Spatially Structured Coordination Games and their Applications in Theoretical Ecology

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The Coordination Game

The coordination game, at its most basic, is a two player game with the following payoff matrix:

	A	B
A	a,a	c,d
B	d,c	b,b

where $a > d$ and $b > c$. It has two pure strategy Nash equilibria, $(A, A), (B, B)$, and a mixed strategy Nash equilibrium where both players play strategy A with probability $p = \frac{b-d}{a+b-c-d}$.

The Coordination Game

The game can also be considered among many players where pairs of players are selected to play uniformly randomly.

- Using myopic best response as a replicator dynamic, a group playing this game with *high inertia* and ε-*noise* will converge to the risk dominant Nash equilibrium [1]
- Changing the way in which the pairs are selected can lead to different equilibrium selection [2]
- Convergence time is very slow [3]

[1] Kandori, Michihiro, et al. "Learning, Mutation, and Long Run Equilibria in Games."

[2] Robson, Arthur J., and Fernando Vega-Redondo. "Efficient equilibrium selection in evolutionary games with random matching."

[3] Ellison, Glenn. "Basins of attraction, long-run stochastic stability, and the speed of step-by-step evolution."

The Structured Coordination Game

If player selection is not uniform random across all players, we call this a structured coordination game.

- Complete Graphs (Identical to unstructured game)
- Circular Cities
- Square Lattice

Each of these previous studies have taken the approach of using symmetry to reduce the state space, considering a transition matrix without noise, and establishing a distance between states through mutation. [4,5]

[4] Ellison, Glenn. "Learning Local Interactions and Coordination."
[5] Weidenholzer, Simon. "Coordination games and local interactions: a survey of the game theoretic literature."

The Structured Coordination Game

Consider this most general setting: for a connected graph $G(V,E)$ each vertex $v \in V$ plays a strategy c from a set of pure strategies C. The payoff for v is given by

$$w(v,c|\,\mathbf{u}) = |\{x \in \Gamma(v); \mathbf{u}_x = c\}| \tag{1}$$

where $\mathbf{u}$ is the strategy profile and $\mathbf{u}_x$ is the strategy that x is using, and $\Gamma(v)$ is the set of neighbors of v. This is the case wherein the payoff matrix is simply I_m. Note that we are not limited to only 2 strategies as before.

The Structured Coordination Game as a Dynamical System

In keeping with the previous work, we use a myopic best response as our replicator dynamic. In this way we construct a sequence of strategy profiles $\mathbf{u}(t)$ with

$$\mathbf{u}_v(t+1) \in \operatorname*{argmax}_{c \in C} \{ w(v, c | \, \mathbf{u}(t)) \} \tag{2}$$

It may be that $|\text{argmax}| > 1$ so we break ties in the following way (ε-inertia)

- if $\mathbf{u}_v(t) \in \text{argmax}_{c \in C}\{w(v, c | \, \mathbf{u}(t))\}$ then $\mathbf{u}_v(t+1) = \mathbf{u}_v(t)$.
- else, select from $\text{argmax}_{c \in C}\{w(v, c | \, \mathbf{u}(t))\}$ uniform randomly.

Goals and Questions

It is clear to see that equilibria in the dynamical system are Nash equilibria of the game. There are many questions that arise from this system:

- Can we characterize equilibria for a general graph (or a particular topology) and determine stability conditions?
- Given a set of "boundary conditions," can we find a strategic interpolation which is a Nash equilibrium?
- What might a strategically continuous version of this game look like, and what can it tell us about cooperative behaviors in ecology?

Equilibria

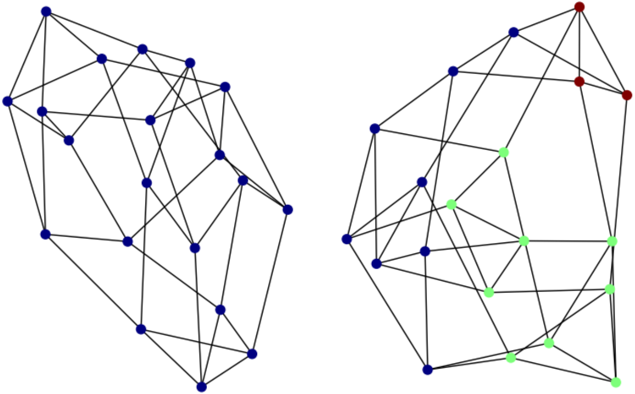


Figure: A trivial and a non-trivial equilibrium in a 4-regular graph

Equilibra: Terminology

In order to discuss these ideas we need a definition

Cluster

A subgraph of G spanned by all of the vertices using a particular strategy.

