Simulating Sample Paths of Stochastic Processes Arising in Financial Engineering

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Outline

Brownian Motion

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How can we generate realisations of this process...
...while preserving probabilities?

Random Walk

Fix two timepoints $t_2 > t_1 > 0$.
If $B(t_1)$ is known, then $B(t_2) \sim \mathcal{N}(B(t_1), t_2 - t_1)$.

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This is the random walk construction. At every time point, the construction "stops and decides" its "direction of travel".

Random Walk

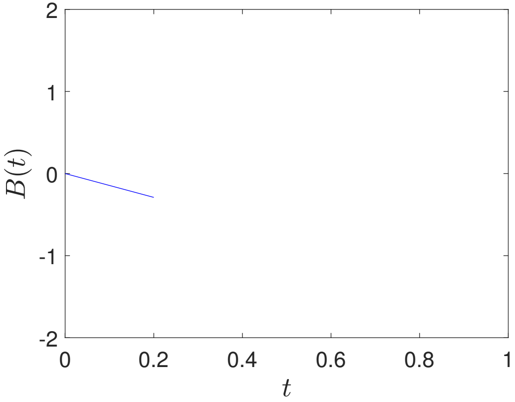

Figure: Standard Brownian Motion simulated with 5 steps. Each displacement is a draw from $\mathcal{N}(0, (1/\sqrt{5})^2)$.

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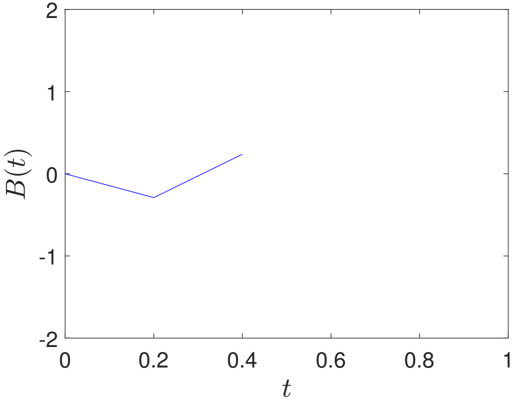

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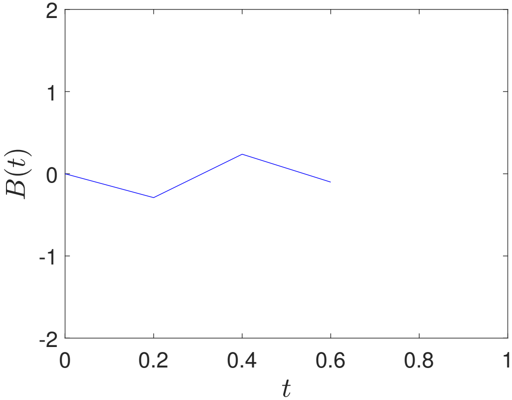

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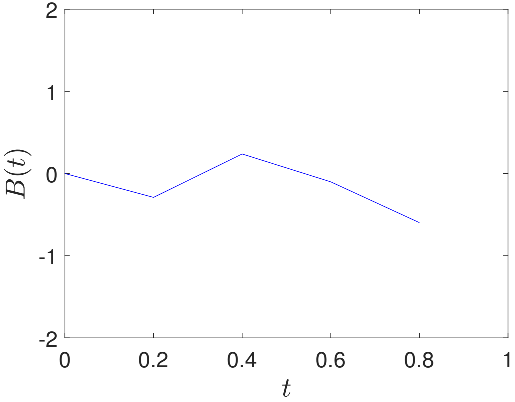

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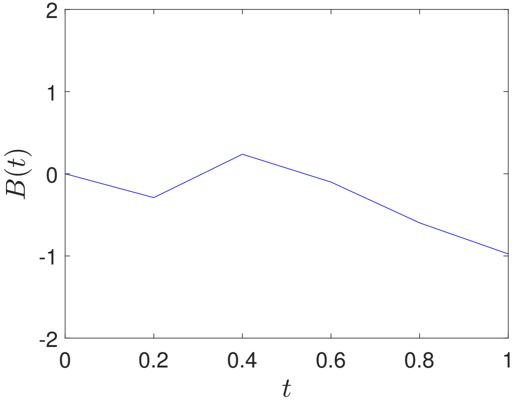

