A Structured Coordination Game With Neutral Options

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The Coordination Game

The coordination game with neutral options, is an incredibly simple game:

	A	B
A	1,1	0,0
B	0,0	1,1

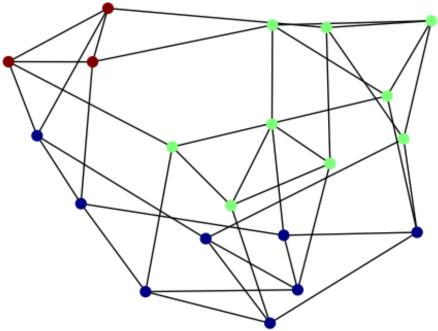
This is a critical case in the general n player coordination game and has been understudied by economists and ecologists[1][2]

[1]Kandori, Michihiro, et al. "Learning, Mutation, and Long Run Equilibria in Games."

[2]Ellison, Glenn. "Basins of attraction, long-run stochastic stability, and the speed of step-by-step evolution."

The Structured Coordination Game

In the case of neutral options there are far more Nash Equilibria



Partitioning

For small graphs, it is easy to catalogue the Nash equilibria. In order to do this we use vertex partitions.

Strategic Partitions

For a strategy profile $\mathbf{u} \in C^{|V|}$ the corresponding partition is $Q := \{q^c\}_{c \in C}$ where $q^c = \{v \in V | \, \mathbf{u}_v = c\}$

Equilibrium Partition

We define a partition to be an equilibrium partition if it corresponds to a strategy profile which is a Nash equilibrium.

Partitioning

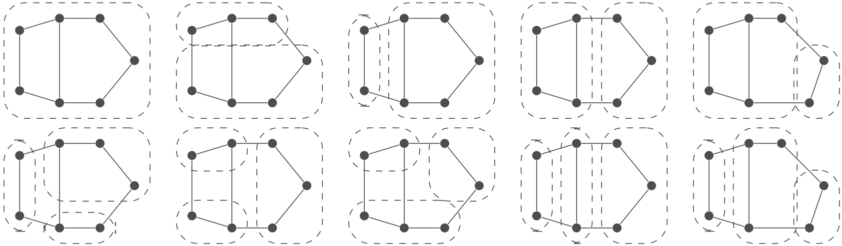


Figure: The ten distinct equilibrium partitions of the graph X_{38} which is graph #445 in the graph atlas.

Some small partitioning results

- For every graph, the trivial partition, $Q = \{V\}$ is an equilibrium partition.
- If $Q = \{q^c\}_{c \in C}$ is an equilibrium partition of a connected graph on at least 2 vertices, then $|q^c| \neq 1$ for all $c \in C$.
- If $Q = \{q^c\}_{c \in C}$ is an equilibrium partition and the subgraph spanned by q^i is disconnected, then, if $q^i_1, \ldots q^i_n$ are the connected components of q^i, $\tilde{Q} = \{q^c\}_{c \neq i} \cup \{q^i_j\}_{j=1}^n$ is also an equilibrium partition.

Methods

We seek to catalogue every equilibrium partition for very small graphs (≤ 7 vertices). We can use a brute force method because Checking if a partition is an equilibrium partition is an $\mathcal{O}(n^2)$ operation.

Challenges:

- Ensure that no partition is a refinement of another disconnected partition
- Ensure no two partitions are isomorphic to one another

Results

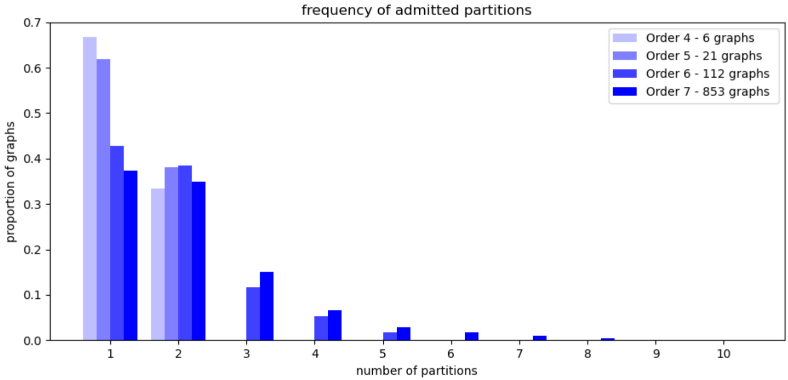


Figure: The frequency of graph which admit n distinct partitions for graphs of size 4, 5, 6, and 7.

