EC442 Problem Set 1 Solution

1 Taking the Solow Model to the Data

In this exercise we are going to solve the canonical Solow in continuous time and try to see if there is hope to explain the income distribution across countries with the model. As seen in the class: without population growth, there will be no growth in the long-run. Hence, we will not be able to see anything about the time-series behavior (after all: most countries did grow in the last 50 years). Hence, we focus on the cross-section of countries in the year 2000.

Consider the standard Solow environment. Time is continuous, the saving rate is s, production in country i is given by

$$Y_i(t) = F(K_i(t), A_i L_i)$$

where F is a neoclassical production function and A_i is a labor-augmenting technology term. Note that L_i and A_i are assumed to be constant over time but allowed to be different for different countries. Capital depreciates at rate δ. Both the saving rate and the rate of depreciation is assumed to be equal across countries.

1. State the accumulation equation for capital-per-capita $k_i(t) = \frac{K_i(t)}{L_i}$ (i.e. the equation for $\dot{k}_i(t)$) and express the level of output per worker $y_i(t)$, wages $W_i(t)$ and capital returns $R_i(t)$ as a function of $k_i(t)$ and A_i. How do $(y_i(t), W_i(t), R_i(t))$ depend on A_i and $k_i(t)$?

2. Derive the condition for a the steady-state capital-labor ratio k_i^*. How does k_i^* depend on A_i? Taking k_i^* as a function of A_i, how does (y^*, W^*, R^*) depend on A_i?

3. Now suppose that F takes the Cobb-Douglas form, i.e.

$$Y_i(t) = F(K_i(t), A_i L_i) = K_i(t)^{\alpha} (A_i L_i)^{1-\alpha}.$$

Derive the expression for the steady state capital-labor ratio k_i^*. Derive an expression for $ln(y_i(t))$ as a function of $(ln(A_i), ln(k_i(t)))$. Derive an expression of $ln(y_i^*)$ as a function of parameters.

4. Now let us go to the data. The most important cross-country dataset are the Penn World Tables[1] . Go to this website and download the newest version of the data PWT 7.1. In particular, download the data on per-capita GDP (to be more precise "PPP Converted GDP Per Capita (Chain Series), at 2005 constant prices") in 2010 for the 190 countries in the world. You also need capital per worker k_i. As capital per worker is not available "off the shelf" and it is a little messy to construct (check Acemoglu, p.97 how to do it), I already calculated it for you. It is available in the file PWT_k.csv on the website.

   (a) To give you a rough idea about the magnitude of income differences across the world, calculate per-capita income relative to the US for the following countries: China, India, France, Vietnam Nigeria. [Hint: Again you can use the `ismember` command to extract country-specific information]. Report your results or plot them in a graph.

   (b) Now we are going to test how well the Solow model does to explain the differences in income across the world. We still assume that $Y(t) = K_i(t)^{\alpha} (A_i L_i)^{1-\alpha}$. Let

$$ln(y_i) = \phi(A_i, ln(k_i))$$

   be the relationship between $ln(y_i)$ and $(A_i, ln(k_i))$ derived in part 3. Suppose that $\alpha = 1/3$.

[1] http://pwt.econ.upenn.edu

i. Assume that technologies are equal across the world, i.e. $A_i = A_j = A^{Solow}$. Hence, the variation in income across countries is fully driven the variation in the capital labor ratio. Let A^{Solow} satisfy

$$(1-\alpha) A^{Solow} = \frac{1}{N}\sum_{i=1}^{N} ln(y_i) - \alpha \frac{1}{N}\sum_{i=1}^{N} ln(k_i).$$

The predicted income by the model is

$$ln\left(\hat{y}_i^{Solow}\right) = \phi\left(A^{Solow}, ln(k_i)\right).$$

Plot $ln\left(\hat{y}_i^{Solow}\right)$ against $ln(y_i)$. Does the model do a good job in predicting income differences across the world? Does it over- or underestimate the inequality across countries? What does this tell you about the assumption that $A_i = A_j$?