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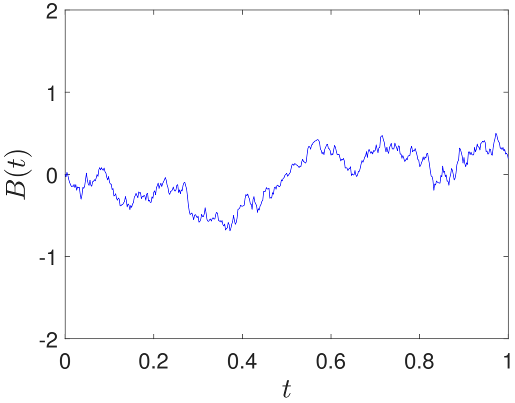

Figure: Standard Brownian Motion with 500 Timesteps

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- We can stop after an arbitrary amount of time and still have a simulation on the entire time interval.
- We can get arbitrarily high resolution by continuing to run the algorithm.

Brownian Bridge

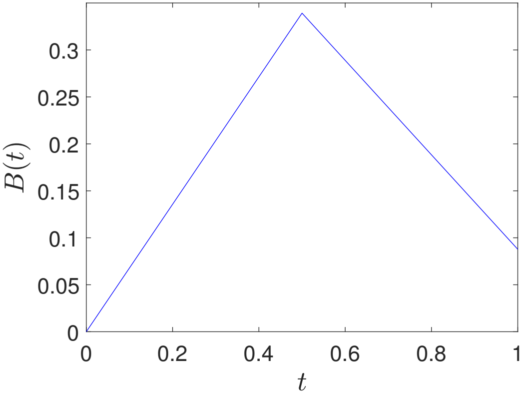

Figure: $BM(1, 0.5^2)$ simulated by 2 steps

Brownian Bridge

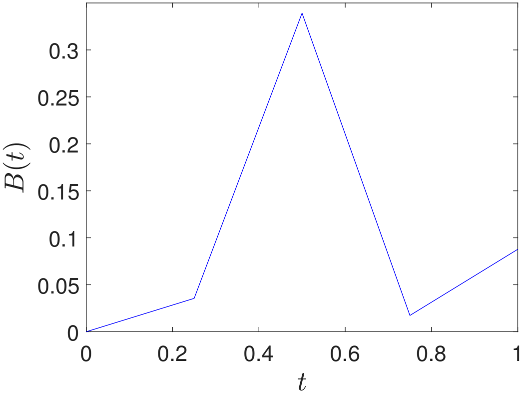

Figure: $BM(1, 0.5^2)$ simulated by 4 steps

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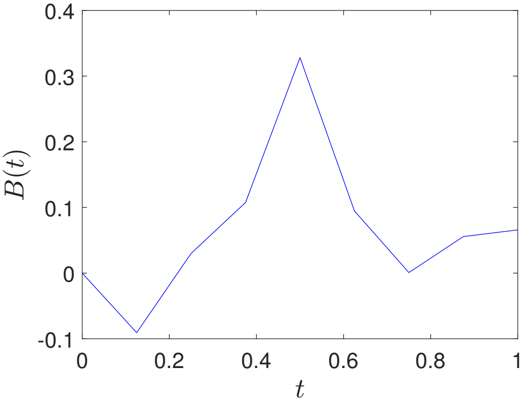

Figure: $BM(1, 0.5^2)$ simulated by 8 steps

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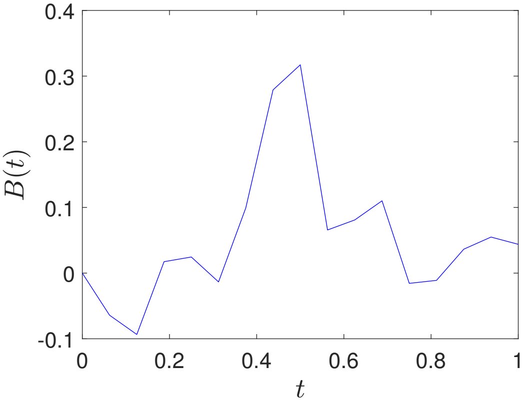

