Exercise 1

A (symmetric) competitive equilibrium (in which all households choose the same per capita variables) is a path of allocations $[c(t), l(t), a(t), k(t)]_t$ and prices $[r(t), w(t)]_t$ such that each household solves

$$
\begin{aligned}
\max_{[c(t),l(t)\in[0,1],a(t)]_t} U(0) &= \int_0^\infty \exp(-\rho t)\, u(c(t), 1-l(t)), && (1)\\
\text{s.t.} \quad & \dot{a}(t) = r(t)\, a(t) + w(t)\, l(t) - c(t), && (2)\\
\text{and} \quad & \lim_{t\to\infty} a(t) \exp\left(-\int_0^t r(s)\, ds\right) \geq 0,
\end{aligned}
$$

firms maximize profits which gives

$$
r(t) = F_K(k(t), A(t)\, l(t)) - \delta,\ w(t) = A(t)\, F_L(k(t), A(t)\, l(t)), \tag{3}
$$

and all markets clear, in particular, $a(t) = k(t)$ for all t.

Exercise 1, Part (a)

Note that Problem 1 is a problem with one state variable, a, and two control variables, c and l. The current value Hamiltonian is

$$
\hat{H}(t, a, c, l, \mu) = u(c, 1-l) + \mu\,(r(t)\, a + w(t)\, l - c).
$$

The first-order conditions are

$$
\begin{aligned}
\hat{H}_c &= 0, \text{ which gives } u_c(c, 1-l) = \mu\\
\hat{H}_l &= 0, \text{ which gives } u_2(c, 1-l) = \mu w(t)\\
\hat{H}_a &= \rho\mu - \dot{\mu}, \text{ which gives } \frac{\dot{\mu}}{\mu} = \rho - r(t).
\end{aligned}
$$

(here, $u_2(c, 1-l) = \partial u_l(c, 1-l) / \partial(1-l)$ denotes the partial derivative of u with respect to leisure choice $1-l$) The first-order conditions can be simplified to

$$
\begin{aligned}
\epsilon_u(c, 1-l)\, \frac{\dot{c}}{c} - \frac{u_{c2}(c, 1-l)\, \dot{l}}{u_c(c, 1-l)} &= r(t) - \rho && (4)\\
u_2(c, 1-l) &= u_c(c, 1-l)\, w(t), && (5)
\end{aligned}
$$

where

$$
\epsilon_u(c, 1-l) = -\frac{u_{cc}(c, 1-l)\, c}{u_c(c, 1-l)}
$$

is the elasticity of the marginal utility u_c with respect to c. Note that the first condition (4) is the intertemporal condition, i.e. the Euler equation, and the second condition (15) is the intratemporal condition, i.e. the labor-leisure trade-off. The strong form of the transversality condition is also necessary in this problem, that is $\lim_{t\to\infty} \exp(-\rho t)\, \mu(t)\, a(t) = 0$. As in the baseline case, the transversality condition can be rewritten as

$$
\lim_{t\to\infty} a(t) \exp\left(-\int_0^t r(s)\, ds\right) = 0. \tag{6}
$$

Note that the maximized Hamiltonian $M(t, a, \mu) = \max_{c,l} \hat{H}(t, a, c, l, \mu)$ is linear and hence concave in a. Note also that for each feasible $\left[\tilde{a}(t), \tilde{c}(t), \tilde{l}(t)\right]_t$, by the no-Ponzi condition, we have $\lim_{t\to\infty} \exp(-\rho t)\, \mu(t)\, \tilde{a}(t) \geq 0$. Then Theorem 7.14 applies and shows that these conditions are sufficient for optimality.

Exercise 1, Part (b)

The social planner solves

$$\begin{aligned}\max_{[c(t),l(t)\in[0,1],k(t)]_t} U(0) &= \int_0^\infty \exp(-\rho t)\, u(c(t), 1-l(t)),\\ \text{s.t. } \dot{k}(t) &= F(k(t), A(t)\, l(t)) - \delta k(t) - c(t) \text{ and } k(t) \geq 0. \end{aligned} \tag{7}$$

