Problem 1

(1) Derive the firm's cost function, show that equilibrium output (given factor prices) is given by

$$y(z) = \left(\gamma \left(\frac{1-\alpha}{w}\right)^{1-\alpha} \left(\frac{\alpha}{r}\right)^{\alpha}\right)^{\frac{\gamma}{1-\gamma}} z^{\frac{1}{1-\gamma}}. \tag{1}$$

and derive the factor demands $(l(z), k(z))$ as a function of z and factor prices. What are the profits of a firm with productivity z?

Solution: The firm's cost minimization problem is

$$\min_{h,k} wl + rk \text{ s.t. } y \leq z\left(k^{\alpha} h^{1-\alpha}\right)^{\gamma}.$$

The optimality condition for h and k is of course given by

$$\frac{k}{h} = \frac{\alpha}{1-\alpha}\frac{w}{r}.$$

Hence,

$$y = z\left(\left(\frac{\alpha}{1-\alpha}\frac{w}{r}h\right)^{\alpha} h^{1-\alpha}\right)^{\gamma} = z\left(\frac{\alpha}{1-\alpha}\frac{w}{r}\right)^{\alpha\gamma} h^{\gamma},$$

so that

$$h = \frac{1-\alpha}{w}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{y}{z}\right)^{1/\gamma} \tag{2}$$

$$k = \frac{\alpha}{r}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{y}{z}\right)^{1/\gamma}. \tag{3}$$

The firm's cost function is therefore given by

$$C = rk + wl = \left(\frac{w}{1-\alpha}\right)^{1-\alpha}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{y}{z}\right)^{1/\gamma}.$$

As all firms produce the same good in this economy, revenues are simply given by y so that profits are

$$\pi(y) = y - C(y) = y - \left(\frac{w}{1-\alpha}\right)^{1-\alpha}\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{y}{z}\right)^{1/\gamma}.$$

Choosing output to maximize profits yields the optimal level of output

$$y(z) = \left(\gamma \left(\frac{1-\alpha}{w}\right)^{1-\alpha} \left(\frac{\alpha}{r}\right)^{\alpha}\right)^{\frac{\gamma}{1-\gamma}} z^{\frac{1}{1-\gamma}}. \tag{4}$$

Profits are therefore given by

$$\begin{aligned}\pi(z) &= (1-\gamma)\left(\gamma\left(\frac{1-\alpha}{w}\right)^{1-\alpha}\left(\frac{\alpha}{r}\right)^{\alpha}\right)^{\frac{\gamma}{1-\gamma}} z^{\frac{1}{1-\gamma}} \\ &= (1-\gamma)\, y(z),\end{aligned} \tag{5}$$

which shows that the entrepreneurial factor captures a fraction $1-\gamma$ of revenues.

(2) Now solve for the equilibrium. State the equilibrium conditions (there should be three as there are three unknowns).

Solution: As we have seen in the recitation, when we discussed the paper by Moll (2010), there are three equilibrium objects which we have to pin down. On the one hand we have to pin down the equilibrium factor prices r and w, on the other hand we have to determine the set of active entrepreneurs, which again will be given by a cutoff rule. To see this, note that profits are increasing in z (see (5)). The "fixed costs" of production are the opportunity costs of being a worker. As the wage per human capital is given by w, the marginal entrepreneur is indifferent between working and opening a firm, i.e. the productivity cutoff $\overline{z}$ is defined by

$$(1-\gamma)\, y(\overline{z}) = wh. \tag{6}$$

Using the factor demands (2) and (3) and the optimal output level (4), it is also easy to see that

$$\begin{aligned} h(z) &= \frac{(1-\alpha)\gamma}{w} y(z) \\ k(z) &= \frac{\alpha\gamma}{r} y(z). \end{aligned}$$

Hence, the market clearing conditions for factor markets are given by

$$hLG(\overline{z}) = L\int_{\overline{z}} h(z)\, dG_Z(z) = LP(z>\overline{z})\int_{\overline{z}} h(z)\frac{g_Z(z)}{1-P(z<\overline{z})}dz \tag{7}$$

$$K = L\int_{\overline{z}} k(z)\, dG_Z(z) = LP(z>\overline{z})\int_{\overline{z}} k(z)\frac{g_Z(z)}{1-P(z<\overline{z})}dz \tag{8}$$

where L denotes the total measure of people in the economy. Note that $hLG(\overline{z})$ is the total labor supply in this economy, because all entrepreneurs (i.e. the people with $z>\overline{z}$) use their human capital to run the firm. The equilibrium is then fully determined by (6), (7) and (8). Note also that

$$\mu(z) \equiv \frac{g_Z(z)}{1-P(z<\overline{z})}$$

is the conditional productivity distribution of active firms.