Figure: $BM(1, 0.5^2)$ simulated by 16 steps

Brownian Bridge

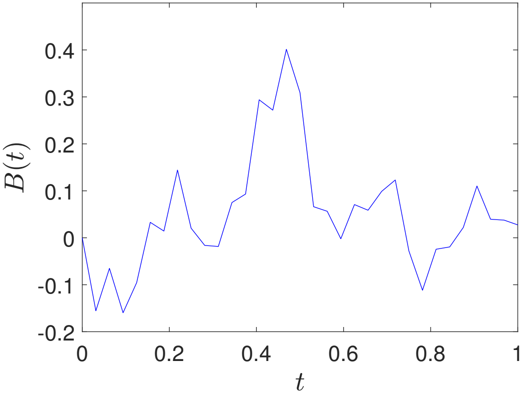

Figure: $BM(1, 0.5^2)$ simulated by 32 steps

Brownian Bridge

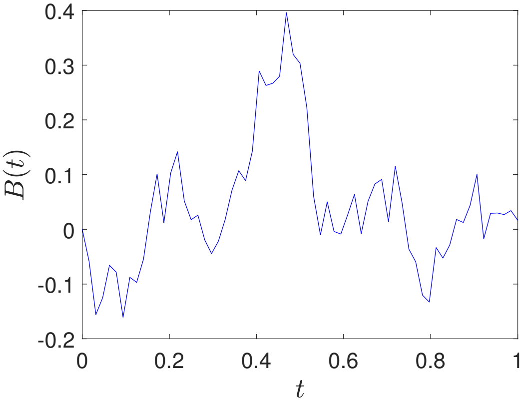

Figure: $BM(1, 0.5^2)$ simulated by 64 steps

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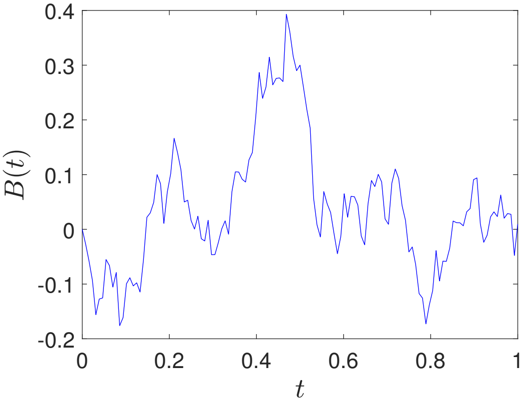

Figure: $BM(1, 0.5^2)$ simulated by 128 steps

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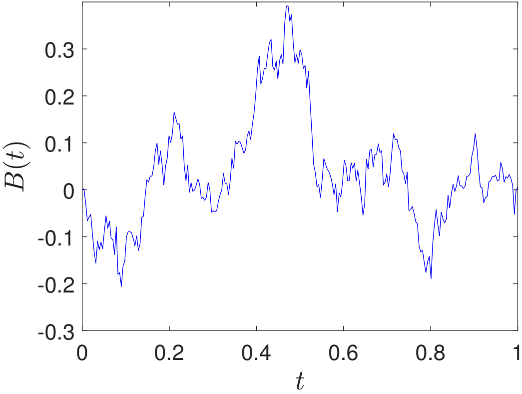

Figure: $BM(1, 0.5^2)$ simulated by 256 steps

Brownian Bridge

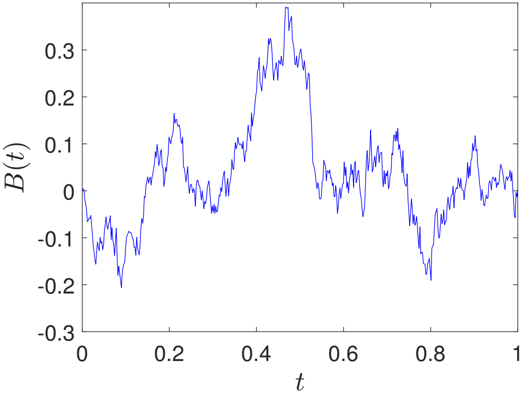

Figure: $BM(1, 0.5^2)$ simulated by 512 steps

Principal Components

We first create the covariance matrix for N equally-spaced timepoints.
Recall that $\mathbf{C}_{ij} = min(i/N, j/N)$.

