

# Trade and the End of Antiquity

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WU Economic and Social History Seminar

24 June 2025

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# What caused the End of Antiquity?

- Antiquity: Roman and Greek civilizations centered around the Mediterranean
- Archaeological evidence points to a shift in economic activity away from the Mediterranean between 5th and 8th century AD (“end of antiquity”)
  - Rise of Northern Europe (Charlemagne).
  - “Golden Age” of Islam

→ *Question:* What caused the End of Antiquity?

- Discussed, among others, by Montesquieu (1734), Voltaire (1756), Gibbon (1789)
- Henri Pirenne (1937) blames the Arab conquests and the emerging Islamic-Christian border for the rupture in the Mediterranean unity

## Political changes in the Mediterranean: 600 AD



## Political changes in the Mediterranean: 600 AD



## Political changes in the Mediterranean: 600 AD



## Political changes in the Mediterranean: 600 AD



## Political changes in the Mediterranean: 632 AD



## Political changes in the Mediterranean: 634 AD



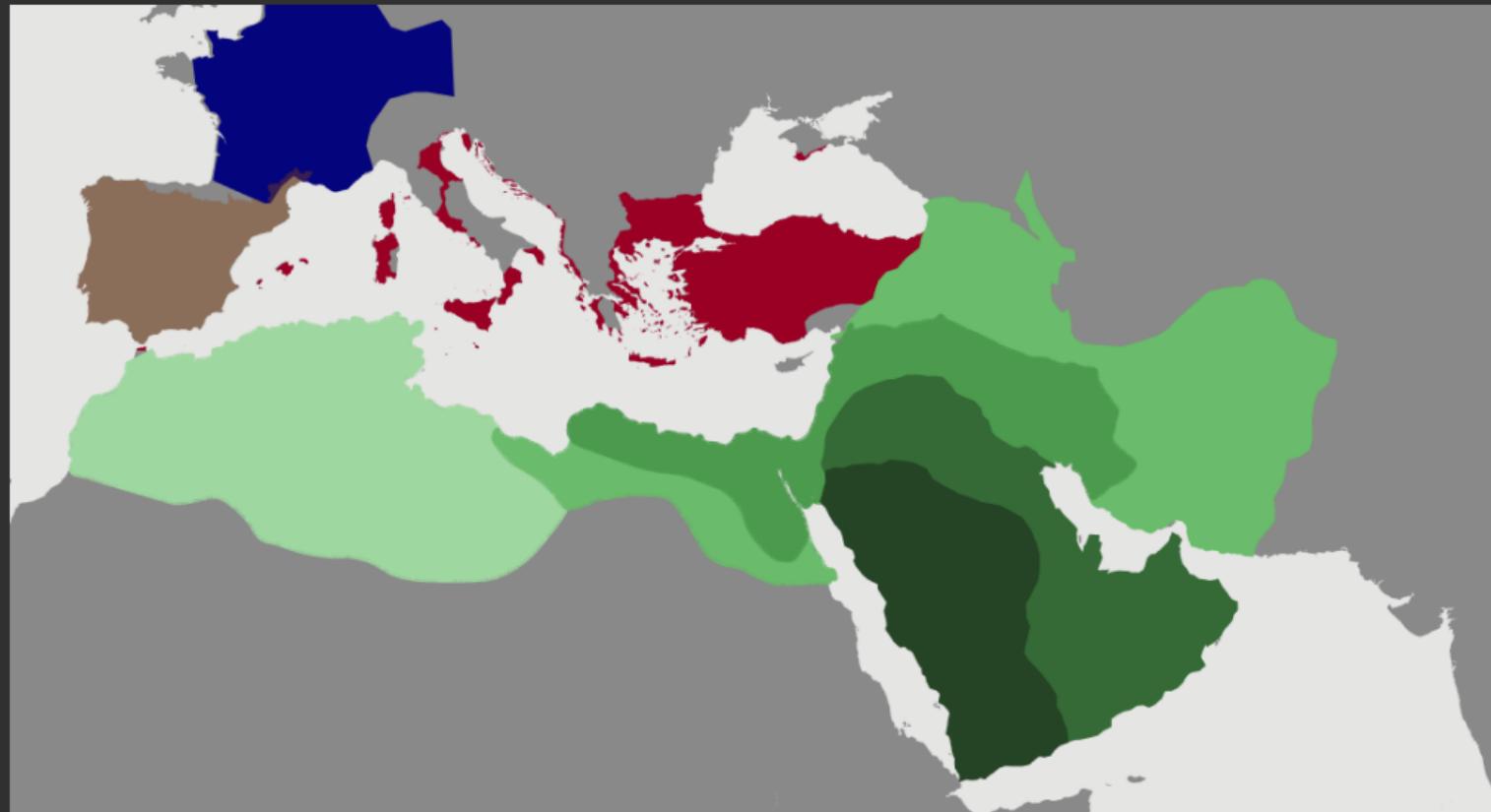
# Political changes in the Mediterranean: 644 AD



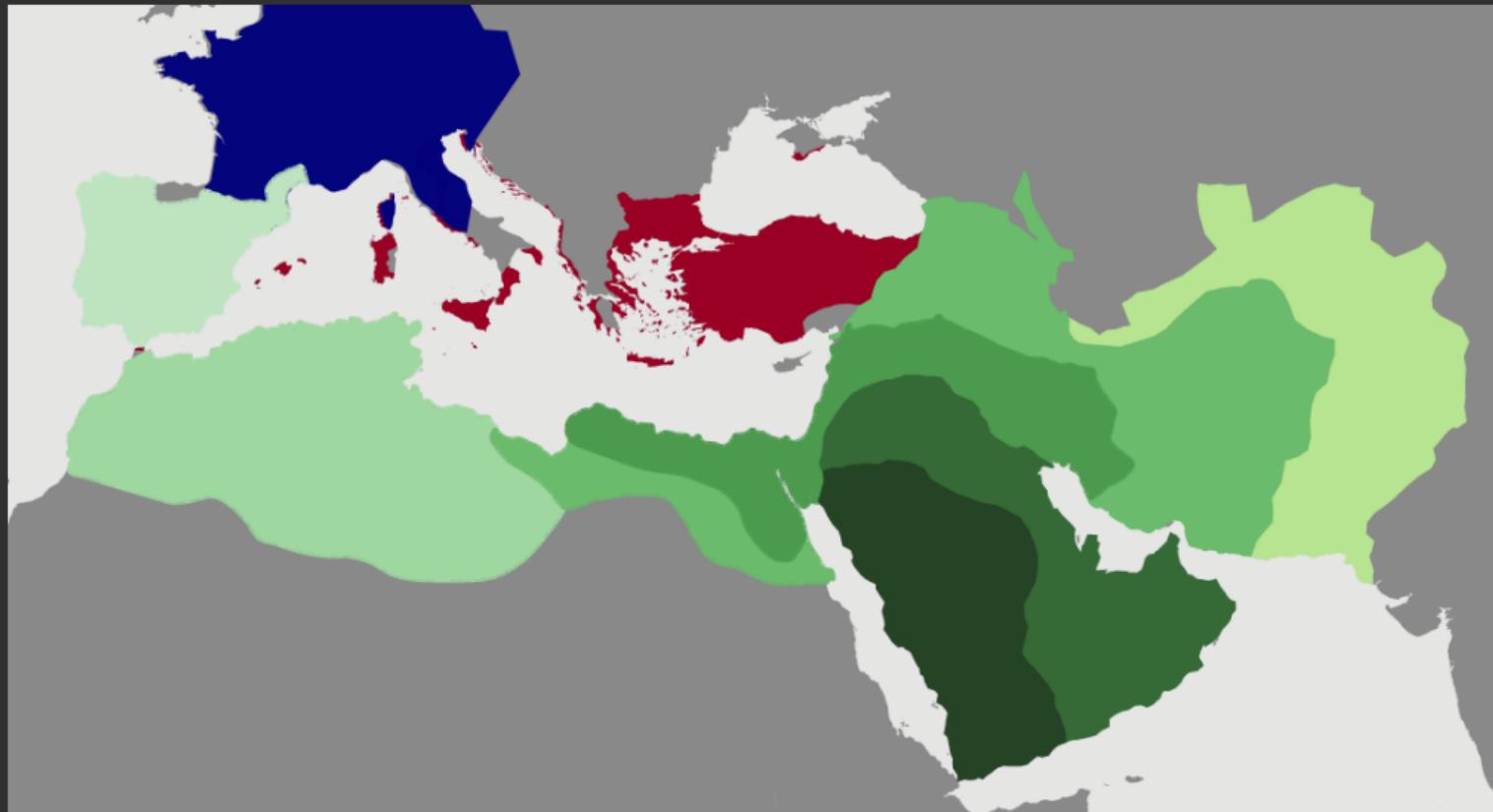
## Political changes in the Mediterranean: 661 AD



# Political changes in the Mediterranean: 661-700 AD



# Political changes in the Mediterranean: 750 AD



This paper: quantitatively investigate changing economic geography

Challenge: virtually no production/consumption/trade data

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⇒ Use data on the movement of coins to study the changing economic geography during Late Antiquity.

- Coins are the main medium of exchange during Late Antiquity, particularly for long-distance trade → informative about trade
- Coins are well studied & documented by historians and numismatists
- Coins have features that help solve econometric identification problems

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- Coins have features that help solve econometric identification problems

⇒ we interpret coin flows through the lens of a trade model and identify

- bilateral trade flows ( $i, j, t$ )
- technology ( $i, t$ )
- trade deficits ( $i, t$ )
- real consumption per capita ( $i, t$ )

in order to better understand what's going on during Late Antiquity

# Findings

- North-western Europe (and Islamic Spain) witness a large increase in relative real consumption in the 7th/8th century;
- Byzantine Heartlands see a large drop in relative real consumption
- Large collapse in trade in the Mediterranean, but (except for Byzantium) relatively small contribution to changes in per-capita consumption
- Most of changing relative real consumption driven by trade deficits (seigniorage) and technology.

# Findings

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Econometric estimates broadly consistent with qualitative empirical evidence:

- Archaeological evidence on Carolingian commerce (Hodges and Whitehouse, 1983, McCormick, 2001, Wickham, 2006)
- Archaeological evidence on Byzantine monetary and economic collapse (“Byzantine dark ages”, Kazhdan, 1954; Pennas, 1996, Haldon, 1997)
- Urbanization patterns in Europe and Asia Minor (Brandes, 1989; Buringh, 2021)

# Outline

Data

Reduced form evidence

Model

Parameterization and Estimation

# Data: Coins around the Mediterranean, AD 325 to AD 950

Assemble a large dataset of coin finds from around the Mediterranean

## 1. FLAME (2023) project:

- “Framing the Late Antique and Early Medieval Economy.”
- Collaboration of >60 historians and numismatists.
- ~200,000 coins with complete records from ~4,600 hoards
- Pre AD 725.

## 2. Hand-coded records from numismatic / archaeological literature:

- 797 coin finds.
- 100,478 coins.
- post AD 725.

Data covers most of published literature on hoard records (and more)

## Coin hoard data: an example from al 'Ush (1972)

No.	MINT	DATE	DIAM.	WEIGHT	NUMB.
51	الأندلس	114	29.	2.93	4
52	"	115	29.5	2.92	1
53	"	116	26.5	2.92	3

Excerpt of an original publication on the Damascus silver hoard:

- record number (51)
- mint (al-Andalus) \*
- mint date (year 114 of the Hijri calendar) \*
- diameter (29mm)
- weight (2.93g)
- number of coins with these attributes (4) \*

The issuing dynasty (Umayyad) is given in the table headings and the denomination and material (silver *dirham*) is stated in the text. \* denotes required attributes

## Coin dataset

- Each coin provides the following information:
  1. Mint location (“birthplace”):  $m$
  2. Mint date (“birthdate”):  $\tau$
  3. Hoard location (“death place”):  $h$
  4. Terminus post quem,  $tpq$  (“death date”):  $T = \sup \tau$

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- 286,035 coins.
- Time:
  - Mint date > AD 325
  - $tpq$  < AD 950.
- Space:
  - Western Europe.
  - Southern Europe.
  - Northern Africa.
  - Middle East++.

