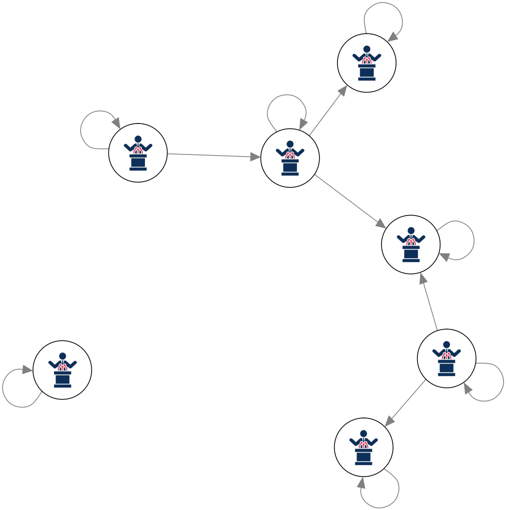
$$\frac{dF}{dt} = f(F)$$

$$\frac{dA}{dt} = f(A, F)$$

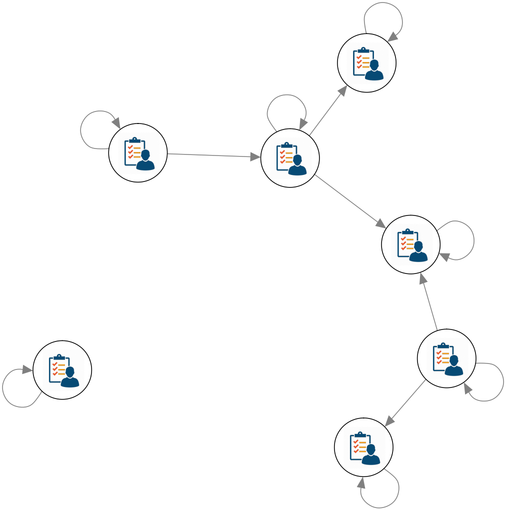
Gene Expressions



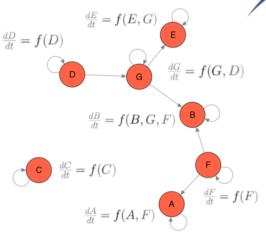
Voting Behavior of Members of Parliament



Symptoms of Mental Disorders



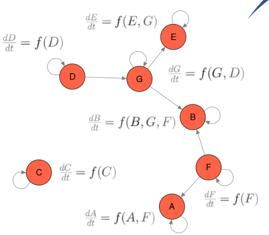
Sample observations



$$\begin{array}{c|ccccccc} & A & B & C & D & E & F & G \\ \hline 1 & 3.45 & 1.11 & -0.91 & 3.18 & -0.88 & 2.28 & 2.12 \\ 2 & 1.72 & 2.14 & 0.78 & 0.82 & 0.36 & 1.03 & 1.28 \\ 3 & 0.11 & -3.12 & 1.21 & 1.80 & 0.57 & 0.99 & 1.32 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ N-2 & -1.29 & 1.65 & 4.62 & 1.74 & -0.44 & -2.80 & 3.13 \\ N-1 & -0.02 & 1.54 & 5.21 & 1.71 & 0.04 & -0.02 & 3.00 \\ N & 1.18 & 0.82 & 1.05 & 8.77 & -0.82 & 0.05 & 2.01 \end{array}$$

Recover the system

Sample observations

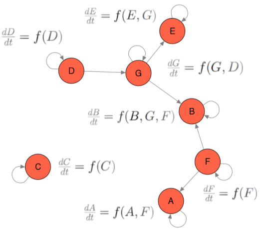


$$\begin{array}{c|ccccccc} & A & B & C & D & E & F & G \\ 1 & 3.45 & 1.11 & -0.91 & 3.18 & -0.88 & 2.28 & 2.12 \\ 2 & 1.72 & 2.14 & 0.78 & 0.82 & 0.36 & 1.03 & 1.28 \\ 3 & 0.11 & -3.12 & 1.21 & 1.80 & 0.57 & 0.99 & 1.32 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ N-2 & -1.29 & 1.65 & 4.62 & 1.74 & -0.44 & -2.80 & 3.13 \\ N-1 & -0.02 & 1.54 & 5.21 & 1.71 & 0.04 & -0.02 & 3.00 \\ N & 1.18 & 0.82 & 1.05 & 8.77 & -0.82 & 0.05 & 2.01 \end{array}$$

