Project Euler, Problem 144:
Investigating Multiple Reflections of a Laser Beam

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2.1 The ellipse $4x^2 + y^2 = 100$ can be rewritten as $y = \pm\sqrt{100 - 4x^2}$ 2

2.2 Given some line, $y = mx + b$ that intersects the ellipse $4x^2 + y^2 = 100$, then the x coordinates of the intersection points are at

$$x_{+,-} = \frac{-2bm \pm \sqrt{(2bm)^2 - 4(4 + m^2)(b^2 - 100)}}{2(4 + m^2)}$$

meaning the points lie at $(x_+, mx_+ + b)$ and $(x_-, mx_- + b)$ 2

2.3 Lemma: if m is the slope of a line, and θ is the angle between that line and the x-axis, then

$$\theta = \arctan(m)$$

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2.4 If m_{l_1}, m_{l_2}, m_t are the slopes of the incoming line, reflected line, and tangent line, respectively, then

$$m_{l_2} = \tan\left(2 \times \arctan(m_t) - \arctan(m_{l_1})\right)$$

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1 Problem Description

2 Theorems whose results aid in my solution

2.1 The ellipse $4x^2 + y^2 = 100$ can be rewritten as $y = \pm\sqrt{100 - 4x^2}$

$$
\begin{aligned}
4x^2 + y^2 &= 100 \\
y^2 &= 100 - 4x^2 \\
y &= \pm\sqrt{100 - 4x^2}
\end{aligned}
$$

2.2 Given some line, $y = mx + b$ that intersects the ellipse $4x^2 + y^2 = 100$, then the x coordinates of the intersection points are at

$$x_{+,-} = \frac{-2bm \pm \sqrt{(2bm)^2 - 4(4 + m^2)(b^2 - 100)}}{2(4 + m^2)}$$

meaning the points lie at $(x_+, mx_+ + b)$ and $(x_-, mx_- + b)$

$$4x^2 + y^2 = 100 \iff y = \pm\sqrt{100 - 4x^2}$$

The intersections happens when the line and ellipse have the same x and y coordinates.

$$
\begin{aligned}
\therefore mx + b = y &= \pm\sqrt{100 - 4x^2} \\
mx + b &= \pm\sqrt{100 - 4x^2} \\
(mx + b)^2 &= (\pm\sqrt{100 - 4x^2})^2 \\
m^2x^2 + 2bmx + b^2 &= 100 - 4x^2 \\
(m^2 + 4)x^2 + (2bm)x + (b^2 - 100) &= 0
\end{aligned}
$$

We can solve for x using the quadradic formula

$$x_{+,-} = \frac{-2bm \pm \sqrt{(2bm)^2 - 4(4 + m^2)(b^2 - 100)}}{2(4 + m^2)}$$

The intersection point are on the line, so if we plug in each x, we get:

$$y_+ = mx_+ + b$$

$$\text{and } y_- = mx_- + b$$

So the intersection points are $(x_+, mx_+ + b)$ and $(x_-, mx_- + b)$

2.3 Lemma: if m is the slope of a line, and θ is the angle between that line and the x-axis, then

$$\tan\theta = m$$

2.4 If m_{l_1}, m_{l_2}, m_t are the slopes of the incoming line, reflected line, and tangent line, respectively, then

$$m_{l_2} = \tan\left(2 \times \arctan(m_t) - \arctan(m_{l_1})\right)$$

3 Application to Code

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