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Now, if we take only the eigenvectors corresponding to the greatest few eigenvalues, we get a good approximation to the overall shape of a sample path.
For an n-step path with equal spacing, $t_{i+1} - t_i = \Delta t$, Åkesson and Lehoczky showed that:

$$v_i(j) = \frac{2}{\sqrt{2n+1}} \sin\left(\frac{2i-1}{2n+1} j\pi\right), \quad j = 1, \ldots, n.$$

and

$$\lambda_i = \frac{\Delta t}{4} \sin^{-2}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right), \quad i = 1, \ldots, n.$$

Principal Components

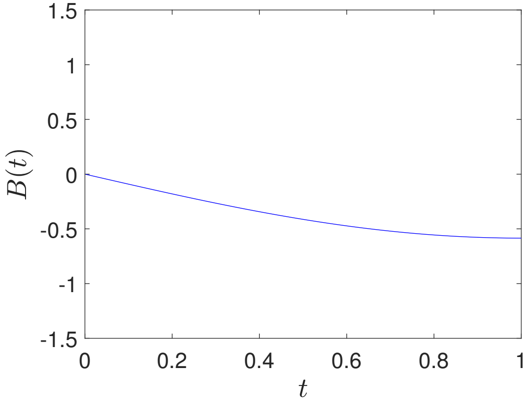

Figure: 1 principal component of standard Brownian motion, simulated with 512 steps

Principal Components

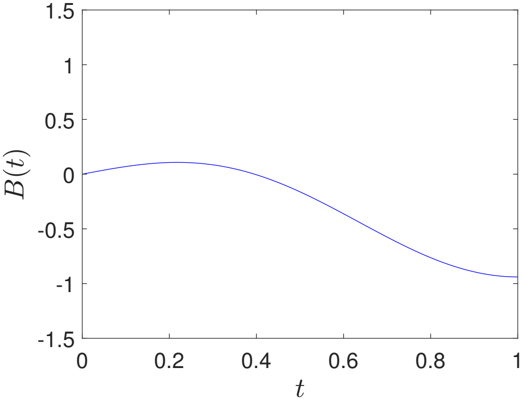

Figure: 2 principal components of standard Brownian motion, simulated with 512 steps

Principal Components

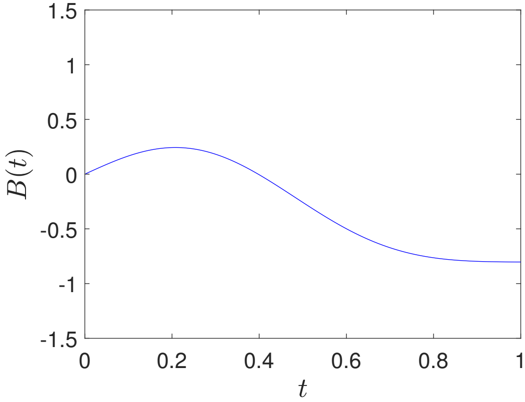

Figure: 3 principal components of standard Brownian motion, simulated with 512 steps

Principal Components

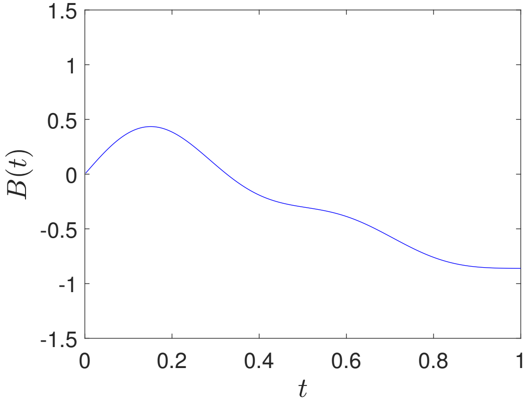

Figure: 5 principal components of standard Brownian motion, simulated with 512 steps