Note that this problem is also an optimal control problem with one state variable $k(t)$ and two control variables $\{c(t), l(t)\}$. The current value Hamiltonian is

$$\hat{H}(t,k,c,l,\mu) = u(c, 1-l) + \mu(F(k, A(t)\, l) - \delta k - c).$$

The first-order conditions are

$$\begin{aligned}\hat{H}_c &= 0, \text{ which gives } u_c(c, 1-l) = \mu\\ \hat{H}_l &= 0, \text{ which gives } u_2(c, 1-l) = \mu A(t) F_L(k, A(t)\, l)\\ \hat{H}_k &= \rho\mu - \dot{\mu}, \text{ which gives } \frac{\dot{\mu}}{\mu} = \rho + \delta - F_K(k, A(t)\, l).\end{aligned}$$

The first-order conditions can once again be simplified to

$$\epsilon_u(c, 1-l)\frac{\dot{c}}{c} - \frac{u_{c2}(c, 1-l)\,\dot{l}}{u_c(c, 1-l)} = F_K(k, A(t)\, l) - \delta - \rho \tag{8}$$

$$u_2(c, 1-l) = u_c(c, 1-l)\, A(t)\, F_L(k, A(t)\, l). \tag{9}$$

The transversality condition can be written as

$$\lim_{t\to\infty} \exp(-\rho t)\, k(t) \int_0^t -(F_K(k(s), A(s)\, l(s)) - \delta)\, ds = 0. \tag{10}$$

Under the parametric restriction $g(1-\theta) < \rho$, there is a unique path that satisfies all of Eqs. (7) – (10).

Assuming that u is jointly concave in c and l, the current value Hamiltonian $\hat{H}(t,k,c,l,\mu)$ is concave and we have $\lim_{t\to\infty} \exp(-\rho t)\,\mu(t)\,\tilde{k}(t) \geq 0$ for all feasible paths since $\tilde{k}(t) \geq 0$. Then Theorem 7.14 applies and shows that these conditions are sufficient for optimality, that is, the path described above is the unique solution to the social planner's problem.

Exercise 1, Part (c)

Note that, after substituting the competitive market prices for $r(t)$ and $w(t)$ from Eq. (3), the household resource constraints (2), first-order conditions (4)–(5), and the transversality condition (6) become equivalent to respectively to their counterparts in the social planner's problem, Eqs. (7),(8) – (9) and (10).

It follows that given any equilibrium allocation $[a(t) \equiv k(t), k(t), c(t), r(t), w(t)]_t$, the allocation $[c(t), k(t)]_t$ solves the social planner's problem. Conversely, consider a solution $[c(t), k(t)]_t$ to the social planner's problem and define the competitive prices $r(t)$ and $w(t)$ as in Eq. (3). From the correspondence that we have noted above, the allocation

$$a^h(t) = k(t),\ c^h(t) = c(t),\ l^h(t) = l(t)$$

solves the household's problem given the path of prices $[r(t), w(t)]_t$ (where we use the superscript h to distinguish between the household's and the social planner's allocations). It follows that the allocation $[a(t) \equiv k(t), k(t), c(t), r(t), w(t)]_t$ is a competitive equilibrium, proving that the two problems are equivalent when the prices are given by Eq. (3).

Exercise 1, Part (d)

Suppose that the equilibrium we have described in Part (d) has constant and equal rates of consumption and output growth, and a constant level of labor supply $l^* \in [0,1]$. From the resource constraints, we have

$$\dot{k}(t) = F(k(t), A(t)l^*) - \delta k(t) - c(t).$$

This equation implies that, $k(t)$ grows at the same constant rate as output and consumption, and that this constant rate must be equal to g, the growth rate of $A(t)$, since F is constant returns to scale. Moreover, in any such BGP, the interest rate is constant since

$$\begin{aligned} r(t) &= F_K(k(t), A(t)l^*) - \delta \\ &= F_K\left(\frac{k(t)}{A(t)}, l^*\right) - \delta = r^*, \end{aligned}$$

where the second line uses the fact that F_K is homogenous of degree 0 and the equality follows from the fact that $k(t)$ and $A(t)$ grow at the same rate g on a BGP. Further, the wages grow at the constant rate g since

$$\begin{aligned} w(t) &= A(t) F_L(k(t), A(t)l^*) \\ &= A(t) F_L(k(t)/A(t), l^*) = A(t) w^*, \end{aligned} \tag{11}$$

where the second line uses linear homogeneity and the last line uses the fact that $k(t)/A(t)$ is constant.

