Structured Coordination in Continuous Spatial and Strategic Domains

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September 2nd, 2024

The Coordination Game

Coordination games are a class of games wherein two players receive a higher payoff when they use the same strategy than when they use different strategies.

	A	B
A	a,a	c,d
B	d,c	b,b

	A	B
A	1,1	0,0
B	0,0	1,1

Figure: **Left** A payoff matrix for the general coordination game whenever $a > d$ and $b > c$. **Right** The payoff matrix for the most simply coordination game

The Structured Coordination Game

Now suppose that the pairwise coordination game is played among many players with a explicit relational structure

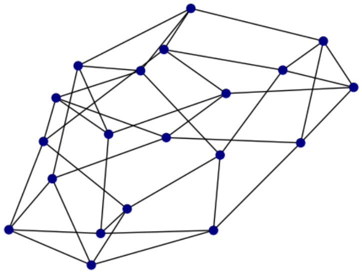

Figure: **Left** Trivial Nash equilibrium which every graph necessarily admits. **Right** Non-trivial Nash equilibrium which is not necessarily admitted by every graph

Discrete Results and Conjectures

Work in the discrete case was pioneered by economists

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Simulation has been used to understand more general structures

- see some of our previous work under revision for DGAA `doi.org/10.48550/arXiv.2406.19273`

Understanding the Discrete Case

In the discrete case, Let $G(V, E, W)$ be a graph with W a (possibly weighted) adjacency matrix. Let $B = \{e_1, e_2, ..., e_m\}$ be the standard basis for $\mathbb{R}^m$ representing each of the m pure strategies. $u : V \to B$ is a strategy profile and $U = [[u(x)]]_{x \in V} \in \mathbb{R}^{m \times n}$. If our payoff matrix is I_m, then

$$w(v|u) = \sum_{i \in V} W_{i,v} u(i) \cdot u(v) \tag{1}$$

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If we are using a general coordination Payoff matrix A then we can still write this as

$$w(v|u) = \sum_{i \in V} W_{i,v} u(i)^T A u(v) = e_v^T U^T A U W e_v \tag{2}$$

Continuous Extension in Space

Now let $\Omega \subset \mathbb{R}^n$. We can replace the sum over the neighbors of v with an integral against the an integrable kernel $K \in \mathcal{L}^1(\mathbb{R}^n)$. Let $u : \Omega \to B$ and we get

$$w(x|u) = \int_\Omega K(x-y)u(y)^T Au(x)dy \tag{3}$$

Continuous Extension in Strategy

Two concepts of continuous strategy space

1. Mixed Strategy Concept
   - Let Δ^{m-1} be the $m-1$ simplex. Let $u : \Omega \to \Delta^{m-1}$
   - $w(x|u) = \int_\Omega K(x-y)u(y)^T Au(x)dy$

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2. Comparable Strategy Concept
   - Replace payoff matrix A with a "recognition function" ρ.
   - Let $u : \Omega \to \mathbb{R}$.
   - $w(x|u) = \int_\Omega K(x-y)\rho(u(x)-u(y))dy$
   - Looking for Nash equilibria is challenging

Non-local Equation

Proposition 1

Under myopic best response, strategy profiles will evolve according to the equation

$$\frac{\partial}{\partial t}u(x,t) = \int_{\Omega} K(x-y)\rho'(u(x,t)-u(y,t))dy \qquad (4)$$

so long as the following hypotheses are met

H1) Players change their strategies in arbitrarily small time steps

H2) The cost of changing strategies increases quadratically with magnitude

Non-local Equation

Proof Outline

1. Argue that if a player changes their strategy by h in a time step Δt, there is an h^* which maximizes fitness at the end of that time step
2. Use properties of $w(x|u)$ to put bounds on h^* in terms of Δt and other constants
3. Take the limit as $\Delta t \to 0$ to achieve the desired result

Proof of proposition 1

Step 1

Each individual $x \in \Omega$, seeking to maximize their own payoff in a Δt time step, will change their strategy by h and achieve a payoff $S^{(x)}(h)$ and incur a cost of $\frac{h^2}{\Delta t}$ (H2).

$$S^{(x)}(h) := \int_{\Omega} K(x-y)\rho(u(x,t)+h-u(y,t))dy$$

This is a coordination game so ρ is bounded above and achieves its maximum at 0. Moreover, if $\rho \in C^{1,1}$, we know $S^{(x)} \in C^{1,1}$ and is bounded above. Thus $\exists h^* \in \mathbb{R}$ a global maximizer of $S^{(x)}(h) - \frac{h^2}{\Delta t}$. Moreover, h^* will satisfy $\frac{d}{dh}S^{(x)}(h^*) = \frac{2}{\Delta t}h^*$.