Results

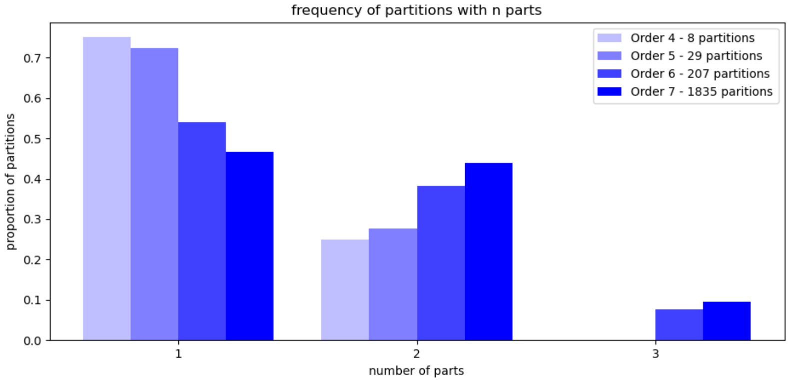


Figure: The frequency of partitions with n parts for graphs of size 4, 5, 6, and 7.

Simulation Methods

On larger graphs, It becomes impractical to check every partition so we take a different approach to finding equilibria

Initial Value Problem

Given a strategy profile $\mathbf{u}(t)$ compute the next strategy profile by computing best responses.

$$\mathbf{u}(t+1) = \left[\operatorname*{argmax}_{c \in C} |\{x \in \Gamma(v); \mathbf{u}_x(t) = c\}|\right]_{v \in V}$$

This gives a (non-unique) solution to the initial value problem resulting in an equilibrium or n-cycle[3].

[3] or the sequence does not converge to a limit

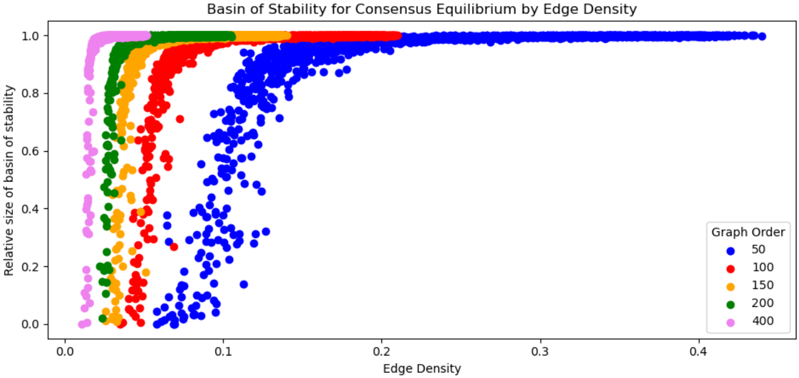
Simulation Methods

Each Nash equilibrium (and corresponding equilibrium partition) therefore has a (possibly overlapping) basin of stability. One crucial question is to understand the size of the basin of stability corresponding to the trivial equilibrium partition.

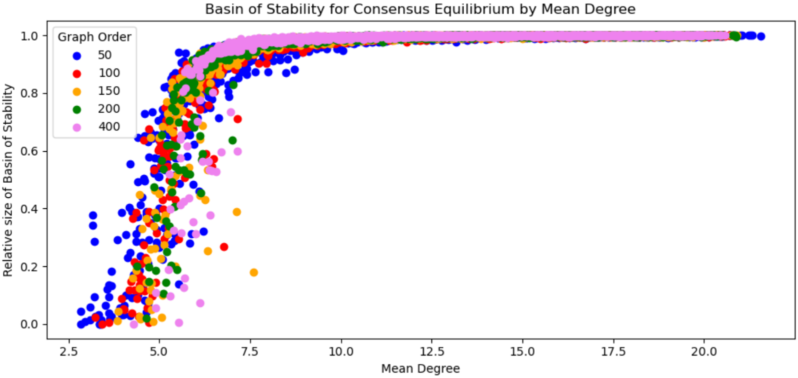
Simulation Design

For each Erdős-Rényi random graph, solve the IVP for 500 random initial conditions and record how many of them result in a trivial partition.