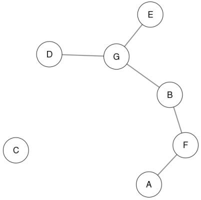
Approximate the system

True Model, Probability Distribution, Graphical Model

True Model



Conditional Independence Network



Approximate

Summarize

$$P(X_1, \ldots, X_p, \theta)$$

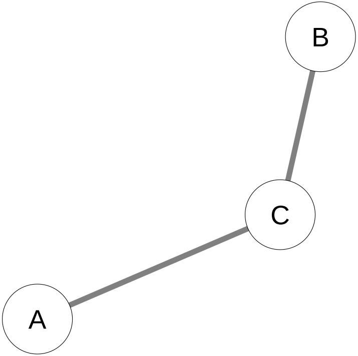
Multivariate Probability Distribution

Conditional Independence Relations in a Graph

$$X_A \perp\!\!\!\perp X_B | X_C$$

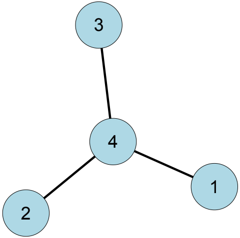
$$X_A \not\!\perp\!\!\!\perp X_C | X_B \qquad \Longleftrightarrow$$

$$X_C \not\!\perp\!\!\!\perp X_B | X_A$$



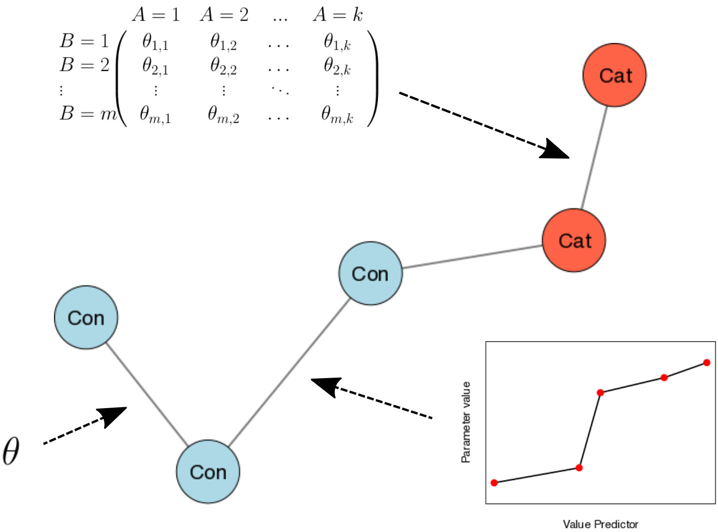
Simple Example: Gaussian Graphical Model

$$\boldsymbol{\Sigma}^{-1} = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \\ X_4 \end{array} \begin{array}{c} \begin{array}{cccc} X_1 & X_2 & X_3 & X_4 \end{array} \\ \begin{pmatrix} 3.45 & 0 & 0 & 3.18 \\ 0 & 2.14 & 0 & 0.82 \\ 0 & 0 & 3.21 & 1.05 \\ 3.18 & 0.82 & 1.05 & 8.77 \end{pmatrix} \end{array} \Longleftrightarrow$$



$$P(X_1, \ldots, X_p) = \frac{1}{\sqrt{(2\pi)^p |\Sigma|}} \exp\left\{ -\frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) \right\}$$

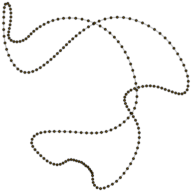
General Graphical Models



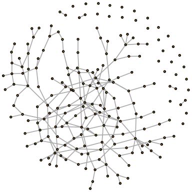
$$\begin{array}{cc} & \begin{array}{cccc} A=1 & A=2 & \ldots & A=k \end{array} \\ \begin{array}{c} B=1 \\ B=2 \\ \vdots \\ B=m \end{array} & \begin{pmatrix} \theta_{1,1} & \theta_{1,2} & \cdots & \theta_{1,k} \\ \theta_{2,1} & \theta_{2,2} & \cdots & \theta_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{m,1} & \theta_{m,2} & \cdots & \theta_{m,k} \end{pmatrix} \end{array}$$

θ

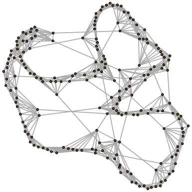
Study multivariate distribution as network