Proof of Proposition 1

Step 2

Let L be the Lipschitz constant for $\frac{d}{dh}S^{(x)}(h)$ In the case that $h^* > 0$ we have that $-Lh^* \leq \frac{d}{dh}S^{(x)}(h^*) - \frac{d}{dh}S^{(x)}(0) \leq Lh^*$ which implies that

$$\frac{d}{dh}S^{(x)}(0) - Lh^* - \frac{2}{\Delta t}h^* \leq 0 \leq \frac{d}{dh}S^{(x)}(0) + Lh^* - \frac{2}{\Delta t}h^*$$

which, when Δt is small enough that $2 - \Delta tL > 0$ (H1), implies

$$\frac{d}{dh}S^{(x)}(0)\frac{\Delta t}{2 + \Delta tL} \leq h^* \leq \frac{d}{dh}S^{(x)}(0)\frac{\Delta t}{2 - \Delta tL}$$

We can do a nearly identical computation when $h^* < 0$.

Proof of Proposition 1

Step 3

Recall that h^* is the amount of strategic change in a single time step so if we make this substitution and divide by Δt we see that

$$\frac{1}{2+\Delta t L}\frac{d}{dh}S^{(x)}(0) \leq \frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} \leq \frac{1}{2-\Delta t L}\frac{d}{dh}S^{(x)}(0)$$

Observe that $\frac{d}{dh}S^{(x)}(0) = \int_\Omega K(x-y)\rho'(u(x,t)-u(y,t))dy$ and take $\Delta t \to 0$ (H1) to see

$$\frac{\partial}{\partial t}u(x,t) = c\int_\Omega K(x-y)\rho'(u(x,t)-u(y,t))dy$$

We will rescale space-time to normalize the constant c to 1.

Non-Local Equation

We have just shown:

- In a continuous player domain and strategic domain, we can express the coordination game with the fitness function

$$w(x|u) = \int_{\Omega} K(x-y)\rho(u(x)-u(y))dy$$

- Under myopic best response, if players update their strategy in arbitrarily small time steps with quadratic cost, the strategic profile evolves according to

$$\frac{\partial}{\partial t}u(x,t) = \int_{\Omega} K(x-y)\rho'(u(x)-u(y))dy$$

Existence and Uniqueness

Existence and uniqueness is supplied by two key lemmas

$$g[u] = \int_{\Omega} K(x-y)\rho'(u(x)-u(y))dy$$

Lemma 1

$g[u]$ is well defined from $C_b^0(\overline{\Omega}_T, \mathbb{R})$ to $C_b^0(\overline{\Omega}_T, \mathbb{R})$

Lemma 2

$g[u]$ is Lipschitz continuous with respect to the sup norm in any compact subset of $C_b^0(\overline{\Omega}_T, \mathbb{R})$ with Lipschitz constant C^g.

Existence and Uniqueness

Theorem 1: The IVP $u_t = g[u] \in \Omega$ and $u(x,0) = u_0 \in C_b^0(\Omega, \mathbb{R})$ has a unique solution

$E_{R,T} := \{u \in C_b^0(\overline{\Omega_T}); u(x,0) = u_0, \|u\| \leq R\}$ for some $R > \|u_0\|$ and T to be chosen later.

$$\Theta u = u_0 + \int_0^t g[u](x,s)ds$$

.

Lemma 1 $\implies \exists T_1$ such that $\Theta : E_{R,T} \to E_{R,T}$ whenever $T < T_1$.
Lemma 2 $\implies \exists T_2$ such that, Θ is a contraction when $T < T_2$.
Contraction Mapping $\implies \exists! u^* \in E_{R,T}$ such that $\Theta u^* = u^*$. This is a solution to the IVP on the interval $[0, \min\{T_1, T_2\})$.

Maximum Principle

Lemma 3 Weak Maximum Principle

If u solves the IVP $u_t = g[u]$ with $u(x,0) = u_0 \in C_0^b(\Omega)$ and if $\rho(z)$ is decreasing in $|z|$ $(z \cdot \rho'(z) < 0)$ then

$$\|u(\cdot, t_2)\|_\infty \leq \|u(\cdot, t_1)\|_\infty$$

whenever $t_1 \leq t_2$.

Theorem 2 Global Existence and Uniqueness

If u_0 and ρ satisfy the conditions of Lemma 3, then the IVP $u_t = g[u]$ with $u(x,0) = u_0$ has a unique solution which is exists for all finite time.