(3) To solve for the equilibrium in closed form, we need to assume a particular distribution for productivity z. Let us assume that z is Pareto with support $[z_0,\infty)$ and shape parameter η, i.e.

$$P(z<\tau) = 1-\left(\frac{z_0}{\tau}\right)^{\eta}. \tag{9}$$

For the remainder assume that η is high enough that the respective moments are well-defined. Express the equilibrium conditions using (9). ***Hint: Recall that if z is Pareto on $[z_0,\infty)$ with shape η, the distribution of z^β conditional on $z>\tau_0>z_0$ is pareto on $[\tau_0^\beta,\infty)$ with shape $\frac{\eta}{\beta}$. Also recall that $E[z]=\frac{\eta}{\eta-1}z_0$.***

Solution: Using the assumption of productivity being pareto, the conditional distribution $\mu(z)\equiv\frac{g_Z(z)}{1-P(z<\overline{z})}$ is also a pareto distribution with shape η and the support $[\overline{z},\infty)$. Hence, it follows that

$$P(z>\overline{z})=\left(\frac{z_0}{\overline{z}}\right)^\eta$$

and

$$\begin{aligned}
\int_{\overline{z}} h(z)\frac{g_Z(z)}{1-P(z<\overline{z})}dz &= \int_{\overline{z}}\frac{(1-\alpha)\gamma}{w}y(z)\frac{g_Z(z)}{1-P(z<\overline{z})}dz \\
&= \frac{(1-\alpha)\gamma}{w}\int_{\overline{z}}\left[\left(\gamma\left(\frac{1-\alpha}{w}\right)^{1-\alpha}\left(\frac{\alpha}{r}\right)^\alpha\right)^{\frac{\gamma}{1-\gamma}}z^{\frac{1}{1-\gamma}}\right]\frac{g_Z(z)}{1-P(z<\overline{z})}dz \\
&= \frac{(1-\alpha)\gamma}{w}\left(\gamma\left(\frac{1-\alpha}{w}\right)^{1-\alpha}\left(\frac{\alpha}{r}\right)^\alpha\right)^{\frac{\gamma}{1-\gamma}}\int_{\overline{z}}z^{\frac{1}{1-\gamma}}\frac{g_Z(z)}{1-P(z<\overline{z})}dz \\
&= \frac{(1-\alpha)\gamma}{w}\left(\gamma\left(\frac{1-\alpha}{w}\right)^{1-\alpha}\left(\frac{\alpha}{r}\right)^\alpha\right)^{\frac{\gamma}{1-\gamma}}E\left[z^{\frac{1}{1-\gamma}}|z>\overline{z}\right] \\
&= \frac{(1-\alpha)\gamma}{w}\left(\gamma\left(\frac{1-\alpha}{w}\right)^{1-\alpha}\left(\frac{\alpha}{r}\right)^\alpha\right)^{\frac{\gamma}{1-\gamma}}\frac{(1-\gamma)\eta}{(1-\gamma)\eta-1}\overline{z}^{\frac{1}{1-\gamma}} \\
&= \frac{(1-\alpha)\gamma}{w}\psi(r,w)\frac{(1-\gamma)\eta}{(1-\gamma)\eta-1}\overline{z}^{\frac{1}{1-\gamma}},
\end{aligned}$$

where $\psi(r,w)=\left(\gamma\left(\frac{1-\alpha}{w}\right)^{1-\alpha}\left(\frac{\alpha}{r}\right)^\alpha\right)^{\frac{\gamma}{1-\gamma}}$. Substituting this in (7) yields

$$\begin{aligned}
hG(\overline{z}) &= \left(\frac{z_0}{\overline{z}}\right)^\eta\frac{(1-\alpha)\gamma}{w}\psi(r,w)\frac{(1-\gamma)\eta}{(1-\gamma)\eta-1}\overline{z}^{\frac{1}{1-\gamma}} \\
&= \frac{(1-\alpha)\gamma}{w}\psi(r,w)\frac{(1-\gamma)\eta}{(1-\gamma)\eta-1}z_0^\eta\overline{z}^{\frac{1-(1-\gamma)\eta}{1-\gamma}}. \qquad (10)
\end{aligned}$$