and some notation

Notation

$Q(\mathbf{u})$	The set of all clusters in a strategy profile $\mathbf{u}$
q^c	A cluster in $Q(\mathbf{u})$ in which vertices use strategy c
∂q^c	Vertices in q_c which have neighbors in other clusters

Simple Graphs with Nice Properties

The only equilibrium in K_n is the trivial equilibrium

Suppose there is a equilibrium strategy profile $\mathbf{u}^\star$ with $m \geq 2$ clusters, $q^{c_1}, ..., q^{c_m}$. Because every pair of vertices shares an edge, $w(v, c_i | \mathbf{u}) = |q^{c_i}|$ if $v \notin q^{c_i}$, and $w(v, c_i | \mathbf{u}) = |q^{c_i}| - 1$ if $v \in q^{c_i}$. Consider a vertex $v_1 \in q^{c_1}$. Because it is at equilibrium,

$$w(v_1, c_1 | \mathbf{u}^\star) = \max_C \{w(v_1, c | \mathbf{u}^\star)\} =: a.$$

Therefore $|q^{c_1}| = a + 1$. Moreover $w(v_1, c_2 | \mathbf{u}^\star) \leq a$ so $|q^{c_2}| \leq a$. Now consider a vertex $v_2 \in q^{c_2}$. $w(v_2, c_2 | \mathbf{u}^\star) = |q^{c_2}| - 1 \leq a - 1$ and $w(v_2, c_1 | \mathbf{u}^\star) = a + 1$. Thus $\mathbf{u}_2^\star = c_2 \notin \text{argmax}\{w(v_2, c | \mathbf{u}^\star)\}$ so $\mathbf{u}^\star$ is not an equilibrium.

Simple Graphs with Nice Properties

$K_{n,m}$ admits an equilibrium with d clusters iff $d|n$ and $d|m$

$\Longleftarrow$ It is easy to construct an equilibrium strategy profile with d clusters
$\Longrightarrow$ I argue by contradiction. $K_{n,m} = E_n + E_m$ Without loss of generality suppose that d is not a divisor of m. Suppose $\mathbf{u}^\star$ is an equilibrium strategy profile with $d > 1$ cliques. d does not divide m so $\exists$ strategies i and j such that $|q^i \cap E_m| > |q^j \cap E_m|$. If $\mathbf{u}^\star$ is at equilibrium it must follow that $q^j \cap E_n = \emptyset$. It then follows $q^j \cap E_m = \emptyset$. If q^j is empty then there are not d cliques in $\mathbf{u}^\star$. This contradiction proves the result.