Next, note that substituting $l(t) = l^*$, the $\dot{l}$ term in Eq. (4) drops out and the Euler equation can be rewritten as

$$\epsilon_u(c(t), 1-l^*)\frac{\dot{c}(t)}{c(t)} = r^* - \rho$$

Since $\dot{c}(t)/c(t)$ is constant on the BGP, it follows that $\epsilon_u(c(t), 1-l^*)$ should be independent of $c(t)$. Since we assume (in the exercise statement) that the function $\epsilon_u(c, 1-l)$ does not depend on l, it follows that it should be a constant function, that is

$$\epsilon_u(c, 1-l) = -\frac{u_{cc}(c, 1-l)\,c}{u_c(c, 1-l)} = \theta \tag{12}$$

for all $c \geq c(0)$ and l, where $\theta \in \mathbb{R}_+$ is some constant. Rewriting Eq. (12) as

$$\frac{\partial \log[u_c(c, 1-l)]}{\partial \log(c)} = -\theta$$

and partially integrating this expression with respect to c, we get

$$\log[u_c(c, 1-l)] = -\theta \log(c) + X(1-l),$$

where $X(1-l)$ is a constant of (partial) integration that could depend on l but not c. Rewriting the previous expression, we have

$$u_c(c, 1-l) = X(1-l)\,c^{-\theta}. \tag{13}$$

Let us now distinguish between two cases.

Case 1, $\theta \neq 1$. Integrating Eq. (13) with respect to c once more, we have

$$u(c, 1-l) = X(1-l)\frac{c^{1-\theta}}{1-\theta} + Y(1-l), \tag{14}$$

where $Y(1-l)$ is a constant of partial integration that could depend on l. Note that the intratemporal first-order condition in Eq. (5) must also hold on a BGP, which, after substituting $w(t) = w^* A(t)$ from Eq. (11), implies

$$u_2(c(t), 1-l^*) = A(t) w^* u_c(c(t), 1-l^*). \tag{15}$$

Plugging in the functional form in Eq. (14), the previous equation can be rewritten as

$$X'(1-l^*)\frac{c(t)^{1-\theta}}{1-\theta}+Y'(1-l^*)=X(1-l^*)A(t)w^*c(t)^{-\theta}.$$

Recall that $c(t)$ and $A(t)$ grow at the same constant rate g. Then, the left hand side and the right hand side grow at the same constant rate only if

$$Y'(1-l^*)=0. \tag{16}$$

In particular, we have

$$Y(1-l)=Y$$

for some constant Y.[1] We define $h(1-l)=X(1-l)$ and take $Y=0$ (which is without loss of generality since it only normalizes the utility function) and conclude that, when $\theta\neq 1$, the only functional form for $u(c,1-l)$ that is consistent with a BGP is

$$u(c(t),1-l(t))=h(1-l(t))\frac{c(t)^{1-\theta}}{1-\theta}. \tag{17}$$

Note also that we should have $h(.)>0$ since otherwise the marginal utility, u_c, would be negative.[2]

Case 2, $\theta=1$. In this case, integrating Eq. (13) gives

$$u(c,1-l)=X(1-l)\log(c)+Y(1-l).$$

Substituting this in the intratemporal condition, we have

$$X'(1-l^*)\log(c(t))+Y'(1-l^*)=X(1-l^*)\frac{A(t)}{c(t)}w^*.$$

This time, since $A(t)/c(t)$ is constant on a BGP, this can be satisfied only if

$$X'(1-l^*)=0,\text{ and } Y'(1-l^*)/X(1-l^*)>0. \tag{18}$$

In particular, we have[3]

$$X(1-l)=X$$

for some X. This time we define $h(1-l)=Y(1-l)$, normalize $X=1$, and conclude that the only functional form for $u(c,1-l)$ that is consistent with a BGP is

$$u(c(t),1-l(t))=\log c(t)+h(1-l(t)), \tag{19}$$

where $h(.)$ is some function with $h'(.)>0$ as desired.

Intuitively, the interest rate is constant only if the intertemporal elasticity of substitution remains constant as c grows, which explains why the utility function must be CES when viewed as a function of c. For the intratemporal trade-off, there are three economic forces. First, income and hence consumption is growing at rate g hence the marginal utility of consumption is shrinking at rate $-\theta g$, which creates a force towards more leisure (the income effect). Second, wages are growing at rate g hence the marginal return to labor is growing at rate g, which creates a force towards more labor (the substitution effect). Third, marginal benefit to leisure might also be changing

[1] This assumes that the restriction in Eq. (16) holds not just for l^* but for any l. This is not entirely correct. Actually, the only restriction we will get will be Eq. (16), since, given l is constant at l^*, we do not really have any information on functional forms away from the BGP value $l=l^*$.