Similarly, we get from (8) that

$$\frac{K}{L}=k=\frac{\alpha\gamma}{r}\psi(r,w)\frac{(1-\gamma)\eta}{(1-\gamma)\eta-1}z_0^\eta\overline{z}^{\frac{1-(1-\gamma)\eta}{1-\gamma}}. \qquad (11)$$

Given $\overline{z}$, these two equations fully determine r and w. Given r and w, we can then find the cutoff from (6), which reads

$$\begin{aligned}
hw &= (1-\gamma)y(\overline{z})=(1-\gamma)\left(\gamma\left(\frac{1-\alpha}{w}\right)^{1-\alpha}\left(\frac{\alpha}{r}\right)^\alpha\right)^{\frac{\gamma}{1-\gamma}}\overline{z}^{\frac{1}{1-\gamma}} \\
&= (1-\gamma)\psi(r,w)\overline{z}^{\frac{1}{1-\gamma}}. \qquad (12)
\end{aligned}$$

(4) Use the equilibrium conditions to solve for the productivity cutoff $\overline{z}$, i.e. the value $\overline{z}$ such that only individuals with $z > \overline{z}$ open a firm. How does the cutoff depend on aggregate factor supplies K and H? What is the intuition for this result and what assumption drives it? How does $\overline{z}$ depend on α and γ and what is the intuition?

Solution: Solving for the cutoff is actually much easier than solving three equations in three unknowns. To see this, simply use (12) and (10) to get

$$hw = (1-\gamma)\psi(r,w)\overline{z}^{\frac{1}{1-\gamma}} = \frac{(1-\alpha)\gamma\psi(r,w)}{G(\overline{z})}\frac{(1-\gamma)\eta}{(1-\gamma)\eta-1}z_0^{\eta}\overline{z}^{\frac{1-(1-\gamma)\eta}{1-\gamma}}.$$

Conveniently, $\psi(r,w)$ fully cancels out from this expression, so that the cutoff is defined by

$$\overline{z}^{\frac{1}{1-\gamma}} = \frac{(1-\alpha)\gamma}{G(\overline{z})}\frac{\eta}{(1-\gamma)\eta-1}z_0^{\eta}\overline{z}^{\frac{1-(1-\gamma)\eta}{1-\gamma}},$$

which implies that

$$\frac{(1-\alpha)\gamma\eta}{(1-\gamma)\eta-1}z_0^{\eta} = \overline{z}^{\eta}G(\overline{z}) = \overline{z}^{\eta}\left(1-\left(\frac{z_0}{\overline{z}}\right)^{\eta}\right).$$

Solving for $\overline{z}$ yields

$$\overline{z} = \left(\frac{(1-\gamma\alpha)\eta-1}{(1-\gamma)\eta-1}\right)^{1/\eta}z_0 \equiv \theta z_0. \quad (13)$$

Hence, the cutoff only depends on exogenous parameters of the model and is proprtional to the minimal productivity z_0. Hence, the cutoff *does not* depend on aggregate factor supplies K or H. The reason for this result is the Cobb-Douglas production function, which implies that both entrepreneurs and workers receive a constant fraction of revenues. This implies that profits and factor payments are proportional (i.e. $\frac{hw}{\pi(z)} = \frac{\gamma(1-\alpha)y(z)}{(1-\gamma)y(z)} = \frac{\gamma(1-\alpha)}{1-\gamma}$) so that the "entry" decisions of entrepreneurs does not depend on factor prices and hence also not on factor supplies. In fact, this result is problematic empirically because the average firm size is increasing in GDP, i.e. richer countries have bigger firms. In the original Lucas (1978) paper there is a discussion of this point. In particular he shows that to get this fact, the elasticity of substitution between capital and labor has to be less than unity. In that case, wages increase when capital accumulates so that over time less and less people start firms (as wages increase relative to profits). If there are less firms, each firm has to employ more people for the labor market to clear. This prediction is in line with the broad facts. In many poor economies there are many many small firms, which get "wiped out" along the development path. As the economy accumulates capital, people transition from running small firms into formal employment contracts with bigger firms. It is also easy to see from (13) that

$$\frac{\partial\overline{z}}{\partial\alpha} < 0 \text{ and } \frac{\partial\overline{z}}{\partial\gamma} > 0,$$

which is intuitive. If the labor share is higher (which occurs then α is low) workers gain so that the marginal entrepreneur has to be more productive. Similarly, if entrepreneurs capture a lower share (which occurs when γ is high as the entrepreneurial share is $1-\gamma$), only more productive firms survive.