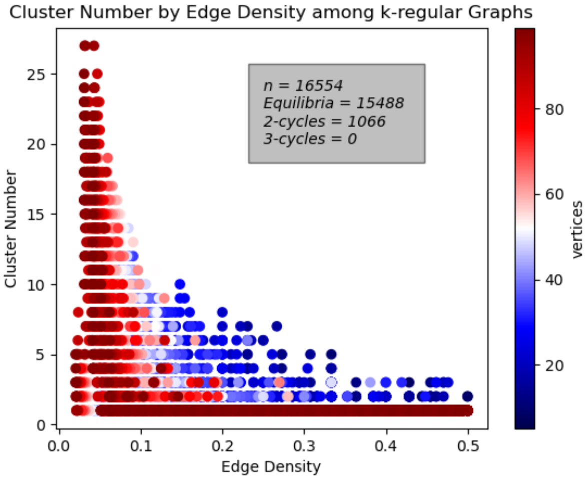
General Remarks on Equilibria

There are a number of ways we may approach our investigation

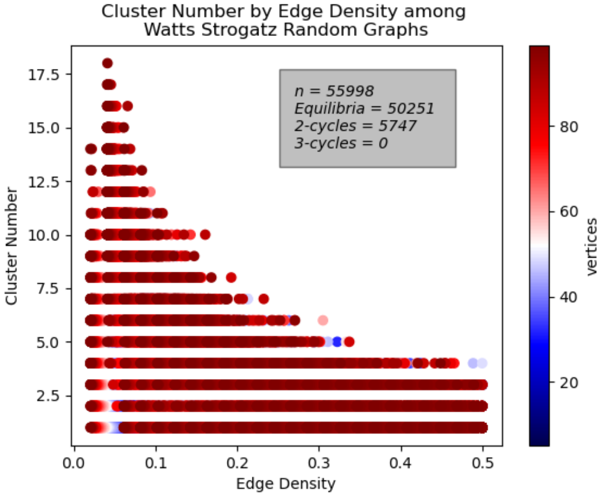
- Graph partitioning: Describing equilibria as graph partitions
- Energy-like estimates: Neither $1 - Q$ (modularity) nor the number of edges which connect different clusters are appropriate energies for this context
- Extremal Techniques: Describing necessary conditions on cluster boundaries by investigating minimal separators [6]

[6] Béla Bollobás. "Extremal Graph Theory"

Numerical Results

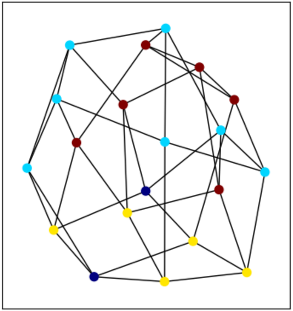


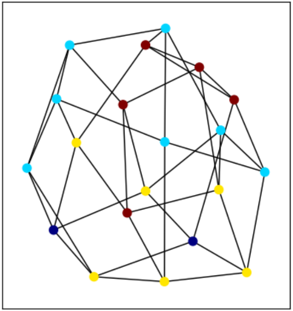
Numerical Results



Cycles

When viewed as a dynamical system, we also see the (common) emergence of cycles. We conjecture that n-cycles with $n > 2$ are impossible.





Stability

When viewed as a dynamical system, stability can be considered in multiple ways.

- Local Stability - A vertex is stable to a single perturbation
- Global Stability - Every vertex is stable to a single perturbation
- Convergent Stability - Every "nearby" strategy profile evolves into this strategy profile
- Structural Stability - A strategy profile is stable to perturbations of the graph structure[7]

[7] Ely, Jeffery C. "Local Conventions"

Goals and Further Questions

We hope to

- Prove the n-cycle conjecture
- Find analytically tractable stability criteria in multiple senses
- Use energy-like estimates to describe basins of stability for equilibria
- Use techniques from extremal graph theory to describe possible equilibria by the structure of the graph by which they are admitted
- Support analytical results with numerical findings

Boundary Value Problem

A vital application of this game is a boundary value conception of the problem.

Boundary Value Problem

Suppose $B \subset V$ is a subset of vertices which are assigned strategies by $f : B \to C$. Is there a strategy profile such that $\mathbf{u}_v = f(v)$ for all $v \in B$ and $\mathbf{u}$ is an equilibrium strategy profile?

Initial Thoughts

This is rather speculative but the direction forward may look like this:

- Trying to prove existence of such an equilibrium interpolation for an admissible set of boundary values.
- (Almost Equivalently) finding a class of boundary values for which an interpolation can be made
- Seeking out an algorithm to build an equilibrium interpolation through graph reductions.
- Seeking out an algorithm to build an equilibrium interpolation through partitioning

Partitioning

As before this becomes a partitioning problem but with an added components

Boundary Value Problem

For a graph $G(V, E)$, $B \subset V$ and $f : B \to C$, find a partition $\mathcal{P} = \{P^c\}_{c \in C}$ such that

$$\begin{aligned} i) \quad & |\Gamma(x) \cap P^{\mathbf{u}_x}| \geq |\Gamma(x) \cap P^c| \, \forall x \in V, c \in C \\ ii) \quad & x \in P^{f(x)} \, \forall x \in B \end{aligned} \tag{3}$$

In this case, we have extra restrictions on the partition which make partitioning results very helpful. Consider the natural correspondence between partitions and strategy profiles.