ii. Now suppose that we wanted to make the model consistent with data. Given the data on $(ln(y_i), ln(k_i))_i$, find the implied values of log productivity $(ln(A_i))_i$, such that the model fits the data perfectly. Plot $ln(A_i)$ against $ln(y_i)$. How does productivity in rich countries compare the one in poor countries? Did you expect this result from your answer in part (i)?

iii. Up to now we have taken the data on $(k_i)_i$ taken as given, i.e. we have not used the model's formula for the steady-state. The steady-state capital-labor ratio you found above follows a relationship

$$k_i^* = \psi\left(\left(\frac{s\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}, A_i\right), \tag{1}$$

i.e. the variation of capital across countries is informative about productivity differences. Let $ln\left(\left(\frac{s\alpha}{\delta}\right)^{\frac{1}{1-\alpha}}\right)$ satisfy

$$ln\left(\left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}\right) = \frac{1}{N}\sum_{i=1}^{N} ln(k_i).$$

Use (1) to find $\left(\tilde{A}_i\right)_i$ such that $k_i^* = k_i$, i.e. that the observed capital-labor ratios are consistent with a steady-state. Now predict per capita income by

$$ln\left(\hat{y}_i^{Full\ Model}\right) = \phi\left(\tilde{A}_i, ln(k_i)\right).$$

Plot $ln\left(\hat{y}_i^{Full\ Model}\right)$ against $ln(y_i)$. Does the model do a good job in predicting income differences across the world?

Solution:

Part 1. The accumulation for the aggregate capital is

$$\begin{aligned}
\dot{K}(t) &= sF(K(t), AL) - \delta K(t) \\
&= sF\left(\frac{K(t)}{L}, A\right)L - \delta\frac{K(t)}{L}L \\
\frac{\dot{K}(t)}{L} &= sF(k(t), A) - \delta k(t).
\end{aligned}$$

As $\dot{k}(t) = \frac{\dot{K}(t)}{L}$, the accumulation equation is

$$\dot{k}(t) = sF(k(t), A) - \delta k(t).$$

Then

$$
\begin{aligned}
y(t) &= \frac{F(K(t), AL)}{L} = F(k(t), A) \\
w(t) &= F_L(K(t), AL) A = F_L(k(t), A) A \\
R(t) &= F_K(K(t), AL) = F_K(k(t), A).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\partial y(t)}{\partial k(t)} &= F_K(k(t), A) > 0 \\
\frac{\partial y(t)}{\partial A} &= F_L(k(t), A) > 0 \\
\frac{\partial R(t)}{\partial k(t)} &= F_{KK}(k(t), A) < 0 \\
\frac{\partial R(t)}{\partial A} &= F_{KL}(k(t), A) > 0 \\
\frac{\partial w(t)}{\partial k(t)} &= F_{LK}(k(t), A) A > 0 \\
\frac{\partial w(t)}{\partial A} &= F_{LL}(k(t), A) + F_L(k(t), A) \gtrless 0.
\end{aligned}
$$

so that wages are not necessarily increasing in labor augmenting progress. This occurs if capital and labor-inputs are very complementarity, so that technological progress A causes firms to substitute away from labor. As example, suppose that production takes place according to Leontief function

$$
Y(t) = \min[K(t), A(t) L].
$$

Now suppose that $A = 1$, $K = 6$ and $L = 5$. Hence, labor is the scarce factor, some capital will be idle so that $R(t) = 0$ and $w(t) L = Y(t)$. Now suppose that A increases to 2. Then capital is the scarce factor and $w(t) = 0$. Hence, labor augmenting technological progress can depress wages. The Leontief example is a special case of the CES production function

$$
Y = \left(K^{\frac{\sigma-1}{\sigma}} + (AL)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}.
$$

For $\sigma = 1$, this is the Cobb-Douglas function, $\sigma = \infty$ is the case of perfect substitutes and $\sigma = 0$ is the Leontief function. For $\sigma < 1$ it is possible that wages are decreasing in A.