(5) Now solve for equilibrium prices. In particular, show that

$$r = \frac{\alpha}{k}\left(h^{1-\alpha}k^{\alpha}\right)^{\gamma}\Theta z_0 \quad (14)$$

$$w = \frac{1-\alpha}{h}\left(h^{1-\alpha}k^{\alpha}\right)^{\gamma}\Theta z_0 \quad (15)$$

where Θ is a constant.

Solution: To solve for equilibrium factor prices, recall that the optimality condition for k and h implies that

$$\frac{k}{h} = \frac{\alpha}{1-\alpha}\frac{w}{r}. \quad (16)$$

Hence,

$$\begin{aligned}
\psi(r,w) &= \left(\gamma\left(\frac{1-\alpha}{w}\right)^{1-\alpha}\left(\frac{\alpha}{r}\right)^{\alpha}\right)^{\frac{\gamma}{1-\gamma}} \\
&= \left(\gamma\left(\frac{1-\alpha}{\frac{1-\alpha}{\alpha}r\frac{k}{h}}\right)^{1-\alpha}\left(\frac{\alpha}{r}\right)^{\alpha}\right)^{\frac{\gamma}{1-\gamma}} \\
&= \left(\gamma\left(\frac{h}{k}\right)^{1-\alpha}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{\alpha}{r}\right)^{\frac{\gamma}{1-\gamma}} \\
&= \left(\gamma h^{1-\alpha}k^{\alpha}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{\alpha}{r}\right)^{\frac{\gamma}{1-\gamma}}k^{-\frac{\gamma}{1-\gamma}}.
\end{aligned}$$

Substituting this and (13) in (11) yields

$$\begin{aligned}
k &= \frac{\alpha}{r}\gamma\left(\gamma h^{1-\alpha}k^{\alpha}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{\alpha}{r}\right)^{\frac{\gamma}{1-\gamma}}k^{-\frac{\gamma}{1-\gamma}}\frac{(1-\gamma)\eta}{(1-\gamma)\eta-1}z_0^{\eta}(\theta z_0)^{\frac{1-(1-\gamma)\eta}{1-\gamma}} \\
&= \gamma^{\frac{1}{1-\gamma}}h^{\frac{(1-\alpha)\gamma}{1-\gamma}}k^{\frac{\alpha\gamma}{1-\gamma}}\left(\frac{\alpha}{r}\right)^{\frac{1}{1-\gamma}}k^{-\frac{\gamma}{1-\gamma}}\frac{(1-\gamma)\eta}{(1-\gamma)\eta-1}(\theta)^{\frac{1-(1-\gamma)\eta}{1-\gamma}}z_0^{\frac{1}{1-\gamma}}.
\end{aligned}$$

Rearraning terms allows us to solve for the interest rate r as

$$\begin{aligned}
r &= \alpha k^{-1}\left(\gamma\left(h^{1-\alpha}k^{\alpha}\right)^{\gamma}\left(\frac{(1-\gamma)\eta}{(1-\gamma)\eta-1}\right)^{1-\gamma}\theta^{1-(1-\gamma)\eta}\right)z_0 \\
&= \frac{\alpha}{k}\left(h^{1-\alpha}k^{\alpha}\right)^{\gamma}\left(\gamma\left(\frac{(1-\gamma)\eta}{(1-\gamma)\eta-1}\right)^{1-\gamma}\theta^{1-(1-\gamma)\eta}\right)z_0
\end{aligned}$$

Using (16) directly implies that

$$w = \left(\frac{1-\alpha}{h}\right)\left(h^{1-\alpha}k^{\alpha}\right)^{\gamma}\left(\gamma\left(\frac{(1-\gamma)\eta}{(1-\gamma)\eta-1}\right)^{1-\gamma}\theta^{1-(1-\gamma)\eta}\right)z_0.$$

This is exactly the required form for

$$\Theta = \gamma\left(\frac{(1-\gamma)\eta}{(1-\gamma)\eta-1}\right)^{1-\gamma}\theta^{1-(1-\gamma)\eta},$$

where θ is defined in (13).