Results


Basin of Stability for Consensus Equilibrium by Edge Density
Relative size of basin of stability
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.1
0.2
0.3
0.4
Edge Density
Graph Order
50
100
150
200
400

Results


Basin of Stability for Consensus Equilibrium by Mean Degree
Graph Order
50
100
150
200
400
Relative size of Basin of Stability
1.0
0.8
0.6
0.4
0.2
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Mean Degree

Simulation Methods

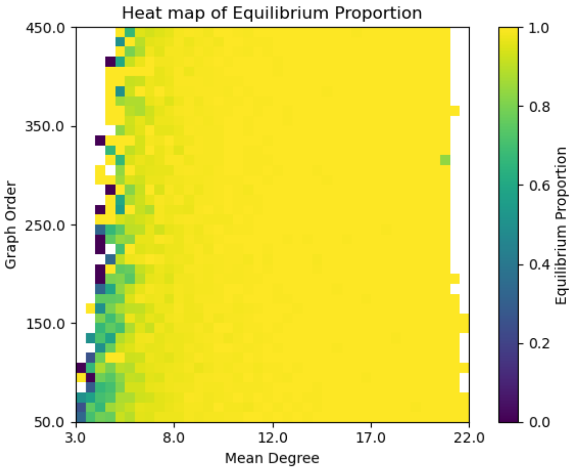
Looking beyond the basin of stability there are many other interesting things we may measure about equilibria

To do this we attempted to find 1×10^6 equilibria in separate graphs using the same random initial value approach and observed things like:

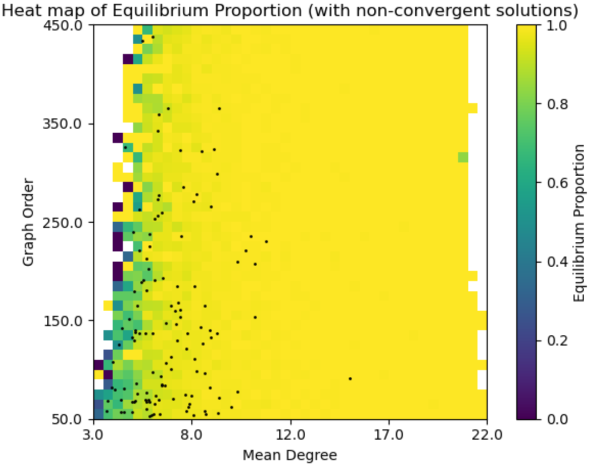
- Cluster number (Number of strategies present in the limit)
- Limit type (equilibrium, 2-cycle, n-cycle, non-convergent)
- Modularity[4]

[4] M.E.J. Newman. "Modularity and community Structure in Networks."

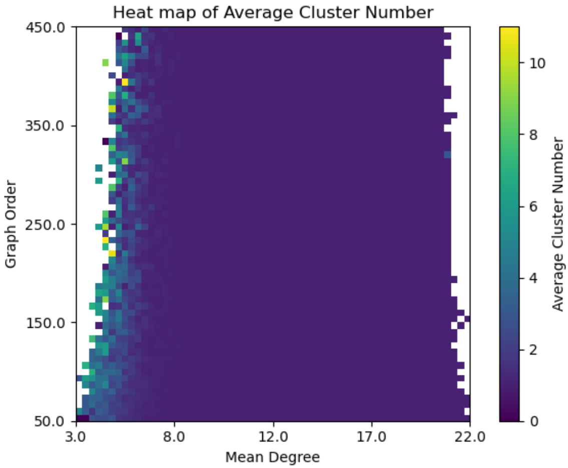
Results


Heat map of Equilibrium Proportion
450.0
350.0
250.0
150.0
50.0
Graph Order
3.0
8.0
12.0
17.0
22.0
Mean Degree
1.0
0.8
0.6
0.4
0.2
0.0
Equilibrium Proportion

Results



Heat map of Equilibrium Proportion (with non-convergent solutions)
Graph Order
450.0
350.0
250.0
150.0
50.0
Mean Degree
3.0
8.0
12.0
17.0
22.0
Equilibrium Proportion
1.0
0.8
0.6
0.4
0.2
0.0