Minimum Cut Partitioning

Let A be the adjacency matrix for the graph G which has m edges.

$$E(\mathbf{u}) = \sum_{v,w} A_{vw}(1 - \delta(\mathbf{u}_v\,\mathbf{u}_w)) = 2m - \sum_{v,w} A_{vw}\delta(u_v, u_w) \tag{4}$$

The minimum cut partition into $|C|$ parts is that partition that which minimizes E.
Note that a minimum cut partition requires exactly $|C|$ parts.
Otherwise, the minimum cut partition would always be the trivial partition.

Minimum Cut Partitioning

Every Minimum Cut Partition with parts of size > 1 corresponds to an equilibrium partition.

Suppose that $\mathcal{P}_\star = \{P_\star^c\}$ is a minimum cut partition of $G(V, E)$ corresponding to a strategy profile $\mathbf{u}^\star$ which is not an equilibrium. Therefore $\exists v \in V$ which is using strategy $r \in C$ which for which $w(v, r|\,\mathbf{u}) < w(v, s|\,\mathbf{u}) \iff |\Gamma(v_\star) \cap P_\star^r| < |\Gamma(v_\star) \cap P_\star^s|$ for some $s \in C$. Let $\hat{\mathbf{u}}$ be a new strategy profile where $\hat{\mathbf{u}}_v = \mathbf{u}_v^\star$ for all $v \neq v_\star$ and $\hat{\mathbf{u}}_{v_\star} = s$. Call the corresponding partition $\hat{\mathcal{P}}$. We can easily compute that

$$E(\mathbf{u}^\star) - E(\hat{\mathbf{u}}) = -2 \sum_{v \neq v_\star} A\delta(\mathbf{u}_v^\star, \mathbf{u}_{v_\star}^\star) + 2 \sum_{v \neq v_\star} A\delta(\hat{\mathbf{u}}_v, \hat{\mathbf{u}}_{v_\star}) \qquad (5)$$

Which is clearly positive, This contradiction proves the result.

Modularity Partitioning

$$Q(\mathbf{u}) = \frac{1}{2m} \sum_{v,w} \left[A_{vw} - \frac{d_v d_w}{2m} \right] \delta(\mathbf{u}_v, \mathbf{u}_w) \qquad (6)$$

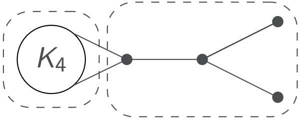
let $B = [[A_{ij} - \frac{d_i d_j}{2m}]]$ be the modularity matrix let $\chi^c(\mathbf{u}) = [\delta(\mathbf{u}_i, c)]_{i=1}^{n}$ then (6) is equivalent to

$$Q(\mathbf{u}) = \frac{1}{2m} \sum_{c \in C} \chi^c(\mathbf{u})^T B \chi^c(\mathbf{u}) \qquad (7)$$

A modularity partition is a partition into $|C|$ parts which maximizes Q. It does not require that every part be non-empty.

Modularity Partitioning

There are modularity partitions which do not correspond to equilibria and there are equilibria which do not correspond to modularity partitions.

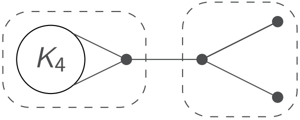


$Q = 0.2809$

Modularity Partition

not an equilibrium

not a min. cut partition



$Q = 0.2603$

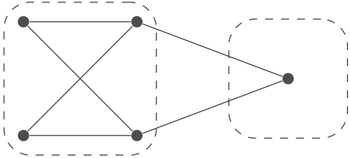
Not a Modularity Partition

equilibrium

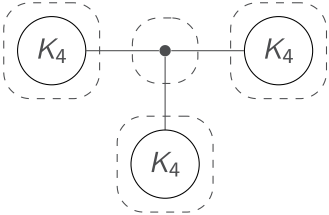
min. cut partition

Partitioning

We can prove that, if a modularity partition has a cluster of size one, that single vertex must share an edge with multiple other clusters.

