Investigating 10-Dimensional Simple Lie Algebras Over GF(2)
An Independent Study with Professor Keith Kearnes

Jonathan Bosnich

CU Boulder - Spring 2020

1 Overview

In this independent study, I first spent a handful of weeks learning the basics of Lie theory before researching an unsolved problem in Lie theory under the mentorship of Professor Keith Kearnes. In our research, the immediate goal was to use methods from the literature to search for new 10-dimensional simple Lie algebras over GF(2). The ultimate goal of this research is to completely classify all 10-dimensional simple Lie algebras over GF(2), which was far beyond the scope of this independent study but may be of interest to a future PhD student. The method we used for searching for a new 10-dimensional simple Lie algebra over GF(2) was inspired by Vaughan-Lee's method in [1]. In conjunction with searching for new simple Lie algebras, we analyzed the only known 10-dimensional simple Lie algebra over GF(2), which Kaplansky found in [2]. In this paper, I present the results of our analysis, make notes on the computational methods used for the analysis, present a few of our observations and conjectures, and describe some next steps for future research.

2 Searching for new 10-dimensional simple Lie algebras over GF(2) using Vaughan-Lee's method

2.1 Method overview

In his paper [1], Vaughan-Lee found multiple new 8-dimensional simple Lie algebras over GF(2) by first considering an element a of some 8-dimensional simple Lie algebra L, and then analyzing the possible characteristic polynomials of $A = \mathrm{ad}(a)$, which is an 8×8 matrix in sl(8,2). After first considering all possible characteristic polynomials of A, Vaughan-Lee rejected polynomials that could not possibly arise from an element of a simple Lie algebra. The first few criteria that Vaughan-Lee used to filter through the characteristic polynomials are below. Note that these criteria are generalized for an n-dimensional simple Lie algebra.

2.2 Criteria for filtering out characteristic polynomials

(C1) By Engel's theorem, we know that a simple Lie algebra cannot have the property that every element is nilpotent. Hence, Vaughan-Lee assumes that A is not nilpotent. This implies that the characteristic polynomial of A cannot be $\chi = x^n$.

(C2) Since A is singular, $\det(A) = 0$. Thus, the constant term in the characteristic polynomial of A is zero.

(C3) Since A has zero trace, the coefficient of x^{n-1} in the characteristic polynomial of A is zero.

As a result of criteria (C1) and (C2), the characteristic polynomial of A has the form $\chi = x^k f(x)$, where $0 < k < n$ and $f(0) \neq 0$. Using this representation, the final criterion that we used from Vaughan-Lee's method is

(C4) In $\chi = x^k f(x)$, $f(x)$ must have repeated nonzero roots.

Note that this is not an exhaustive list of the criteria used in Vaughan-Lee's paper; it is only the criteria that we were able to get through during the semester.

2.3 Preliminary results from applying criteria (C1)-(C4) to the 10-dimensional case

Now, consider a 10-dimensional simple Lie algebra L over GF(2) and an element $a \in L$ with matrix representation $A = \text{ad}(a)$, which is a 10×10 matrix in sl(10,2). After filtering every possible characteristic polynomial for A through criteria (C1)-(C3), we are left with 255 polynomials. Then using criterion (C4) to filter out all characteristic polynomials where $f(x)$ does not have repeated nonzero roots, we are left with 85 polynomials. Each one of these 85 characteristic polynomials is listed in its fully expanded form in the appendix.

2.4 Notes on the MATLAB code used for these computations

The MATLAB program used for the computations in this section is "char_eqns.m". First, this program constructs a large matrix of all of the possible characteristic polynomials, where each row is composed of the polynomial coefficients. Filtering this list through criteria (C1)-(C3) was relatively straightforward to code; however, applying (C4) took some work as there is not a convenient built-in MATLAB function to find polynomial roots over GF(2). Therefore, I wrote my own function "gcd_GF2.m", which computes the gcd of f and f' over GF(2) using the Euclidean algorithm. If $\gcd(f, f') = 1$, the roots of f are distinct, which implies the characteristic polynomial doesn't satisfy (C4). For more notes on the code, consult the comments made directly in the program files.