Heat map of Average Cluster Number
450.0
350.0
250.0
150.0
50.0
Graph Order
3.0
8.0
12.0
17.0
22.0
Mean Degree
10
8
6
4
2
0
Average Cluster Number

Conjectures and further directions

- Non-trivial equilibria are prevalent among "almost" disconnected graphs
- The threshold function[5] for indecomposability $A(n)$ is

$$\frac{1}{2} n \log n \leq A(n)$$

- The computation time for solving the IVP is linear in graph size
- n-cycles occur with probability zero for $n > 2$.

[5] P. Erdős and A. Rényi "On Random Graphs I"

Thank You

- Thank you to Nina Fefferman and all the members of the Fefferman Lab who commented on this work
- Thank you to Graham Derryberry for help using the cluster
- The computation for this work was performed on the National Institute for Modeling Biological Systems (NIMBioS) computational resources at the University of Tennessee

- All the code is available on the Fefferman Lab github: github.com/feffermanlab/JSM_2024_StructuredCoordination

Checking for Isomorphisms

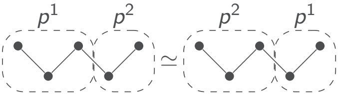
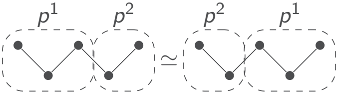
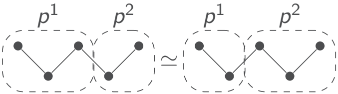
Type	Partition Diagram	Vector View
Isomorphic by way of relabeling		$[1, 1, 1, 2, 2] \simeq [2, 2, 2, 1, 1]$
Isomorphic by way of symmetry		$[1, 1, 1, 2, 2] \simeq [2, 2, 1, 1, 1]$
Isomorphic by way of symmetry and relabeling		$[1, 1, 1, 2, 2] \simeq [1, 1, 2, 2, 2]$

Figure: An example of the three different ways two labeled partitions of labeled graphs can be isomorphic to one another and the "vector view" which is how the computer stores the partition information.

Notes on Complexity

Checking if a partition is an equilibrium partition is $\mathcal{P}$ hard.

We need only calculate the best response of each vertex in graph. If every vertex is playing its best response, then it is a Nash equilibrium and thus an equilibrium partition. This can be done with in at most n^2 operations.

Corollary

Finding an equilibrium partition of a graph given certain compatible constraints in $\mathcal{NP}$ hard.

Results

	number of graphs with n partitions									
Graph size	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0
3	2	0	0	0	0	0	0	0	0	0
4	4	2	0	0	0	0	0	0	0	0
5	13	8	0	0	0	0	0	0	0	0
6	48	43	13	6	2	0	0	0	0	0
7	319	297	128	56	25	15	8	3	1	1
Total	399	350	141	62	27	15	8	3	1	1

Table: A table showing the number of connected graphs which admit n different equilibrium partitions for n from 1 to 10 up to isomorphism. Among graphs of size less or equal to seven, there are no graphs which admit more than 10 different partitions.

Results

	number of partitions with n parts					
Graph Size	1		2		3	
1	1	100%	0	0%	0	0%
2	1	100%	0	0%	0	0%
3	2	100%	0	0%	0	0%
4	6	75%	2	25%	0	0%
5	21	72%	8	27%	0	0%
6	112	54%	79	38%	16	8%
7	853	46%	808	44%	174	9%
Total	996	48%	897	43%	190	9%

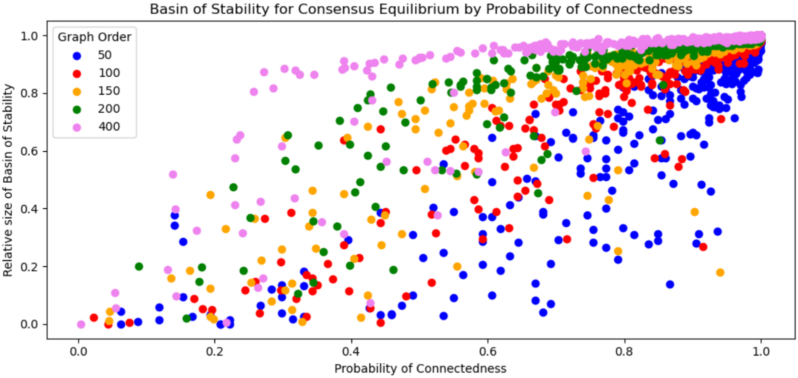
Table: A table showing the number of distinct partitions with n parts for each graph size from 1 to 7 vertices.

