EC321: Problem Set 4

Johannes Boehm

London School of Economics

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Q2 Optimal Monetary Policy

- Central bank minimizes

$$L_t = \frac{1}{2}\left[(\pi_t - \pi^*)^2 + a x_t^2\right]$$

- Quadratic loss function (symmetric!). Parameter a governs relative weight of the output target.
- What affects inflation target π^*?
  - Shoe-leather costs, supports deflation
  - Avoiding menu costs, supports price stability
  - Avoiding relative price distortions, supports price stability
  - Upward bias in measured inflation: supports $\pi^* > 0$
  - Lack of indexation in tax system: supports price stability
  - Avoid risk of a liquidity trap (zero lower bound on i_t), supports $\pi^* > 0$
  - Downward nominal wage rigidity, supports price stability

- Let $\pi^* = 0$. Central bank minimizes

$$L_t = \frac{1}{2}\left[\pi_t^2 + a x_t^2\right]$$

subject to NKPC and IS

$$\begin{aligned} \pi_t &= E_t \pi_{t+1} + \kappa x_t + u_t \\ x_t &= E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1} - \hat{r}_t) \end{aligned}$$

In period t we learn that from $t+1$ onwards the cost-push shock will be higher:

$$u_t = 0, \quad u_{t+1} = u_{t+2} = \cdots = \bar{u} > 0$$

- Solve optimal policy problem in period $t+1$ (periods $t+2$ etc will be the same), this gives us π_{t+1}, then go back to period t and use this as $E_t \pi_{t+1}$.

$$\begin{aligned} & \max_{x_{t+1}, \pi_{t+1}} \frac{1}{2}\left[\pi_{t+1}^2 + a x_{t+1}^2\right] \\ \text{s.t. } \pi_{t+1} &= E_{t+1} \pi_{t+2} + \kappa x_{t+1} + \bar{u} \end{aligned}$$

- First-order condition:

$$\frac{\partial L_{t+1}}{\partial x_{t+1}} = \kappa\pi_{t+1} + a x_{t+1} = 0$$

- Inflation is entirely forward-looking, and periods $t+1, t+2, \ldots$ are exactly the same, therefore

$$\pi_{t+1} = \pi_{t+2} = \pi_{t+3} = \cdots$$

and $\pi_{t+1} = E_{t+1}\pi_{t+2}$. Plugging this into the NKPC:

$$x_{t+1} = -\frac{1}{\kappa}\bar{u}$$

Use this in the first-order condition to get

$$\pi_{t+1} = -\frac{a}{\kappa}x_{t+1} = \frac{a}{\kappa^2}\bar{u}$$

- Having found π_{t+1}, we have also found $E_t\pi_{t+1}$,

$$E_t\pi_{t+1} = \pi_{t+1} = \frac{a}{\kappa^2}\bar{u}$$

so we can use this in the period t problem:

$$\begin{aligned} \max L_t &= \frac{1}{2}\left[\pi_t^2 + a x_t^2\right] \\ \text{s.t. } \pi_t &= E_t\pi_{t+1} + \kappa x_t + 0 \end{aligned}$$

- The first-order condition is again

$$\frac{\partial L_t}{\partial x_t} = \kappa\pi_t + a x_t = 0$$

Two equations, two unknowns ($E_t\pi_{t+1}$ is known). Solve for π_t and x_t:

$$x_t = -\frac{a}{a+\kappa^2}\frac{1}{\kappa}\bar{u}, \qquad \pi_t = \frac{a}{a+\kappa^2}\frac{a}{\kappa^2}\bar{u}$$

Optimal policy spreads burden of cost-push shock over output and inflation.

- (graph)

THE VERY IMPORTANT SLIDE

- **How to solve optimal policy problems** (minimize loss fct subject to NKPC, IS):
- To solve for optimal x_t, π_t you need three components
  - The first-order condition. Plug NKPC into objective function and take first-order conditions w.r.t. x_t to get it.
  - The NKPC. You have that.
  - Expectations (particularly $E_t\pi_{t+1}$). This is the tricky bit. You might need to solve the optimal policy problem in the next period to get π_{t+1}, then take expectations E_t of that. Or, make an educated guess and verify it afterwards.
- Once you have these three components, plug expectations into the NKPC, then you have two equations (FOC and NKPC) in two variables x_t and π_t. Solve for this.
- If the question asks you to solve for the actual optimal interest rate i_t, you then need to plug the optimal x_t together with expectations $E_t\pi_{t+1}$ and $E_t x_{t+1}$ into the IS curve and solve for i_t. That should be easy once you have solved the above.

Q3 Optimal Monetary Policy with persistent CP shocks

- Central bank minimizes

$$L_t = \frac{1}{2}\left[\pi_t^2 + a x_t^2\right]$$

subject to the NKPC and IS curves,

$$\begin{aligned} \pi_t &= E_t \pi_{t+1} + \kappa x_t + u_t \\ x_t &= E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1} - \hat{r}_t). \end{aligned}$$

All variables in percentage (log) deviations from steady state.