## Spatial distribution: the (extended) Mediterranean

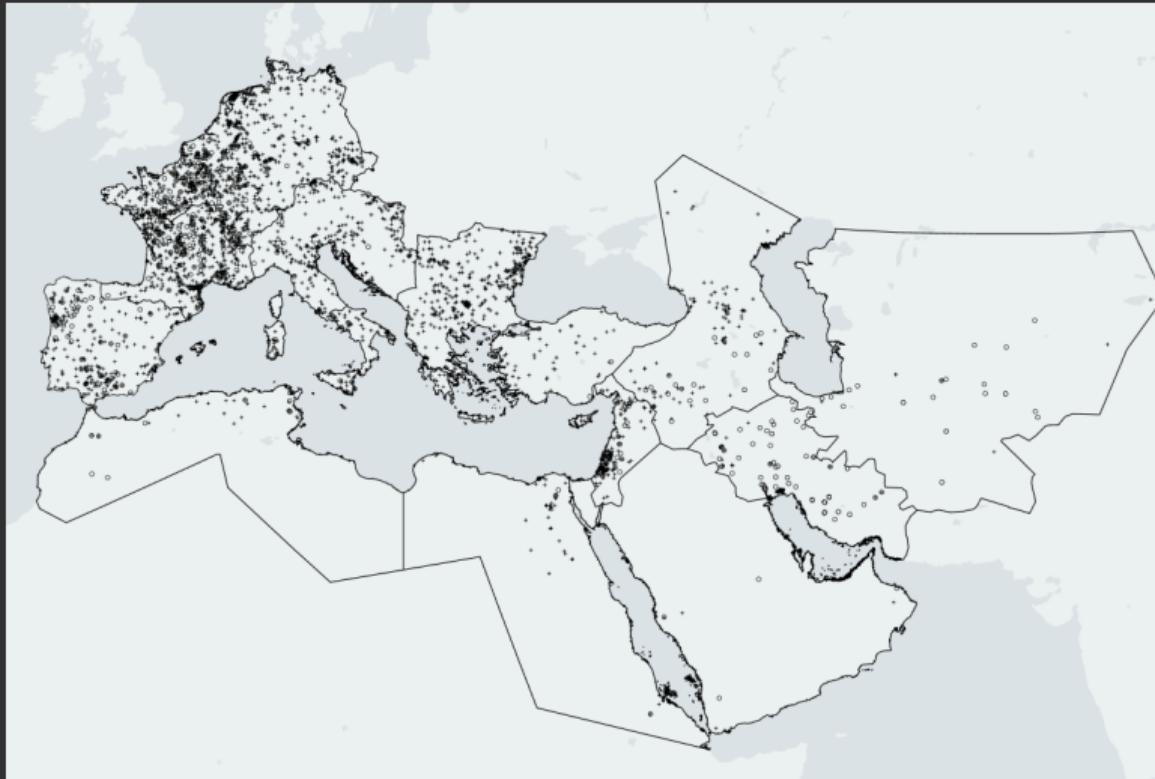


Figure 1: Region Definitions

# Distribution of coin “death dates” ( $tpq$ )

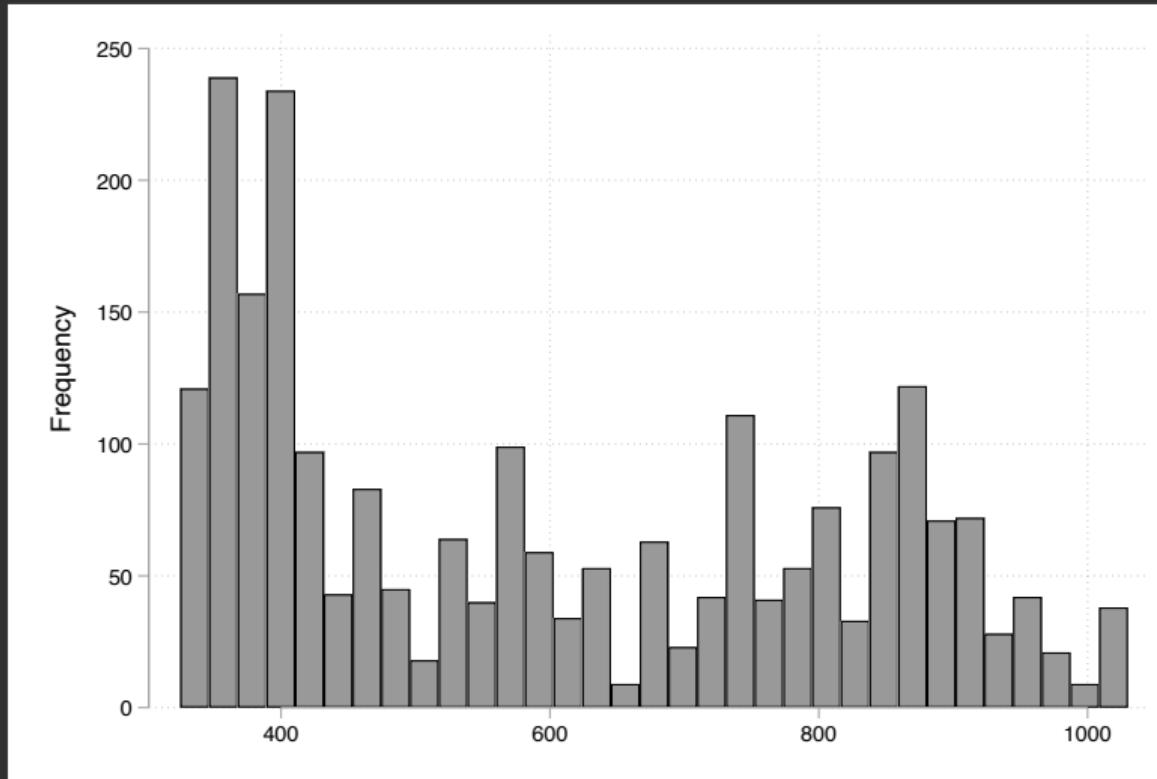


Figure 2: Terminus Post Quem ( $tpq$ ) of hoards

## Distribution of coin ages ( $tpq$ minus mint date)

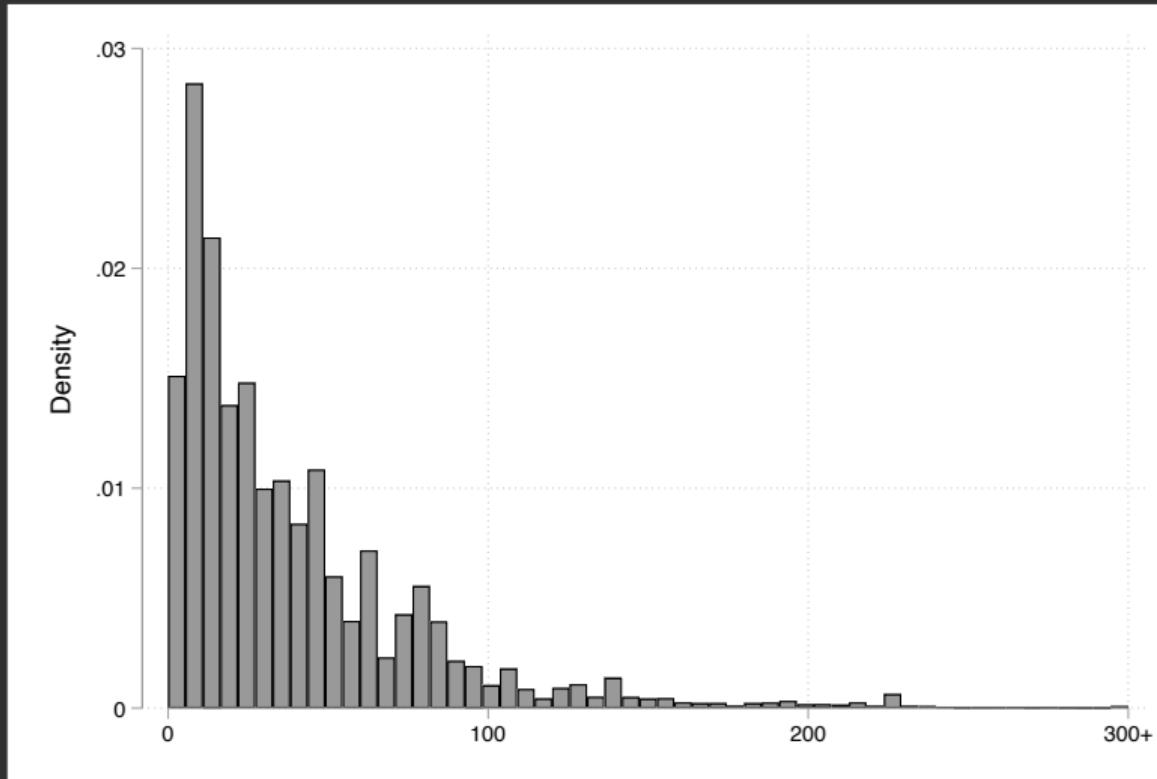


Figure 3: Coin age at time of deposit ( $tpq$ ), in years

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# Stylized facts

We document 3 main stylized facts:

1. Older coins travel further.
2. Distance *and* politics impede coin travels (gravity).
3. The geography of coin flows changes sharply around the Arab conquests.

# Fact #1: within a hoard, older coins have travel farther

Table 1: Within-hoard distance travelled and age of coin at deposit

Dependent variable: Log Distance between Mint and Hoard					
	(1)	(2)	(3)	(4)	(5)
Log Age of Coin	0.146*** (0.044)	0.0831*** (0.026)	0.0749** (0.031)	0.160*** (0.043)	0.0485** (0.020)
Sample				No non-hoards	No non-hoards
Hoard FE	Yes	Yes	Yes	Yes	Yes
Mint × 50-year-interval FE		Yes			
Mint × 25-year-interval FE			Yes		Yes
R <sup>2</sup>	0.762	0.863	0.869	0.775	0.898
Observations	287243	287029	286873	250156	249830

Standard errors in parentheses, clustered at the hoard level.

\*  $p < 0.10$ , \*\*  $p < 0.05$ , \*\*\*  $p < 0.01$

Age of coin =  $tpq - \text{mint date}$

⇒ coins diffuse across space over time.

## Fact #2: distance and political borders impede coin travels

Construct  $1^\circ \times 1^\circ$  cells for mint and hoard locations and calculate flows  $\#coins_{mdh}$

**Table 2:** Gravity and Border Effects in Coin Flows

	Dependent variable: # Coins <sub>mdh</sub>				Dep. var.: Value <sub>mdh</sub>	
	(1)	(2)	(3)	(4)	(5)	(6)
Log Distance	-1.137*** (0.12)	-1.002*** (0.13)	-1.135*** (0.10)	-0.951*** (0.076)	-1.144*** (0.075)	-0.989*** (0.068)
Political border		-1.945*** (0.62)		-2.073*** (0.47)		-1.516*** (0.27)
Hoard Cell FE	Yes	Yes	Yes	Yes	Yes	Yes
Mint × Empire Cell FE	Yes	Yes	Yes	Yes	Yes	Yes
Sample		Gold only	Gold only	Gold and Silver	Gold and Silver	
Estimator	PPML	PPML	PPML	PPML	PPML	PPML
Pseudo-R <sup>2</sup>	0.767	0.778	0.808	0.824	0.800	0.810
Observations	217748	217748	57287	57287	146767	146767

Standard errors in parentheses, clustered at mint cell × empire and hoard cell level.

\*  $p < 0.10$ , \*\*  $p < 0.05$ , \*\*\*  $p < 0.01$

Estimating eqn:  $\text{count}_{mdh} = \exp(\gamma_1 \log \text{distance}_{mh} + \gamma_2 \text{withinBorder}_{dh} + \alpha_{md} + \alpha_h + \varepsilon_{mhd})$

## Fact #2: distance has a weaker impact on older coin flows

$\text{logcount}_{mth\tau} =$

$$\sum_{\tau' \in T} \beta_{\tau'} \log \text{distance}_{mh} \\ \times \mathbf{1}(t - \tau = \tau') \\ + \alpha_{mt} + \alpha_{h\tau} + \varepsilon_{mth\tau}$$

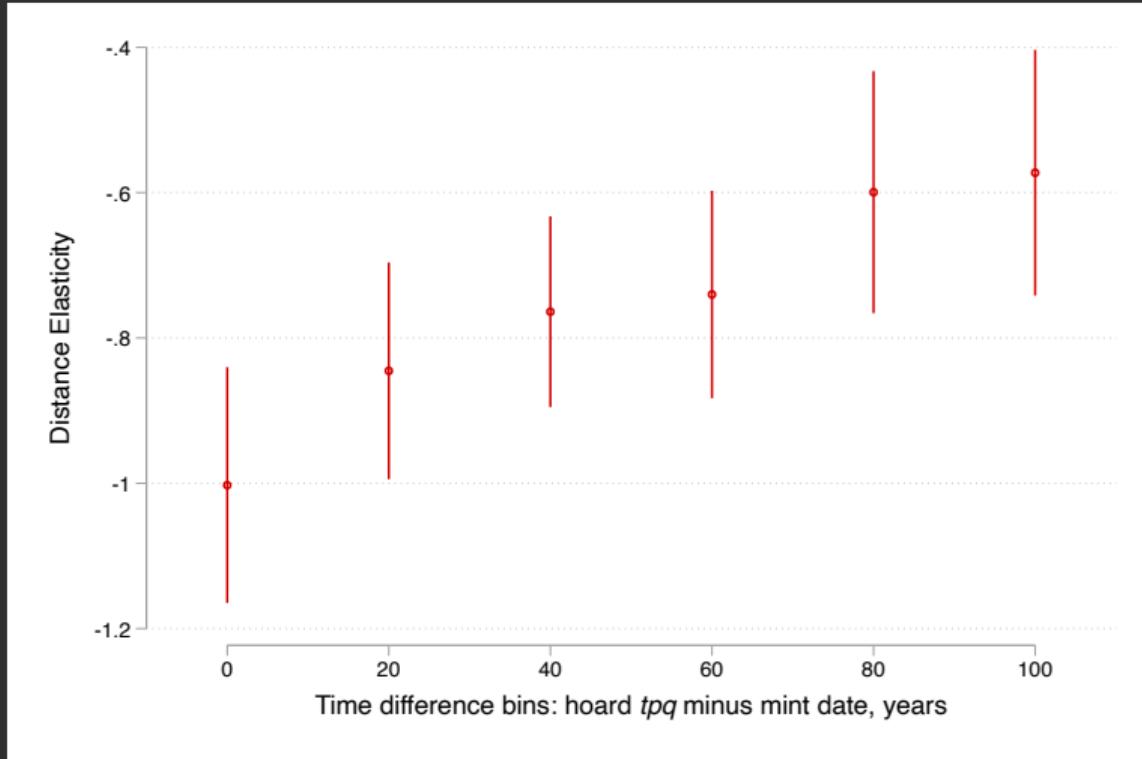


Figure 4: The distance elasticity declines as coins get older

## Fact #3: Coin flows before/after the Arab conquests

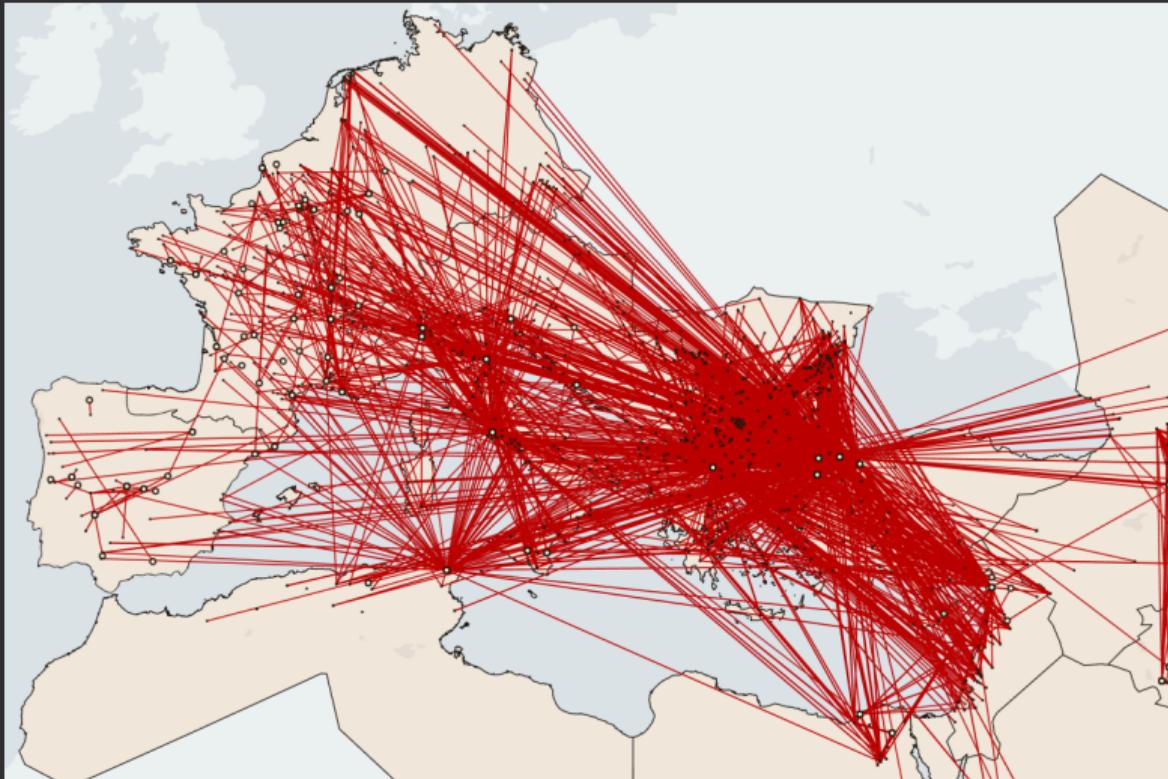


Figure 5: Before the Arab conquests: 450-630 AD

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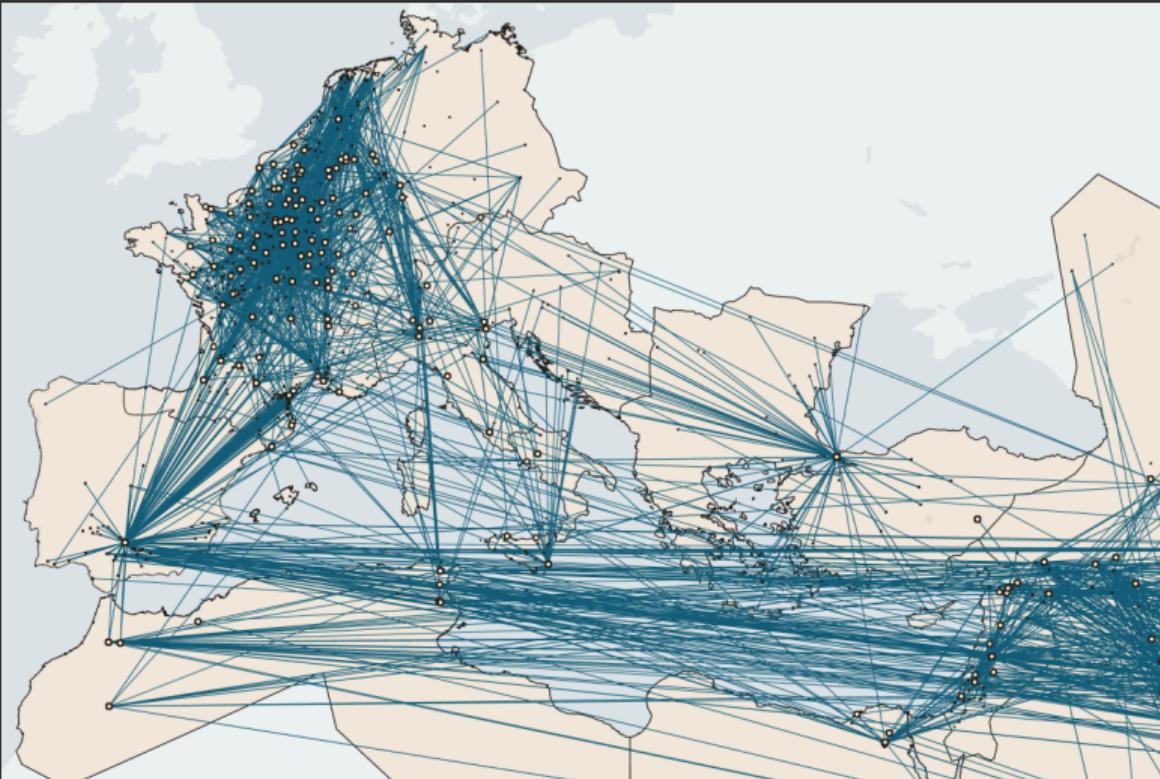


Figure 6: After the Arab conquests: 713-900 AD

► Cross-med: reg

► Cross-med: plot

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# Model

## Objective:

Quantify impact of trade barriers (and other changes) on market access / trade / welfare

## Approach:

Dynamic model of trade flows (gravity), where coins diffuse alongside trade and are thus informative about trade flows.

## Key Assumption:

Traders are blind to coin *types* (mint location and date).

- ⇒ Coins diffuse in proportion to trade flows
- ⇒ Recover trade *flows* from stocks of coin *types* in hords in different locations/time periods.