Connected Probability

Let $f(n, p)$ be the probability that $G_{n,p}$, generated by the Erdős-Rényi Algorithm, is connected then clearly $f(1, p) = 1$ and

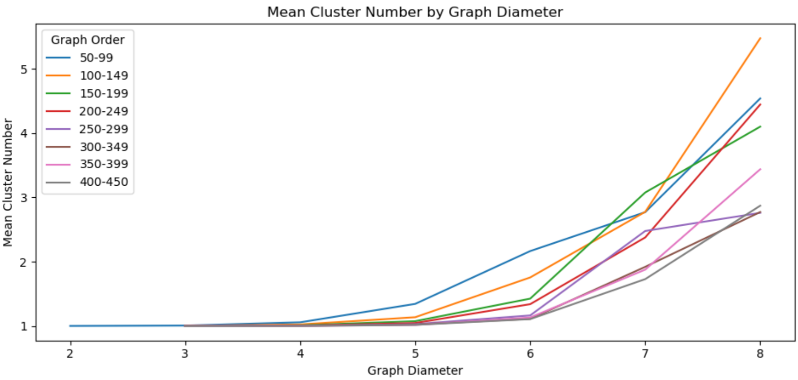
$$f(n, p) = 1 - \sum_{i=1}^{n-1} f(i, p) \binom{n-1}{i-1} (1-p)^{i(n-i)} \tag{1}$$

Results



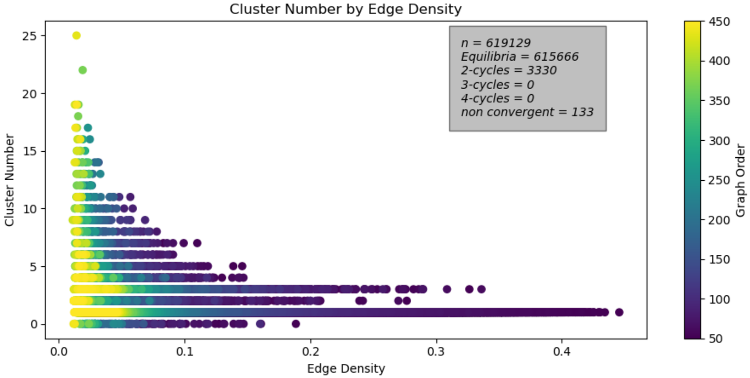
Basin of Stability for Consensus Equilibrium by Probability of Connectedness
Graph Order
50
100
150
200
400
Relative size of Basin of Stability
Probability of Connectedness
0.0
0.2
0.4
0.6
0.8
1.0

Results

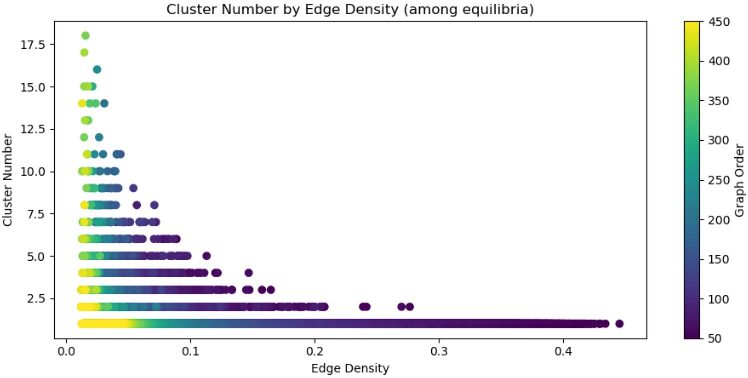


Mean Cluster Number by Graph Diameter
Graph Order
50-99
100-149
150-199
200-249
250-299
300-349
350-399
400-450
Mean Cluster Number
1
2
3
4
5
Graph Diameter
2
3
4
5
6
7
8