Numerical Examples

Some times solutions evolve towards a consensus equilibrium

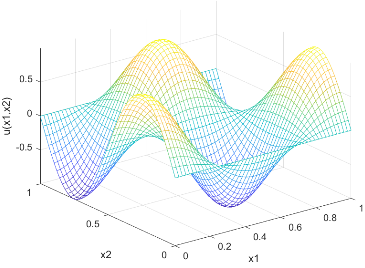
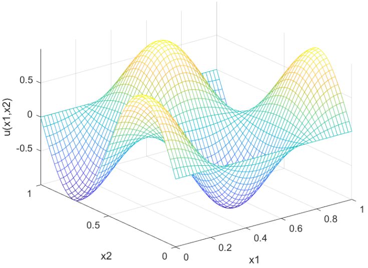


Figure: **Left** Continuous and bounded initial condition, u_0, on $\Omega = [0,1]^2$ **Right** Solution to $u_t = g[u]$ with $u(x,0) = u_0$ at time $t = 1000$. In this case $\rho' \neq 0$ on $\mathbb{R} \setminus \{0\}$

Some times solutions evolve towards non-consensus equilibria



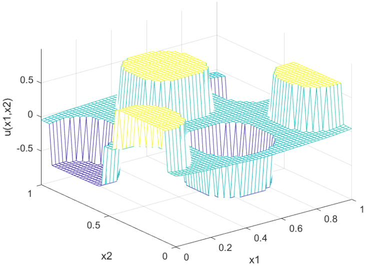


Figure: **Left** Continuous and bounded initial condition, u_0, on $\Omega = [0,1]^2$ **Right** Solution to $u_t = g[u]$ with $u(x,0) = u_0$ at time $t = 1000$.In this case $\rho' = 0$ on $\mathbb{R} \setminus B_{1/4}(0)$

Stationary Solutions

Lemma 5

If u satisfies $g[u] = 0$ in Ω and attains its maximum, and if $\rho' > 0$ supported on $[-2\|u\|_\infty, 2\|u\|_\infty]$ and K is supported on $B_{2diam\Omega}(0)$, then $u(\Omega)$ has measure 0.

Propositions 2

If u is a Nash equilibrium to the game with players Ω and payoff function $\int_\Omega K(x-y)\rho(u(x)-u(y))dy$ then u satisfies $g[u] = 0$.

Classifying Stationary solutions to of the form $0 = g[0]$ will allow us to narrow our search for Nash equilibria in the classical game.

What's next?

There are still many questions to be addressed.

- Prove that if $u_t = g[u]$, then $\lim_{t\to\infty} u(x,t)$ exists.
- Describe gradient thresholds which determine whether $\|Du(\cdot,t)\|_{L^\infty}$ increases or decreases with time.
- Describe a boundary value problem using a position dependent kernel.
- Consider inhomogeneous versions of this problem.

Thank you

Thank you to

- Dr. Nina Fefferman
- Dr. Tadele Mengesha
- All the members of the Fefferman Lab
- All of you

Contact me

Find my previous work here: feffermanlab.org/JohnMcA.html
Explore the numerical results here: github.com/feffermanlab/JSM_2024_ContinuousCoordination

Previous work

Code

Email me at jmcalis6@vols.utk.edu

Maximum Principle and its relationship to game theory

$\rho(z)$ decreases with $|z|$

- Coordination game implies ρ attains its maximum at 0
- assume mutual intelligibility depends on distance. As strategies get further away they provide smaller mutual benefit

Game Theoretical Interpretation

Innovation is never beneficial in a coordination game
When the only payoff comes from coordination, adopting a strategy outside the bounds of those already being used always results in a decrease in fitness.

Regularity and Numerical Results

Theorem 3 Regularity Estimates

If u solves the IVP $u_t = g[u]$ with $u_0 \in C_b^1(\Omega)$ with bounded derivative, then $u \in W^{1,\infty}(\Omega)$ (so long as Ω has a smooth boundary). Furthermore, There is a positive c such that

$$\|D_x u(\cdot, t)\|_{L^\infty(\Omega)} \leq e^{ct}\|Du_0\|_\infty \tag{5}$$

Theorem 4 Convergence of Forward Euler numerical scheme

The Forward Euler scheme approximates solution to the IVP

$$w(\mathbf{x}, t_{i+1}) = w(\mathbf{x}, t_i) + \tau \sum_{\mathbf{y} \in^- \Omega h} K(\mathbf{x} - \mathbf{y})\rho'(w(\mathbf{x}, t_i) - w(\mathbf{y}, t_i))h^n$$