General Proofs

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Contents

1 Even and Odd Proofs

1.1 Lemma: $\forall n \in \mathbb{N}$, $\exists k \in \mathbb{N} : n = 2k$ or $n = 2k + 1$

Proof by induction over n:

case $n = 0$:	case $n = 1$:
$0 = 2 \cdot 0$	$1 = 0 + 1$
	$1 = 2 \cdot 0 + 1$
Let $k = 0$	Let $k = 0$
$0 = 2k$	$1 = 2k + 1$

Assume $n - 1 = 2k$, or $n - 1 = 2k + 1$

case $n - 1 = 2k$:	case $n - 1 = 2k + 1$:
$n = 2k + 1$	$n = 2k + 2$
	$n = 2(k + 1)$
	Let $k' = k + 1$
	$n = 2k'$

1.2 Theorem: o is odd $\iff \exists n \in \mathbb{N} : o = 2n + 1$, and e is even $\iff \exists n \in \mathbb{N} : e = 2n$

$$
\begin{aligned}
&\text{Suppose } e \text{ is even.}\\
&\iff e \text{ is divisible by } 2\\
&\iff \frac{e}{2} = n \text{ for some } n \in \mathbb{N}\\
&\iff e = 2n
\end{aligned}
$$

$$
\begin{aligned}
&\text{Suppose } o \text{ is odd.}\\
&\iff o \text{ is not divisible by } 2\\
&\iff o \text{ is not even.}\\
&\iff o \neq 2n\\
&\iff o = 2n + 1 \text{ by lemma 1.1}
\end{aligned}
$$

1.3 Theorem: An odd number plus an even number equals an odd number, and an odd number plus an odd number equals an even number

$$
\begin{aligned}
&\text{Consider an odd number plus an even number.}\\
&\iff (2n_1 + 1) + (2n_2) \text{ for some } n_1, n_2 \in \mathbb{N}\\
&= 2n_1 + 2n_2 + 1\\
&= 2(n_1 + n_2) + 1\\
\text{Let } n_3 &= n_1 + n_2\\
&= 2n_3 + 1\\
&\text{which is odd by lemma 2.}\\
&\therefore \text{ an odd plus an even equals an odd.}
\end{aligned}
$$

Now consider an odd number plus an odd number.

$$\begin{aligned}
&\iff (2n_1+1)+(2n_2+1) \text{ for some } n_1, n_2 \in \mathbb{N} \\
&= 2n_1 + 2n_2 + 1 + 1 \\
&= 2n_1 + 2n_2 + 2 \\
&= 2(n_1 + n_2 + 1) \\
\text{Let } n_3 &= n_1 + n_2 + 1 \\
&= 2n_3
\end{aligned}$$

which is even by theorem 1.2.

$\therefore$ an odd plus an odd equals an even.

2

2.1 $\sum\limits_{n=1}^{N} n = \frac{N(N+1)}{2}$

Proof by induction:
Case $N = 1$:

$$\begin{aligned}
\sum_{n=1}^{N} n &= \sum_{n=1}^{1} n \\
&= 1 \\
&= \frac{2}{2} \\
&= \frac{1 \cdot (1+1)}{2} \\
&= \frac{N(N+1)}{2}
\end{aligned}$$

Assume $\sum\limits_{n=1}^{N} n = \frac{N(N+1)}{2}$:

$$\begin{aligned}
\sum_{n=1}^{N} n &= \frac{N(N+1)}{2} \\
\left(\sum_{n=1}^{N} n\right) + (N+1) &= \frac{N(N+1)}{2} + (N+1) \\
\sum_{n=1}^{N+1} n &= \frac{N(N+1)}{2} + \frac{2(N+1)}{2} \\
\sum_{n=1}^{N+1} n &= \frac{N(N+1) + 2(N+1)}{2} \\
\sum_{n=1}^{N+1} n &= \frac{(N+1)\cdot(N+2)}{2} \\
\sum_{n=1}^{N+1} n &= \frac{(N+1)\cdot((N+1)+1)}{2}
\end{aligned}$$

Therefore by induction, $\forall N \in \mathbb{N}, \sum\limits_{n=1}^{N} n = \frac{N(N+1)}{2}$