Not used/needed for identification: total *quantities* of coins found

## Model components (simplified case: exogenous saving)

$N$  locations with labor endowments  $L_n$ , Ricardian trade as in Eaton and Kortum (2002)

- Households finance consumption expenditure using saving  $S_n(t - 1)$ , exogenous newly minted coins  $M_n(t)$ ; end-of-period income is saving for next period
- Goods market clearing, denominated in coins

$$\underbrace{w_i L_i[t]}_{income_i[t]} = \sum_n \pi_{ni}[t] \overbrace{\left( (1 - \lambda) w_n L_n[t - 1] + M_n[t] \right)}^{expenditure_n[t]}, \forall i, t \quad (1)$$

$$\Leftrightarrow S_i[t] = \sum_n \pi_{ni}[t] \left( (1 - \lambda) S_n[t - 1] + M_n[t] \right), \forall i, t \quad (2)$$

$w_i$ : wages;  $L_i$ : labor;  $\pi_{ni}$ : expenditure shares;  $\lambda$ : coin loss;  $M_n$ : minting;  $S_i$ : coin stocks

- Fraction  $\pi_{ni}$  of  $n$ 's expenditure allocated to goods from  $i$

$$\pi_{ni}[t] = \frac{(T_i[t] w_i^{-\theta}[t]) (d_{ni}[t])^{-\theta}}{\sum_k (T_k[t] w_k^{-\theta}[t]) (d_{nk}[t])^{-\theta}}, \forall i, n, t \quad (3)$$

$T_i$ : technology;  $d_{ni}$ : trade cost;  $\theta$ : trade elasticity

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## Composition of coin stocks

- Stock  $S_i [T]$  composed of different coin types,  $S_i [T] = \sum_{m=1}^N \sum_{t \leq T} S_{mi} [t, T]$
- Coins start their ‘coin life’ when they are minted,  $S_{mm} [t, t] = M_m [t]$
- Then stocks evolves recursively,

$$S_{mi} [t, \tau] = (1 - \lambda) \sum_{n=1}^N \pi_{ni} [\tau] S_{mn} [t, \tau - 1] \quad (4)$$

- Recursive solution in matrix form (coin origin  $\times$  coin destination),

$$\mathbf{S} [t, T] \quad (4')$$

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- Recursive solution in matrix form (coin origin  $\times$  coin destination),

$$\mathbf{S} [t, t + 2] = (1 - \lambda)^2 \mathbf{M} [t] \mathbf{\Pi} [t + 1] \mathbf{\Pi} [t + 2] \quad (4')$$

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- Recursive solution in matrix form (coin origin  $\times$  coin destination),

$$\mathbf{S} [t, T] = (1 - \lambda)^{T-t} \mathbf{M} [t] \mathbf{\Pi} [t+1] \mathbf{\Pi} [t+2] \cdots \mathbf{\Pi} [T] \quad (4')$$

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## Data generating process

- In  $n$  at  $t$ , a fraction  $\eta_n(t)$  of the coins in circulation  $S_n(t)$  is lost and subsequently found by archaeologists  
⇒ For coin shares within a hoard, the  $\eta$ 's cancel out.  $\eta_n(t)$  not used in identification!

## Taking the model to the data

- 13 regions around the Mediterranean [▶ details](#)
- 20-year time intervals
- Assume constant  $\lambda$  and estimate as exponential decay parameter in within-hoard age distribution:

$$\hat{\lambda}_{20y} = 0.301$$

(or 1.7% per year)

- Parameterize trade frictions,

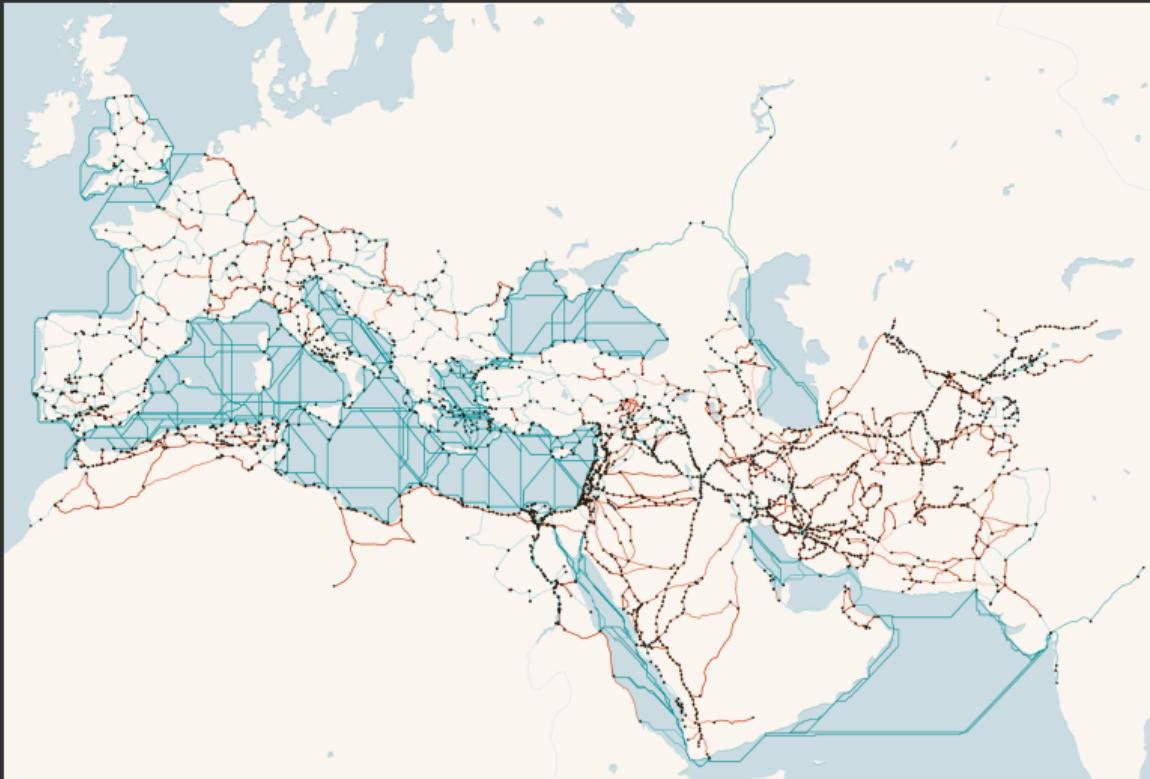
$$d_{ni}(t)^\theta = \exp(\gamma_0 + \zeta \ln(\text{TravelTime}_{ni}) + \kappa_1 \text{PoliticalBorder}_{ni}(t) + \kappa_2 \text{ReligiousBorder}_{ni}(t))$$

if  $n \neq i$  and  $d_{nn}(t) = 1$ .

- How to split domestic trade vs saving? Calibrate  $\bar{s}_n(t) = 1.5\%$  (Scheidel, 2020).
- For counterfactuals, assume  $\theta = 4$  (Simonovska and Waugh, 2014).

[▶ Intuition for identification](#)

Trade costs *only* depend on travel times (and politics/religion)



Note: Combined geospatial models from Orbis (Scheidel, 2015) and al-Turayyā (Romanov and Seydi, 2022). ▶ al-Maqdisi

## Maximum likelihood estimation

Assume coins in our data are a random sample of coin types in each location  $\times$  time.

- Multinomial distribution of coin types,

$$P(\dots, X_i^{(m,\tau)}(T) = x_i^{(m,\tau)}, \dots) = \frac{N_i(T)!}{\prod_{(m,\tau)} x_i^{(m,\tau)}!} \prod_{(m,\tau)} [p_i^{(m,\tau)}(T)]^{x_i^{(m,\tau)}}$$

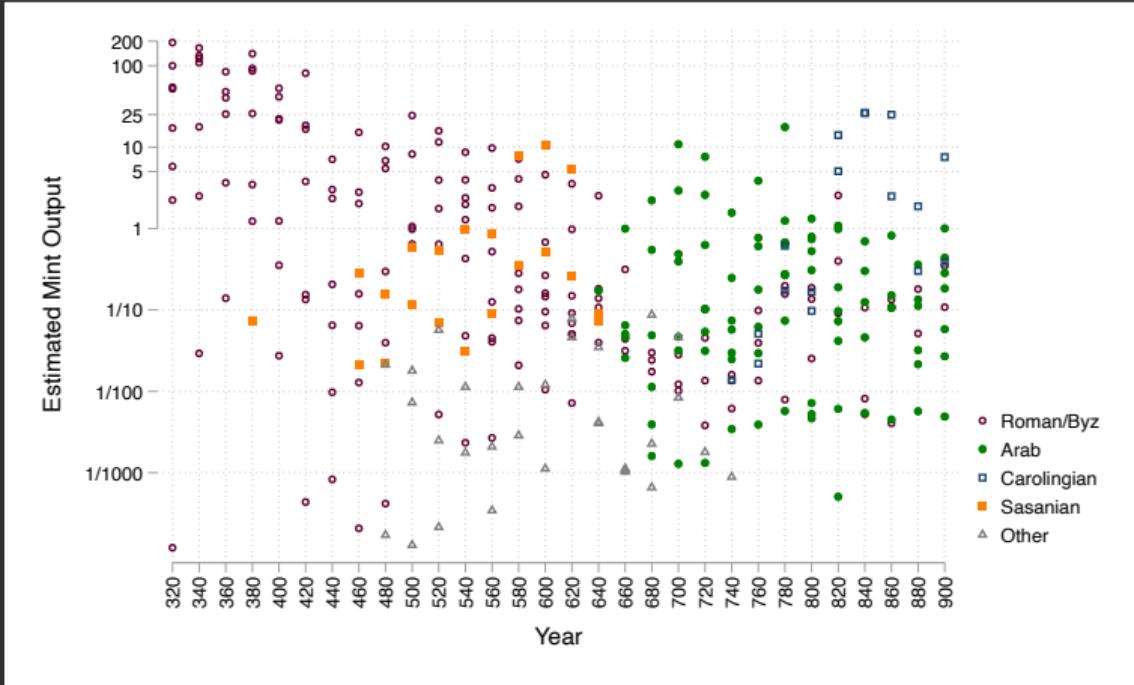
with the probability of drawing a coin of type  $(m, \tau)$ ,

$$p_i^{(m,\tau)}(T) = \frac{S_i^{(m,\tau)}(T)}{\sum_{(m',\tau')} S_i^{(m',\tau')}(T)} = \frac{S_i^{(m,\tau)}(T)}{S_i(T)}.$$

- Likelihood of observing a sample of coins given parameters  $\theta$ ,

$$\ell(X; \theta) = \sum_i \sum_T \sum_{(m,\tau)} x_i^{(m,\tau)} [\log S_i^{(m,\tau)}(T; \theta) - \log S_i(T; \theta)] + \text{constant}$$

## Estimation results: Mint output



Normalization:  $M_{n_0}[t_0] = 100$  (Northern Italy, AD 320-40).

Discussion on Byzantine monetary output: Kazhdan (1954), Pennas (1996)

## Estimation results: Determinants of trade costs

$$\ln((d_{ni}[t])^{-\theta}) = \text{constant}$$

$$- 2.98 \underset{(0.02)}{\ln(TravelTime_{ni})} - 0.3 \underset{(0.02)}{\ln(PoliticalBorder_{ni}[t])} - 4.05 \underset{(0.11)}{\ln(ReligiousBorder_{ni}[t])}$$

- Travel time elasticity similar to estimates on ancient trade.  
Roman trade: Flückiger et al. (2022); Bronze Age trade: Barjamovic et al. (2019).
- Political border tax: 8%  
(with  $\theta = 4$ ,  $d_{between}/d_{within} = e^{0.3/4} \approx 1.08$ )
- Religious border tax: 175%  
(with  $\theta = 4$ ,  $d_{between}/d_{within} = e^{4.05/4} \approx 2.75$ )
- Anderson and van Wincoop (2003) US-Canada border tax: 49%  
(with  $\theta = 4$ ,  $d_{between}/d_{within} = e^{1.59/4} \approx 1.49$ )

# Welfare and counterfactuals

Real consumption depends on a combination of  $L$  and  $T$  (that's not separately identified):

$$X_n/p_n = \gamma^{-1} (\pi_{nn})^{-1/\theta} (L_n T_n^{1/\theta}) \left(1 + \frac{M_n - \lambda w_n L_n}{w_n L_n}\right)$$

or equivalently in per capita terms

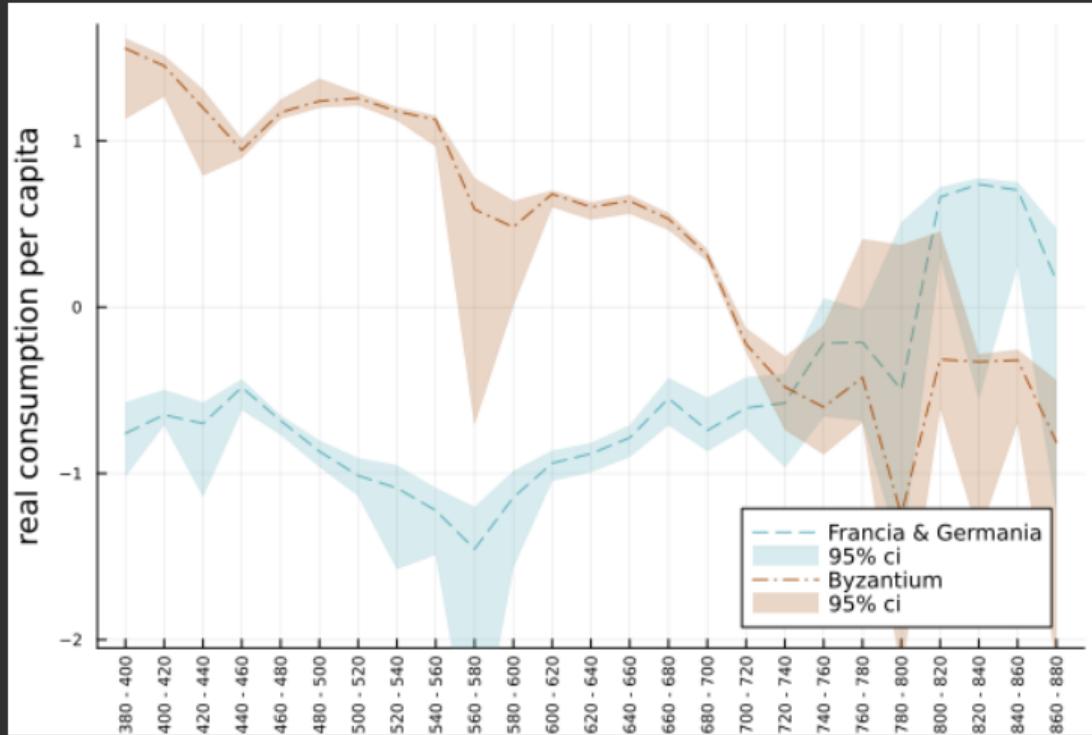
$$\underbrace{\frac{X_n/p_n}{L_n}}_{\text{Real Consumption}} = \underbrace{\gamma^{-1} (\pi_{nn})^{-1/\theta}}_{\text{Openness}} \underbrace{(T_n)^{1/\theta}}_{\text{Technology}} \underbrace{\left(1 + \frac{M_n - \lambda w_n L_n}{w_n L_n}\right)}_{\text{Trade Deficit}}$$

Note:  $T$  and  $L$  are not separately identified. We separate  $L$  and  $T$  through a Malthusian assumption

$$L_n = T_n \quad \forall n$$

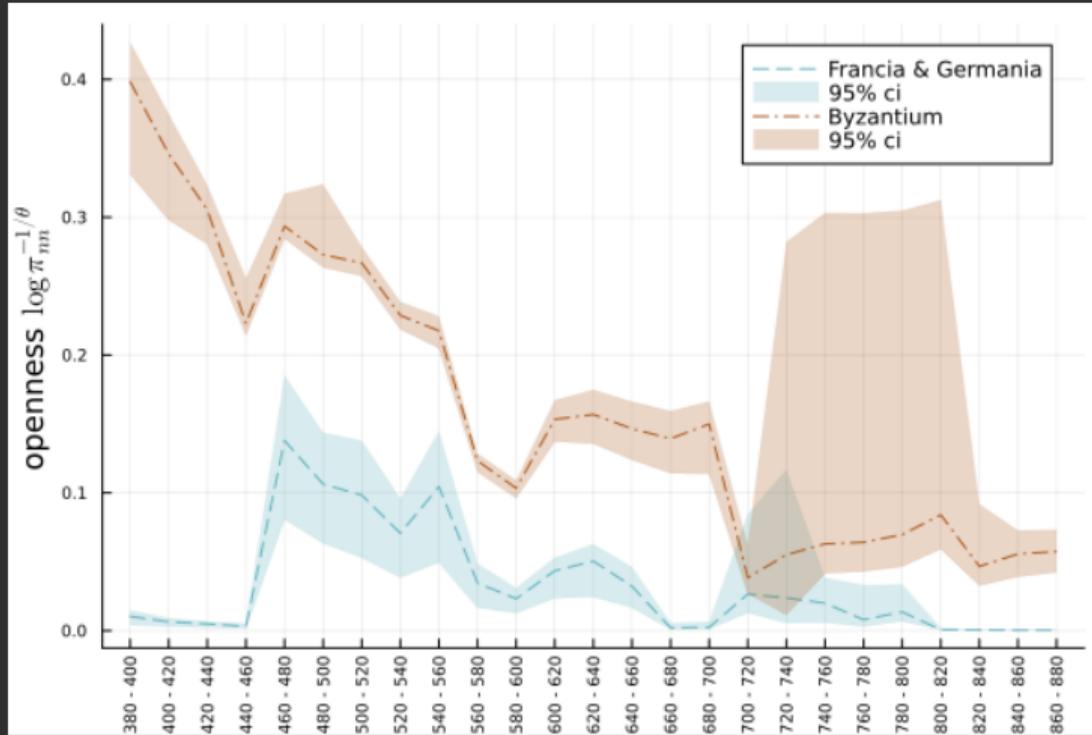
and decompose per-capita real consumption into the three components

