

Econ 510a General Economic Theory: Macroeconomics

Problem Set 2 (due on 09/21)

Fall 2012

1 Complete Market and Risk-Sharing

Consider an economy that lasts for one period and consists of two individuals $i = A, B$. There are two state of the worlds, $s = 1, 2$, where state s occurs with probability π_s (clearly $\pi_1 + \pi_2 = 1$). Let y_s^i be the endowment of individual i in state s . The endowments are given by

$$\begin{aligned}y_1^A &= \theta \\y_2^A &= 0 \\y_1^B &= 0 \\y_2^B &= \theta\end{aligned}$$

The consumers preferences are given by

$$u^i(c) = -\frac{1}{\gamma_i} e^{-\gamma_i c}. \quad (1)$$

These preferences are called CARA preference (where CARA stands for constant absolute risk aversion). In particular: consumption can be negative given those preferences. Note that there are two sources of heterogeneity: The two people differ in their tastes (as $\gamma_A \neq \gamma_B$) and in their income profiles (they receive income in different states and $\pi_1 \neq \pi_2$).

1. A standard measure of risk aversion is the *coefficient of absolute risk aversion* $AR(c) = -\frac{u''(c)}{u'(c)}$. Show that CARA is not a misnomer.
2. Assume that there are complete markets in this economy. Let λ^i be the multiplier on the budget constraint of individual i . Write down the maximization problem of individual i .
 - (a) Solve for the optimal consumption level c_s^i as a function of prices p_s , state probabilities π_s and the multiplier on the budget constraint λ^i .
 - (b) Solve for the optimal consumption level c_s^i as a function the multipliers on the budget constraint $[\lambda^A, \lambda^B]$ and the aggregate endowment $Y_s = y_s^A + y_s^B$. How does c_s^i depend on the state?

3. Show that the equilibrium prices satisfy

$$\frac{p_1}{p_2} = \frac{\pi_1}{\pi_2}.$$

4. We can normalize one of the prices (you will see this in your micro class). Hence, let us normalize $p_1 = 1$. Use the individuals' budget constraint to solve for the consumption levels c_s^i only as a function of aggregate income Y_s and parameters. What determines the inequality of consumption between people (i.e. $\frac{c_s^A}{c_s^B}$)? Does it depend on tastes γ^i ? Does it depend on the differences in the income process? Interpret.
5. [EXTRA CREDIT] Now suppose there is a representative consumer in this economy with preferences

$$u(c) = -\frac{1}{\gamma_R} e^{-\gamma_R c}.$$

What is the level of risk-aversion γ_R such that this representative consumer chooses $c_s^R = Y_s$ at the equilibrium prices of the heterogeneous economy? Can you give an intuition for this result?

Solution

Part 1. If $u(c) = -\frac{1}{\gamma}e^{-\gamma c}$, then $u'(c) = e^{-\gamma c}$ and $u''(c) = -\gamma e^{-\gamma c}$. Hence, $AR(c) = \gamma$, so that the coefficient of absolute risk aversion is constant.

Part 2. The maximization problem is

$$\max_{\{c_1^i, c_2^i\}} \sum_{s=1}^2 \pi_s \left(-\frac{1}{\gamma_i} e^{-\gamma_i c_s^i} \right) \text{ s.t. } \sum_{s=1}^2 p_s c_s^i \leq \sum_{s=1}^2 p_s y_s^i.$$

The first order condition is

$$u'_i(c_s^i) P_s = e^{-\gamma_i c_s^i} \pi_s = p_s \lambda^i,$$

where λ^i is the multiplier on the budget constraint. Rearranging terms yields

$$c_s^i = \ln \left(\frac{\pi_s}{p_s} \frac{1}{\lambda^i} \right) \frac{1}{\gamma_i} = \frac{1}{\gamma_i} \ln \left(\frac{\pi_s}{p_s} \right) - \frac{1}{\gamma_i} \ln(\lambda^i).$$

Combining the two first order conditions for individual A and B for state s yields

$$\frac{e^{-\gamma_A c_s^A}}{e^{-\gamma_B c_s^B}} = \frac{\lambda^A}{\lambda^B},$$

so that

$$c_s^B = \frac{\gamma_A}{\gamma_B} c_s^A + \frac{1}{\gamma_B} \ln \left(\frac{\lambda^A}{\lambda^B} \right).$$

