Project Euler, Problem 144:
Investigating Multiple Reflections of a Laser Beam

John Butler

Contents

1 Problem Description

Consider an ellipse with the equation $4x^2 + y^2 = 100$ with the section $x \in [-0.01, 0.01]$ missing from the top. A straight lightbeam is shined in at $(0, 10.1)$ and hits the ellipse at $(1.4, -9.6)$. The light then reflects off the inside until it exits through the 0.02 wide opening it entered from.
It's given that the slope of the tangent of the ellipse at any point is $\frac{-4x}{y}$ and that the angle of incidence equals the angle of reflection, that is the angle between the incoming beam and the tangent line equals the angle between the outgoing beam and the tangent.

2 Theorems whose results aid in my solution

2.1 The ellipse $4x^2 + y^2 = 100$ can be rewritten as $y = \pm\sqrt{100 - 4x^2}$

$$\begin{aligned} 4x^2 + y^2 &= 100 \\ y^2 &= 100 - 4x^2 \\ y &= \pm\sqrt{100 - 4x^2} \end{aligned}$$

2.2 Given some line, $y = mx + b$ that intersects the ellipse $4x^2 + y^2 = 100$, then the x coordinates of the intersection points are at

$$x_{+,-} = \frac{-2bm \pm \sqrt{(2bm)^2 - 4(4 + m^2)(b^2 - 100)}}{2(4 + m^2)}$$

meaning the points lie at $(x_+, mx_+ + b)$ and $(x_-, mx_- + b)$

$$4x^2 + y^2 = 100 \iff y = \pm\sqrt{100 - 4x^2}$$

The intersections happens when the line and ellipse have the same x and y coordinates.

$$\begin{aligned} \therefore mx + b = y &= \pm\sqrt{100 - 4x^2} \\ mx + b &= \pm\sqrt{100 - 4x^2} \\ (mx + b)^2 &= (\pm\sqrt{100 - 4x^2})^2 \\ m^2x^2 + 2bmx + b^2 &= 100 - 4x^2 \\ (m^2 + 4)x^2 + (2bm)x + (b^2 - 100) &= 0 \end{aligned}$$

We can solve for x using the quadradic formula

$$x_{+,-} = \frac{-2bm \pm \sqrt{(2bm)^2 - 4(4 + m^2)(b^2 - 100)}}{2(4 + m^2)}$$

Truthfully, this can be simplified, but I believe this is simpler to compute.

The intersection point must be on the line, so if we plug in each x, we get:

$$y_+ = mx_+ + b$$

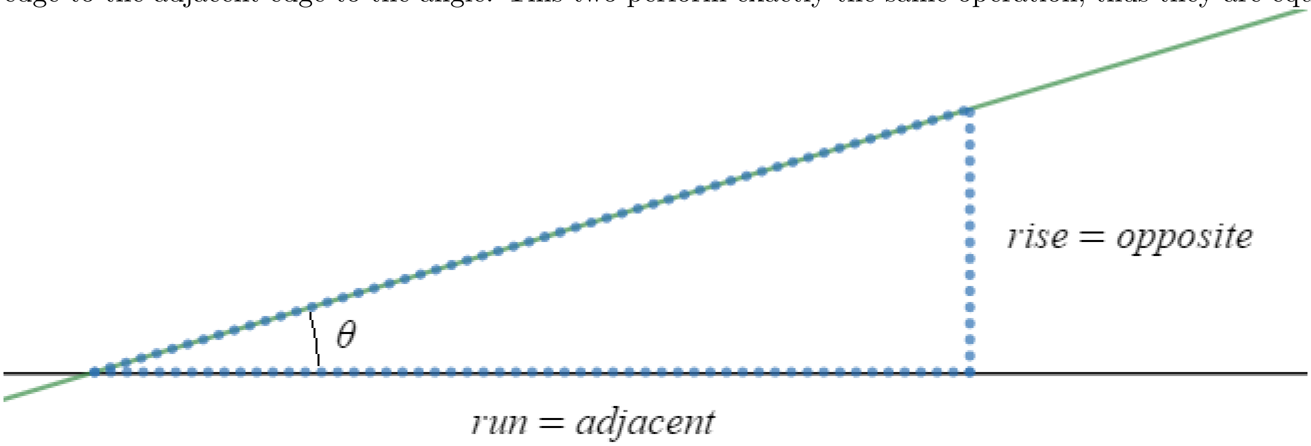
$$\text{and } y_- = mx_- + b$$

So the intersection points are $(x_+, mx_+ + b)$ and $(x_-, mx_- + b)$

2.3 Lemma: if m is the slope of a line, and θ is the angle between that line and the x-axis, then

$$\tan\theta = m$$

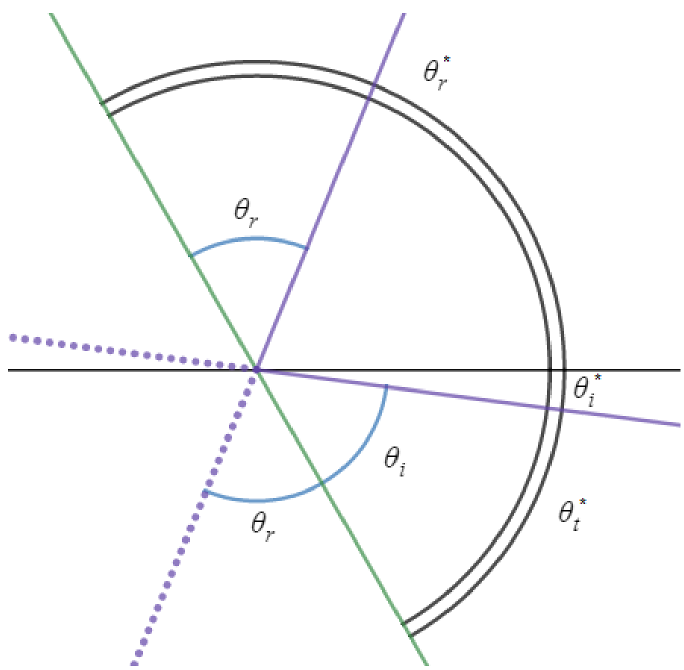
Consider the figure below. I started by taking an arbitrary line, then I drew a vertical line from which I constructed a right triangle. To calculate the slope of the line, I divided the "rise" and the "run" of the line over this triangle. The rise measures the change in the y-coordinate, and since the opposite edge of the triangle is exactly vertical, the length of that line is the rise. Similarly, the bottom edge of the triangle is exactly horizontal, so its length is the run. The slope is the ratio of those to edges. Now let's consider what the tangent of the angle between the line and the x-axis is. Well it's the ratio of the lengths of the opposite edge to the adjacent edge to the angle. This two perform exactly the same operation, thus they are equal.



2.4 If m_i, m_r, m_t are the slopes of the incoming line, reflected line, and tangent line, respectively, then

$$m_r = \tan\left(2\times\arctan(m_t) - \arctan(m_i)\right)$$

Consider θ_i and θ_r, the angle of incidence and the angle of reflection. These angles are defined as the angle between the tangent line and either the line of incidence and reflection. Therefore, in an absolute sense, $\theta_i = |\theta_t^* - \theta_i^*|$ and $\theta_r = |\theta_t^* - \theta_r^*|$, where θ^* represents the angle between a line and the x-axis. Applying lemma 2.3, we get $\begin{cases}\theta_i = |\arctan m_i - \arctan m_t| \\ \theta_r = |\arctan m_r - \arctan m_t|\end{cases}$. Since both lines are on the same side of the tangent line, one angle will be positive and one will be negative, so we get $\begin{cases}\theta_i = \arctan m_i - \arctan m_t \\ \theta_r = \arctan m_t - \arctan m_r\end{cases}$ or $\begin{cases}\theta_i = \arctan m_t - \arctan m_i \\ \theta_r = \arctan m_r - \arctan m_t\end{cases}$. If we remember that $\theta_i = \theta_r$, in either case, we get



$$\begin{aligned}(\arctan m_i - \arctan m_t) &= -(\arctan m_r - \arctan m_t)\\ \arctan m_i - \arctan m_t &= \arctan m_t - \arctan m_r\\ \arctan m_r &= 2\times\arctan m_t - \arctan m_i\\ m_r &= \tan\left(2\times\arctan m_t - \arctan m_i\right)\end{aligned}$$

3 Application to Code

The core of this problem is being able to find the reflection of arbitrary lines at arbirary points on the ellipse. So how do we do that? Well there's certainly a few ways to go about it, but I did something along these lines

```java
abstract class P144 {

    private static Point intersectionPoint(Line line, Point previousIntersection);
    private static Line reflectLine(Line incidence, Point reflectionPoint);
    private boolean exits(Point intersectionPoint);

    public static void main(String[] args){
        /* * Initialize everything * */
        do{
            // 2.2
            intersectionPoint = insersectsEllipseAt(line, intersectionPoint);
            // 2.4
            line = reflectLine(line, intersectionPoint);
        } while(!exits(intersectionPoint));
    }
}
```

4 Complexity Analysis / Further Optimizations and Considerations