# Byzantium vs northern Europe (380-880): *real consumption per capita*



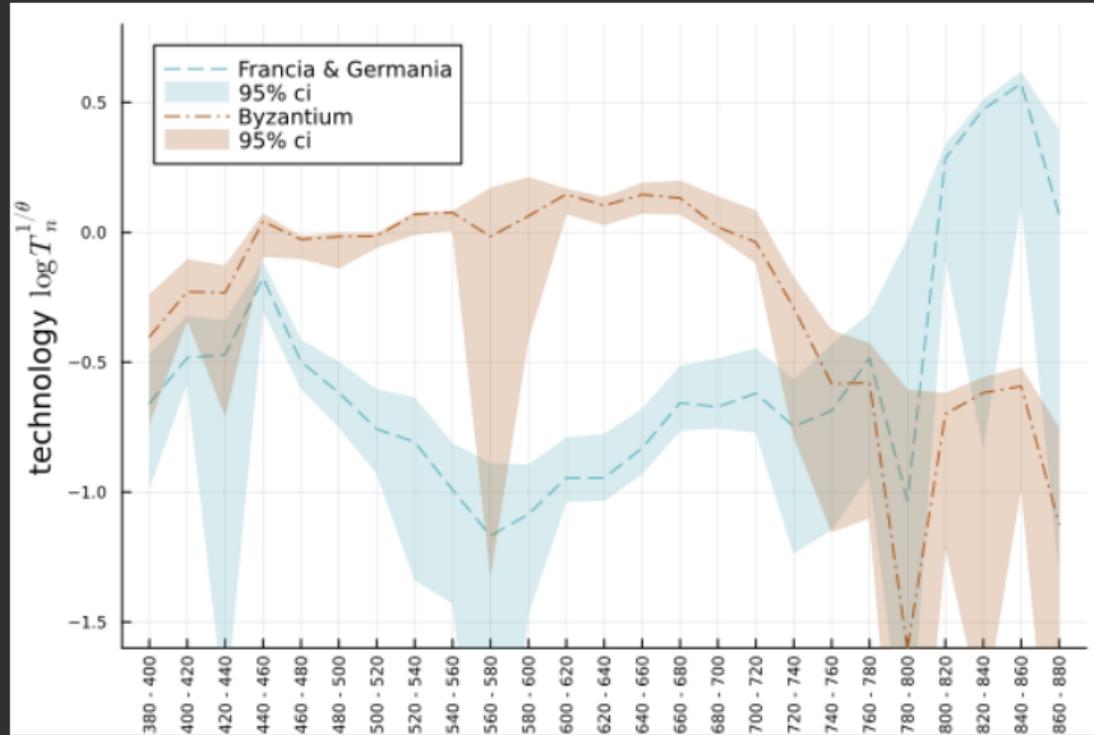
Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Byzantium vs northern Europe (380-880): *trade openness*



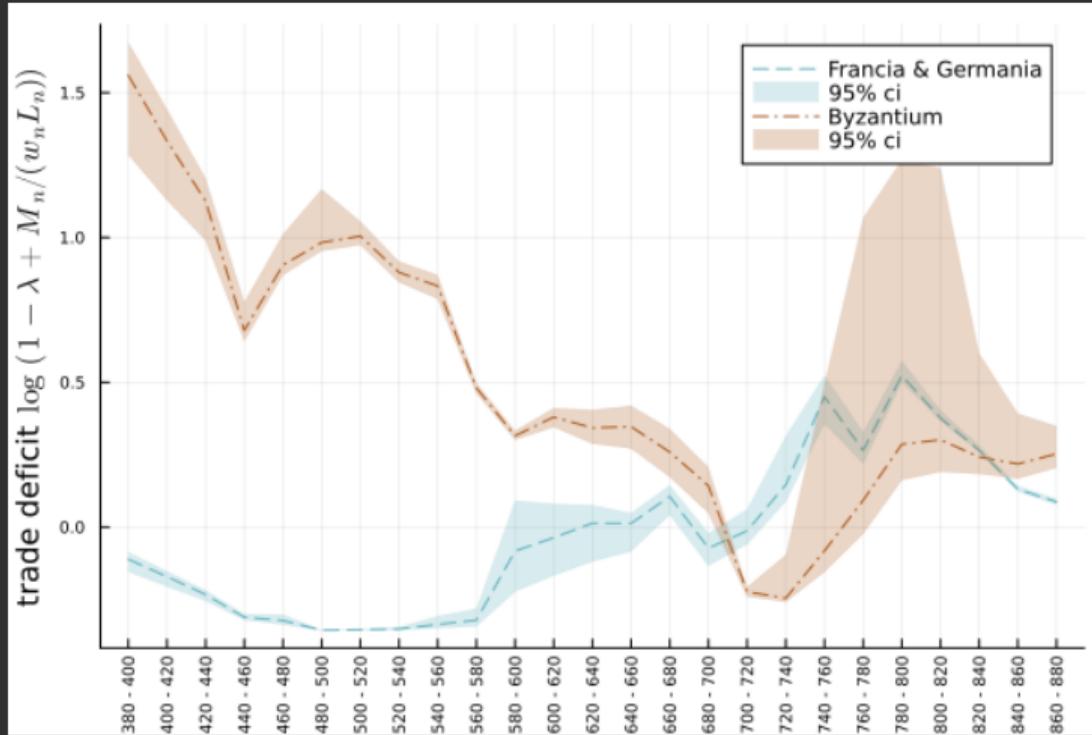
Bootstrapped 95% confidence intervals.

# Byzantium vs northern Europe (380-880): technology



Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Byzantium vs northern Europe (380-880): *trade deficits*



Bootstrapped 95% confidence intervals.

# Real consumption per capita: technology, geography, and trade (deficits)

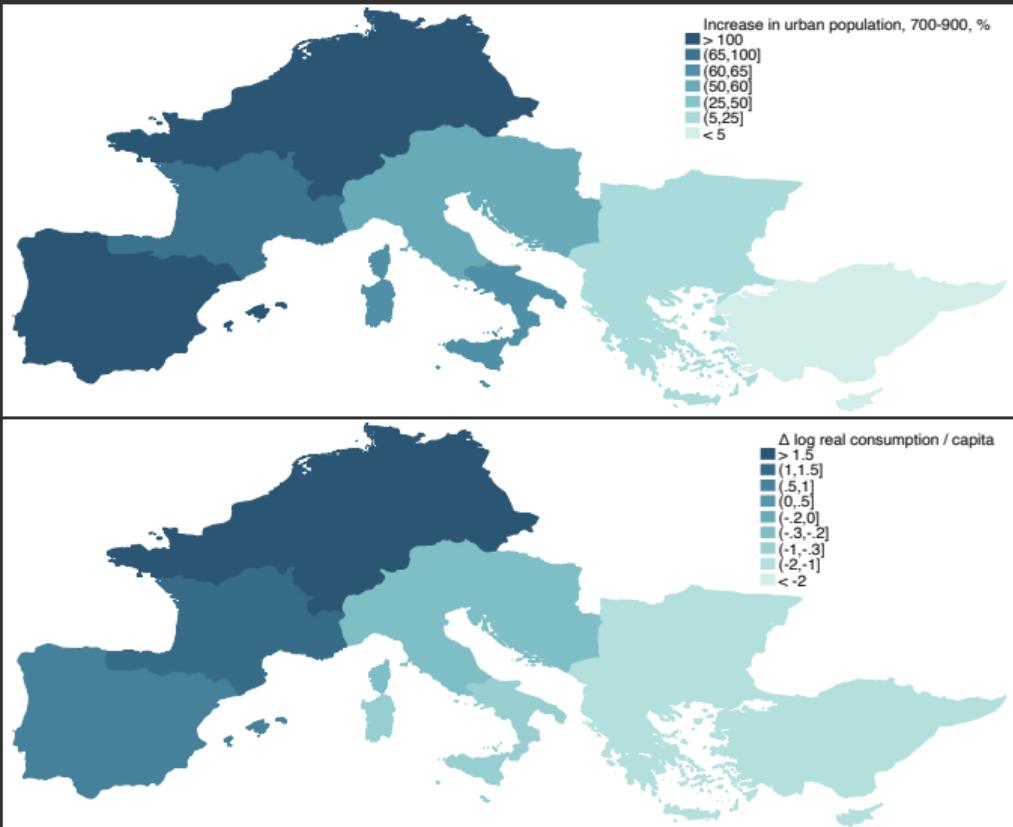
Table 3: Real consumption in the ancient world from AD 460-620 to AD 700-900

	Consumption $\Delta \log \left( \frac{x_n / p_n}{L_n} \right)$ (1)	Openness $\Delta \log \left( \pi_{nn}^{-1/\theta} \right)$ (2)	Technology $\Delta \log \left( r_n^{1/\theta} \right)$ (5)	Trade Deficits $\Delta \log \left( 1 + \frac{M_n - \lambda w_n L_n}{w_n L_n} \right)$ (7)	
al-Andalus	0.53 ( 0.08 )	-0.04 ( 0.01 )	0.65 ( 0.10 )	-0.08 ( 0.05 )	
Francia and Germania	1.99 ( 0.14 )	-0.07 ( 0.02 )	1.94 ( 0.16 )	0.12 ( 0.04 )	
Byzantine Heartlands	-1.56 ( 0.22 )	-0.16 ( 0.06 )	-0.74 ( 0.13 )	-0.66 ( 0.25 )	
Arabian Peninsula	1.12 ( 0.28 )	-0.02 ( 0.04 )	0.98 ( 0.37 )	0.15 ( 0.23 )	

## Compare to relative urbanization rates, 700–900 AD

Top: Change in total urban population (urban: > 1k inhabitants), data from Buringh (2021)

Bottom:  $\Delta W_n/L_n$



# Counterfactuals

Table 4: Counterfactual changes in real consumption per capita after AD 700

	Log consumption		Counterfactual log consumption change if:					
	All parameters AD 460-620		Religious border AD 700-900		Technology AD 700-900		Minting AD 700-900	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
al-Andalus	-0.66	( 0.03 )	0.07	( 0.01 )	0.53	( 0.05 )	1.15	( 0.18 )
Francia and Germania	-1.53	( 0.08 )	-0.09	( 0.02 )	1.78	( 0.11 )	6.70	( 0.54 )
Byzantine Heartlands	1.17	( 0.03 )	-0.61	( 0.03 )	-0.64	( 0.06 )	-1.34	( 0.05 )
Arabian Peninsula	-1.50	( 0.22 )	0.13	( 0.04 )	1.05	( 0.37 )	2.03	( 0.50 )

## Taking stock

- Clear pattern of change in economic geography before vs after conquest
- Trade disruption can account for the relative decline in the eastern Mediterranean
- Change in trade cost *alone* not able to account for urbanization in Muslim Spain, or in Carolingian empire
- In conjunction with changes in technology  $T_i$  and mint output, can account for urbanization patterns.

Back to Pirenne:

- Yes, new political and religious borders change market access, quant'ly relevant
- But unlikely to account for entire shift towards north-east
- Seigniorage and technical change seem to be more important drivers

# Conclusion

*“Simply looking at the Mediterranean cannot of course explain everything about a complicated past created by human agents, with varying doses of calculation, caprice and misadventure. But this is a sea that patiently recreates for us scenes from the past, breathing new life into them, locating them under a sky and in a landscape that we can see with our own eyes, a landscape and sky like those of long ago. A moment’s concentration or daydreaming, and that past comes back to life.”*

*Fernand Braudel, Les Mémoires de la Méditerranée*

THANK YOU!

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## BACKUP SLIDES

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# Regions



## Fact #3: Coin flows before/after the Arab conquests

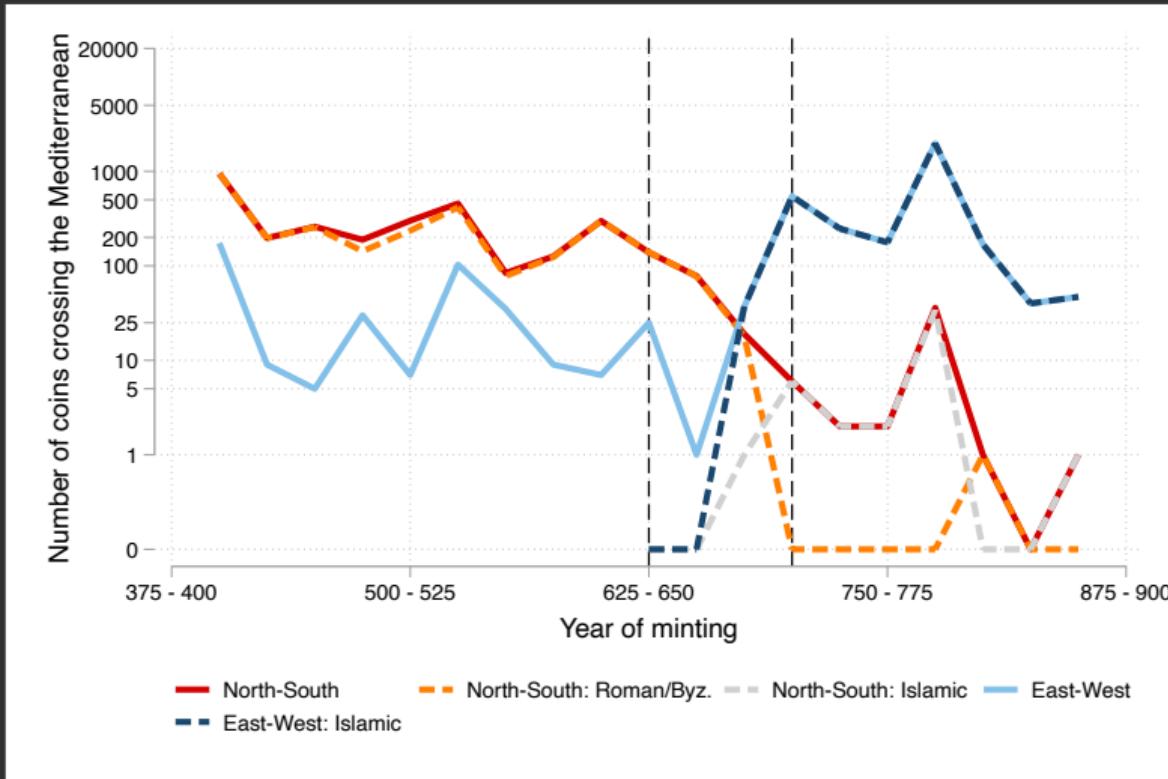


Figure 7: Number of coins flowing across the Mediterranean

## Fact #3: Coin flows before/after the Arab conquests

Table 5: The Mediterranean Before and After the Arab Conquest

	Dependent variable: Number of Coins			
	(1)	(2)	(3)	(4)
Crossing Mediterranean × After Conquests	-1.893*** (0.48)	-3.246*** (0.53)	-0.662 (0.63)	-1.736 (1.27)
Crossing Mediterranean × After Conquests × Islamic Coin		7.267*** (0.90)	4.789*** (0.95)	7.545*** (0.89)
Crossing Mediterranean × After Conquests × Roman Coin			-3.287*** (0.75)	-2.893*** (0.61)
Mint Cell × Empire FE	Yes	Yes	Yes	Yes
Mint Cell × Hoard Cell FE	Yes	Yes	Yes	Yes
After Conquests FE	Yes	Yes	Yes	
Mint Cell × After Conquests FE				Yes
Hoard Cell × After Conquests FE				Yes
Estimator	PPML	PPML	PPML	PPML
Observations	10480	10480	10480	6208

Standard errors in parentheses, clustered at the hoard × era and mint × era level.

+  $p < 0.10$ , \*  $p < 0.05$ , \*\*  $p < 0.01$

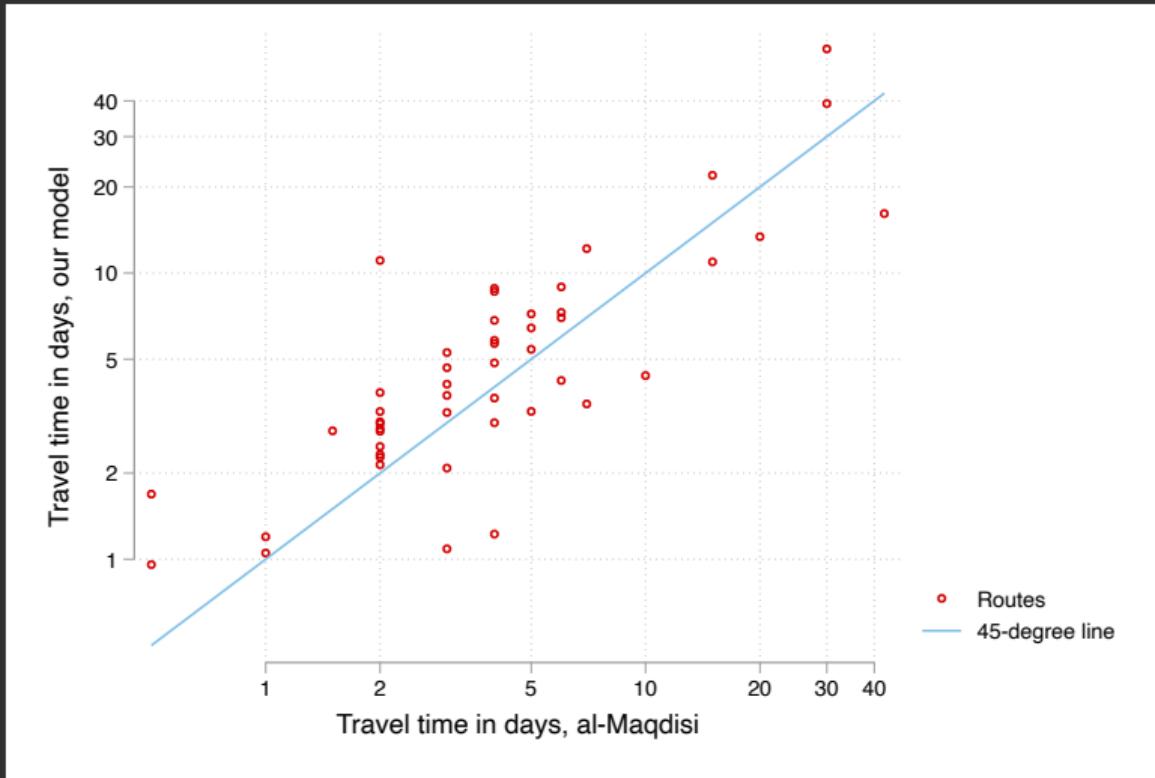
Estimating eqn:  $\text{count}_{mdht} = \exp(\gamma_1 \text{mediterranean}_{mh} \times \text{after}_t + \dots + \alpha_{md} + \alpha_{mh} + \varepsilon_{mhdt})$

▶ Back

▶ Gold

# Validating Travel Times

Al-Maqdisi (c. 945–991): *The Best Divisions for Knowledge of the Regions*



## Intuition for identification of $\Pi_t$

When every region mints some new coins in every period ( $N \times T$  types of coins)

1. Calibrate savings rate  $s$  (to 1.5%)
2. Distribution of coins of type  $(m, t)$  in hoards at  $t + 1$  inform about trade matrix  $\Pi_{t+1}$

When some regions sometimes don't mint new coins, older vintages of coins become relevant:

- When distribution of coin shares don't change much over time  $\rightarrow$  region does not export much (*conditional on s*)
- When distribution of coin shares changes to become more similar to another region  $\rightarrow$  high exports to that region

Let  $R$  be the number of coin types  $(m, t)$ . Then we have  $\approx N \times T \times R - 1$  share observations and  $N \times N - 1 \times T$  trade matrix entries to identify.