Market clearing requires that

$$Y_s = \sum_{i=A,B} y_s^i = \sum_{i=A,B} c_s^i.$$

Hence

$$\begin{aligned} Y_s &= c_s^B + c_s^A = \frac{\gamma_A}{\gamma_B} c_s^A + \frac{1}{\gamma_B} \ln \left(\frac{\lambda^A}{\lambda^B} \right) + c_s^A \\ &= \frac{\gamma_A + \gamma_B}{\gamma_B} c_s^A + \frac{1}{\gamma_B} \ln \left(\frac{\lambda^A}{\lambda^B} \right), \end{aligned}$$

so that

$$\begin{aligned} c_s^A &= \frac{\gamma_B}{\gamma_B + \gamma_A} Y_s - \frac{1}{\gamma_A + \gamma_B} \ln \left(\frac{\lambda^A}{\lambda^B} \right) \\ c_s^B &= \frac{\gamma_A}{\gamma_B + \gamma_A} Y_s + \frac{1}{\gamma_A + \gamma_B} \ln \left(\frac{\lambda^A}{\lambda^B} \right). \end{aligned}$$

As the aggregate endowment is constant, i.e. $Y_1 = Y_2 = \theta$, consumption levels are constant across states

Part 3. Consider individual i . Combining the first order conditions for states $s = 1, 2$ we get

$$\frac{e^{-\gamma_A c_1^i} \pi_1}{e^{-\gamma_A c_2^i} \pi_2} = \frac{p_1 \lambda^i}{p_2 \lambda^i},$$

which implies that

$$c_2^i - c_1^i = \frac{1}{\gamma_i} \ln \left(\frac{\pi_2 p_1}{\pi_1 p_2} \right).$$

As $c_2^i = c_1^i = c^i$ we get that

$$\frac{p_1}{p_2} = \frac{\pi_1}{\pi_2}.$$

Part 4. Consider person A . We get that

$$p_1 c_1^A + p_2 c_2^A = p_1 y_1^A + p_2 y_2^A = p_1 \theta.$$

Using $p_1 = 1$ and $p_2 = \frac{\pi_2}{\pi_1}$ we get that

$$\begin{aligned} c^A &= \pi_1 \theta = E[y_s^A] \\ c^B &= \pi_2 \theta = E[y_s^B]. \end{aligned}$$

Hence: consumption inequality is fully determined from income inequality. In particular, each individual receives his average income every period. Hence, the risk in this economy is perfectly shared! Note also that the entire allocation is independent of tastes γ^i ! I.e. prices and allocations do not depend on the risk-aversion of individuals. The reason is that *in equilibrium* nobody in the economy has to hold any risk.

Part 5. The representative agent has a first order condition

$$e^{-\gamma_R c_s^R} \frac{\pi_s}{p_s} = \lambda_R,$$

where λ_R is the multiplier on the budget constraint of the representative agent. Using the two first order conditions for states $s = 1, 2$, we get that

$$\begin{aligned} e^{-\gamma_R c_1^R} \frac{\pi_1}{p_1} &= e^{-\gamma_R c_2^R} \frac{\pi_2}{p_2} \\ e^{\gamma_R (c_s^R - c_1^R)} &= \frac{\pi_2 p_1}{\pi_1 p_2} \\ c_s^R - c_1^R &= \frac{1}{\gamma_R} \ln \left(\frac{\pi_2 p_1}{\pi_1 p_2} \right). \end{aligned}$$

At prices $p_1 = 1$ and $p_2 = \frac{\pi_2}{\pi_1}$ we get that $c_s^R - c_1^R = 0$ as required. In particular, this holds true irrespective of γ_R . Because there is no risk in the aggregate, the representative consumer has a constant consumption stream. Hence, the attitudes towards risk γ_R do not matter for the equilibrium allocations.

Problem 2

Note: this solution assumes CRRA period utility.