Part 2. The steady state k^* is given by

$$
F(k^*, A) = \frac{\delta}{s} k^*
$$

Hence,

$$
\begin{aligned}
F_K(k^*, A) dk^* + F_L(k^*, A) dA^* &= \frac{\delta}{s} dk^* \\
\frac{dk^*}{dA^*} &= \frac{F_L(k^*, A)}{\frac{\delta}{s} - F_K(k^*, A)} \\
&= \frac{F_L(k^*, A)}{\frac{F(k^*, A)}{k^*} - F_K(k^*, A)} > 0,
\end{aligned}
$$

as the average product of capital $\frac{F(k^*, A)}{k^*}$ exceeds the marginal product $F_K(k^*, A)$. This also implies that

$$
\frac{dy^*}{dA} = \frac{dF(k^*, A)}{dA} = \frac{\partial F(k^*, A)}{\partial k} \frac{dk^*}{dA} + \frac{\partial F(k^*, A)}{\partial A} > 0.
$$

For interest rates

$$\frac{dR^*}{dA} = \frac{dF_K\left(k^*, A\right)}{dA} = \frac{dF_K\left(\frac{k^*}{A}, 1\right)}{dA}.$$

But in the steady state

$$\frac{\delta}{s} = \frac{F\left(k^*, A\right)}{k^*} = \frac{F\left(\frac{k^*}{A}, 1\right)A}{\frac{k^*}{A}A} = \frac{F\left(\frac{k^*}{A}, 1\right)}{\frac{k^*}{A}},$$

so that $\frac{k^*}{A}$ is fully determined from $\frac{\delta}{s}$. Hence,

$$\frac{dR^*}{dA} = 0.$$

Once capital adjusts, the long-run interest rate will not be affected by A. For wages:

$$\frac{dw}{dA} = \frac{dF_L\left(k^*, A\right)A}{dA} = \frac{dF_L\left(\frac{k^*}{A}, 1\right)A}{dA} = \frac{dF_L\left(\frac{k^*}{A}, 1\right)}{dA} + F_L\left(\frac{k^*}{A}, 1\right) = F_L\left(\frac{k^*}{A}, 1\right) > 0$$

because again $\frac{k^*}{A}$ is constant.

Part 3. Accumulation

$$\dot{k}\left(t\right) = sf\left(k\left(t\right)\right) - \delta k\left(t\right) = sk\left(t\right)^{\alpha} A_i^{1-\alpha} - \delta k\left(t\right)$$

Hence

$$k_i^* = \left(\frac{sA_i^{1-\alpha}}{\delta}\right)^{\frac{1}{1-\alpha}} = k_i^* = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}} A_i$$

so that

$$y_i^* = \left(k_i^*\right)^{\alpha} A_i^{1-\alpha} = \left(\frac{s}{\delta}\right)^{\frac{\alpha}{1-\alpha}} A_i$$

so that

$$ln\left(y_i^*\right) = \alpha ln\left(k_i^*\right) + \left(1-\alpha\right) ln\left(A_i\right)$$

Also

$$ln\left(y_i\left(t\right)\right) = \left(1-\alpha\right) ln\left(A_i\right) + \alpha ln\left(k_i\left(t\right)\right).$$

Part 4.

1. See Figure 1.
2. See Figure 2. According to the model

$$\begin{aligned} ln\left(y_i\right) &= \left(1-\alpha\right) ln\left(A_i\right) + \alpha ln\left(k_i\right) \\ &= \left(1-\alpha\right) ln\left(A^{Solow}\right) + \alpha ln\left(k_i\right). \end{aligned}$$

   The model underestimates the inequality across countries. Hence, differences in capital are not enough to generate the observed dispersion in income per-capita.