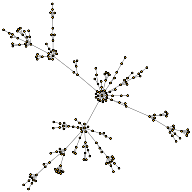
Ring Network



Random Network

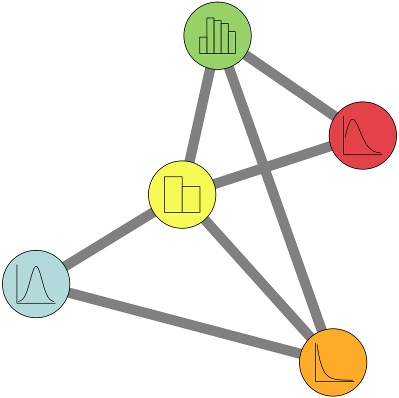


Small World Network



Scale−free Network

Mixed Exponential Graphical Models



Mixed Exponential Graphical Models: Construction

Conditional univariate members of the exponential family

$$P(X_s|X_{\setminus s}) = \exp\left\{E_s(X_{\setminus s})\phi_s(X_s) + C_s(X_s) - \Phi(X_{\setminus s})\right\},$$

factorize to a global multivariate distribution which factors according the graph defined by the conditional distributions if and only if $E_s(X_{\setminus s})$ has the form:

$$\theta_s + \sum_{t \in N(s)} \theta_{st}\phi_t(X_t) + ... + \sum_{t_2,...,t_k \in N(s)} \theta_{t_2,...,t_k} \prod_{j=2}^{k} \phi_{t_j}(X_{t_j})$$

(Yang and colleagues, 2014)

Mixed Exponential Graphical Models: Construction

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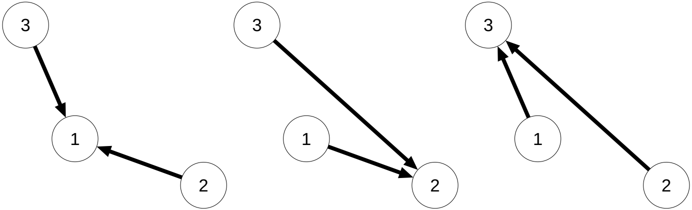
(Yang and colleagues, 2014)

Nodewise Graph Estimation



(Meinshausen & Buehlmann, 2006)

Algorithm: Estimating MGMs

For each node s :

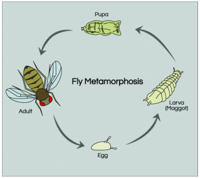
1. Regress $X_{\setminus s}$ on X_s
   - $\min_{(\theta_0,\theta)\in\mathbb{R}^p}\left[\frac{1}{N}\sum_{i=1}^{N}(y_i-\theta_0-X_{\setminus s;i}^{T}\theta)^2+\lambda_n||\theta||_1\right]$
   - Select λ_n using EBIC
2. Threshold Parameter Estimates
   - $\tau_n \asymp \sqrt{d}||\theta||_2\sqrt{\frac{\log p}{n}}$

Combine Estimates from both regressions

- AND-rule: Edge present if both parameters are nonzero
- OR-rule: Edge present if at least one parameter is nonzero

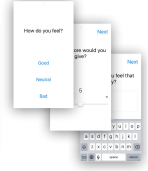
(Loh & Wainwright, 2013; Haslbeck & Waldorp, 2016)

Back to Applications



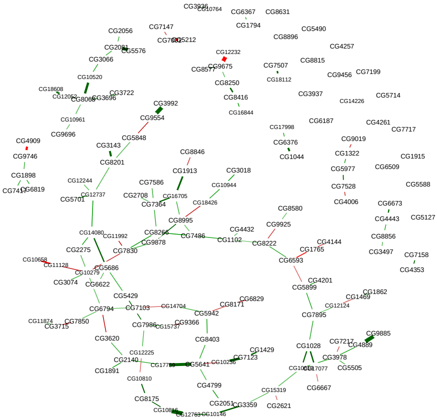
67 measurements of 150 gene expressions related to the immune system of Drosophila melanogaster (fruit fly) over its full life cycle

Votes of 623 members of the German parliament on 136 bills from Nov 2013 - April 2015



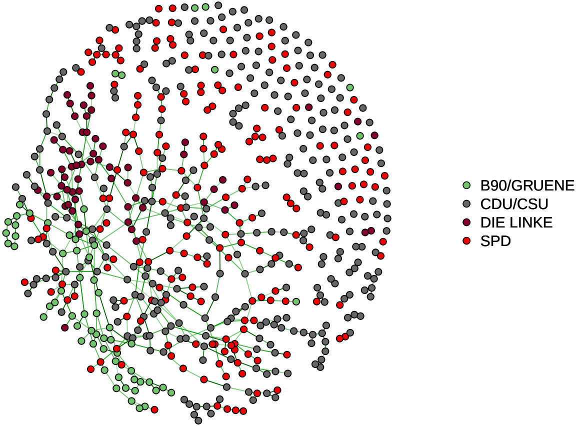
1476 measurements of 16 mood related variables of one individual over 238 consecutive days