Part 1

The maximization problem of the representative household is given by

$$\begin{aligned} \max_{[c(t), K(t)]_{t=0}^{\infty}} & \int_0^{\infty} \exp(-\rho t) \left[\frac{c(t)^{1-\theta} - 1}{1-\theta} + G(t) \right] dt \\ \text{s.t. } \dot{k}(t) &= (1 - \tau(t)) (f(k(t)) - c(t)) - \delta k(t), \end{aligned}$$

where we again defined all variables as per capita variables and already substituted that $c(t) + i(t) = y(t)$ (which will of course hold with equality). Note especially that the household does *not* internalize that $g(t) = \tau(t) i(t)$, i.e. takes $g(t)$ as given. The corresponding current-value Hamiltonian for this problem is given by

$$\hat{H}(c, k, \mu) = \frac{c(t)^{1-\theta} - 1}{1-\theta} + G(t) + \mu(t) ((1 - \tau(t)) (f(k(t)) - c(t)) - \delta k(t)),$$

which yields the necessary conditions

$$\hat{H}_c(c, k, \mu) = c(t)^{-\theta} - (1 - \tau(t))\mu(t) = 0 \quad (2)$$

$$\hat{H}_k(c, k, \mu) = \mu(t) [(1 - \tau(t))f'(k(t)) - \delta] = -\dot{\mu}(t) + \rho\mu(t). \quad (3)$$

From (2) we get that

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta} \left(\frac{\dot{\tau}(t)}{1 - \tau(t)} - \frac{\dot{\mu}(t)}{\mu(t)} \right),$$

so that by substituting (3) we get the modified Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta} \left(\frac{\dot{\tau}(t)}{1 - \tau(t)} + (1 - \tau(t))f'(k(t)) - \delta - \rho \right). \quad (4)$$

The intuition for (4) is straightforward. As usual, this equation describes the consumer's intertemporal consumption behavior. This however now takes the tax sequence the consumer faces into account. If the tax schedule is increasing over time, i.e. $\dot{\tau}(t) > 0$, the consumer will tilt his consumption schedule more as investing today is relatively cheap. Hence, an increasing tax schedule acts like a higher interest rate, as the returns of investing today are higher than doing so tomorrow.

Part 2

If $\lim_{t \rightarrow \infty} \tau(t) = \tau$ we can characterize the steady state of this economy. As taxes are constant asymptotically, the $\frac{\dot{\tau}(t)}{1 - \tau(t)}$ term vanishes in (4) so that asymptotically, consumption behavior is described by

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta} ((1 - \tau)f'(k(t)) - \delta - \rho).$$

As consumption has to be constant in the steady state, the steady state capital stock k^* is implicitly defined by

$$f'(k^*) = \frac{\delta + \rho}{1 - \tau}. \quad (5)$$

By the concavity of f , the steady state capital-labor ratio is unique. The steady state levels of consumption and investment can then be backed out from the capital accumulation equation and the resource constraint as

$$\begin{aligned} i^* &= \frac{\delta}{1 - \tau} k^* \\ c^* &= f(k^*) - \frac{\delta}{1 - \tau} k^*. \end{aligned} \quad (6)$$

The steady state per capita level of the public good is given by

$$G^* = \tau i^* = \tau \frac{\delta}{1 - \tau} k^*. \quad (7)$$

Part 3

To study the optimal steady state tax rate, suppose the economy is in the steady state. The utility level of the representative consumer is given by

$$U^{SS}(\tau) = \int_0^\infty \exp(-\rho t) \left[\frac{(c^*)^{1-\theta} - 1}{1 - \theta} + G^* \right] dt = \left[\frac{(c^*)^{1-\theta} - 1}{1 - \theta} + G^* \right] \frac{1}{\rho},$$

where $U^{SS}(\tau)$ stresses the fact that we consider steady state utility and explicitly denote the dependence on the tax rate τ via the steady state levels of consumption c^* and the public good G^* given in (6) and (7). Substituting those expressions, the optimal tax rate τ^{SS} is given by

$$\tau^{SS} = \arg \max_{\tau} \frac{(f(k^*) - \frac{\delta}{1-\tau}k^*)^{1-\theta} - 1}{1-\theta} + \tau \frac{\delta}{1-\tau}k^*.$$