3 Conjecture regarding the number of characteristic polynomials that satisfy these criteria

From Vaughan-Lee's paper, where an 8-dimensional simple Lie algebra was considered, a similar reduction pattern emerged. For the 8-dimensional case, the number of characteristic polynomials that satisfied criteria (C1)-(C3) was 63, and the number of characteristic polynomials that satisfied criteria (C1)-(C4) was 21. For both the $n = 10$ and $n = 8$ cases, the following ratio emerges

$$\frac{\#\text{ of polynomials that satisfy (C1)-(C4)}}{\#\text{ of polynomials that satisfy (C1)-(C3)}} = \frac{1}{3}.$$

After observing this phenomenon, we did the same computations for a few more dimensions. The results are summarized in the table below.

algebra dimension	# of polynomials that satisfy (C1)-(C3)	# of polynomials that satisfy (C1)-(C4)
3	1	1
4	3	1
5	7	3
6	15	5
7	31	11
8	63	21
10	255	85

From this data, we state the following conjecture.

Conjecture 1. *Let L be an n-dimensional simple Lie algebra over GF(2), and let $a \in L$ have the matrix representation $A = ad(a)$. Then, considering all possible characteristic polynomials of A with respect to the criteria (C1)-(C4), we have the following equation*

$$\#\textit{ of polynomials that satisfy (C1)-(C4)} = \textit{ceiling}\left(\frac{\#\textit{ of polynomials that satisfy (C1)-(C3)}}{3}\right).$$

Note that when n is even, the *ceiling* function may be omitted.

4 The Kaplansky 10-dimensional simple Lie algebra over GF(2)

The only known 10-dimensional simple Lie algebra over GF(2) was determined by Irving Kaplansky in [2], and it will henceforth be referred to as the "Kaplansky algebra." In this text, Kaplansky constructs finite-dimensional simple Lie algebras over GF(2) from finite-dimensional J-systems over GF(2). In particular, the Kaplansky algebra is constructed from a 5-dimensional J-system of the third type. For more notes on Kaplansky's procedure, refer to [2].

4.1 Graph, notation, and Lie bracket of the Kaplansky algebra

Given the complete graph on five vertices below, the vertices represent the basis of the 5-dimensional J-system that generates the Kaplansky algebra, and the edges represent the basis of the Kaplansky algebra.

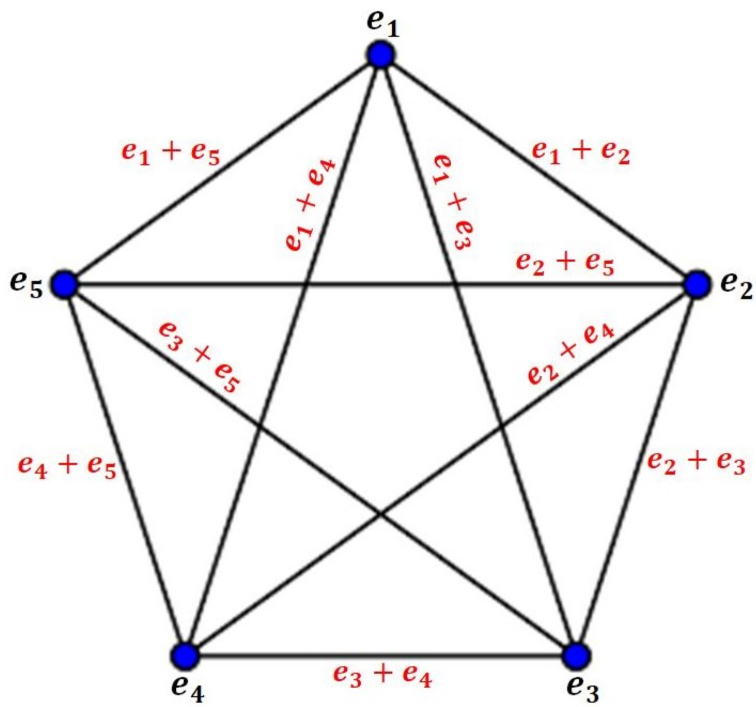


Figure 1: The complete graph on five vertices, which represents the basis of the J-system (vertices) and the basis of the Kaplansky algebra (edges)

The basis of the Kaplansky algebra is ordered as follows:

$$\mathcal{B}_{algebra} = \begin{bmatrix} e_1 + e_2 \\ e_1 + e_3 \\ e_1 + e_4 \\ e_1 + e_5 \\ e_2 + e_3 \\ e_2 + e_4 \\ e_2 + e_5 \\ e_3 + e_4 \\ e_3 + e_5 \\ e_4 + e_5 \end{bmatrix}.$$

The rules for the Lie bracket operation on the Kaplansky algebra are:

(R1) $[e_i + e_j, e_j + e_k] = e_i + e_k$.