Gene Expressions of Fruit Fly



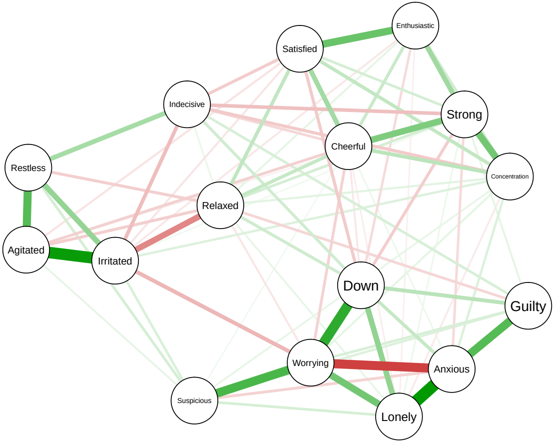
67 Measurements of 150 genes expressions related to immune system of the fruit fly (Lebre et al., 2010)

Voting Behavior of Members of German Parliament



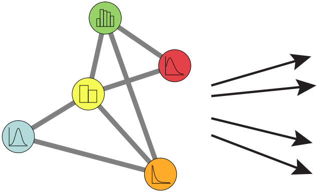
136 public votes, 623 members of parliament of 4 parties

Symptoms of Mental Disorder



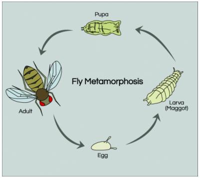
1476 measurements of 16 mood related variables of one individual over 238 consecutive days (Kossakowski et al., 2017)

Does the Structure of the System change over time?



$$\begin{array}{l} \mathit{Time} \\ 1 \\ 2 \\ \vdots \\ T-1 \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \end{array} \\ \left(\begin{array}{ccccc} 3.45 & 1 & 0.98 & 3 & 1 \\ 1.11 & 3 & 0.82 & 3 & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.12 & 2 & 0.71 & 2 & 2 \\ -0.78 & 1 & 0.18 & 1 & 1 \end{array} \right) \end{array}$$

Back to Applications: time-varying?



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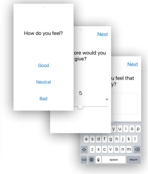
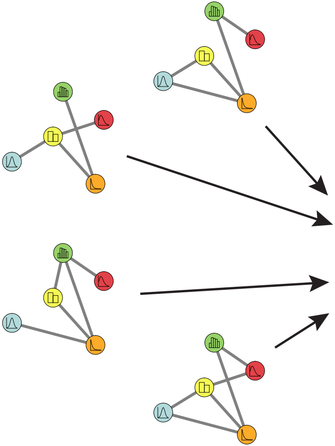


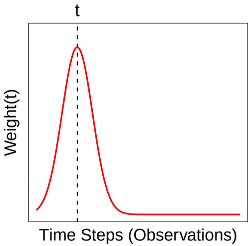
1476 measurements of 16 mood related variables of one individual over 238 consecutive days

How to Estimate a time-varying Model?

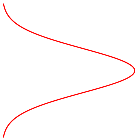


$$\begin{array}{l|ccccc}
Time & X_1 & X_2 & X_3 & X_4 & X_5 \\
1 & 3.45 & 1 & 0.98 & 3 & 1 \\
2 & 1.11 & 3 & 0.82 & 3 & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
T-1 & 0.12 & 2 & 0.71 & 2 & 2 \\
T & -0.78 & 1 & 0.18 & 1 & 1
\end{array}$$

Time-varying Model via weighted nodewise regression



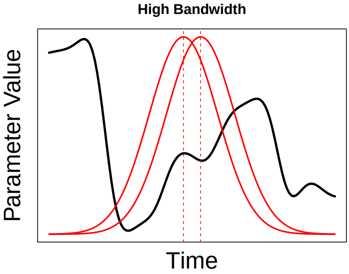
$$\begin{array}{cc}
\textit{Time} & \begin{array}{ccccc} X_1 & \dots & X_5 & \textit{Weight}(t) \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ \vdots \\ T \end{array} &
\left(\begin{array}{cccc}
0.92 & \dots & -1.47 & 0 \\
0.78 & \dots & -0.48 & 0.13 \\
0.07 & \dots & 0.42 & 0.60 \\
-1.99 & \dots & 1.36 & 1.00 \\
0.62 & \dots & -0.10 & 0.60 \\
-0.06 & \dots & 0.39 & 0.13 \\
-0.21 & \dots & 0.99 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
-0.16 & \dots & 0.18 & 0
\end{array}\right)
\end{array}$$