Principal Components

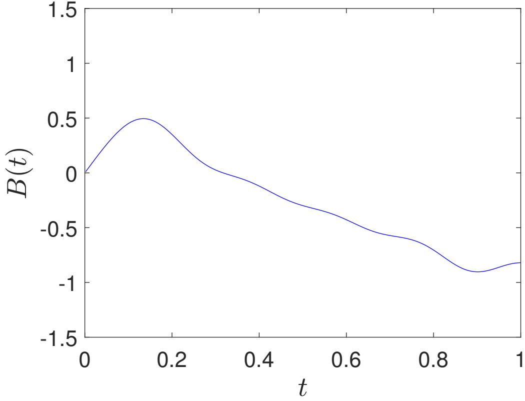

Figure: 10 principal components of standard Brownian motion, simulated with 512 steps

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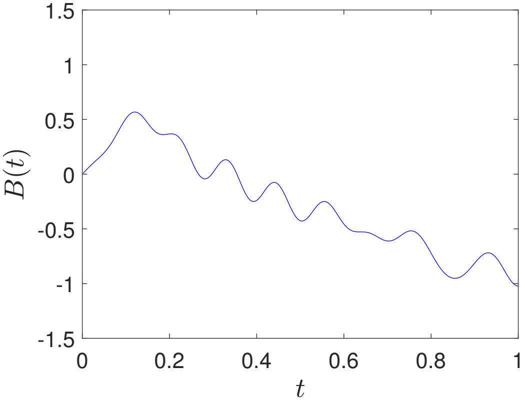

Figure: 20 principal components of standard Brownian motion, simulated with 512 steps

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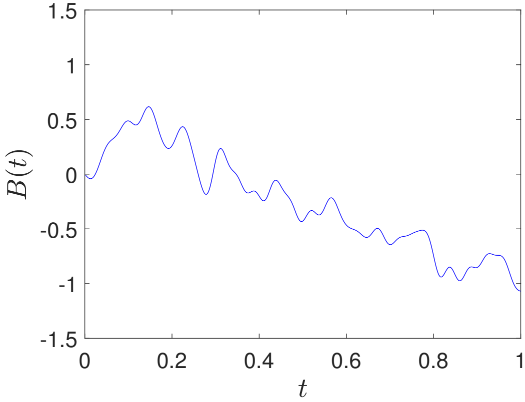

Figure: 50 principal components of standard Brownian motion, simulated with 512 steps

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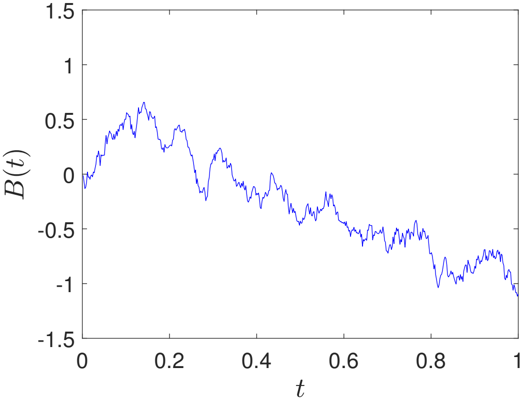

Figure: 512 principal components of standard Brownian motion, simulated with 512 steps

Geometric Brownian Motion (One Dimension)

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$$B_t \mapsto z_0 e^{\mu t + \sigma B_t}$$

Geometric Brownian Motion (One Dimension)