(R2) $[e_i + e_j, e_k + e_p] = 0$, given i, j, k, p are distinct (i.e. disjoint edges bracket to zero).

4.2 Characteristic polynomials of the Kaplansky algebra

In conjunction with using Vaughan-Lee's method, it will be useful to know all of the characteristic polynomials from the Kaplansky algebra. Since these polynomials come from a known 10-dimensional simple Lie algebra over GF(2), we know that they cannot be removed from the list of 85 polynomials in the appendix, which will allow us to focus only on the other possible polynomials.

To find all of the characteristic polynomials of the Kaplansky algebra, we must compute the characteristic polynomial of the matrix representation of each of the 34 subgraphs of the graph from Fig. 1. These 34 subgraphs are listed in order in the appendix. The method for computing the matrix representation of each one of these subgraphs is to (1) find the matrix representation of each of the basis elements (single edges), and then (2) add together the matrices of the single edges that construct any particular multi-edged graph. For example, the matrix representation of the two-edge, shared vertex graph is simply $\mathrm{ad}(e_1 + e_2) + \mathrm{ad}(e_2 + e_3)$. After computing the characteristic polynomial of the matrix representation of each of the 34 subgraphs, we found that the unique characteristic polynomials of the Kaplansky algebra are

$$\begin{aligned}
\chi_1 &= x^{10} \\
\chi_2 &= x^{10} + x^6 \\
\chi_3 &= x^{10} + x^8 + x^6 + x^4 \\
\chi_4 &= x^{10} + x^8 + x^4 + x^2.
\end{aligned}$$

Unfortunately, the above list is not long. Additionally, the characteristic polynomial of x^{10} did arise, implying that there exists at least one nilpotent element (which we did not assume when working through Vaughan-Lee's procedure). Therefore, only the above polynomials χ_2, χ_3, and χ_4 are present in the list of the 85 polynomials in the appendix. In conclusion, the number of possible polynomials that require further interrogation was only reduced from 85 to 82.

4.3 Hypothesis on the automorphism group of the Kaplansky algebra

Given that the Kaplansky algebra has so few unique characteristic polynomials, we suspect that the Kaplansky algebra has a large automorphism group. While we did not have time to determine the automorphism group, we do have a few ideas of what possible automorphisms might be. To be able to more easily identify automorphisms of the Kaplansky algebra, we first define the action of an edge on a vertex to be

$$[e_1 + e_2, e_i] = \begin{cases} 0, & i \neq 1, 2 \\ e_j, & \{i, j\} = \{1, 2\} \end{cases}.$$

Since the Kaplansky algebra is now acting on the five vertices of the graph, if a is an element of the Kaplansky algebra, $\mathrm{ad}(a)$ will be a 5×5 matrix. This 5-dimensional representation yields more obvious automorphisms. For example, in GF(2) conjugating $\mathrm{ad}(a)$ by an orthogonal matrix M is an automorphism. Therefore, we know that $O_5(2)$, the set of all 5×5 orthogonal matrices over GF(2), is contained in the automorphism group of the Kaplansky algebra. We also noted that there exists $M \in O_5(2)$ that sends a single-edge graph to a two-edge, shared vertex graph via conjugation, which explains why these two graphs share the same characteristic polynomial.

4.4 Subgraphs grouped together by characteristic polynomial

As noted above, the single-edge graph and the two-edge, shared vertex graph share the same polynomial. Below, we group together each one of the 34 subgraphs by their characteristic polynomial. Note that the number indicating the graph corresponds to the numbered list of the graphs in the appendix.

Polynomial	Graphs
x^{10}	1, 5, 8, 17, 19, 21, 24, 26, 28
$x^{10} + x^6$	2, 6, 7, 18, 20, 22, 23, 25, 27
$x^{10} + x^8 + x^6 + x^4$	3, 4, 9, 10, 11, 12, 15, 16, 29, 30, 32, 33
$x^{10} + x^8 + x^4 + x^2$	13, 14, 31, 34

4.5 Notes on the MATLAB code used for these computations

The MATLAB program used for the computations in this section is "Kap_char.m". In this program, I coded the procedure described in Section 4.2 to find all of the characteristic polynomials of the Kaplansky algebra. The coefficients of the polynomial of each of the 34 subgraphs are stored in the rows of the array "kap_char". The order of the polynomials is exactly the order of the 34 subgraphs in the appendix. For example, the command >>kap_char(5,:) returns the coefficients of the characteristic polynomial of the fifth graph in the list (two-edge, shared vertex). I then deleted duplication in the list of the 34 polynomials, which yielded the four unique polynomials. Finally, I grouped together the subgraphs by their polynomials. For more notes on the code, consult the comments made directly in the program file.