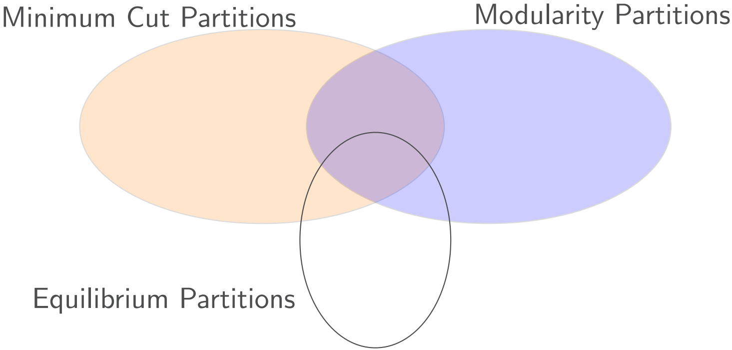


not a modularity partition
not an equilibrium
min. cut partition



Modularity partition
not an equilibrium
min. cut partition

Partitioning



We can use the relationships between these three partitions to get a better understanding of what equilibrium partitions look like and how they can be approximated.

A note on complexity class

Finding a modularity partition with more than 2 clusters can be shown to be $\mathcal{NP}$ complete[8]. We can say certainly that finding an equilibrium partition is $\mathcal{NP}$ hard

Finding an Equilibrium Partition is $\mathcal{NP}$ hard

The process to check if a partition is an equilibrium partition can be done in $n \cdot |C| \leq n^2$ operations. It is checkable in polynomial time so the problem is certainly in $\mathcal{NP}$.

[8] Brandes, U. Delling, D. Gaertler, M. Görke, R. Hoefer, M. Nikoloski Z. Wagner, D. "On Modularity - NP-Completeness and Beyond"

Sympatric Evolution of Cooperative Behavior

Sympatric speciation is the process of speciation without geographic isolation. This system, modified to be continuous, gives us some tools to think about sympatric evolution of cooperative behavior.

Sympatric Solution

A Sympatric Solution is a Nash equilibrium with a strategic gradient. That is, a continuous, non-constant Nash equilibrium.

Mixed Strategy Concept

Continuous Player Space with Mixed Strategies

Let Ω be a domain and Δ_m be the m-simplex in which every mixed strategy lies. Let $\Phi : \Omega \to \Delta_m$ be a strategy profile, where payoffs are calculated as

$$w(x|\Phi) = \sum_{i=1}^{m} \Phi_i(x) \int_{\Omega} K(x-y)\Phi_i(y)dy =: B(\Phi(x), \Phi(x)) \quad (8)$$

Where K is a familiarity kernel and B is a bilinear form.

Are there any non-constant, continuous Φ which are Nash equilibria?

Mixed Strategy Concept

Are there any non-constant, continuous Φ which are Nash equilibria?

Optimization Challenge

We are looking for continuous Φ such that for any x,

$$w(x|\Phi) \geq w(x|\tilde{\Phi}) \quad \forall \tilde{\Phi}(y) = \Phi(y) \text{ for } y \neq x \tag{9}$$

.

For the game posed above, there are no continuous non constant Φ which satisfy this condition.

Mixed Strategy Concept

There are no nonconstant continuous solutions to (9)

Suppose that $\Phi^\star \in C^0(\Omega)$ satisfies (9), and is non-constant The convolution $K * \Phi_i^\star$ is clearly continuous and non-constant for any i for which $\Phi_i^\star$ is not constant. There exists an $\hat{x} \in \Omega$ such that$(K * \Phi_i^\star)(\hat{x}) \geq (K * \Phi_j^\star)(\hat{x})$ for all j and in particular $(K * \Phi_i^\star)(\hat{x}) > (K * \Phi_l^\star)(\hat{x}) > 0$ for some l, lest $\Phi^\star$ be constant in all of Ω. Thus $\mathbf{a} \cdot (K * \Phi^\star)(\hat{x})$ is maximized when $\mathbf{a} = e_i$. Let

$$\tilde{\Phi}(x) = \begin{cases} e_i & x = \hat{x} \\ \Phi^\star(x) & x \neq \hat{x} \end{cases} \tag{10}$$

Observe $w(\hat{x}, \tilde{\Phi}) = (K * \Phi_i^\star)(\hat{x}) > w(\hat{x}, \Phi^\star)$, so Φ^* does not solve (9).

Mixed Strategy Concept

This is not altogether surprising:

- When strategies are discrete and the payoff matrix is the identity, then being around "nearby" strategies is not beneficial.
- When strategies are discrete, the model is not entirely biologically relevant.