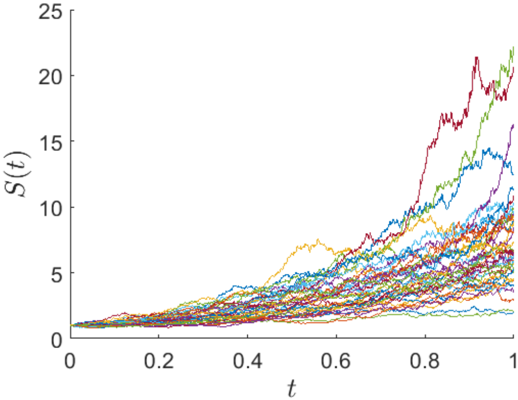

Figure: 40 sample paths of geometric Brownian motion. The respective Brownian motion has $\mu = 2$, $\sigma = 0.5$. Each path has 1000 timesteps and were generated using the random walk construction.

Central Limit Theorem

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The error converges to zero rather slowly: at a rate of $O(n^{-1/2})$.

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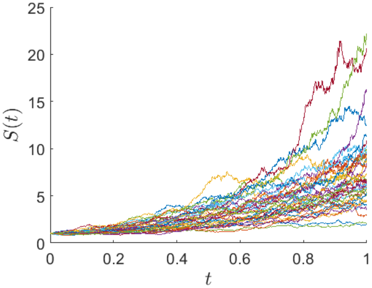

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Lookback Option (with floating strike)

Definition (Lookback Option)

A floating lookback put (resp. call) option gives the holder the right to sell (resp. buy) an asset at the highest (resp. lowest) realised price. Hence, a lookback put option has payoff:

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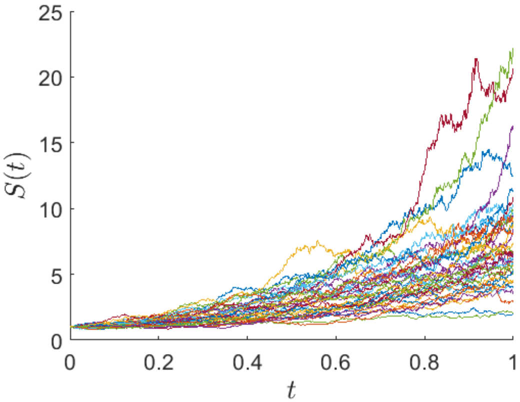

We price the option on each sample path.
Then, we take the mean of these prices.

Lookback Option (with floating strike)

We have to be careful about discretisation error:

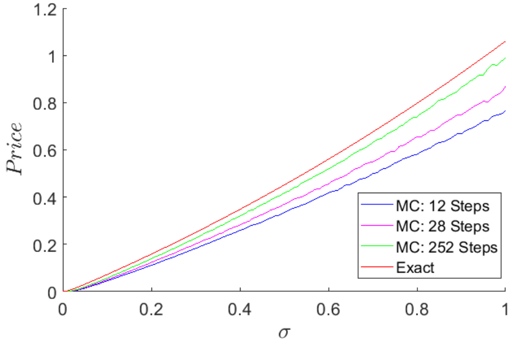

Figure: Monte Carlo pricing of a lookback option. The underlying GBM has changing σ. $\mu = 0.05$, $r = 0.02$, $S(0) = 1$. We compare various discrete monitoring schema and exact continuous monitoring.

Short Rate Models

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Is there a way to model stochastic interest rates?

Vasicek Model

In 1977, Vasicek proposed modelling the short rate by an Ornstein-Uhlenbeck process:

$$dr(t) = \alpha(b(t) - r(t))dt + \sigma dB(t)$$

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$$dr(t) = \alpha(b(t) - r(t))dt + \sigma dB(t)$$

This captures mean-reversion, and allows for negative interest rates. We can solve this SDE at time t, given $0 \leq u < t$

$$r(t) = e^{-\alpha(t-u)}r(u) + \alpha \int_u^t e^{-\alpha(t-s)}b(s)ds + \sigma \int_u^t e^{-\alpha(t-s)}dB(s).$$

We can discretise this, similar to Euler's method on ODEs.