[2] It turns out that the condition $h'(.)>0$ is not necessary in this case. Note that we have

$$u_2(c(t),1-l(t))=h'(1-l(t))\frac{c(t)^{1-\theta}}{1-\theta}.$$

Hence, to ensure that $u_2>0$ so that the individual enjoys leisure, we need $h'(.)>0$ when $\theta<0$ and $h'(.)<0$ when $\theta>1$.

[3] The same caveat above applies here as well. The only restriction we get is Eq. (18). Given that l is constant at l^* on a BGP, we do not have any information on the shape of the function away from the BGP level $l=l^*$.

as consumption grows, depending on whether consumption or leisure are complements or substitutes. To have a constant labor choice l^* on a BGP, we must have the functional form such that the third force exactly balances the first two forces. In particular, when $\theta > 1$, we need the leisure and consumption to be substitutes with the functional form in (17) so that with more consumption marginal value for leisure decreases just enough that the individual keeps leisure choice constant. When $\theta < 1$, we need the leisure and consumption to be complements with exactly the functional form in (17) so that with more consumption marginal value for leisure increases just enough that the individual keeps leisure choice constant. With $\theta = 1$, the first two effects (income and substitution) cancel so we want consumption and labor to be separable (neither substitutes nor complements) as in Eq. (19).

Problem 3: Growth with Non-Life-cycle Consumers [35]

It is hard to believe, but rumor has it that some consumers out there do not conform with the life-cycle hypothesis. Instead, they simply adopt an ad-hoc rule-of-thumb consumption behavior where they simply consume their entire labor-income each period and do not save. In this exercise we are going to study what the long-run implications of the presence of such individuals are. In particular, suppose the economy consists of a continuum of individuals of size 1. A fraction μ of these consumers are the *savers*, which behave according to the neoclassical model, i.e. they choose consumption and savings to maximize their utility function

$$U = \sum_{t=0}^{\infty} \beta^t u\left(c^S(t)\right),$$

where $c^S(t)$ is the per-capita consumption level of a saver. The remaining $1-\mu$ individuals are *hand-to-mouth consumers*, who simply set their consumption $c^{HM}(t)$ equal to their income. Each individual has one unit of labor efficiency units, that she can supply to the market (and hence aggregate labor supply is equal to 1). Technology is given by a representative firm with Cobb-Douglas technology, i.e.

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha}$$

and the aggregate resource constraint is

$$K(t+1) + C(t) = Y(t) + (1-\delta) K(t),$$

where

$$C(t) = C^S(t) + C^{HM}(t) = \mu c^S(t) + (1-\mu) c^{HM}(t)$$

is aggregate consumption and $C^S(t)$ and $C^{HM}(t)$ is aggregate consumption of the savers and hand-to-mouth consumers respectively. The initial capital stock $K(0)$ is entirely owned by the savers.

1. Define an equilibrium in this economy. In particular, state the budget constraints faced by savers and hand-to-mouth consumers and derive the static optimality conditions for the firms.

2. Show that in equilibrium, labor income and consumption of the group of hand-to-mouth consumers is a constant fraction λ of aggregate output $Y(t)$ (i.e. $C^{HM}(t) = \lambda Y(t)$). Solve for λ. How does λ depend on α? Interpret.

3. Now solve for the steady-state capital-level in this economy. How does it compare to the one in the standard neoclassical growth model? How does it depend on μ?

4. Now suppose that $u\left(c^S\right) = ln\left(c^S\right)$ and that capital depreciates fully, i.e. $\delta = 1$. Guess and verify that in equilibrium capital accumulates according to

$$K(t+1) = sK(t)^{\alpha}.$$

   Determine the constant s. How does the introduction of hand-to-mouth consumers affect the equilibrium dynamics of capital relative to the neoclassical model? [HINT: SHOW THAT THE GUESS $K(t+1) = sK(t)^{\alpha}$ SOLVES THE NECESSARY CONDITIONS FOR AN EQUILIBRIUM AND FIND s.]

Problem 3

Part 1 There is a market for labor, capital and the final good. An equilibrium is a sequence of prices $(w(t), r(t))$ and allocations $\left(c^{HM}(t), c^S(t), K(t)\right)$such that all markets clear, the savers behave optimally, firms maximize profits and the hand-to-mouth consumers consumer their entire labor income each period.