5 Future research

With respect to our approach and the progress we made over the semester, there are some immediate next steps to continue this research.

(1) As noted in Section 2, we did not work through all of Vaughan-Lee's method for the 10-dimensional case. Therefore, working through the rest of Vaughan-Lee's procedure with the remaining possible characteristic polynomials is a clear next step. However, Vaughan-Lee's procedure for filtering out polynomials becomes increasingly nuanced and time-consuming. Therefore, unless another significant reduction is made to the 82 polynomials that require further scrutiny, this process will take a considerable amount of time and effort.

(2) Another next step is to further investigate the automorphism group of the Kaplansky algebra. While we did group together graphs that shared the same characteristic polynomial, we did not have time to extensively analyze the similarities between graphs in the same group. Determining why these graphs share the same characteristic polynomial should give good insight into the automorphism group of the Kaplansky algebra.

(3) Finally, we did not attempt to prove the conjecture in Section 3. Proving the relationship may or may not be difficult, but it would be fun to find out why this pattern emerges.

References

[1] Vaughan-Lee, Michael (2006). Simple Lie Algebras of Low Dimension Over GF(2). *LMS J. Comput. Math.*, 9, 174-192.

[2] Kaplansky, Irving (1982). Some Simple Lie Algebras of Characteristic 2. In: Lie Algebras and Related Topics (New Brunswick, NJ, 1981). *Lecture Notes in Math*, 933, 127-129.

6 Appendix

6.1 List of possible characteristic polynomials that satisfy criteria (C1)-(C4)

Below is the list of the 85 possible characteristic polynomials of a 10-dimensional simple Lie algebra over GF(2) that satisfy criteria (C1)-(C4).

$$\chi_1 = x^{10} + x^2$$
$$\chi_2 = x^{10} + x^3 + x^2 + x$$
$$\chi_3 = x^{10} + x^4$$
$$\chi_4 = x^{10} + x^4 + x^2$$
$$\chi_5 = x^{10} + x^4 + x^3 + x$$
$$\chi_6 = x^{10} + x^5 + x^2 + x$$
$$\chi_7 = x^{10} + x^5 + x^3 + x^2$$
$$\chi_8 = x^{10} + x^5 + x^4 + x$$
$$\chi_9 = x^{10} + x^5 + x^4 + x^3$$
$$\chi_{10} = x^{10} + x^5 + x^4 + x^3 + x$$
$$\chi_{11} = x^{10} + x^6$$
$$\chi_{12} = x^{10} + x^6 + x^2$$
$$\chi_{13} = x^{10} + x^6 + x^3 + x$$
$$\chi_{14} = x^{10} + x^6 + x^4$$
$$\chi_{15} = x^{10} + x^6 + x^4 + x^2$$
$$\chi_{16} = x^{10} + x^6 + x^4 + x^3 + x^2 + x$$
$$\chi_{17} = x^{10} + x^6 + x^5 + x$$
$$\chi_{18} = x^{10} + x^6 + x^5 + x^3$$
$$\chi_{19} = x^{10} + x^6 + x^5 + x^3 + x^2 + x$$
$$\chi_{20} = x^{10} + x^6 + x^5 + x^4 + x^2 + x$$
$$\chi_{21} = x^{10} + x^6 + x^5 + x^4 + x^3 + x^2$$
$$\chi_{22} = x^{10} + x^7 + x^2 + x$$
$$\chi_{23} = x^{10} + x^7 + x^3 + x^2$$
$$\chi_{24} = x^{10} + x^7 + x^4 + x$$
$$\chi_{25} = x^{10} + x^7 + x^4 + x^3$$
$$\chi_{26} = x^{10} + x^7 + x^5 + x^2$$
$$\chi_{27} = x^{10} + x^7 + x^5 + x^3 + x^2 + x$$
$$\chi_{28} = x^{10} + x^7 + x^5 + x^4$$
$$\chi_{29} = x^{10} + x^7 + x^5 + x^4 + x^3$$
$$\chi_{30} = x^{10} + x^7 + x^5 + x^4 + x^3 + x$$
$$\chi_{31} = x^{10} + x^7 + x^6 + x$$
$$\chi_{32} = x^{10} + x^7 + x^6 + x^2 + x$$
$$\chi_{33} = x^{10} + x^7 + x^6 + x^3$$
$$\chi_{34} = x^{10} + x^7 + x^6 + x^4 + x^2 + x$$
$$\chi_{35} = x^{10} + x^7 + x^6 + x^4 + x^3 + x$$
$$\chi_{36} = x^{10} + x^7 + x^6 + x^4 + x^3 + x^2$$