The necessary first-order condition is given by

$$(c^*)^{-\theta} \left[\left(f'(k^*) - \frac{\delta}{1-\tau} \right) \frac{\partial k^*}{\partial \tau} - \frac{\delta}{(1-\tau)^2} k^* \right] + \frac{\tau \delta}{1-\tau} \frac{\partial k^*}{\partial \tau} + \frac{\delta k^*}{(1-\tau)^2} = 0. \quad (8)$$

Although this might look daunting, recall that from (5) we get that

$$f'(k^*) - \frac{\delta}{1-\tau} = \frac{\delta + \rho}{1-\tau} - \frac{\delta}{1-\tau} = \frac{\rho}{1-\tau}. \quad (9)$$

Additionally we have that

$$\frac{\partial k^*}{\partial \tau} = \frac{\delta + \rho}{(1-\tau)^2 f''(k^*)},$$

so that

$$\frac{\tau \delta}{1-\tau} \frac{\partial k^*}{\partial \tau} + \frac{\delta k^*}{(1-\tau)^2} = \frac{\tau \delta}{1-\tau} \frac{\delta + \rho}{(1-\tau)^2 f''(k^*)} + \frac{\delta k^*}{(1-\tau)^2} = \frac{\delta}{(1-\tau)^2} \left[\frac{\tau}{1-\tau} \frac{\delta + \rho}{f''(k^*)} + k^* \right].$$

Using this and (9), we can write (8) as

$$(c^*)^{-\theta} \left[\frac{\rho}{\delta(1-\tau)} \frac{\delta + \rho}{f''(k^*)} - k^* \right] + \left[\frac{\tau}{1-\tau} \frac{\delta + \rho}{f''(k^*)} + k^* \right] = 0,$$

which defines the optimal tax rate implicitly.

Although this tax rate maximizes the steady state utility of the representative consumer, it will not maximize the utility of the representative household if the economy starts away from the steady state. The reason is that the sequence of taxes $[\tau(t)]_{t=0}^{\infty}$ determines the investment behavior of the household and hence the whole sequence of the capital stock $[k(t)]_{t=0}^{\infty}$. In particular taxes therefore determine the speed of adjustment to the steady state capital stock and the consumption level during the transitional dynamics. This is not taken into account when taxes are chosen to maximize the steady state utility of the representative consumer.

Problem 3: Growth with Overlapping Generations and Borrowing Constraints

Consider the following OLG economy, where individuals live for *three* periods. The preferences of an individual born at time t are given by

$$U(t) = \log(c_Y(t)) + \beta \log(c_M(t+1)) + \rho \ln(c_O(t+2)),$$

where $c_Y(t)$ denotes the level of consumption when the individual is young, $c_M(t+1)$ when individual is middle-aged and $c_O(t+2)$ when it is old. Individuals are born with zero assets. Their labor supply is $(0, 1, 0)$, i.e. individuals can only work when they are middle-aged. They can borrow and save at interest rate $r(t+1)$. In particular, if someone saves s units at time t , she receives $(1+r(t+1))s$ units in period $t+1$. Let $s_Y(t)$ and $s_M(t+1)$ denote the level of savings when the consumer is young and middle-aged respectively (if $s_i(\tau) < 0$, the individual borrows at time τ).

1. Let the consumers face a sequence of wages and interest rates given by $\{w(t), r(t)\}_t$. State the consumers' maximization problem and solve for $(c_Y(t), c_M(t), c_O(t), s_Y(t), s_M(t))$ as a function of $\{w(t), r(t)\}_t$ and parameters.

2. Now suppose that young consumers face a borrowing constraint of the form

$$s_Y(t) \geq -\phi \frac{w(t+1)}{1+r(t+1)}, \quad (10)$$

i.e. they can borrow only up to a fraction $\phi > 0$ of their present discounted value of income. Derive a condition on ϕ such that the borrowing constraint is binding. How does this value depend on $(w(t+1), r(t+1), r(t+2), \beta, \delta)$? Why?