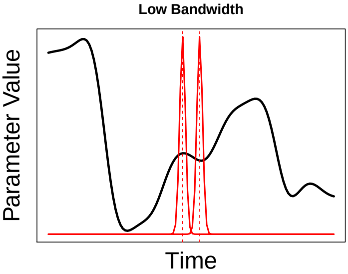
Weighted cost function:

$$\min_{(\theta_0,\theta)\in\mathbb{R}^p}\left[\frac{1}{N}\sum_{i=1}^{N} w_{i;t}(y_{i;t}-\theta_{0;t}-X_{\setminus s;i}^{T}\theta_t)^2+\lambda_n||\theta_t||_1\right]$$

What is the right bandwidth?



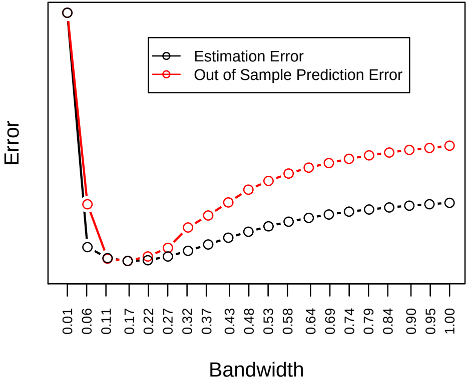
More Information for estimation



Higher sensitivity to change

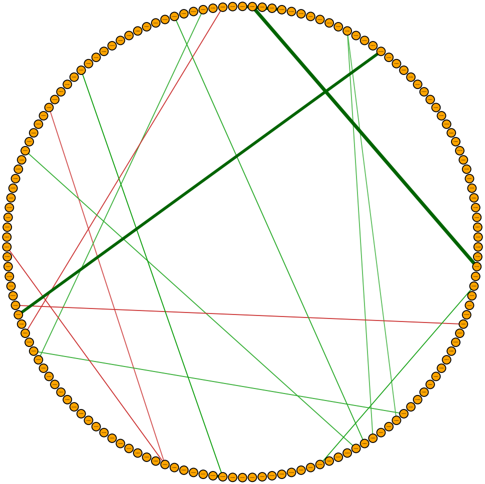
Scaling: $\tau_n \asymp \sqrt{d}||\theta||_2\sqrt{\frac{\log p}{n}}$

Data driven bandwidth selection



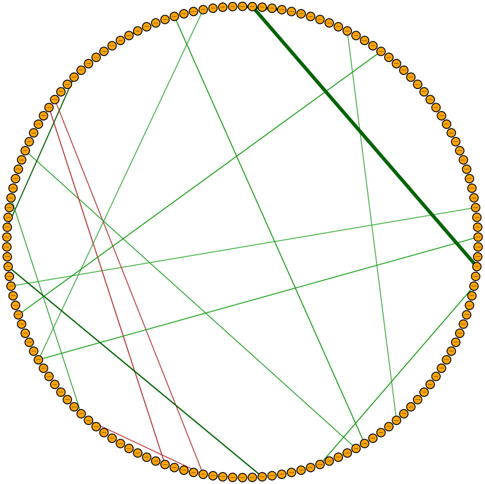
True Model: 6 variables, 4 time-varying parameters, $N = 1000$