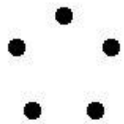
$\chi_{37} = x^{10} + x^7 + x^6 + x^5$
$\chi_{38} = x^{10} + x^7 + x^6 + x^5 + x^3 + x$
$\chi_{39} = x^{10} + x^7 + x^6 + x^5 + x^3 + x^2$
$\chi_{40} = x^{10} + x^7 + x^6 + x^5 + x^4 + x^2$
$\chi_{41} = x^{10} + x^7 + x^6 + x^5 + x^4 + x^2 + x$
$\chi_{42} = x^{10} + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x$
$\chi_{43} = x^{10} + x^8$
$\chi_{44} = x^{10} + x^8 + x^2$
$\chi_{45} = x^{10} + x^8 + x^3 + x$
$\chi_{46} = x^{10} + x^8 + x^4$
$\chi_{47} = x^{10} + x^8 + x^4 + x^2$
$\chi_{48} = x^{10} + x^8 + x^4 + x^3 + x^2 + x$
$\chi_{49} = x^{10} + x^8 + x^5 + x$
$\chi_{50} = x^{10} + x^8 + x^5 + x^3$
$\chi_{51} = x^{10} + x^8 + x^5 + x^4 + x^2 + x$
$\chi_{52} = x^{10} + x^8 + x^5 + x^4 + x^3 + x^2$
$\chi_{53} = x^{10} + x^8 + x^5 + x^4 + x^3 + x^2 + x$
$\chi_{54} = x^{10} + x^8 + x^6$
$\chi_{55} = x^{10} + x^8 + x^6 + x^2$
$\chi_{56} = x^{10} + x^8 + x^6 + x^3 + x^2 + x$
$\chi_{57} = x^{10} + x^8 + x^6 + x^4$
$\chi_{58} = x^{10} + x^8 + x^6 + x^4 + x^2$
$\chi_{59} = x^{10} + x^8 + x^6 + x^4 + x^3 + x$
$\chi_{60} = x^{10} + x^8 + x^6 + x^5 + x^2 + x$
$\chi_{61} = x^{10} + x^8 + x^6 + x^5 + x^3 + x$
$\chi_{62} = x^{10} + x^8 + x^6 + x^5 + x^3 + x^2$
$\chi_{63} = x^{10} + x^8 + x^6 + x^5 + x^4 + x$
$\chi_{64} = x^{10} + x^8 + x^6 + x^5 + x^4 + x^3$
$\chi_{65} = x^{10} + x^8 + x^7 + x$
$\chi_{66} = x^{10} + x^8 + x^7 + x^3$
$\chi_{67} = x^{10} + x^8 + x^7 + x^4 + x^2 + x$
$\chi_{68} = x^{10} + x^8 + x^7 + x^4 + x^3 + x^2$
$\chi_{69} = x^{10} + x^8 + x^7 + x^5$
$\chi_{70} = x^{10} + x^8 + x^7 + x^5 + x^3 + x$
$\chi_{71} = x^{10} + x^8 + x^7 + x^5 + x^4 + x$
$\chi_{72} = x^{10} + x^8 + x^7 + x^5 + x^4 + x^2$
$\chi_{73} = x^{10} + x^8 + x^7 + x^5 + x^4 + x^3 + x^2$
$\chi_{74} = x^{10} + x^8 + x^7 + x^5 + x^4 + x^3 + x^2 + x$
$\chi_{75} = x^{10} + x^8 + x^7 + x^6 + x$
$\chi_{76} = x^{10} + x^8 + x^7 + x^6 + x^2 + x$

$$\chi_{77} = x^{10} + x^8 + x^7 + x^6 + x^3 + x^2$$
$$\chi_{78} = x^{10} + x^8 + x^7 + x^6 + x^3 + x^2 + x$$
$$\chi_{79} = x^{10} + x^8 + x^7 + x^6 + x^4 + x$$
$$\chi_{80} = x^{10} + x^8 + x^7 + x^6 + x^4 + x^3$$
$$\chi_{81} = x^{10} + x^8 + x^7 + x^6 + x^5 + x^2$$
$$\chi_{82} = x^{10} + x^8 + x^7 + x^6 + x^5 + x^3$$
$$\chi_{83} = x^{10} + x^8 + x^7 + x^6 + x^5 + x^3 + x^2 + x$$
$$\chi_{84} = x^{10} + x^8 + x^7 + x^6 + x^5 + x^4$$
$$\chi_{85} = x^{10} + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x$$