3. *For the remainder of this exercise assume that the borrowing constraint (10) is binding.* Consider the aggregate economy at time t . On the production side there is a representative firm with a Cobb Douglas production function

$$Y(t) = K(t)^\alpha L(t)^{1-\alpha}.$$

The population is constant, i.e. each period t , L individuals are born. Assume also that capital fully depreciates after one period (i.e. $\delta = 1$). Each period, there is a market for capital and individuals can borrow from each other. The interest rate will be equal to $r(t)$ on both markets because of arbitrage. Describe (no math needed) the interactions on the asset market, i.e. who borrows from whom and who saves in which type of assets.

4. Derive the accumulation equation for the capital labor ratio $k(t) = \frac{K(t)}{L}$, i.e. express $k(t+1)$ as a function of past values and solve for the steady-state capital-labor ratio k^* .
5. How does the steady-state capital-labor ratio k^* and the steady-state return to capital $f'(k^*)$ depend on ϕ ? What is the intuition? Now suppose that ϕ varies across countries (with lower values of ϕ corresponding to worse financial markets) and all countries are in their steady-state. Is this model qualitatively consistent with data?

Solution

Problem 3

Part 1. Consumers solve

$$\max_{(c_1, c_2, c_3, b, s)} \ln(c_1) + \beta \ln(c_2) + \rho \ln(c_3)$$

subject

$$\begin{aligned} c_1 &= b \\ c_2 + s &= w(t+1) - (1+r(t+1))b \\ c_3 &= (1+r(t+2))s \end{aligned}$$

Hence,

$$\max_{b, s} \ln(b) + \beta \ln(w(t+1) - (1+r(t+1))b - s) + \rho \ln((1+r(t+2))s)$$

so that

$$\begin{aligned} \frac{1}{b} &= \beta \frac{(1+r(t+1))}{w(t+1) - (1+r(t+1))b - s} \\ \frac{\rho}{s} &= \beta \frac{1}{w(t+1) - (1+r(t+1))b - s} \end{aligned}$$

Then

$$\frac{\frac{1}{b}}{\frac{\rho}{s}} = \frac{s}{\rho b} = (1+r(t+1))$$

so that

$$s = (1+r(t+1))\rho b.$$

Hence,

$$\begin{aligned} \beta(1+r(t+1))b &= w(t+1) - (1+r(t+1))b - s \\ &= w(t+1) - (1+r(t+1))b(1+\rho) \end{aligned}$$

so that

$$(1+r(t+1))b = \frac{w(t+1)}{1+\rho+\beta}$$

and hence

$$s = \frac{\rho w(t+1)}{1+\rho+\beta}.$$

Hence, the solution is

$$\begin{aligned} c_Y(t) &= \frac{1}{(1+\beta+\rho)} \frac{w(t+1)}{1+r(t+1)} \\ c_M(t+1) &= w(t+1) - \frac{w(t+1)}{1+\rho+\beta} - \frac{\delta w(t+1)}{1+\rho+\beta} \\ &= w(t+1) \frac{\beta}{1+\rho+\beta} \\ c_O(t+2) &= \frac{\rho w(t+1)(1+r(t+2))}{1+\rho+\beta}. \\ s_Y(t) &= -\frac{w(t+1)}{1+\rho+\beta} \frac{1}{1+r(t+1)} \\ s_M(t+1) &= \frac{\rho w(t+1)}{1+\rho+\beta}. \end{aligned}$$

Part 2. Borrowing constraint

$$s_Y(t) \geq -\phi \frac{w(t+1)}{(1+r(t+1))}.$$

Optimal borrowing

$$s_Y(t) = -\frac{1}{1+\rho+\beta} \frac{w(t+1)}{(1+r(t+1))}.$$

Borrowing constraint is binding whenever

$$\begin{aligned} -\frac{1}{1+\rho+\beta} \frac{w(t+1)}{(1+r(t+1))} &\leq -\phi \frac{w(t+1)}{(1+r(t+1))} \\ \frac{1}{1+\rho+\beta} &\geq \phi \end{aligned}$$

Clearly, if β and ρ are high, you want to borrow less up front, i.e. impatient consumers (β and ρ low) are more likely to be borrowing constrained. Also: neither $w(t)$ nor $r(t+1)$ or $r(t+2)$ matter, which is due to the log preferences.