Gene Expressions of Fruit Fly: Time-varying



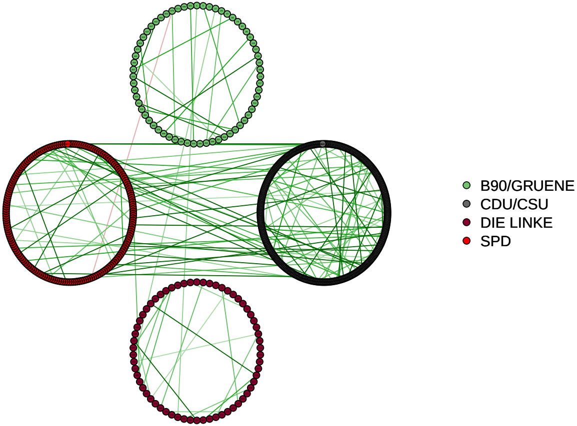
1 h

Gene Expressions of Fruit Fly: Time-varying



8 h

Voting Behavior: time-varying Model



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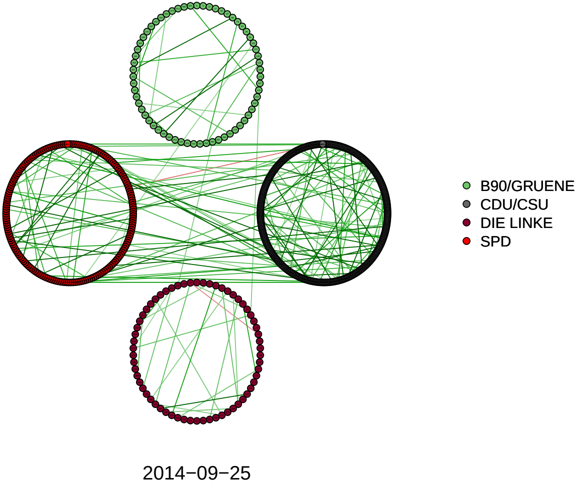
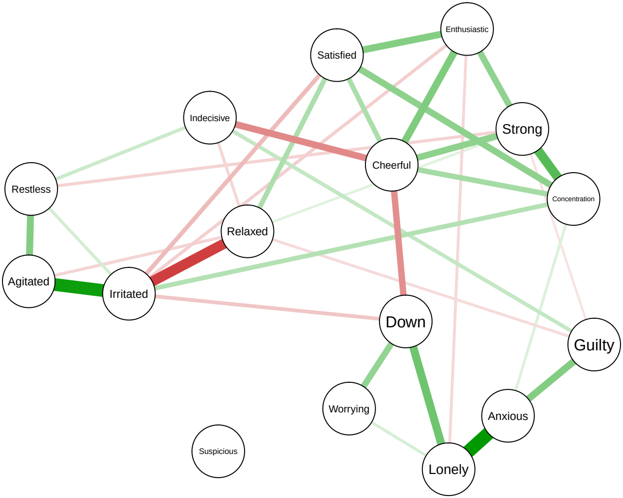
Voting Behavior: time-varying Model



Symptoms of Mental Disorder: Time-varying



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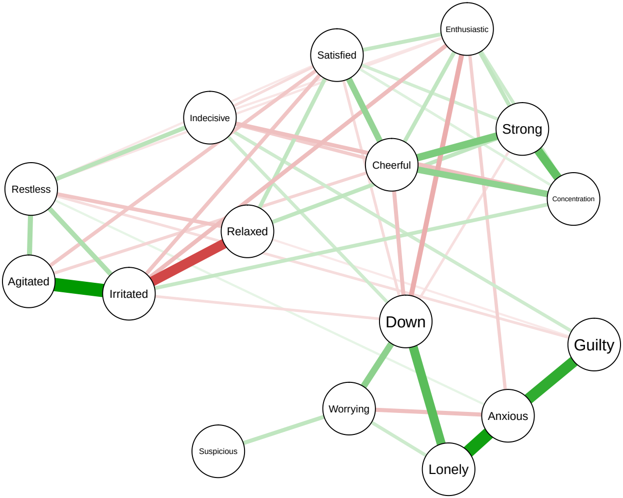
Network of Mental Disorder: Time-varying



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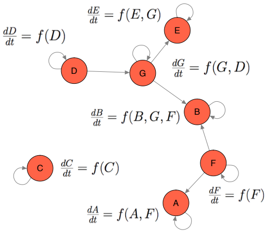
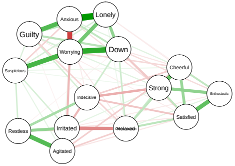
On the direction of Influence

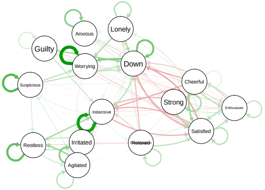
True Structure:



Instantaneous Influence



Influence over time (1h)



Time-varying Mixed Graphical Models

Summary

- Powerful way to gain insights into multivariate datasets
- Allows for mixed variables (e.g. categorical and continuous)
- Scales well for large p and allows for $p > n$
- Available via R-package *mgm* on CRAN

Contact

- Email: `jonashaslbeck@gmail.com`
- Website: `http://jmbh.github.io`

References

- Haslbeck, J., & Waldorp, L. J. (2015). mgm: Structure Estimation for time-varying Mixed Graphical Models in high-dimensional Data. arXiv preprint arXiv:1510.06871.
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