3. See Figure 3. The implies productivity level is

$$ln\left(A_i\right) = \frac{ln\left(y_i\right) - \alpha ln\left(k_i\right)}{1-\alpha}.$$

   Clearly, A and y are positively correlated, i.e. richer countries have better technology. This is expected from the results above, as capital accumulation alone is not able to explain the dispersion in per-capita income - rich countries do not have enough capital to justify their high level of income and poor countries have too much capital to be consistent with their low level of income.

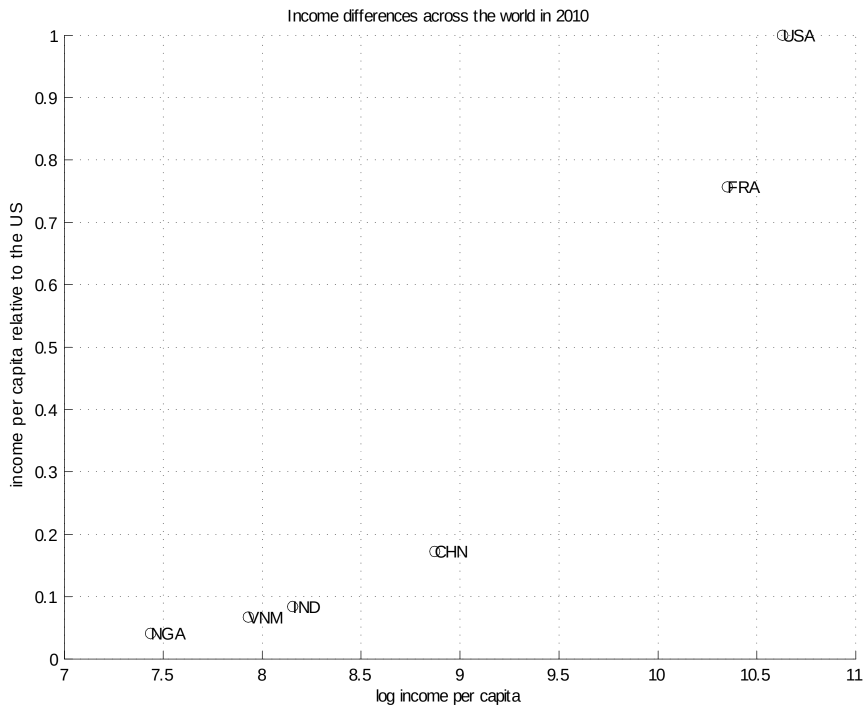

Figure 1: Figure 1

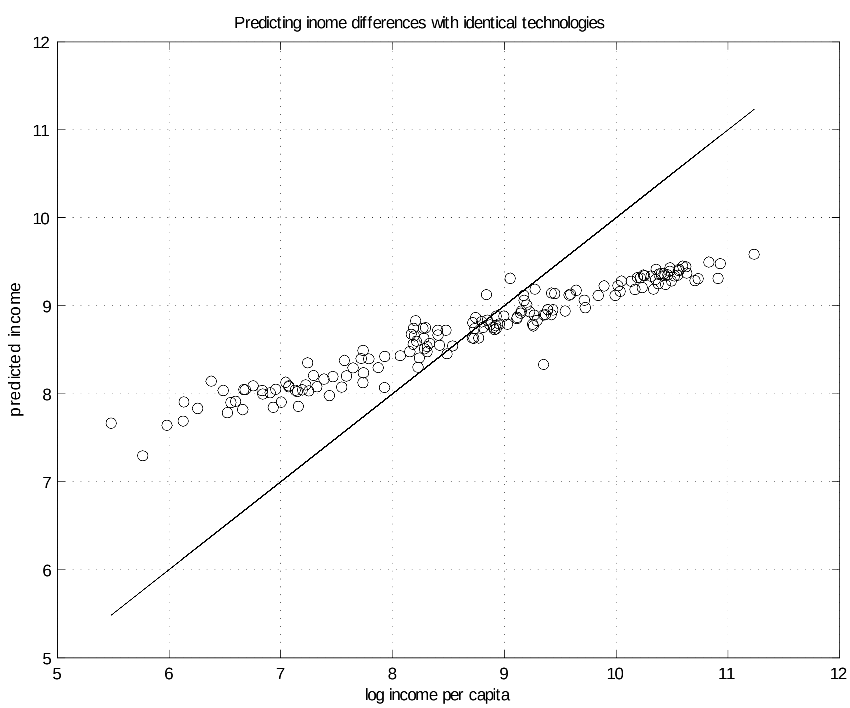

Figure 2: Figure 2

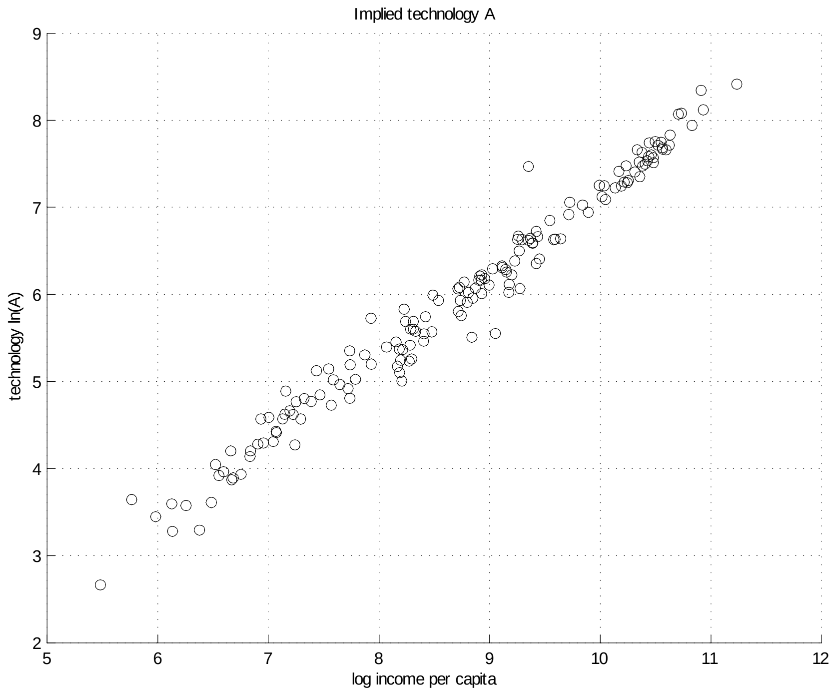

Figure 3: Figure 3.

4. According to the model
$$ln\left(k_i^*\right) = ln\left(\left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}\right) + ln\left(A_i\right) \tag{2}$$
so that we can solve for $ln\left(A_i\right)$. Then we can predict
$$ln\left(y_i\right) = (1-\alpha)\, ln\left(\overline{A}_i\right) + \alpha ln\left(k_i\right)$$
where $\overline{A}_i$ solves (2). In fact, it is useful to take these $ln\left(y_i\right)$ and normalize them to the mean of the income data. The result is depicted in Figure 4. The model does a pretty good job to predict income differences across the world - the predicted income lies pretty much on the 45 degree line. Note that this is not mechanically the case. We simply take data on k_i and use the Solow model to infer A_i. We then plug k_i and A_i into the production function and predict y_i pretty succesfully. From that point of view, we could say: if we want to understand income differences across the world, we just need a theory of A_i, why is technology so different across counties. If we had a theory of A_i, we could then use the Solow model to predict the distribution of capital and income across the world.

