Fredholm Integral Equations & State Estimation

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Definitions

(1) Integral Equation (IE): an *integral equation* is an equation where an unknown function is in the integrand.

The most general form is

$$\int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)dt - \lambda u(x) = f(x),$$

where the kernel $k(x,t)$, the function $f(x)$, and the constant parameter λ are known, and $u(x)$ is the unknown you're solving for.

Definitions

(2) First Kind and Second Kind IEs: an integral equation is said to be of the *first kind* if $\lambda = 0$, i.e.

$$\int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)dt - \cancelto{0}{\lambda u(x)} = f(x).$$

If $\lambda \neq 0$, then the integral equation is said to be of the *second kind*.

Definitions

(3) Fredholm and Volterra IEs: An integral equation is said to be a *Fredholm integral equation* (FIE) if the bounds of integration are constant, i.e.

$$\int_a^b k(x,t)u(t)dt - \lambda u(x) = f(x).$$

If the bounds of integration are variable, i.e. $\alpha(x)$ and $\beta(x)$, then the integral equation is said to be a *Volterra integral equation*.

What we'll use

For this research, we are concerned with linear FIEs of the first kind with a multivariate kernel $k(\vec{x}, \gamma)$, i.e.

$$\int_a^b k(\vec{x}, \gamma)\psi(\gamma)d\gamma = f(\vec{x})$$

Motivation

The BPRL Static State Estimation Method® (BPRLSSEM):

1. Get measured data from a system of distributed sensors (high-dimensional, messy)
2. Choose a set of basis functions
3. Project sensory data onto each basis function using the inner product map (yields a scalar-valued parameter)

Problem Statement

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Solutions:

- Bio-inspired basis functions (Sean's paper)

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- Experimentally generated basis functions (Mantas’ research)

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- Bio-inspired basis functions (Sean's paper)
- Fourier basis functions (Mike's research)
- Experimentally generated basis functions (Mantas' research)
- FIE generated basis functions??? (my research)

Case Study (Revisited)

Reconsider the inner product projection equation:

$$\langle \dot{Q}, F_i \rangle = \frac{1}{\pi} \int_0^{2\pi} \dot{Q}(\vec{x}, \gamma) \cdot F_i(\gamma) d\gamma$$

Case Study (Revisited)

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The FIE Solution

Given:

(i) $\vec{x} = [x_1, x_2, \dots, x_n]$, n-dimensional state vector

(ii) $k(\vec{x}, \gamma)$, measured sensory data (kernel)

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Method: For each $i \in \{1, 2, \dots, n\}$, prescribe the inner product of the i^{th} projection equation to be the i^{th} state variable, $\langle k, \psi_i \rangle \equiv x_i$, and solve for ψ_i.

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$$\langle k, \psi_i \rangle = \int_a^b k(\vec{x}, \gamma)\psi_i(\gamma)d\gamma \rightarrow x_i = \int_a^b k(\vec{x}, \gamma)\psi_i(\gamma)d\gamma$$

Solving This FIE

Now, we only need to solve

$$x_i = \int_a^b k(\vec{x},\gamma)\psi_i(\gamma)d\gamma$$

for $\psi_i(\gamma)$.

But this can be challenging, i.e.

$$\dot{Q}(\vec{x},\gamma) = -\dot{\psi} + \left(\frac{\dot{x}_b}{a-y}\right)\sin^2\gamma\cos\psi + \left(\frac{\dot{x}_b-\dot{y}_b}{a-y}\right)\sin\gamma\cos\gamma\cos\psi - \left(\frac{\dot{x}_b}{a-y}\right)\cos^2\gamma\sin\psi$$

Degenerate Kernel Method

Def: A *degenerate (separable) kernel* is a kernel $k(x, t)$ that can be represented as

$$k(x,t) = \sum_{j=1}^{n} \alpha_j(x)\beta_j(t), \qquad n < \infty \; (n = \# \; of \; terms)$$

Method (overview):

(i) Manipulate the FIE into an algebraic equation

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\vec{c} = \vec{g}$$

(ii) Finding a solution for $\vec{c}$ yields a solution for the original FIE

Degenerate Kernel Method

Notes:

- Must be a FIE of the second kind
- Thus, we must "regularize" our FIE of the first kind
  - Regularization: ill-posed first kind FIE → well-posed second kind FIE via a regularization parameter μ
- Doesn't work for multivariate FIEs that don't have "relatively simple domains of integration"
- Reference: "Montana State Notes", and others

Operator Theoretic Method

Integral operator representation:

Given a FIE

$$g(x) = \int_a^b k(x,t)f(t)dt, \qquad c \le t \le d$$

we can represent this equation as a *Fredholm integral operator equation*

$$g(x) = (Kf)(x) = \int_a^b k(x,t)f(t)dt\,,$$

where $K: L^2[a,b] \to L^2[c,d]$.

Note: In general, $K: H_1 \to H_2$.

Singular Value Expansion

Necessary condition: $K: H_1 \to H_2$ must be compact.

Theorem: If the kernel of K is square integrable, i.e. $k(x,t) \in L^2[a,b] \times L^2[c,d]$, then K is compact.

Singular Value Expansion

Singular Value Expansion: Given a compact operator $K: H_1 \to H_2$ defined as

$$g(x) = (Kf)(x) = \int_a^b k(x,t)f(t)dt,$$

then there exists a *singular system* $\{v_j, u_j; \mu_j\}_{j=1}^{\infty}$ such that

$$Kf = \sum_{j=1}^{\infty} \mu_j \langle f, u_j \rangle v_j\,,$$

and Moore-Penrose pseudo-inverse

$$f = K^{\backslash\text{dagger}} g = \sum_{j=1}^{\infty} \frac{\langle g, u_j \rangle}{\mu_j} v_j$$

Method of Moments

Hansen's paper

Future Research

1. Find best solution method for the FIEs/operators
2. Investigate regularization (Tikhonov)
3. Training to get new “best” basis functions for different environments

Preguntas?