6.2 List of the 34 subgraphs of the complete graph on five vertices

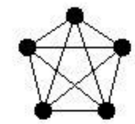
$5K_1 = \overline{K_5}$

1)



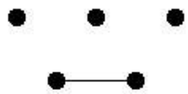
K_5

2)



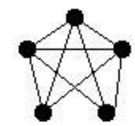
$\overline{K_5 - e} = 5K_1 + e = K_2 \cup 3K_1$

3)



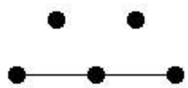
$K_5 - e$

4)



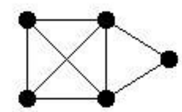
$P_3 \cup 2K_1$

5)



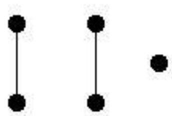
$\overline{P_3 \cup 2K_1}$

6)



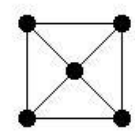
$\overline{W_4}$

7)



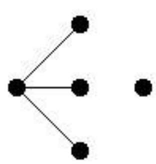
W_4

8)



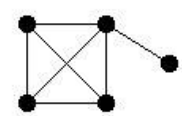
claw $\cup K_1$

9)

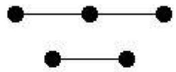


$\overline{\text{claw} \cup K_1}$

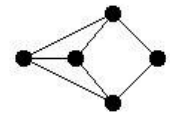
10)



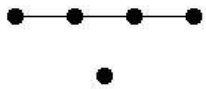
$P_2 \cup P_3$

11) 

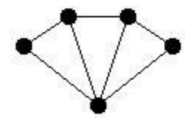
$\overline{P_2 \cup P_3}$

12) 

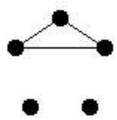
co-gem

13) 

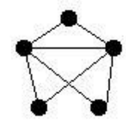
gem = 3-fan

14) 

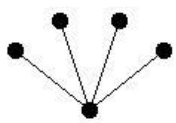
$K_3 \cup 2K_1$

15) 

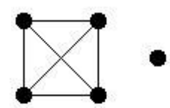
$\overline{K_3 \cup 2K_1}$

16) 

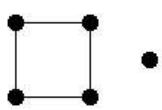
$K_{1,4}$

17) 

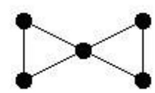
$\overline{K_{1,4}} = K_4 \cup K_1$

18) 

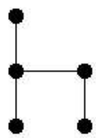
co-butterfly = $C_4 \cup K_1$

19) 

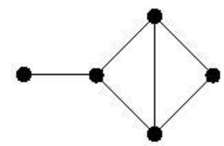
butterfly = hourglass

20) 

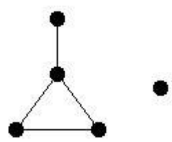
fork = chair

21) 

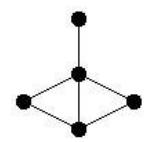
co-fork = kite = co-chair = $\overline{\text{chair}}$

22) 

co-dart

23) 

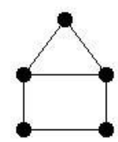
dart

24) 

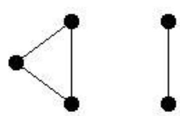
P_5

25) 

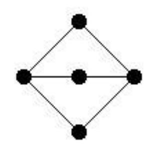
house = $\overline{P_5}$

26) 

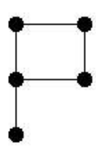
$K_2 \cup K_3 = \overline{K_{2,3}}$

27) 

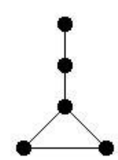
$K_{2,3}$

28) 

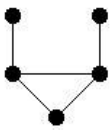
P = 4-pan = banner

29) 

$\overline{P}$

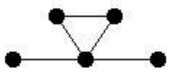
30) 

bull g6: D{O

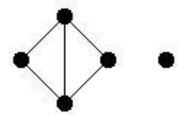
31) 

Self complementary

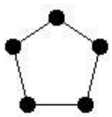
cricket = $K_{1,4}$+e g6: DIS

32) 

co-cricket = diamond ∪ K_1 g6:

33) 

C_5 g6: Dhc

34) 

Self complementary