# General Proofs

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# 1 Even and Odd Proofs

## 1.1 Lemma: $\forall n \in \mathbb{N}$, $\exists k \in \mathbb{N} : n = 2k$ or $n = 2k + 1$

Proof by induction over $n$:

$$\begin{array}{c|c}
\underline{\text{case } n = 0\text{:}} & \underline{\text{case } n = 1\text{:}} \\
0 = 2 \cdot 0 & 1 = 0 + 1 \\
 & 1 = 2 \cdot 0 + 1 \\
\text{Let } k = 0 & \text{Let } k = 0 \\
0 = 2k & 1 = 2k + 1
\end{array}$$

Assume $n - 1 = 2k$, or $n - 1 = 2k + 1$

$$\begin{array}{c|c}
\underline{\text{case } n - 1 = 2k\text{:}} & \underline{\text{case } n - 1 = 2k + 1\text{:}} \\
n = 2k + 1 & n = 2k + 2 \\
 & n = 2(k + 1) \\
 & \text{Let } k' = k + 1 \\
 & n = 2k'
\end{array}$$

## 1.2 Theorem: $o$ is odd $\iff \exists n \in \mathbb{N} : o = 2n + 1$, and $e$ is even $\iff \exists n \in \mathbb{N} : e = 2n$

$$\begin{aligned}
&\text{Suppose } e \text{ is even.} \\
&\iff e \text{ is divisible by } 2 \\
&\iff \frac{e}{2} = n \text{ for some } n \in \mathbb{N} \\
&\iff e = 2n
\end{aligned}$$

$$\begin{aligned}
&\text{Suppose } o \text{ is odd.} \\
&\iff o \text{ is not divisible by } 2 \\
&\iff o \text{ is not even.} \\
&\iff o \neq 2n \\
&\iff o = 2n + 1 \text{ by lemma 1.1}
\end{aligned}$$

## 1.3 Theorem: An odd number plus an even number equals an odd number, and an odd number plus an odd number equals an even number

$$\begin{aligned}
&\text{Consider an odd number plus an even number.} \\
&\iff (2n_1 + 1) + (2n_2) \text{ for some } n_1, n_2 \in \mathbb{N} \\
&= 2n_1 + 2n_2 + 1 \\
&= 2(n_1 + n_2) + 1 \\
\text{Let } n_3 &= n_1 + n_2 \\
&= 2n_3 + 1 \\
&\text{which is odd by lemma 2.} \\
\therefore\ &\text{an odd plus an even equals an odd.}
\end{aligned}$$

Now consider an odd number plus an odd number.

$$
\begin{aligned}
&\iff (2n_1 + 1) + (2n_2 + 1) \text{ for some } n_1, n_2 \in \mathbb{N} \\
&= 2n_1 + 2n_2 + 1 + 1 \\
&= 2n_1 + 2n_2 + 2 \\
&= 2(n_1 + n_2 + 1) \\
\text{Let } n_3 &= n_1 + n_2 + 1 \\
&= 2n_3 \\
&\text{which is even by theorem 1.2.}
\end{aligned}
$$

$\therefore$ an odd plus an odd equals an even.