How to derive the FOC in Topic 1

Johannes Boehm

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We use the Leibniz rule for the derivative of an integral.

Theorem 1 (Leibniz rule)

$$\frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)dx = \frac{db(\alpha)}{d\alpha} f(b(\alpha),\alpha) - \frac{da(\alpha)}{d\alpha} f(a(\alpha),\alpha) + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha} f(x,\alpha)dx \tag{1}$$

if f and $\partial f/\partial x$ are both continuous.

From slide 16, we have that

$$E(B_j') = B_j - i(I_j+P_j) + i_d \int_{-\infty}^{B_j+I_j+P_j} (B_j+I_j+P_j-T_j) f(T_j) dT_j + i_b \int_{B_j+I_j+P_j}^{\infty} (B_j+I_j+P_j-T_j) f(T_j) dT_j$$

and we need to minimize this expression w.r.t I_j and P_j. So we take derivatives and set them equal to zero:

$$\frac{\partial E(B_j')}{\partial I_j} = 0 \tag{2}$$

The second FOC is the same as the first one, so I omit it. The difficult thing is how to take the derivative of the integrals, in particular because I_j, the variable w.r.t. which we differentiate, is in the limits of the integral. This is where the Leibniz rule comes in; it tells us how to differentiate this integral. Applying the formula (1) with $\alpha = I_j$, $b(\alpha) = B_j + I_j + P_j$, we have

$$\begin{aligned}
\frac{\partial E(B_j')}{\partial I_j} &= -i + i_d \left(\begin{array}{c} \left(\frac{d}{dI_j}(B_j+I_j+P_j)\right)(B_j+I_j+P_j-(B_j+I_j+P_j)) f(B_j+I_j+P_j) + \\ +0 + \int_{-\infty}^{B_j+I_j+P_j} \left(\frac{d}{dI_j}(B_j+I_j+P_j-T_j) f(T_j)\right) dT_j \end{array} \right) \\
&\quad + i_b \left(\begin{array}{c} 0 - \left(\frac{d}{dI_j}(B_j+I_j+P_j)\right)(B_j+I_j+P_j-(B_j+I_j+P_j)) f(B_j+I_j+P_j) + \\ + \int_{-\infty}^{B_j+I_j+P_j} \left(\frac{d}{dI_j}(B_j+I_j+P_j-T_j) f(T_j)\right) dT_j \end{array} \right)
\end{aligned}$$

Because $B_j + I_j + P_j - (B_j + I_j + P_j) = 0$, we can simplify this further,

$$\begin{aligned}
\ldots &= -i + i_d \int_{-\infty}^{B_j+I_j+P_j} \left(\frac{d}{dI_j}(B_j+I_j+P_j-T_j) f(T_j)\right) dT_j + i_b \int_{-\infty}^{B_j+I_j+P_j} \left(\frac{d}{dI_j}(B_j+I_j+P_j-T_j) f(T_j)\right) dT_j \\
&= -i + i_d \int_{-\infty}^{B_j+I_j+P_j} (1 \cdot f(T_j))\, dT_j + i_b \int_{-\infty}^{B_j+I_j+P_j} (1 \cdot f(T_j))\, dT_j = \ldots
\end{aligned}$$

Now remember that the integral of a density f up to a certain point is the cumulative distribution function (CDF) F of the distribution, so

$$\int_{-\infty}^{B_j+I_j+P_j} f(T_j)dT_j = F(B_j + I_j + P_j)$$

and we get

$$\cdots = -i + i_d F(B_j + I_j + P_j) + i_b(1 - F(B_j + I_j + P_j)) = 0$$

and the fact that it's zero comes from (2). This is the last equation on slide 17 in the lecture notes.