Part 3 and 4. Now consider the aggregate economy. The economy-wide savings stem from the middle aged individuals. In particular, aggregate savings at period t are given by

$$S(t) = s_M(t) L = \frac{\delta w(t)(1-\phi)}{\beta+\rho} L.$$

Note that this stems from

$$\begin{aligned} &L \left[\overbrace{w(t) - (1+r(t)) s_Y(t-1) - \hat{c}_M(t)}^{\text{Income after repaying debt}} \right] \\ &= L \left[w(t) - \phi w(t) - \frac{\beta}{\beta+\rho} w(t+1)(1-\phi) \right], \end{aligned}$$

where \hat{c}_M is the optimal consumption level of the middle aged when the constraint is binding.¹ These savings are spent on future capital $K(t+1)$ and total borrowing from the young $B(t)$. The young borrow

$$B(t) = -s_Y(t) L = \phi \frac{w(t+1)}{1+r(t+1)} L.$$

Hence

$$\begin{aligned} K(t+1) &= \frac{\rho w(t)(1-\phi)}{\beta+\rho} L - \phi L \frac{w(t+1)}{(1+r(t+1))} \\ k(t+1) &= \frac{\rho w(t)(1-\phi)}{\beta+\rho} - \phi \frac{w(t+1)}{(1+r(t+1))} \end{aligned}$$

Now

$$\begin{aligned} w(t+1) &= (1-\alpha) k(t+1)^\alpha \\ 1+r(t+1) &= 1+\alpha k(t+1)^{\alpha-1} - \delta \end{aligned}$$

With $\delta = 1^2$ we get that

$$1+r(t+1) = \alpha k(t+1)^{\alpha-1}$$

¹Note that $\hat{c}_M \neq c_M = w \frac{\beta}{1+\delta+\beta}$ calculated above, because this implicitly assumed that the constraint is not binding! But now the middle aged have more money (as they borrowed less when they were young!)

²Sorry taking δ to be both the depreciation rate and the parameter of the utility function. Bad notation.

so that

$$\frac{w(t+1)}{(1+r(t+1))} = \frac{(1-\alpha)k(t+1)^\alpha}{\alpha k(t+1)^{\alpha-1}} = \frac{1-\alpha}{\alpha} k(t+1).$$

Hence,

$$k(t+1) = \frac{\delta(1-\phi)(1-\alpha)k(t)^\alpha}{\rho+\beta} - \phi \frac{1-\alpha}{\alpha} k(t+1)$$

so that

$$k(t+1) = \frac{1}{1+\phi \frac{1-\alpha}{\alpha}} \frac{\rho(1-\phi)(1-\alpha)}{\rho+\beta} k(t)^\alpha.$$

Steady state

$$k(t+1) - k(t) = \frac{1}{1+\phi \frac{1-\alpha}{\alpha}} \frac{\rho(1-\phi)(1-\alpha)}{\rho+\beta} k(t)^\alpha - k(t) = sk(t)^\alpha - k(t)$$

so that

$$k^* = s^{\frac{1}{1-\alpha}} = \left(\frac{1}{1+\phi \frac{1-\alpha}{\alpha}} \frac{\rho(1-\phi)(1-\alpha)}{\rho+\beta} \right)^{\frac{1}{1-\alpha}}.$$

Part 5. From above

$$\frac{\partial k^*}{\partial \phi} < 0 \text{ and } \frac{\partial f'(k^*)}{\partial \phi} > 0$$

i.e. tighter borrowing constraints increase the capital stock and reduces the return to capital. The reason is intuitive: The savings of the middle class have two ways to go. They could go into capital and they could go into loans for the young. If the young are constrained, the middle class puts their money into real capital. This increases the capital stock and drives down the return to capital. Additionally, borrowing constraints also increase the amount of money the middle class actually has - because they borrowed less when they were young. Hence, both the supply of savings go up and the amount of borrowing from the young goes down and both these margins increase the amount of capital.

If we were to look at the data, we could conclude that Nigeria (which has a low capital-labor ratio) had *better* financial markets as the US (which has a high capital-labor ratio). This is not consistent with the data.