2 The Solow Model with Labor Market Policies

Consider the canonical Solow model discussed above, i.e. population and technology is constant. Output is given by
$$Y(t) = F\left(K(t), L(t)\right) = K(t)^{\alpha} L(t)^{1-\alpha}.$$
The saving rate is s and capital depreciates at rate δ. While the capital market is perfect, i.e. capital owners receive a rental rate $R(t)$ per unit of capital, the labor market is subject to a minimum wage regulation, i.e. workers are not allowed to be paid less than $\overline{w}$. If labor demand at this wage falls short of L, employment is equal to the amount of labor demanded by firms, L^d and the unemployed do not contribute to production and earn zero. Assume that

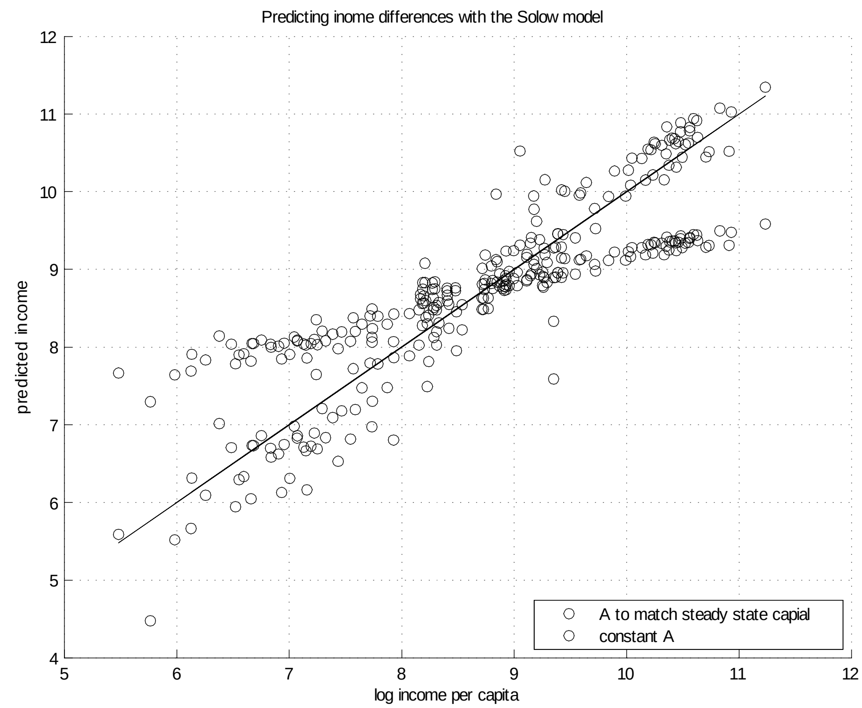

Figure 4: Figure 4

$\overline{w} > (1-\alpha)(k^*)^\alpha$, where $k^* = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}$ is the steady-state capital-labor ratio of the basic Solow model. Characterize the dynamic equilibrium path of this economy starting with some amount of physical capital $K(0)$ (feel free to assume that $\frac{K(0)}{L} < \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}$).

Solution: Again denote by $k(t) = \frac{K(t)}{L(t)}$ the capital-labor ratio in the population. The marginal product of labor is given by

$$MPL(K,L) = (1-\alpha)\left(\frac{K}{L}\right)^\alpha,$$

i.e. is increasing in $\frac{K}{L}$. Hence,

$$MPL(K(t), L(t)) = (1-\alpha)k(t)^\alpha < (1-\alpha)(k^*)^\alpha < \overline{w}$$

so that the minimum wage exceeds the marginal product of labor if there was full employment. Hence, $L^D(t)$ is given by

$$(1-\alpha)\left(\frac{K(t)}{L^D(t)}\right)^\alpha = \overline{w}$$

i.e. the marginal product of firms' is equal to the minimum wage. This implies that

$$L^D(t) = \left(\frac{1-\alpha}{\overline{w}}\right)^{1/\alpha} K(t)$$

so that total output in this economy is given by

$$Y(t) = K(t)^\alpha L^D(t)^{1-\alpha} = K(t)\left(\frac{1-\alpha}{\overline{w}}\right)^{\frac{1-\alpha}{\alpha}}.$$