Vasicek Model

Hence, given some $r(u)$, with $0 \leq u < t$

$$r(t) \sim \mathcal{N}\left(e^{-\alpha(t-u)}r(u) + \alpha\int_u^t e^{-\alpha(t-s)}b(s)ds,\ \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(t-u)})\right)$$

And so we can simulate this process step-by-step as a random walk.

Vasicek Model

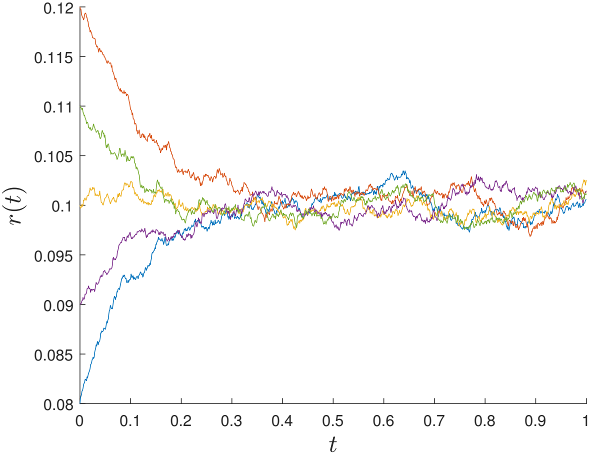

Figure: Five simulation of the Vasicek model with $b = 0.1$, $\alpha = 10$, $\sigma = 0.02$ and different values of $r(0)$.

Bonds

An investment earning interest at a rate $r(u)$ at time u grows from a value of 1 at time 0 to a value of

$$\beta(t) = \exp\left(\int_0^t r(u)du\right)$$

at time t.
We take this as the numeraire.

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We take this as the numeraire.
Hence, the time-0 discounted price of a bond paying 1 at T is

$$B(0, T) = \mathbb{E}\left[\exp\left(-\int_0^T r(u)du\right)\right].$$

Vasicek Bond Pricing

If we suppose that r follows the Vasicek model, we can use the sampled values of $r(u)$ at each timepoint to estimate the integral using the trapezium rule.

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Then, the average result from repeatedly evaluating this expression on sample paths will converge to the expected value (i.e. the price of the bond).

Vasicek Bond Pricing

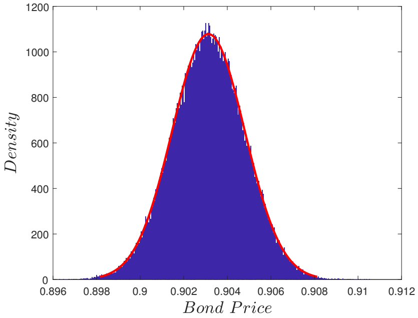

Figure: 10^5 trials of pricing a bond paying 1 at $T = 1$, with $b = 0.1$, $\alpha = 10$, $\sigma = 0.02$, $r(0) = 0.12$.

```
Normal distribution
mu = 0.903134 [0.903124, 0.903145]
sigma = 0.00166463 [0.00165737, 0.00167196]
```

Jump Processes

So far, all of our models have been (almost surely) continuous, but empirical evidence suggests the presence of "jumps" in real-life stock prices. The existence of jumps also aids in explaining unexpected leptokurtosis which has been observed in asset returns.

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So far, all of our models have been (almost surely) continuous, but empirical evidence suggests the presence of "jumps" in real-life stock prices. The existence of jumps also aids in explaining unexpected leptokurtosis which has been observed in asset returns. How can we model these jumps?

Pure Poisson Process

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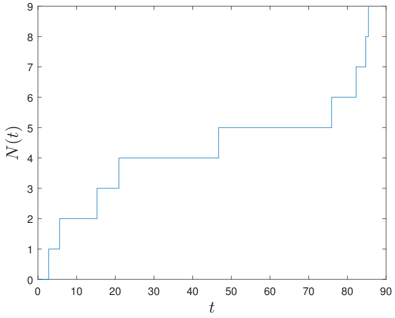

Figure: One realisation of the pure Poisson process over the interval $[0, 90]$. The exponential distribution has rate 10.

Compound Poisson Process

Now, we can allow the jumps to vary. In particular, Merton proposed a that the jump sizes follow a lognormal distribution.