- If  $R \gg N$ , we're pretty safe
- Parameterizing  $\Pi$  further reduces the parameter space

## Fact #3: Arab conquests disrupt Mediterranean trade: Gold only

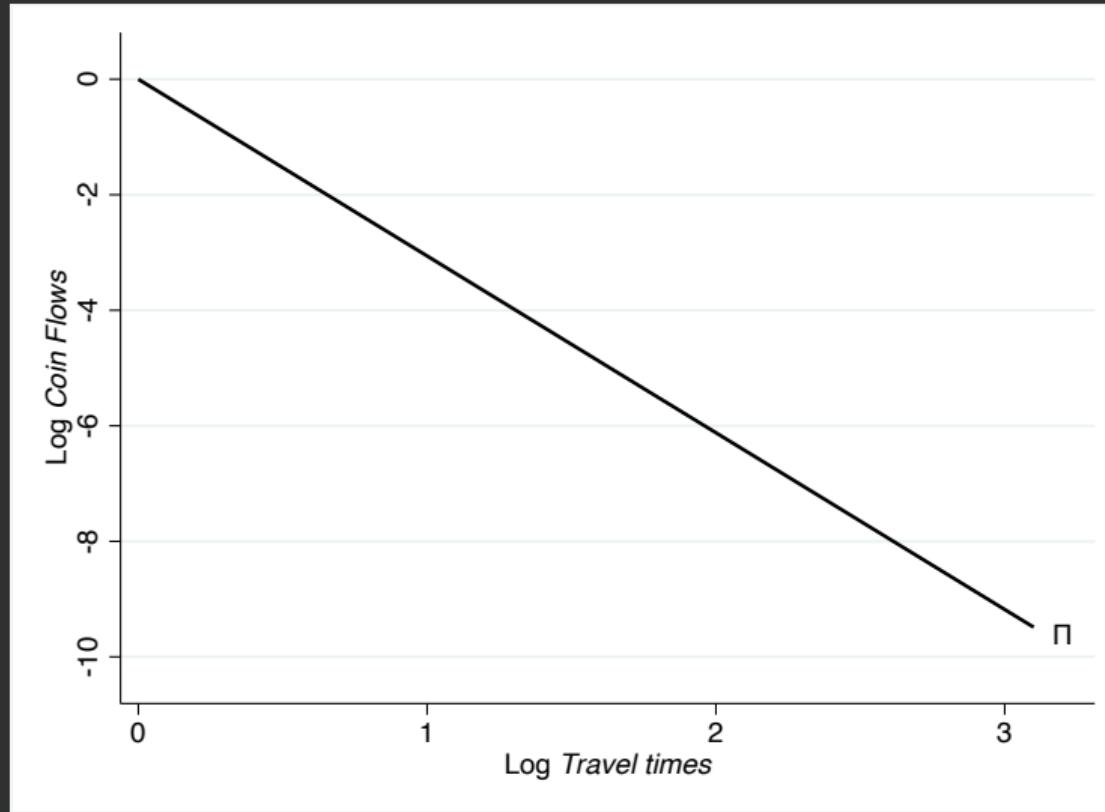
	Dependent variable: Number of Coins				
	(1)	(2)	(3)	(4)	(5)
Log Distance	-1.516*** (0.13)	-1.541*** (0.13)	-1.544*** (0.13)	-1.189*** (0.15)	-1.193*** (0.15)
Crossing Mediterranean	0.298 (0.40)	0.307 (0.39)	0.320 (0.39)	0.0942 (0.31)	0.122 (0.31)
Crossing Mediterranean × After Conquests	-1.600** (0.70)	-2.858*** (0.68)	-1.719** (0.69)	-2.576*** (0.98)	-3.379*** (1.13)
Crossing Mediterranean × After Conquests × Islamic Coin		3.020*** (0.71)	1.864** (0.76)		2.985** (1.20)
Crossing Mediterranean × After Conquests × Roman Coin			-1.699 (1.04)		
Mint Cell × Empire FE	Yes	Yes	Yes	Yes	Yes
Hoard Cell × After Conquests FE	Yes	Yes	Yes	Yes	Yes
Sample				Gold only	Gold only
Observations	172442	172442	172442	32024	32024

Standard errors in parentheses, clustered at the hoard × era and mint × era level.

+  $p < 0.10$ , \*  $p < 0.05$ , \*\*  $p < 0.01$

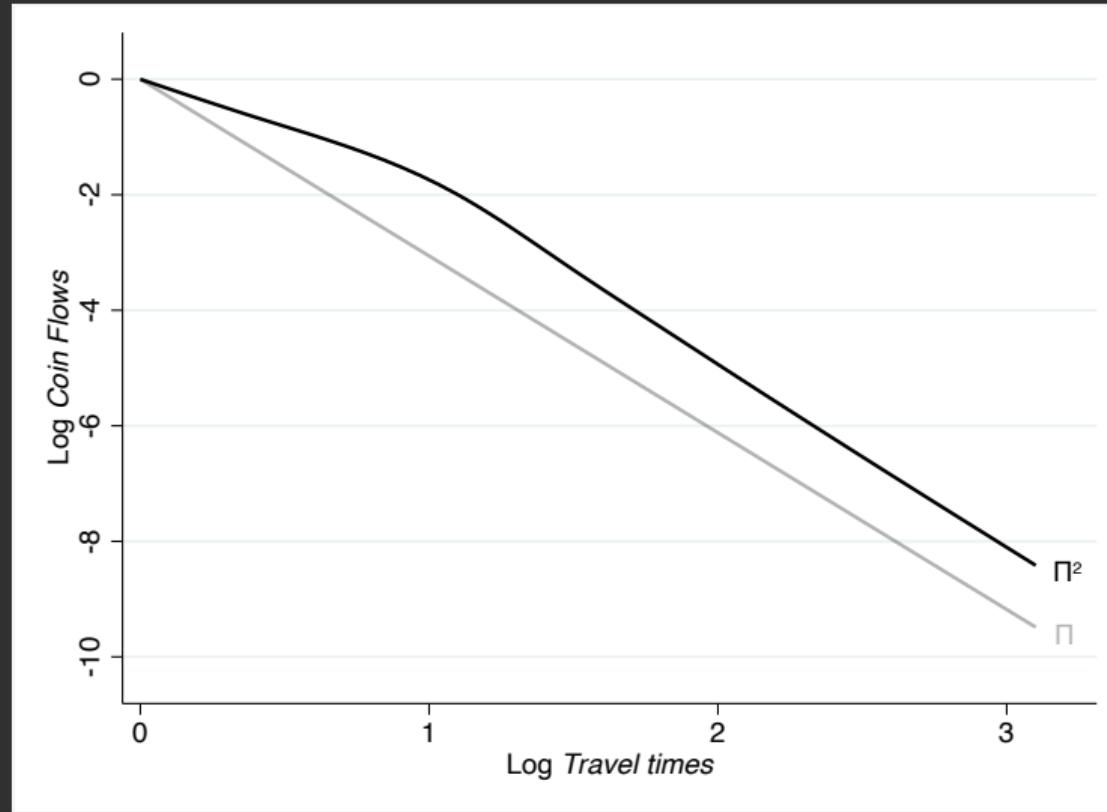
Before: 400–630; after: 713–950

## Pitfall #2: stocks vs flows (numerical example)



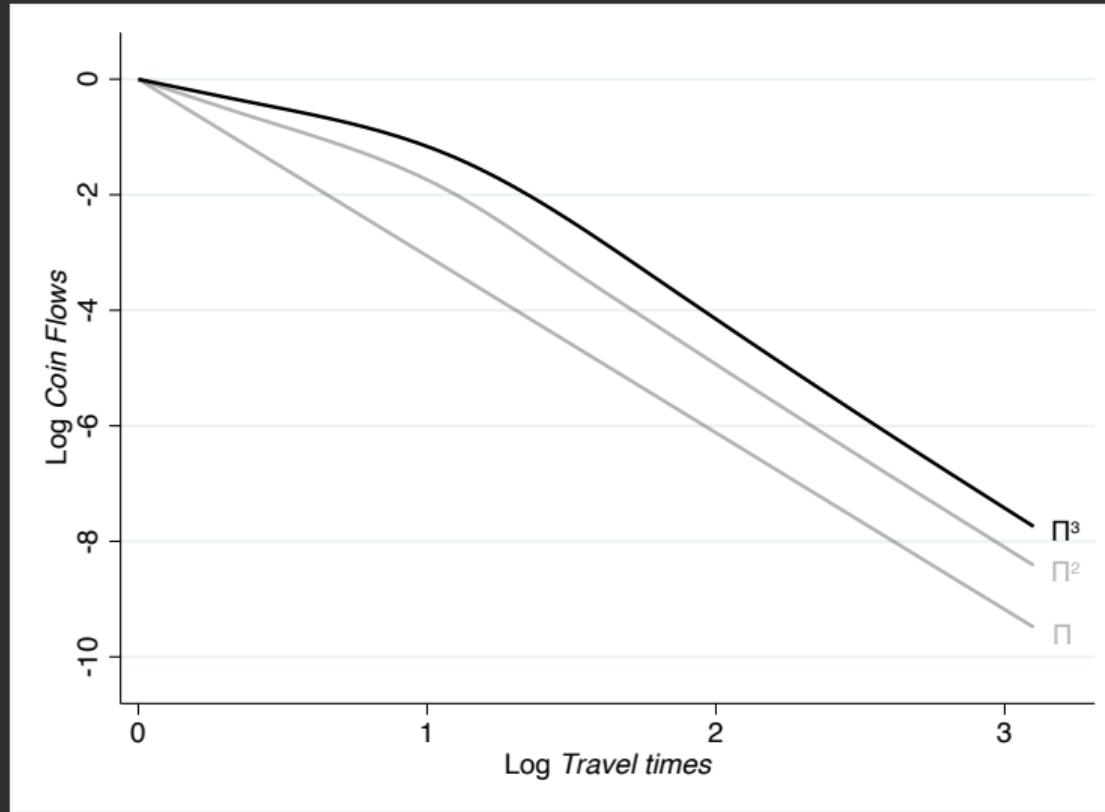
- Flow of coins: age 1 (same as trade flows  $\Pi$ )

