1 Solow with Investment-Specific Technological Progress (Sketch)

1. We have
$$\dot{K} = q(t)I(t) - \delta K(t). \tag{1}$$
Denote the growth rates of K, L, and q by γ_K, γ_L, and γ_q, respectively (sorry, notation change, but this is a bit simpler to read). Then we have
$$\dot{K} = \gamma_K K, \qquad \dot{L} = \gamma_L L, \qquad \dot{q} = \gamma_q q$$
and thus (1) is equivalent to
$$\begin{aligned} \gamma_K K(t) &= q(t)I(t) - \delta K(t) \\ \gamma_K K(0)e^{\gamma_K t} &= q(0)se^{\gamma_q t}Y(t) - \delta K(0)e^{\gamma_K t} \\ (\gamma_K + \delta) K(0)e^{\gamma_K t} &= q(0)se^{\gamma_q t}Y(t) \end{aligned} \tag{2}$$
For a BGP to exist, this function has to have a solution for all $t \in [0, \infty)$. Thus, $Y(t) = F(K, L)$ has to grow at a constant rate itself. By taking total derivatives of F and using Euler's property of CRS functions, we have
$$\begin{aligned} \dot{Y} &= \frac{\partial F}{\partial K}\dot{K} + \frac{\partial F}{\partial L}\dot{L} = \frac{\partial F}{\partial K}K\frac{\dot{K}}{K} + \frac{\partial F}{\partial L}L\frac{\dot{L}}{L} \\ \frac{\dot{Y}}{Y} &= \frac{\frac{\partial F}{\partial K}K}{\frac{\partial F}{\partial K}K + \frac{\partial F}{\partial L}L}\frac{\dot{K}}{K} + \left(1 - \frac{\frac{\partial F}{\partial K}K}{\frac{\partial F}{\partial K}K + \frac{\partial F}{\partial L}L}\right)\frac{\dot{L}}{L} \end{aligned}$$
We know that $\dot{K}/K = \gamma_K$ and $\dot{L}/L = \gamma_L$, thus the term $\frac{\partial F}{\partial K}K/\left(\frac{\partial F}{\partial K}K + \frac{\partial F}{\partial L}L\right)$ would need to be a constant for a BGP to exist. This is generally not the case.

One can show that the C-D is the only function that satisfies this: from $\frac{\partial F}{\partial K}K/\left(\frac{\partial F}{\partial K}K + \frac{\partial F}{\partial L}L\right) = \alpha$ for some $\alpha \in R$, we have that
$$\begin{aligned} \frac{1}{1 + \frac{\frac{\partial F}{\partial L}L}{\frac{\partial F}{\partial K}K}} &= \alpha \\ \left(\frac{1}{\alpha} - 1\right)\frac{\partial F}{\partial K}K &= \frac{\partial F}{\partial L}L \\ \frac{1}{\alpha}\frac{\partial F}{\partial K}K &= Y \\ \frac{\partial \log Y}{\partial \log K} &= \alpha \end{aligned}$$

thus, solving this PDE, $Y = K^\alpha g(L)$. Analogously, $Y = L^{1-\alpha} h(K)$, thus Y must be Cobb-Douglas.

2. In the case of a Cobb-Douglas production function, $\frac{\partial F}{\partial K} K / \left(\frac{\partial F}{\partial K} K + \frac{\partial F}{\partial L} L \right) = \alpha$. Thus (2) becomes

$$(\gamma_K + \delta)\, K(0) e^{\gamma_K t} = q(0) s e^{\gamma_q t} Y(0) e^{\alpha \gamma_K t + (1-\alpha)\gamma_L t}$$

for all t, and we have to have that

$$\gamma_K = \gamma_q + \alpha \gamma_K + (1-\alpha)\gamma_L$$

which yields

$$\gamma_K = \frac{1}{1-\alpha}\gamma_q + \gamma_L.$$

This gives us the growth rate of the factor by which we need to normalize so that K becomes stationary. Thus,

$$\hat{k}(t) \equiv \frac{K(t)}{q(t)^{\frac{1}{1-\alpha}} L(t)}$$

is stationary, and $\gamma_Y = \alpha\gamma_K + (1-\alpha)\gamma_L = \frac{\alpha}{1-\alpha}\gamma_q + \alpha\gamma_L + (1-\alpha)\gamma_L = \frac{\alpha}{1-\alpha}\gamma_q + \gamma_L$ is the growth rate of Y.

3. From above, $\gamma_Y < \gamma_K$, thus $K/Y \to 1$ as $t \to \infty$. Note that in the model, K is measured in units of output (i.e. in real terms), whereas in the data it would be in nominal terms (typically, the sum of companies' fixed assets). Since the price of capital is decreasing, this is consistent with the aggregate capital-output ratios calculated from nominal (undeflated) data being roughly constant.