- **(a) Why can we regard the CB as setting the output gap x_t?** Nominal interest rate only appears in the IS curve, so there is a one-to-one relationship between interest rate i_t and output gap x_t. So we can say the CB is setting x_t optimally (actually get optimal i_t from the IS curve). Only constraint is NKPC.

- **(b) Find the first-order condition for optimal policy (under discretion).** Plug NKPC for π_t into the objective function, take the derivative w.r.t x_t and set it equal to zero: (as in the previous question)

$$\frac{\partial L_t}{\partial x_t} = \kappa \pi_t + a x_t = 0 \tag{1}$$

- **(c) + (d) Assume shock is unpredictable, i.e.** $E_t u_{t+1} = 0$. **Find the optimal policy** i_t **and the implied** π_t **and** x_t**.** Again, we first assume that the CB can control x_t perfectly, then calculate π_t and x_t under the optimal policy and then finally solve for the optimal i_t using the IS equation.
- Start off by guessing that $E_t \pi_{t+1}$. We will later verify that this is indeed the case.
- Substitute the first-order condition (1) into the NKPC to get

$$\pi_t = 0 - \kappa \frac{\kappa}{a} \pi_t + u_t$$

and solve for π_t, and then use this to get x_t from the FOC

$$\pi_t = \frac{a}{a + \kappa^2} u_t, \qquad x_t = -\frac{\kappa}{a + \kappa^2} u_t.$$

$$\pi_t = \frac{a}{a+\kappa^2} u_t, \qquad x_t = -\frac{\kappa}{a+\kappa^2} u_t \tag{2}$$

- Period $t+1$ is exactly the same as period t (apart from the realization of the u shock, of course), so

$$\pi_{t+1} = \frac{a}{a+\kappa^2} u_{t+1}, \qquad x_{t+1} = -\frac{\kappa}{a+\kappa^2} u_{t+1}.$$

  Taking expectations, and noting that $E_t u_{t+1} = 0$ (see text),

$$E_t \pi_{t+1} = \frac{a}{a+\kappa^2} E_t u_{t+1} = 0, \qquad E_t x_{t+1} = -\frac{\kappa}{a+\kappa^2} E_t u_{t+1} = 0.$$

  So we have verified that indeed $E_t \pi_{t+1} = 0$.
- Optimal policy i_t: plug the solution (2) together with $E_t \pi_{t+1} = E_t x_{t+1} = 0$ into the IS curve

$$x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1} - \hat{r}_t)$$

  to get

$$i_t = \hat{r}_t + \frac{1}{\sigma} \frac{\kappa}{a+\kappa^2} u_t.$$

- **(e) Assume the cost-push shock is persistent, i.e. $E_t u_{t+1} = \delta u_t$, where $0 < \delta < 1$. Guess that in equilibrium $\pi_t = \psi u_t$, for some coefficient ψ, and find inflation π_t and output gap x_t under optimal policy.**
- Note that the first-order condition is the same as above, it does not depend on $E_t u_{t+1}$ (try it! good exercise!), so

$$\kappa \pi_t + a x_t = 0.$$

- Guess $\pi_t = \psi u_t$ and plug this into the FOC to get

$$x_t = -\frac{\kappa}{a}\pi_t = -\frac{\kappa}{a}\psi u_t$$

Next period: $\pi_{t+1} = \psi u_{t+1}$, so

$$E_t \pi_{t+1} = \psi E_t u_{t+1} = \psi \delta u_t$$

Plug all this into the NKPC to get

$$\pi_t = E_t \pi_{t+1} + \kappa x_t + u_t = \psi \delta u_t - \frac{\kappa^2}{a}\psi u_t + u_t = \left(\psi\delta - \frac{\kappa^2}{a}\psi + 1\right) u_t$$

$$\pi_t = \left(\psi\delta - \frac{\kappa^2}{a}\psi + 1\right) u_t$$

- But notice also that $\pi_t = \psi u_t$, so the coefficients of u_t in the two equations must be the same! Thus,

$$\psi = \psi\delta - \frac{\kappa^2}{a}\psi + 1$$

Solve for ψ to obtain

$$\psi = \frac{a}{a - \delta a + \kappa^2}$$

so this means that

$$\pi_t = \psi u_t = \frac{a}{a - \delta a + \kappa^2} u_t, \qquad x_t = -\frac{\kappa}{a - \delta a + \kappa^2} u_t$$

- **(f) Calculate the loss function L_t as a function of δ. Increasing or decreasing in δ? Intuition?**
- Plug optimal solution x_t, π_t into the loss function

$$L_t = \frac{1}{2}\left[\pi_t^2 + a x_t^2\right]$$

  and do some tedious simplifications to get

$$L_t = \frac{a}{2}\left(\frac{a+\kappa^2}{\left(\left(1-\delta\right)a+\kappa^2\right)^2}\right)u_t^2$$

  This is increasing in δ. Persistent cost-push shock means higher inflation for a longer period of time, adverse shift of the NKPC (higher inflation expectations $\longrightarrow$ higher current inflation).
- **GOOD LUCK FOR THE EXAM!** (and don't panic!)