## Pitfall #2: stocks vs flows (numerical example)



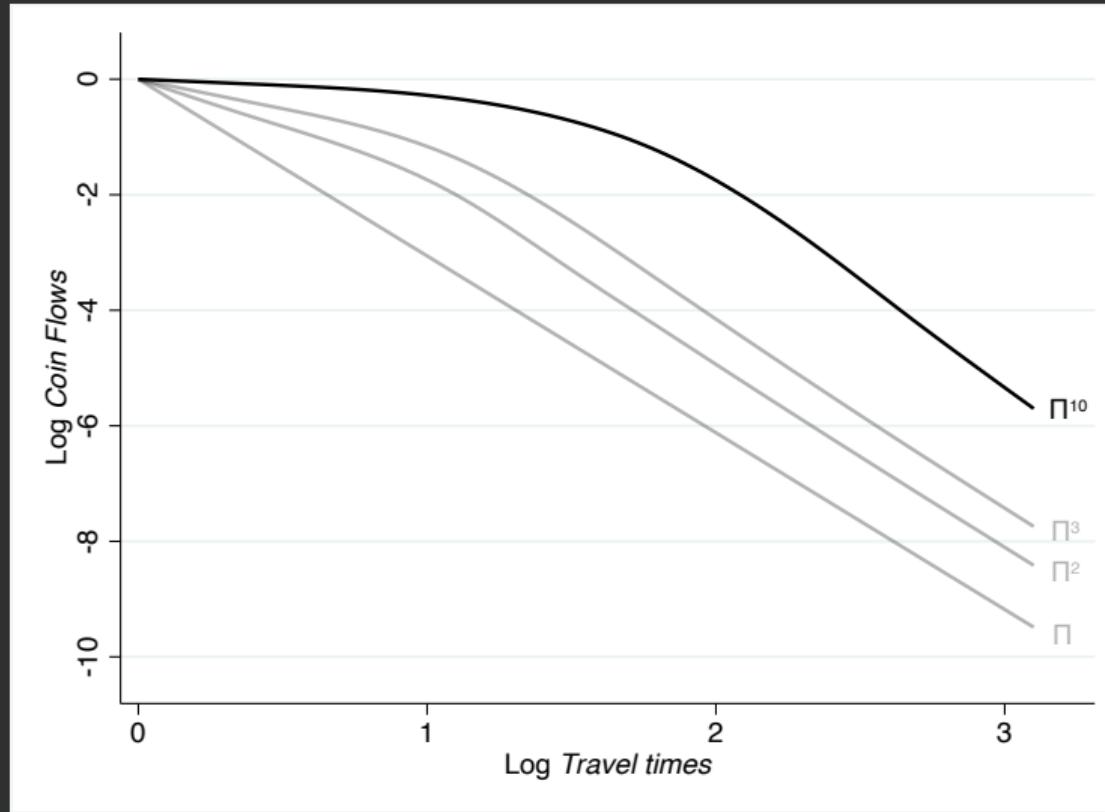
- Flow of coins: age 1, age 2

## Pitfall #2: stocks vs flows (numerical example)



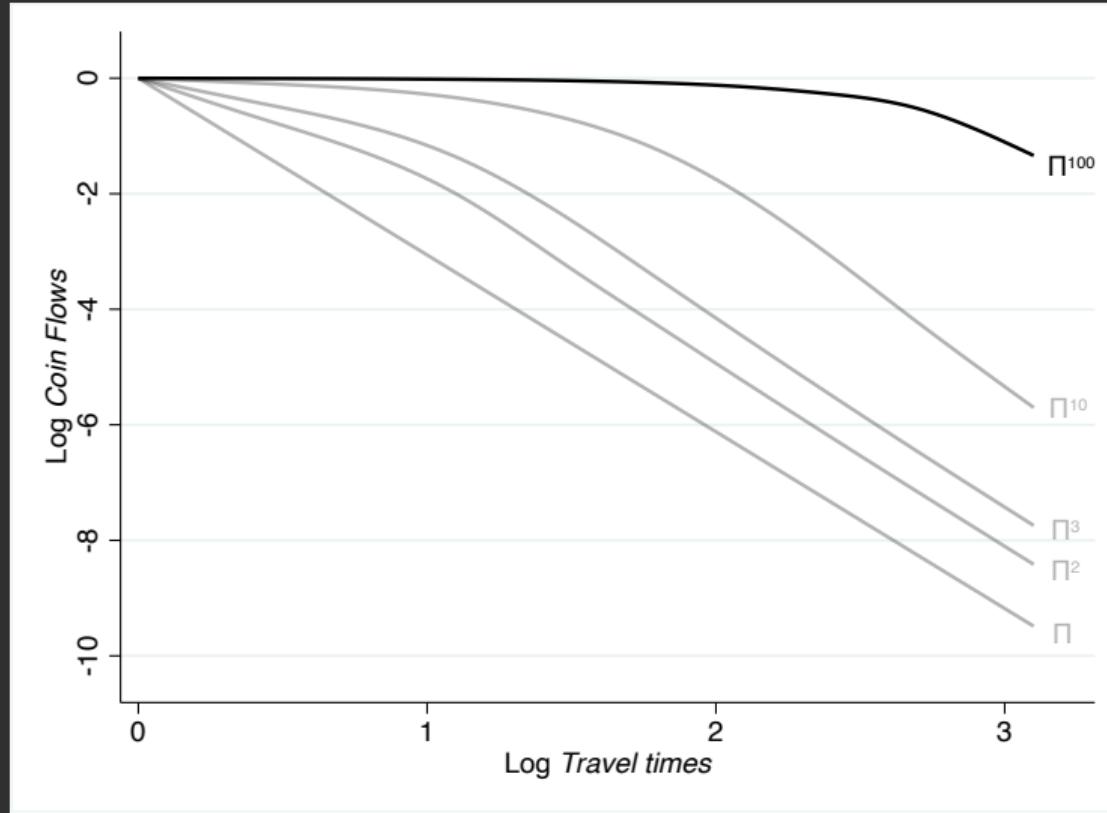
- Flow of coins: age 1, age 2, age 3

## Pitfall #2: stocks vs flows (numerical example)



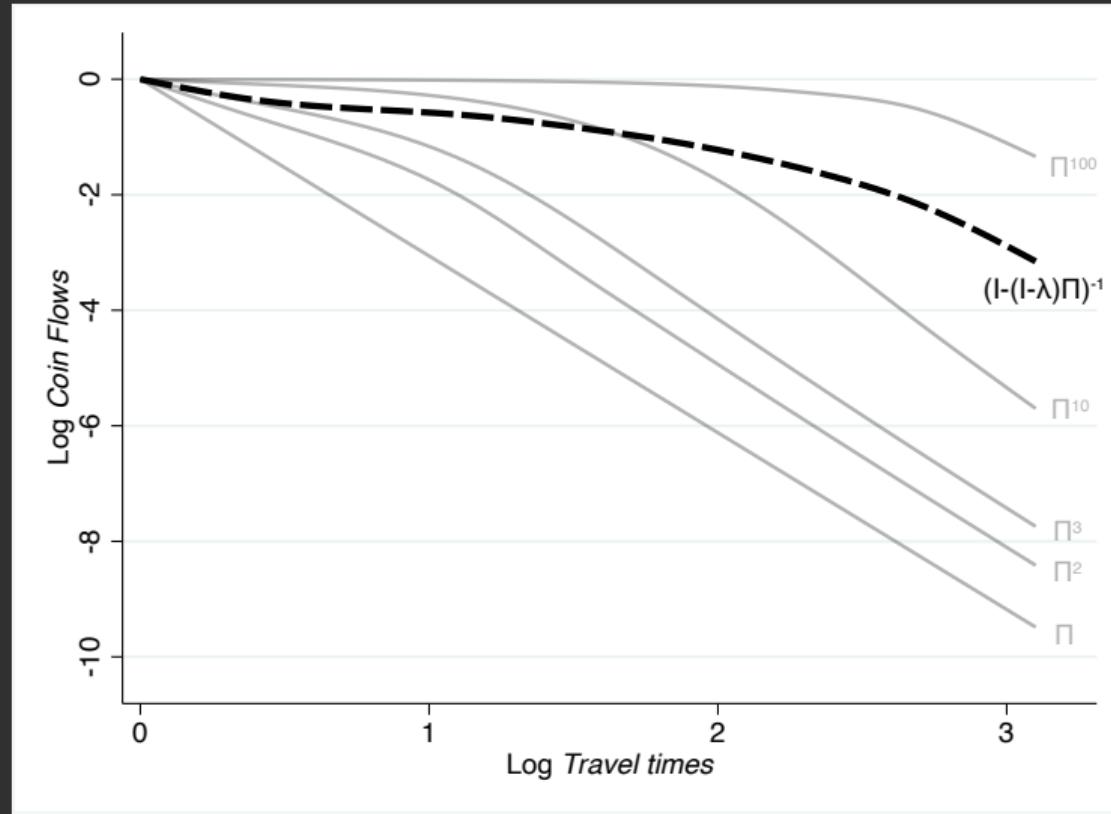
- Flow of coins: age 1, age 2, age 3, age 10

## Pitfall #2: stocks vs flows (numerical example)



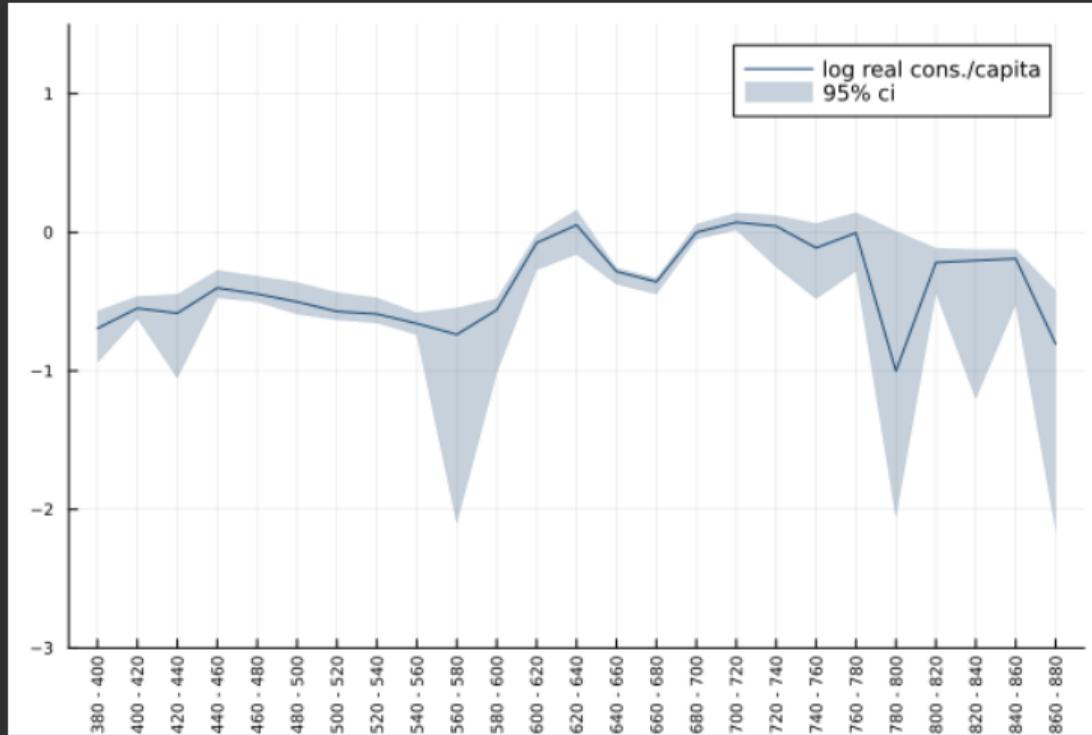
- Flow of coins: age 1, age 2, age 3, age 10, age 100

## Pitfall #2: stocks vs flows (numerical example)



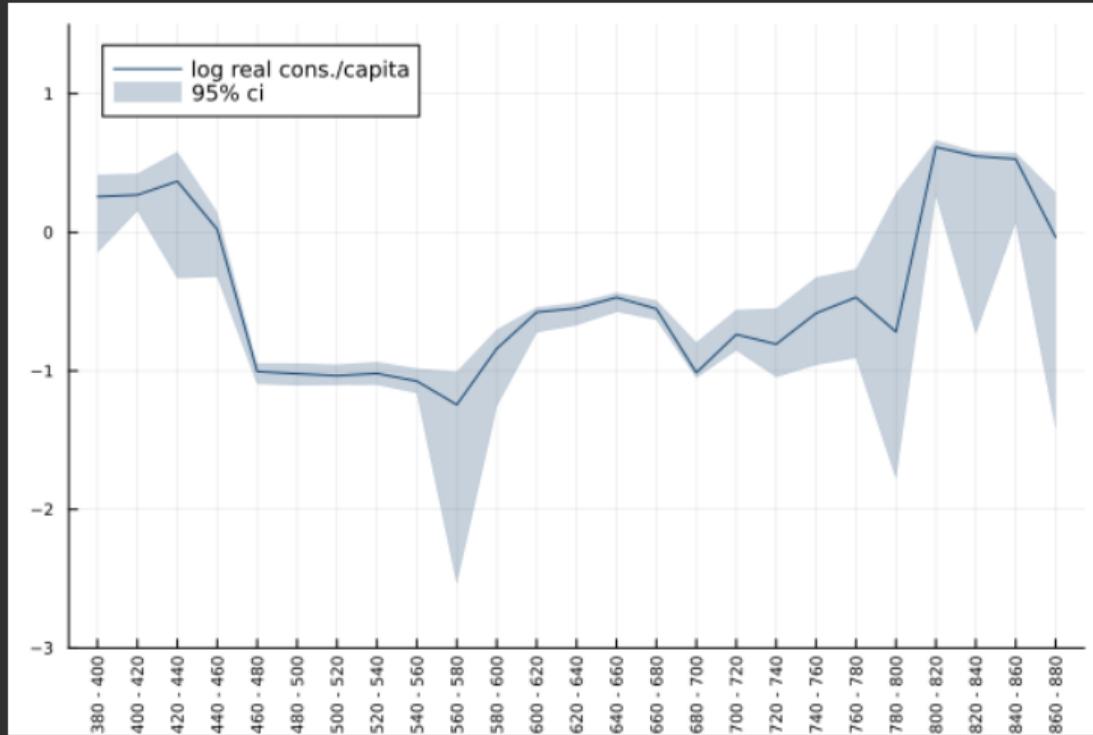
- Flow of coins: age 1, age 2, age 3, age 10, age 100, all ages

# Real consumption per capita (380-880): al-Andalus (Spain)



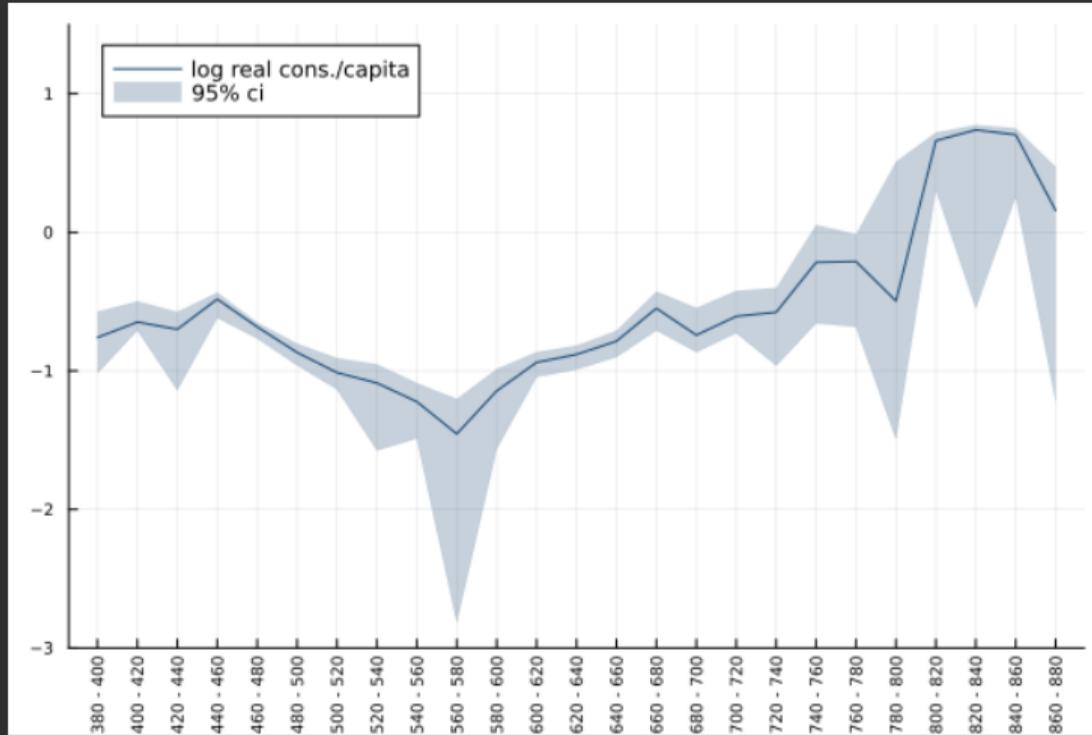
Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Real consumption per capita (380-880): Aquitaine (South France)



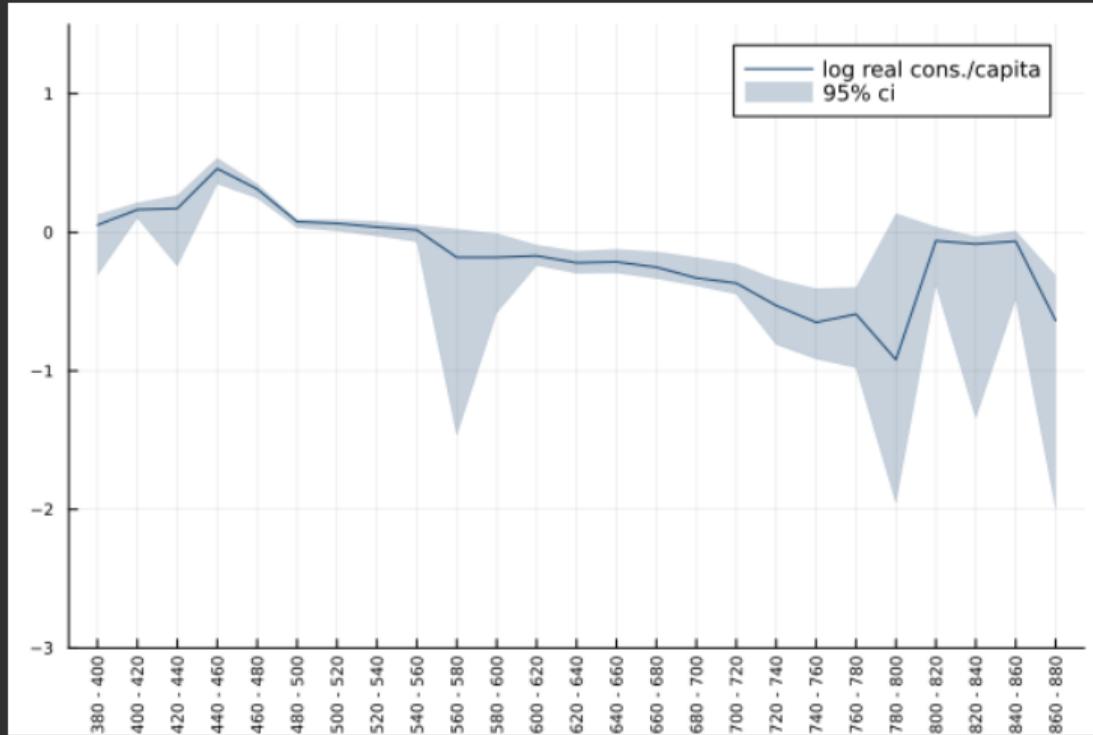
Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Real consumption per capita (380-880): Francia and Germania



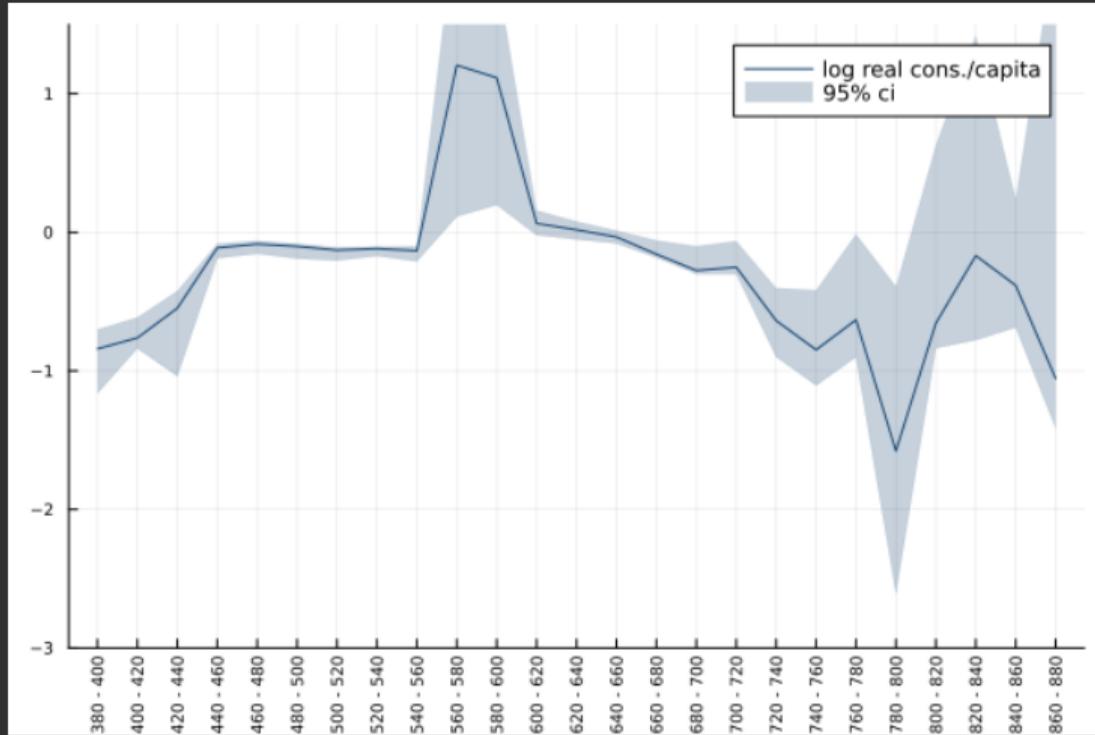
Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Real consumption per capita (380-880): Northern Italy