The budget constraint of hand-to-mouth consumers is

$$Y^{HM}(t) = (1-\mu) w(t) = (1-\mu) c^{HM}(t).$$

The budget constraint of savers is

$$Y^S(t) = \mu w(t) + K(t)(1 + r(t)) = (1-\mu) c^S(t) + K(t+1).$$

The optimality conditions for firms are

$$\begin{aligned} R(t) &= \alpha K(t)^{\alpha-1} \\ w(t) &= (1-\alpha) K(t)^{\alpha} \end{aligned}$$

Finally, the interest return is the net-return on capital

$$r(t) = R(t) - \delta.$$

Part 2. Total Wages

$$w(t) = (1-\alpha) Y(t).$$

The hand to mouth workers receive

$$Y^{HM}(t) = (1-\mu)(1-\alpha) Y(t).$$

Hence,

$$\lambda = (1-\mu)(1-\alpha).$$

Clearly, λ is decreasing in α as hand-to-mouth consumers only supply their labor to the market. If the capital-share α is higher, the total wagebill of the economy declines and hand-to-mouth consumers receive less.

Part 3. The steady-state capital-stock is still given by the Euler equation of the savers. This Euler equation is

$$\begin{aligned} u'(c_t) &= \beta (1 + r_t) u'(c_{t+1}) \\ 1 &= \beta \left(1 + \alpha \left(\frac{K(t)}{L}\right)^{\alpha-1} - \delta\right), \end{aligned}$$

so that the steady-state is identical to the one of the neoclassical growth model. Intuitively: in the steady-state, the consumption level of the savers cannot grow. Hence, the gross interest rate has to be equal to the (inverse of) the discount factor. This marginal condition does not depend on the level of consumption and hence not on μ. Irrespective of how many hand-to-mouth consumers there are in the economy, the steady-state capital stock is exactly the same as if all individuals were "rational".

Part 4. Problem of the savers

$$\max_{c(t)} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$c(t) + a(t+1) = (1 + r(t)) a(t) + y(t).$$

A necessary condition for an optimum is the Euler equation, which - with log preferences - reads

$$\frac{1}{c(t)} = \beta (1 + r(t+1)) \frac{1}{c(t+1)}.$$

With $\delta = 1$ we have

$$1 + r(t+1) = R(t+1) = \alpha K(t+1)^{\alpha-1}.$$

The budget constraint reads

$$\begin{aligned} c(t) + K(t+1) &= R(t+1)K(t) + w(t)\mu \\ &= \alpha K(t)^{\alpha-1} K(t) + (1-\alpha) K(t)^{\alpha} \mu \\ &= (\alpha + (1-\alpha)\mu) K(t)^{\alpha}. \end{aligned}$$

Now suppose that $K(t+1) = sK(t)^{\alpha}$. Then

$$c(t) = (\alpha + (1-\alpha)\mu - s) K(t)^{\alpha}.$$

Substituting this in the Euler equation gives

$$\begin{aligned} \frac{1}{(\alpha + (1-\alpha)\mu - s) K(t)^{\alpha}} &= \beta\alpha K(t+1)^{\alpha-1} \frac{1}{(\alpha + (1-\alpha)\mu - s) K(t+1)^{\alpha}} \\ \frac{1}{K(t)^{\alpha}} &= \beta\alpha \frac{1}{K(t+1)} \\ K(t+1) &= \beta\alpha K(t)^{\alpha}. \end{aligned}$$

Hence,

$$K(t+1) = \beta\alpha K(t)^{\alpha}$$

so that the evolution of the capital-stock is independent of μ! μ only affects the consumption level of the savers, i.e.

$$c^{S}(t) = [\alpha + (1-\alpha)\mu - s] K(t)^{\alpha}.$$

Hence, the presence of hand-to-mouth consumers affects the consumption level of the savers but not the speed of capital accumulation. Given an initial condition $K(0)$, the entire path of capital $[K(t)]_t$ and income $[Y(t)]_t$ is entirely independent of μ and hence identical to the one of the neoclassical growth model with a representative agent.

For the aggregate resource constraint

$$\begin{aligned} K(t+1) + (1-\mu) c^{HM}(t) + \mu c^{S}(t) &= sK(t)^{\alpha} + (1-\mu)(1-\alpha) Y(t) + (\alpha + (1-\alpha)\mu - s) K(t)^{\alpha} \\ &= sY(t) + (1-\mu)(1-\alpha) Y(t) + (\alpha + (1-\alpha)\mu - s) Y(t) \\ &= [s + (1-\mu)(1-\alpha) + \alpha + (1-\alpha)\mu - s] Y(t) \\ &= Y(t), \end{aligned}$$

which is consistent with full depreciation.