Results

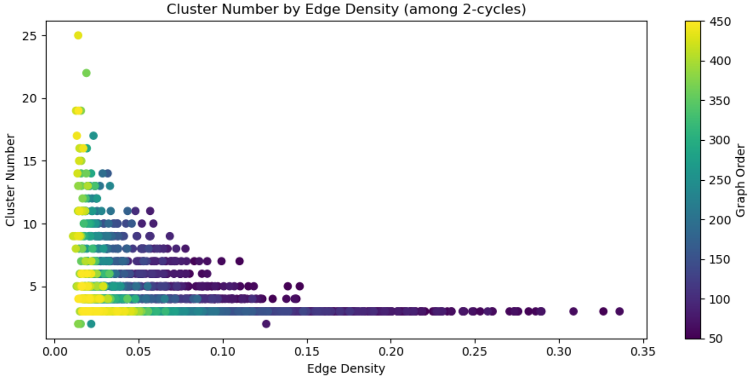

Cluster Number by Edge Density
n = 619129
Equilibria = 615666
2-cycles = 3330
3-cycles = 0
4-cycles = 0
non convergent = 133
Cluster Number
0
5
10
15
20
25
Edge Density
0.0
0.1
0.2
0.3
0.4
Graph Order
50
100
150
200
250
300
350
400
450

Results



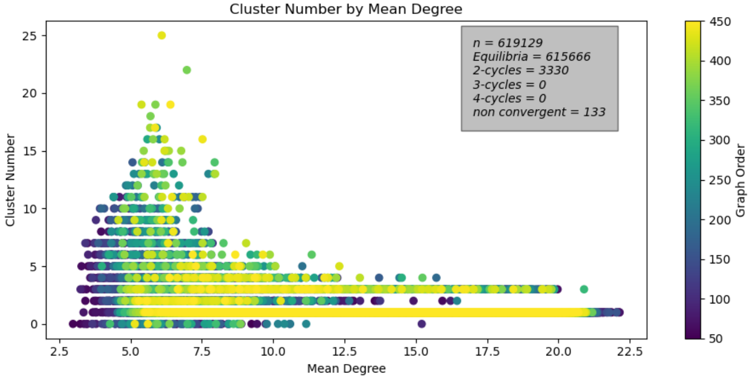
Cluster Number by Edge Density (among equilibria)
Cluster Number
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
0.1
0.2
0.3
0.4
Edge Density
Graph Order
450
400
350
300
250
200
150
100
50

Results



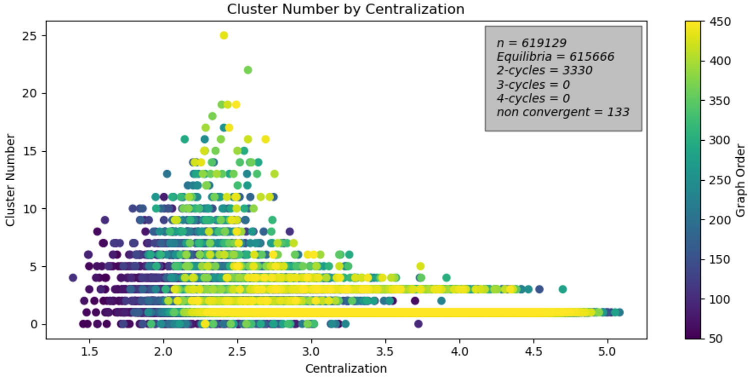
Cluster Number by Edge Density (among 2-cycles)
Cluster Number
Edge Density
Graph Order
25
20
15
10
5
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
450
400
350
300
250
200
150
100
50

Results



Cluster Number by Mean Degree
n = 619129
Equilibria = 615666
2-cycles = 3330
3-cycles = 0
4-cycles = 0
non convergent = 133
Cluster Number
0
5
10
15
20
25
Mean Degree
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
22.5
Graph Order
50
100
150
200
250
300
350
400
450

Results



Cluster Number by Centralization
n = 619129
Equilibria = 615666
2-cycles = 3330
3-cycles = 0
4-cycles = 0
non convergent = 133
Cluster Number
Centralization
Graph Order
0
5
10
15
20
25
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
50
100
150
200
250
300
350
400
450