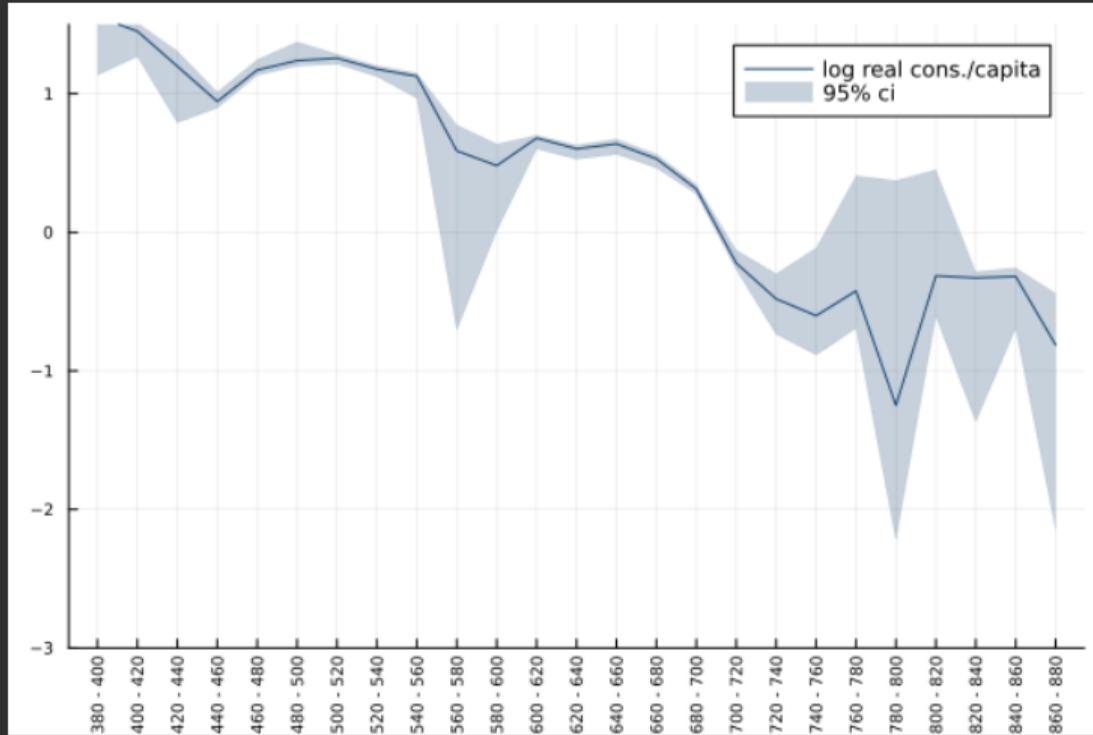
Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Real consumption per capita (380-880): Southern Italy



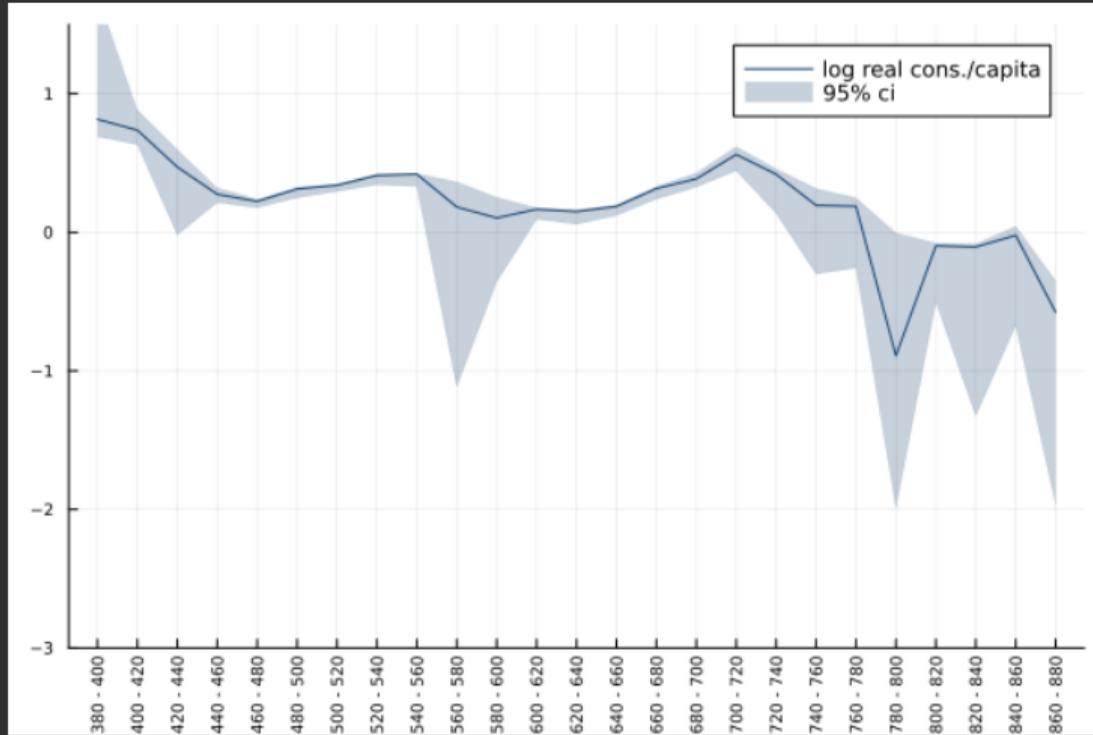
Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Real consumption per capita (380-880): Byzantine Heartlands



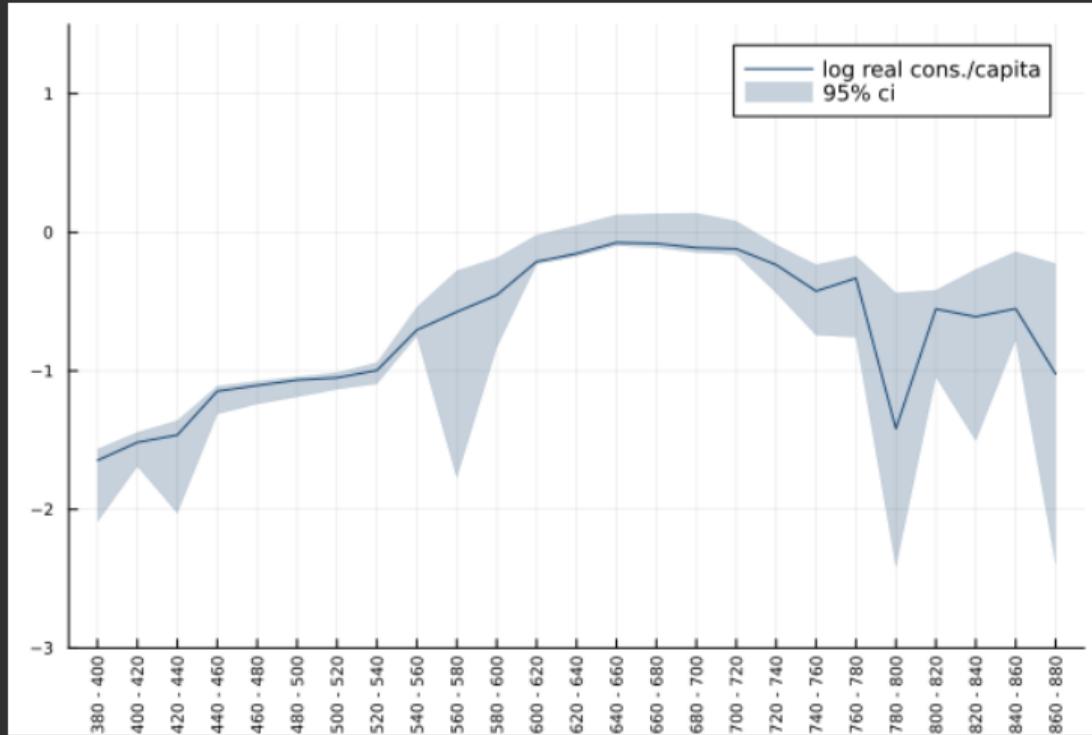
Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Real consumption per capita (380-880): al-Sham (Greater Syria)



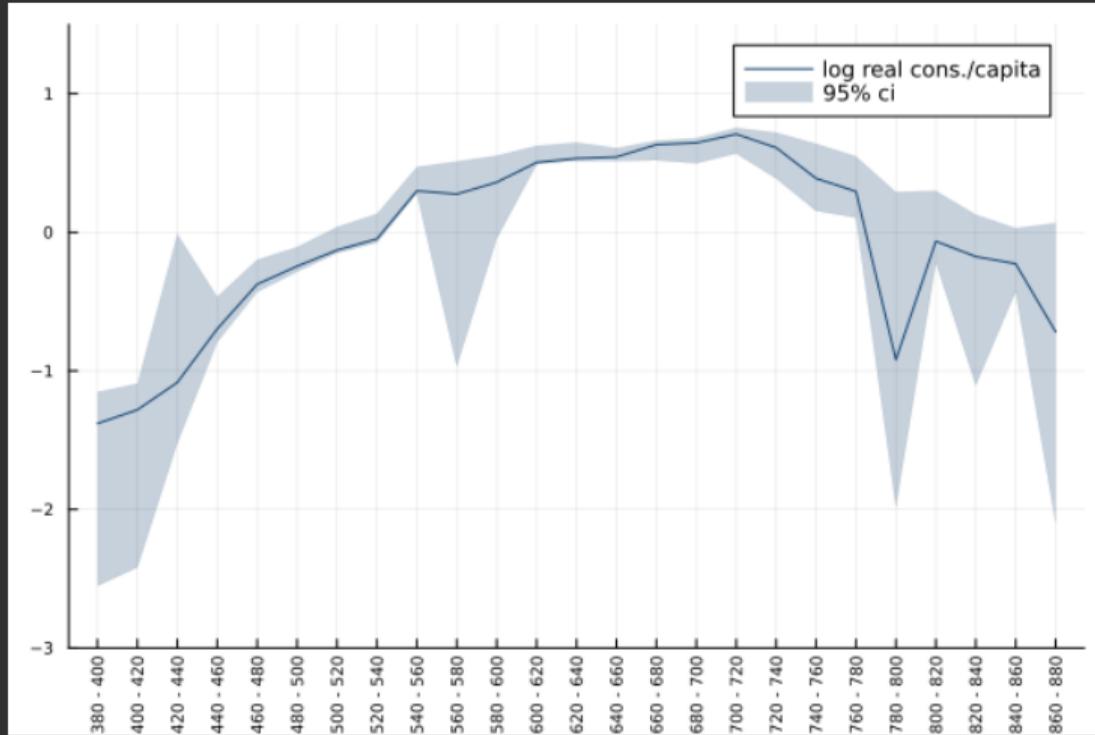
Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Real consumption per capita (380-880): Northern Syria, Caucasus



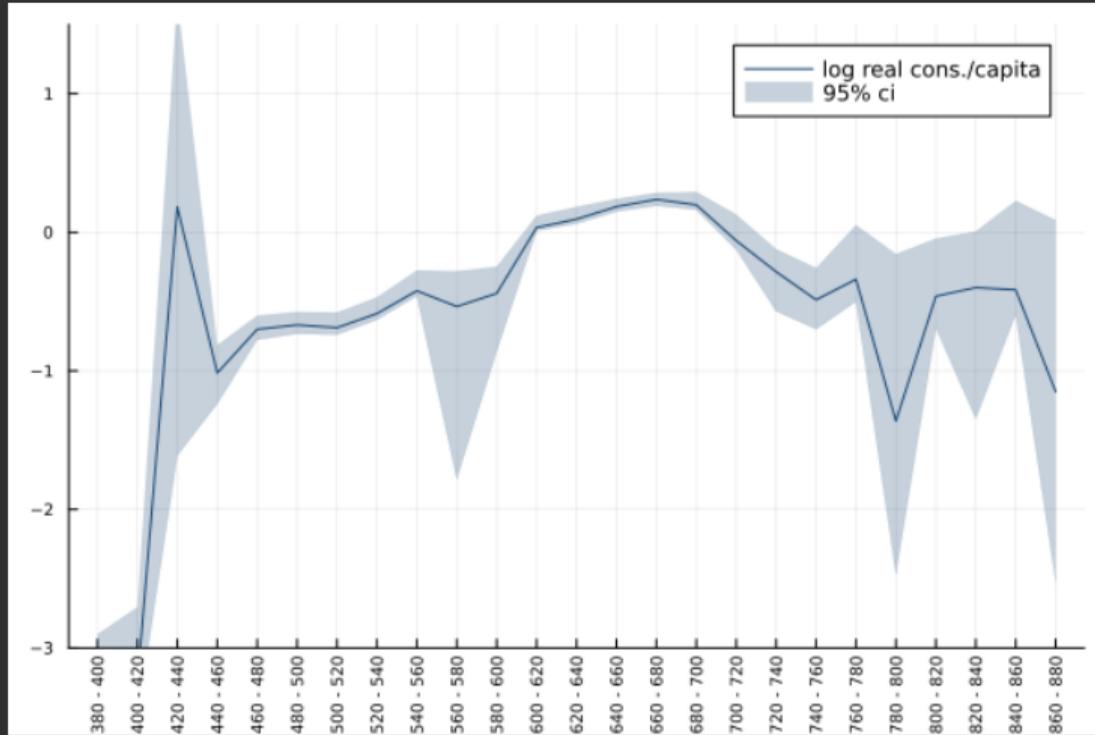
Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Real consumption per capita (380-880): Iraq, Iran



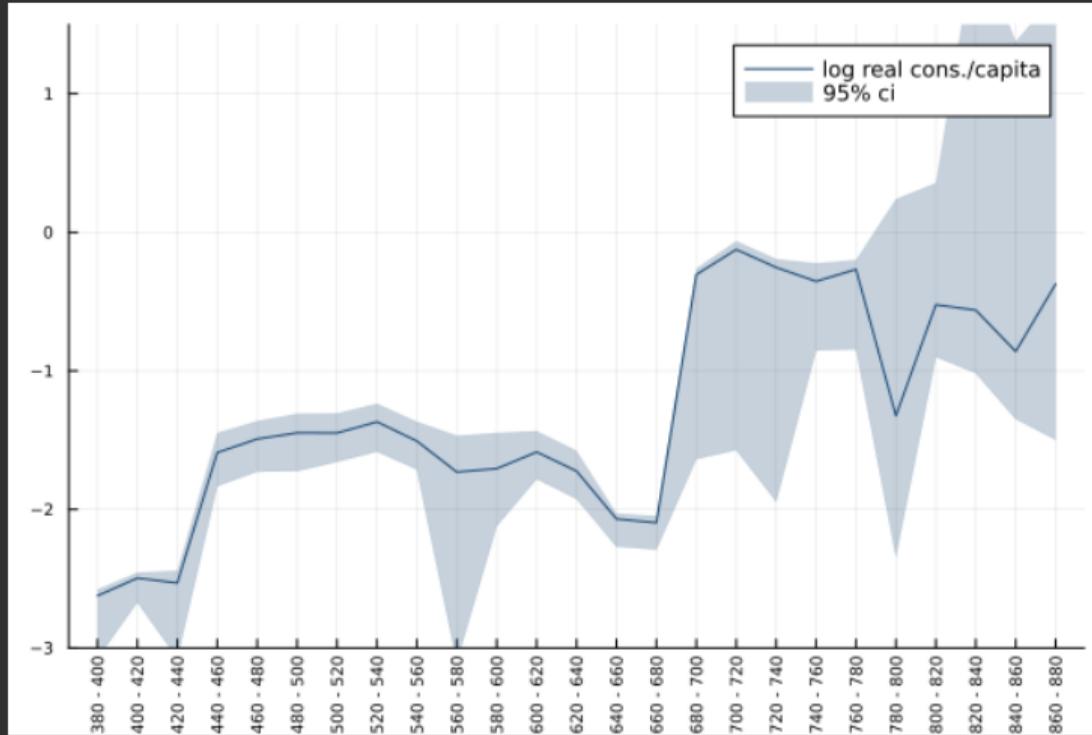
Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Real consumption per capita (380-880): Eastern Caliphate



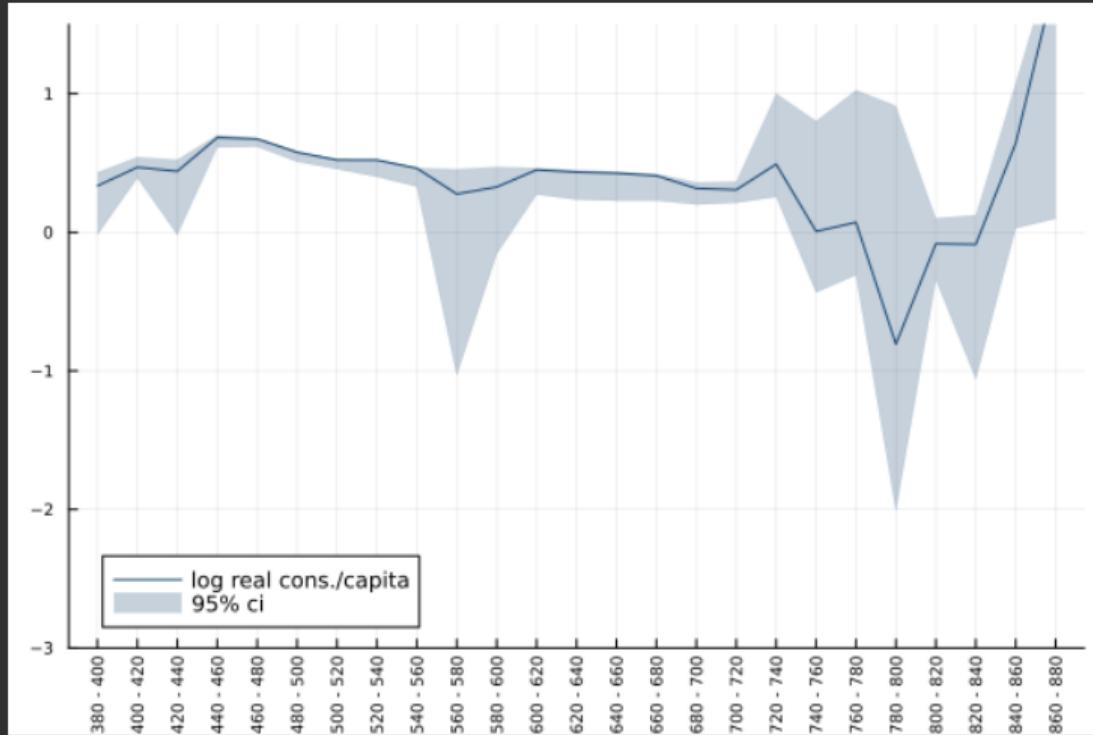
Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Real consumption per capita (380-880): Arabian Peninsula