Instead we ought to look at strategies which are comparable under a "blurred" payoff matrix.

Comparable Strategy Concept

Consider this symmetric payoff matrix

$$\begin{bmatrix} 1 & \alpha & 0 \\ \alpha & 1 & \alpha \\ 0 & \alpha & 1 \end{bmatrix} \tag{11}$$

Figure: This represents a Nash equilibrium if and only if $\alpha \geq \frac{1}{2}$

This poses the question: In a more general setting are there "recognition thresholds" which define critical regions where sympatric evolution of cooperative behavior can occur? Are there critical domain sizes?

Comparable Strategy Concept

Now instead of using linear combinations of pure incomparable pure strategies, consider the game played in the domain Ω where a strategy profile $\Phi : \Omega \to M$ gives the payoff

$$w(x|\Phi) = \int_\Omega K(x-y)\rho(d(\Phi(y),\Phi(x)))dy \tag{12}$$

Where M is a metric space equipped with the metric d, $\rho(y)$ is a certain recognition function which has $\rho(0) = 1$ and $\rho(r) = 0$ for some $r > r^\star$ a recognition threshold.

Easy Example

Let $\Omega \subset \mathbb{R}^n$, let $\rho(r) = 1$ for $r \leq r^\star$ and 0 otherwise and let $K(x) = \frac{1}{\alpha(n)\kappa}$ for $|x| < \kappa$ and 0 otherwise.

$$w(x|\Phi) = \int_\Omega K(x-y)\rho(|\Phi(y) - \Phi(x)|)dy \tag{13}$$

$\Phi : \Omega \to M$ is a Nash Equilibrium if $\Phi \in C^{0,1}(\Omega; M)$ with Lipschitz constant $L \leq \frac{r^\star}{\kappa}$

We can show that under these conditions $w(x|\Phi) = \frac{1}{\alpha(n)\kappa} vol(\Omega \cap supp_y K(x-y))$ which is the max possible fitness and so it is trivially a Nash equilibrium.

Limiting Example

- If $r^\star \to 0$ then a continuous non-constant Nash Equilibrium is impossible.
- If $\kappa > diam(\Omega)$ then again every continuous solution must have $diam(\Phi(\Omega)) < r^\star$

Consider κ describing the edge density of the graph. This second limiting example is analogous to the result about equilibria in K_n.

Conclusion

- Chapter 1 - Understanding the structured coordination game with a discrete player space through time
- Chapter 2 - Constructing equilibrium solutions to the structured coordination game with a discrete player space
- Chapter 3 - Characterizing equilibrium of the structured coordination game in a continuous player space and continuous strategy space

Thank you

Questions?

Partitioning and the IVP approach

In general the process of finding equilibria is equivalent to finding this kind of vertex partition.

$$\mathcal{P} = \{P^c\}_{c\in C} \text{ where } x \in P^c \Rightarrow |\Gamma(x)\cap P^c| \geq |\Gamma(x)\cap P^d|\, \forall c, d \in C \tag{14}$$

There are other similar kinds of graph partitions

- Modularity Partitions
- Minimum Cut Partitions

Problems in community detection rely heavily on these partitions but the game theory literature has not relied on insights from this field.

Barriers to Partitioning for the IVP

In the light of graph partitioning, this question is a special kind of community detection. Without more restrictions on the partition we run into problems:

- We do not know a priori how many strategies will be present in an equilibrium (equivalently how many parts the graph will be partitioned into)[9]
- It is not assumed that a non-trivial equilibrium (non-trivial partition) exists
- Sub-optimal equilibria are still of interest to us but may be missed by partitioning algorithms.[10]

[9] Newman, M. E. J. "Modularity and Community Structure in Networks."

[10] Clauset, Aaron, Newman, M. E. J., Moore, Cristopher. "Finding Community Structure in very large networks

Boundary Value Problem Intuition

Suppose that in a signaling network G

- Each vertex uses exactly one language at any one time to send a receive signals.
- A subset of vertices are assigned languages which they cannot change
- Translation of the signal is costly relative to transmission.

How can we assign a "language" to each vertex so that each vertex is minimizing their own "translation" burden?