(6) We are now going to look at the empirical implications of this model, in particular the distribution of employment in the cross section of active firms. Given the equilibrium factor prices, solve for $l(z)$, i.e. the employment of a z-firm as a function of parameters and aggregate factor supplies. How is employment distributed in the cross-section of active firms, i.e. what is the equilibrium employment distribution? How does the average firm size (i.e. mean employment) vary with aggregate factor supplies K and L? Is this implication consistent with the facts? What is the size of the smallest firm? Can you give an intuition why there are no "atomistic" firms in this economy, i.e. why is the firm-size bounded away from zero?

Solution: To solve for the equilibrium firm size, i.e. the distribution of employment, note that

$$\begin{aligned} l(z) &= \frac{1-\alpha}{hw}\gamma y(z) \\ &= \frac{1-\alpha}{hw}\frac{\gamma}{1-\gamma}\pi(z) \\ &= \frac{1-\alpha}{hw}\frac{\gamma}{1-\gamma}\pi(\overline{z})\frac{\pi(z)}{\pi(\overline{z})} \\ &= \frac{(1-\alpha)\gamma}{1-\gamma}\frac{y(z)}{y(\overline{z})}, \end{aligned}$$

where the last line used that $hw = \pi(\overline{z})$ and $\pi(z) = (1-\gamma)y(z)$. As $y(z) \propto z^{\frac{1}{1-\gamma}}$, it follows directly that

$$l(z) = \frac{(1-\alpha)\gamma}{1-\gamma}\left(\frac{z}{\overline{z}}\right)^{\frac{1}{1-\gamma}} = \frac{(1-\alpha)\gamma}{1-\gamma}\left(\frac{1}{\overline{z}}\right)^{\frac{1}{1-\gamma}} z^{\frac{1}{1-\gamma}}.$$

Hence, employment is proportional to $z^{\frac{1}{1-\gamma}}$. In the population of active firms $z^{\frac{1}{1-\gamma}}$ is distributed pareto with shape $(1-\gamma)\eta$ and has a lower bound of $\overline{z}^{\frac{1}{1-\gamma}}$. Hence, the employment distribution is given by

$$\begin{aligned} P(l(z) < \tau | z > \overline{z}) &= P\left(\frac{(1-\alpha)\gamma}{1-\gamma}\left(\frac{1}{\overline{z}}\right)^{\frac{1}{1-\gamma}} z^{\frac{1}{1-\gamma}} < \tau | z > \overline{z}\right) \\ &= P\left(z^{\frac{1}{1-\gamma}} < \tau\frac{1-\gamma}{(1-\alpha)\gamma}\overline{z}^{\frac{1}{1-\gamma}} | z > \overline{z}\right) \\ &= 1 - \left(\frac{\overline{z}^{\frac{1}{1-\gamma}}}{\tau\frac{1-\gamma}{(1-\alpha)\gamma}\overline{z}^{\frac{1}{1-\gamma}}}\right)^{(1-\gamma)\eta} \\ &= 1 - \left(\frac{\frac{(1-\alpha)\gamma}{1-\gamma}}{\tau}\right)^{(1-\gamma)\eta}, \end{aligned}$$

i.e. employment is distributed pareto with shape $(1-\gamma)\eta$ and support $[\frac{(1-\alpha)\gamma}{1-\gamma}, \infty)$. The equilibrium employment distribution is therefore *entirely* determined by exogenous parameters of the model. In particular, it does not depend on aggregate factor supplies (which is - empirically - a problem as argued above). The smallest firm employs $\frac{(1-\alpha)\gamma}{1-\gamma}$ employees. That the firm size is bounded from zero follows from the fact that there is a fixed cost of production. As you could always earn the equilibrium wage wh, all active firms need to have profits of at least wh. And as profits are increasing in output y, each firm needs to produce at least a minimal level of output for which a non-infinitesimal number of workers have to be hired. For too small firms, profits will simply be too low to make running the firm worthwhile.

References

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