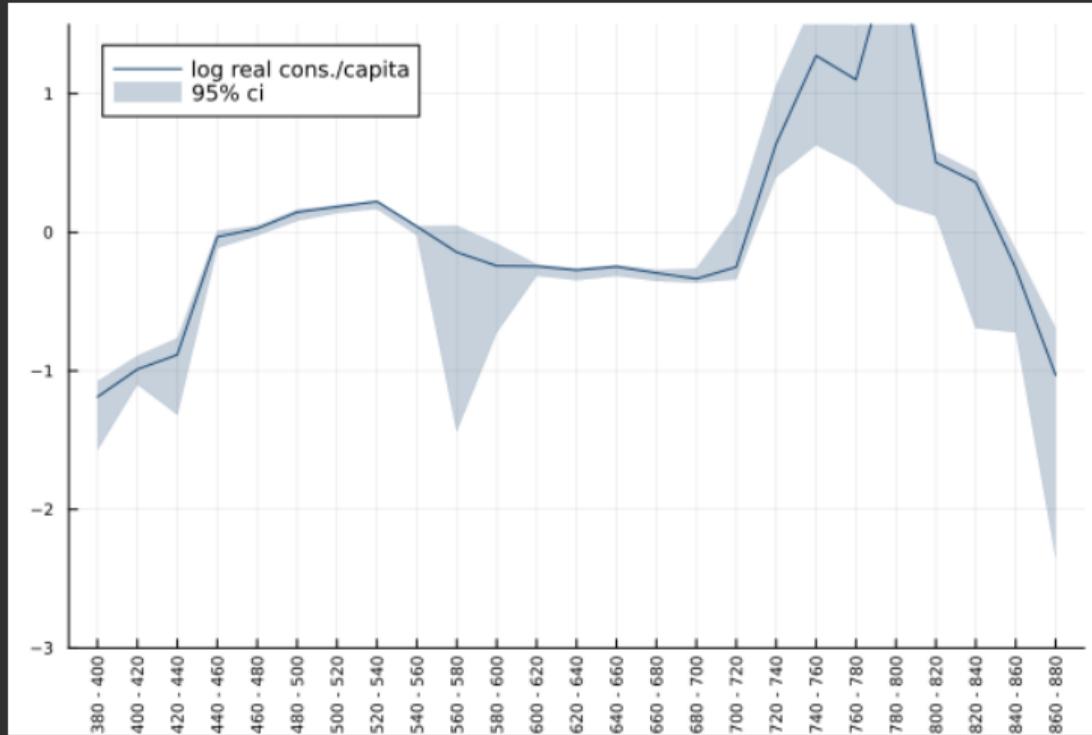
Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Real consumption per capita (380-880): Misr (Egypt)



Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Real consumption per capita (380-880): al-Maghrib



Bootstrapped 95% confidence intervals. Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t.$

# Detailed estimates

	Log Trade Costs	
	(1)	(2)
Log Travel Time	2.98 (0.02)	3.03 (0.02)
Political Border	0.3 (0.02)	0.49 (0.02)
Religious Border	4.05 (0.11)	
Religious Border: East		1.97 (0.12)
Religious Border: West		4.59 (0.22)
Religious Border: Mediterranean		5.2 (0.18)
Sample	All	All
Coin Accounting	Number	Number
Estimator	MLE	MLE
Observations	4,413	4,413

# Realized vs counterfactual changes in real consumption per capita

Realized changes, from AD 460-620 to AD 700-900

	Real consumption	Openness		Technology		Trade Deficit		
	$\Delta \log \left( \frac{X_n/p_n}{L_n} \right)$	$\Delta \log \left( \pi_{nn}^{-1/\theta} \right)$	$\Delta \log \left( T_n^{1/\theta} \right)$	$\Delta \log \left( 1 + \frac{M_n - \lambda w_n L_n}{w_n L_n} \right)$				
al-Andalus (Spain)	0.62	( 0.25 )	-0.06	( 0.04 )	0.77	( 0.32 )	-0.09	( 0.18 )
Aquitaine (South France)	1.28	( 0.23 )	-0.05	( 0.01 )	1.22	( 0.23 )	0.11	( 0.06 )
Francia and Germania	1.96	( 0.24 )	-0.05	( 0.01 )	1.80	( 0.26 )	0.20	( 0.04 )
Northern Italy	-0.31	( 0.24 )	-0.08	( 0.03 )	-0.10	( 0.26 )	-0.13	( 0.10 )
Southern Italy	-0.20	( 0.34 )	0.19	( 0.18 )	-0.94	( 0.37 )	0.55	( 0.40 )
Byzantine Heartlands	-1.56	( 0.33 )	-0.23	( 0.14 )	-0.44	( 0.41 )	-0.89	( 0.54 )
al-Sham (Greater Syria)	-0.32	( 0.27 )	-0.04	( 0.02 )	-0.11	( 0.29 )	-0.17	( 0.11 )
Northern Syria, Caucasus	0.22	( 0.30 )	-0.01	( 0.03 )	0.15	( 0.37 )	0.08	( 0.12 )
Iraq, Iran	0.06	( 0.27 )	-0.00	( 0.01 )	0.06	( 0.29 )	-0.00	( 0.04 )
Eastern Caliphate	0.37	( 0.33 )	-0.00	( 0.00 )	0.39	( 0.34 )	-0.02	( 0.04 )
Arabian Peninsula	1.16	( 0.34 )	-0.01	( 0.04 )	0.66	( 0.45 )	0.51	( 0.26 )
Misr (Egypt)	-0.36	( 0.72 )	0.09	( 0.23 )	-0.82	( 0.50 )	0.37	( 0.90 )
al-Maghrib	0.28	( 0.33 )	0.13	( 0.07 )	-0.49	( 0.27 )	0.65	( 0.30 )

Normalizations:  $E_t \left[ \frac{X_n[t]/P_n[t]}{L_n[t]} \right] \equiv 1, \forall t$ . Bootstrapped s.e.'s in parentheses (100 bootstraps).

▶ back

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# Realized vs counterfactual changes in real consumption per capita

Counterfactual changes relative to AD 700-900

	Initial $\log\left(\frac{X_n/p_n}{L_n}\right)$		Counterfactual $\Delta \log\left(\frac{X_n/p_n}{L_n}\right)$ if:					
	All parameters		Religious border		Technology		Minting	
	AD 460-620	AD 700-900	AD 700-900	AD 700-900	AD 700-900	AD 700-900	AD 700-900	AD 700-900
al-Andalus (Spain)	-0.70	( 0.10 )	0.09	( 0.02 )	0.55	( 0.10 )	1.57	( 0.31 )
Aquitaine (South France)	-1.04	( 0.08 )	-0.15	( 0.03 )	0.99	( 0.09 )	3.93	( 0.30 )
Francia and Germania	<b>-1.55</b>	<b>( 0.09 )</b>	<b>-0.07</b>	<b>( 0.02 )</b>	<b>1.68</b>	<b>( 0.11 )</b>	<b>6.17</b>	<b>( 0.47 )</b>
Northern Italy	0.07	( 0.04 )	-0.24	( 0.05 )	-0.24	( 0.08 )	-0.21	( 0.07 )
Southern Italy	-0.25	( 0.06 )	-0.11	( 0.02 )	-0.60	( 0.13 )	-0.03	( 0.02 )
Byzantine Heartlands	<b>1.22</b>	<b>( 0.11 )</b>	<b>-0.69</b>	<b>( 0.08 )</b>	<b>-0.57</b>	<b>( 0.13 )</b>	<b>-1.41</b>	<b>( 0.19 )</b>
al-Sham (Greater Syria)	0.30	( 0.04 )	0.04	( 0.01 )	-0.18	( 0.10 )	-0.22	( 0.08 )
Northern Syria, Caucasus	-0.34	( 0.11 )	0.02	( 0.02 )	0.15	( 0.22 )	0.19	( 0.19 )
Iraq, Iran	0.28	( 0.08 )	0.01	( 0.00 )	0.03	( 0.08 )	0.03	( 0.06 )
Eastern Caliphate	-0.44	( 0.08 )	0.01	( 0.00 )	0.38	( 0.16 )	0.34	( 0.26 )
Arabian Peninsula	<b>-1.80</b>	<b>( 0.18 )</b>	<b>0.26</b>	<b>( 0.09 )</b>	<b>0.66</b>	<b>( 0.40 )</b>	<b>2.71</b>	<b>( 0.84 )</b>
Misr (Egypt)	0.32	( 0.07 )	0.02	( 0.00 )	-0.71	( 0.24 )	-0.09	( 0.02 )
al-Maghrib	0.12	( 0.06 )	0.01	( 0.00 )	-0.46	( 0.17 )	-0.05	( 0.06 )

▶ back

Normalizations:  $E_t \left[ \frac{X_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t$ . Bootstrapped s.e.'s in parentheses (100 bootstraps).

# Realized vs counterfactual changes in real consumption per capita

Realized changes, from AD 460-620 to AD 700-900

	Real consumption $\Delta \log \left( \frac{x_n/p_n}{L_n} \right)$	Openness $\Delta \log \left( \pi_{nn}^{-1/\theta} \right)$	Technology $\Delta \log \left( T_n^{1/\theta} \right)$	Trade Deficit $\Delta \log \left( 1 + \frac{M_n - \lambda w_n L_n}{w_n L_n} \right)$
Francia and Germania	1.96 ( 0.24 )	-0.05 ( 0.01 )	1.80 ( 0.26 )	0.20 ( 0.04 )
Byzantine Heartlands	-1.56 ( 0.33 )	-0.23 ( 0.14 )	-0.44 ( 0.41 )	-0.89 ( 0.54 )
Arabian Peninsula	1.16 ( 0.34 )	-0.01 ( 0.04 )	0.66 ( 0.45 )	0.51 ( 0.26 )

Counterfactual changes relative to AD 700-900

	Initial $\log \left( \frac{x_n/p_n}{L_n} \right)$	Counterfactual $\Delta \log \left( \frac{x_n/p_n}{L_n} \right)$ if:					
		All parameters AD 460-620		Religious border AD 700-900		Technology AD 700-900	Minting AD 700-900
Francia and Germania	-1.55 ( 0.09 )	-0.07 ( 0.02 )	1.68 ( 0.11 )	6.17 ( 0.47 )			
Byzantine Heartlands	1.22 ( 0.11 )	-0.69 ( 0.08 )	-0.57 ( 0.13 )	-1.41 ( 0.19 )			
Arabian Peninsula	-1.80 ( 0.18 )	0.26 ( 0.09 )	0.66 ( 0.40 )	2.71 ( 0.84 )			

Normalizations:  $E_t \left[ \frac{x_n[t]/p_n[t]}{L_n[t]} \right] \equiv 1, \forall t$ . Bootstrapped s.e.'s in parentheses (100 bootstraps).

## Setup

Period  $t$ , location  $n$ : homogeneous mass  $L_n(t)$  of workers

- Start with  $S_n(t)$  coins saved from period  $t - 1$  to  $t$ .
- Beginning of period  $t$ :
  - Fraction  $\lambda_n(t)$  of saved coins  $S_n(t)$  is lost (turned into hoards  $H_n(t) = \lambda_n(t) S_n(t)$ )
  - $M_n(t) \geq 0$ , fresh new coins minted
- Middle of period  $t$ : divide coins into consumption and saving
  - $X_n(t)$ , expenditure on consumption
  - $S_n(t + 1)$ , saving for period  $t + 1$ .
- End of period  $t$ :
  - Workers produce and sell goods in exchange for coins.
  - $w_n(t)$ , competitive wage, in coins.
  - $Y_n(t) = w_n(t) L_n(t)$ , aggregate labor income.

## Setup

Location  $n$ , period  $t$ : homog. mass  $L_n(t)$  of workers. Four sub-periods  $t_{sub1}, t_{sub2}, t_{sub3}, t_{sub4}$

$t_{sub1}$  Start with  $S_n(t)$  coins saved from period  $t - 1$

$t_{sub2}$  A fraction  $\lambda_n(t)$  of those saved coins is lost

$M_n(t) \geq 0$  fresh new coins are minted

$t_{sub3}$   $X_n(t)$ , expenditure on consumption

Budget constraint:

$$X_n(t) \leq (1 - \lambda_n(t)) S_n(t) + M_n(t)$$

$t_{sub4}$   $L_n(t)$  workers produce and sell goods in exchange for coins

$w_n(t)$ , competitive wage,  $w_n(t)L_n(t)$ , aggregate labor income

$S_n(t+1)$  coins saved for  $t + 1$

$$\underbrace{(1 - \lambda_n(t)) \overbrace{S_n(t)}^{t_{sub1}} + M_n(t)}_{t_{sub2}} - \underbrace{X_n(t)}_{t_{sub3}} + \underbrace{w_n(t) L_n(t)}_{t_{sub4}} = \underbrace{S_n(t+1)}_{(t+1)_{sub1}}$$

## Intra-temporal allocations

- Fraction  $\pi_{ni}$  of expenditure  $X_n$  allocated to goods from  $i$ :

$$\pi_{ni}(t) = \frac{T_i(t)(w_i(t)d_{ni}(t))^{-\theta}}{\sum_k T_k(t)(w_k(t)d_{nk}(t))^{-\theta}}, \quad (5)$$

as in Eaton and Kortum (2002).

## Intertemporal preferences

- Intertemporal utility  $U_n$ , within period welfare  $W_n$ ,

$$U_n(t) = \mathbb{E} \left[ \sum_{\tau \geq t} \beta^{\tau-t} \ln \left( \frac{x_n(\tau)}{p_n(\tau)} \right) \right],$$

$$\text{with } p_n(t) = \gamma \left( \sum_k T_k(t) (w_k(t) d_{nk}(t))^{-\theta} \right)^{1/\theta}$$

# Dynamic optimization

- Optimal coin savings dynamics,

$$\max_{\{S_n(\tau)\}_{\tau \geq t}} \left[ \sum_{\tau \geq t} \beta^{\tau-t} \ln \left( \frac{X_n(\tau)}{p_n(\tau)} \right) \right]$$

$$X_n(\tau) = w_n(\tau)L_n(\tau) + M_n(\tau) + (1 - \lambda_n(\tau))S_n(\tau) - S_n(\tau+1),$$

$$S_n(\tau+1) \geq w_n(\tau)L_n(\tau), \forall \tau \geq t,$$

$$\lim_{\tau \rightarrow \infty} \beta^\tau S_n(\tau+1)/X_n(\tau) = 0$$

- Dynamic equilibrium wages clear markets,

$$w_i L_i = \sum_n \pi_{ni}(T, d; w) [w_n L_n + M_n + (1 - \lambda_n) S_n - S'_n]$$

Savings  $S_n(T, d, \delta, L, M; w)$  depend on parameters and wages, which depend on wages etc.

# Optimal consumption/saving

Under log utility:

- price level  $p_n(t)$  dynamics irrelevant (i.e. separates out)
- when unconstrained, consumption declines exponentially:

$$\frac{X_n(t+1)}{X_n(t)} = \beta(1 - \lambda_n(t)) < 1$$

- when constrained, consume as much as you can:

$$S'_n = w_n(t)L_n$$

Define *net saving*:

$$s_n(t) = \frac{(1 - \lambda_n(t))S_n(t) + M_n(t) - X_n(t)}{(1 - \lambda_n(t))S_n(t) + M_n(t)}$$

**Useful benchmark: zero net savings,  $S_n(t+1) = S_n(t)$ ,  $\forall t$**

- Wages and trade shares jointly determined,

$$w_i L_i = \sum_n \pi_{ni} ((1 - \lambda_n) w_n L_n + M_n)$$
$$\pi_{ni} = \frac{T_i (w_i d_{ni})^{-\theta}}{\sum_k T_k (w_k d_{nk})^{-\theta}}$$

- Constant aggregate stock of coins in circulation,

$$\sum_n M_n = \lambda_n \sum_n w_n L_n$$

- Note: trade deficits as in Dekle, Eaton and Kortum (2007), but endogenous:

$$D_n = X_n - Y_n = M_n - \lambda_n S_n$$

# Introducing and tracking coins of different vintages

Coin stocks  $S_n(\tau)$  consist of coins of different vintage:

$$S_n(\tau) = \sum_{m=1}^N \sum_{t < \tau} S_{mn}(t, \tau)$$

Coin stocks start their life when minted:  $S_{mm}(t, t) = M_m(t)$ .

Traders are ‘blind’ to coin types, draw coins with equal probability:

$$S_{mi}(t, \tau + 1) = \sum_{n=1}^N (1 - \lambda_n(\tau)) (1 - s_n(\tau)) S_{mn}(t, \tau) \pi_{ni}(\tau) + (1 - \lambda_i(\tau)) s_i(\tau) S_{mi}(t, \tau), \forall \tau \geq t$$

In compact matrix form:

$$\mathbf{S}(t, T) = \mathbf{M}(t) \left( \prod_{\tau=t}^{T-1} (\mathbf{I} - \boldsymbol{\lambda}(\tau)) \left( (\mathbf{I} - \mathbf{s}(\tau)) \boldsymbol{\Pi}(\tau) + \mathbf{s}(\tau) \right) \right)$$