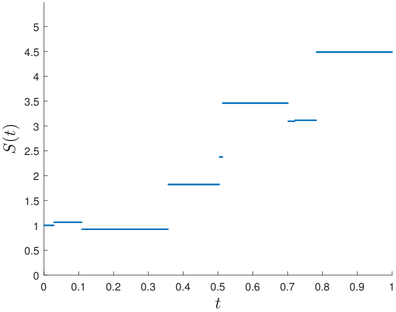

Figure: One realisation of the compound Poisson process over the interval $[0, 1]$. The exponential distribution has rate 0.1. The lognormal distribution has parameters $\mu_{jump} = 0.1$ and $\sigma_{jump} = 0.2$.

Merton's Jump-Diffusion Model

In 1976, Merton proposed an extension to the existing Black-Scholes-Merton framework. He said that we should add a jump term to the normal geometric Brownian motion SDE:

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dB(t) + dJ(t)$$

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$J(t) = \sum_{j=1}^{N(t)}(Y_j - 1)$ where the $Y_1, Y_2, \ldots$ are random variables.

Merton's Jump-Diffusion Model

In 1976, Merton proposed an extension to the existing Black-Scholes-Merton framework. He said that we should add a jump term to the normal geometric Brownian motion SDE:

$$\frac{dS(t)}{S(t-)} = \mu dt + \sigma dB(t) + dJ(t)$$

$J(t) = \sum_{j=1}^{N(t)} (Y_j - 1)$ where the $Y_1, Y_2, \ldots$ are random variables. $N(t)$ is a counting process: there exist random arrival times $0 < \tau_1 < \tau_2 < \ldots$ and $N(t) = \sup\{n \mid \tau_n \leq t\}$ counts the number of arrivals in $[0, t]$.

Merton's Jump-Diffusion Model

Now, letting the Y_i follow a lognormal distribution, we can simulate Merton's jump-diffusion model simply by taking the pointwise product of the appropriate geometric Brownian motion and compound Poisson process sample paths. This construction will satisfy the Merton jump-diffusion SDE.

Merton's Jump-Diffusion Model

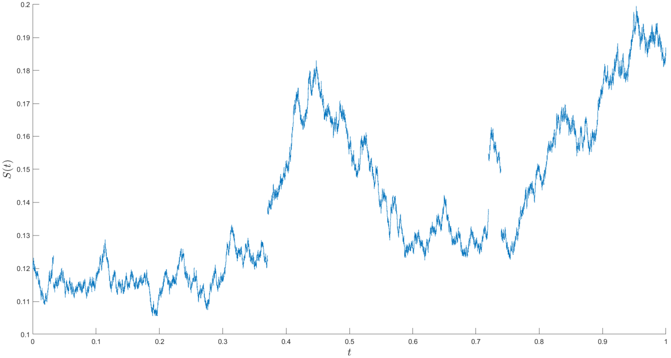

Figure: One realisation of the Merton jump-diffusion model over the interval $[0, 1]$ with $S(0) = 0.12$. The exponential distribution has rate 0.15. The lognormal distribution has parameters $\mu_{jump} = 0.05$ and $\sigma_{jump} = 0.08$. $\mu_{BM} = 0.1$ and $\sigma_{BM} = 0.4$.

Summary

What have we covered in this presentation?

- Three methods for simulating Brownian motion.
- Simulation of geometric Brownian motion.
  - In one dimension.
  - In multiple dimensions.
- Monte Carlo pricing of options.
- Simulating a short rate model (Vasicek).
  - Monte Carlo bond pricing.
- Simulating Merton's jump-diffusion model.

Summary

Other topics covered in the project:

- An alternative to GBM: the CEV model.
- Variance reduction in Monte Carlo estimates:
  - Antithetic variates
- Monte Carlo pricing of more exotic multi-asset options:
  - Asian options
  - Barrier options
  - Spread options
- More short rate models:
  - Ho-Lee model
  - CIR (square-root diffusion) model
- Monte Carlo option pricing with jumps.