Hence, the accumulation of the capital stock is given by

$$\begin{aligned}\dot{K}(t) &= sY(t) - \delta K(t) \\ &= L(t)\,\delta k(t)\left(\frac{s}{\delta}\left(\frac{1-a}{\overline{w}}\right)^{\frac{1-\alpha}{\alpha}} - 1\right).\end{aligned}$$

But $k^* = \left(\frac{s}{\delta}\right)^{\frac{1}{1-\alpha}}$ so that

$$\begin{aligned}\frac{s}{\delta}\left(\frac{1-a}{\overline{w}}\right)^{\frac{1-\alpha}{\alpha}} &= (k^*)^{1-\alpha}\left(\frac{1-\alpha}{\overline{w}}\right)^{\frac{1-\alpha}{\alpha}} \\ &= \left[\frac{(1-\alpha)(k^*)^{\alpha}}{\overline{w}}\right]^{\frac{1-\alpha}{\alpha}} \\ &< 1.\end{aligned}$$

Hence,

$$\dot{K}(t) < 0$$

so that

$$\begin{aligned}K(t) &\to 0 \\ Y(t) &\to 0 \\ L^D(t) &\to 0.\end{aligned}$$

In fact, the evolution of the capital-labor ratio is given by

$$k(t) = k(0)\,e^{-\delta\left(1-\frac{s}{\delta}\left(\frac{1-a}{\overline{w}}\right)^{\frac{1-\alpha}{\alpha}}\right)t} \to 0.$$

Hence, in the long-run, the capital stock converges to zero and nobody is employed. The intuition is the following: If the minimum wages are too high, the economy will feature unemployment. Hence, production and savings are strictly lower than in the economy without the minimum wage, so there is less accumulation. The capital stock however still depreciates. And: the depreciations always exceed the new accumulation. To see why this is the case, consider a general production function and note that

$$\begin{aligned}\dot{K} &= sF\left(K(t), L^D(t)\right) - \delta K(t) \\ &= sK(t)\left[\frac{F\left(K(t), L^D(t)\right)}{K(t)} - \frac{\delta}{s}\right] \\ &= sK(t)\left[\frac{F\left(K(t), L^D(t)\right)}{K(t)} - \frac{F(K^*(t), L(t))}{K^*(t)}\right]\end{aligned}$$

Hence, the capital stock increases, if the average product of capital is higher than at the steady-state. In the canonical Solow model this is the case whenever $K(t) < K^*$ by the concavity of the production function. But in this economy with a minimum wage, the average product of capital is low because only few people are employed, i.e. the effective capital-labor ratio $\frac{K(t)}{L^D(t)}$ is very large. To see this formally note that

$$F\left(K(t), L^D(t)\right) = \frac{1}{\frac{L(t)}{L^D(t)}}F\left(\frac{L(t)}{L^D(t)}K(t), \frac{L(t)}{L^D(t)}L^D(t)\right) = \frac{1}{\frac{L(t)}{L^D(t)}}F\left(\frac{L(t)}{L^D(t)}K(t), L(t)\right)$$

so that

$$\dot{K}(t) = sK(t)\left[\frac{F\left(z(t), L^D(t)\right)}{z(t)} - \frac{F(K^*(t), L(t))}{K^*(t)}\right],$$

where $z(t) = \frac{L(t)}{L^D(t)} K(t)$. Hence, we need to show that $z(t) > K^*(t)$ as the average product of capital is decreasing in the capital input. But

$$\overline{w} = MPL\left(K(t), L^D(t)\right) = MPL(z(t), L(t))$$

as the marginal product is homogenous of degree zero and

$$MPL(K^*(t), L(t)) < \overline{w}$$

by assumption (for the Cobb-Douglas case: $\overline{w} = (1-\alpha)(k^*)^{\alpha}$). As MPL is increasing in the capital input, we have

$$MPL(z(t), L(t)) > MPL(K^*(t), L(t)) \Rightarrow z(t) > K^*(t),$$

as required.