This document contains observations and questions about the possible relationship between

- Hilbert series of Koszul algebras,
- ribbon-positive homomorphisms, and
- the Hadamard closure of the set of Pólya Frequency series.

Notation.

- Let Λ denote the ring of symmetric functions, with the usual notation as in EC2.
- If $f(t) = \sum_{k\geq 0} a_k t^k \in \mathbb{R}[[t]]$ is a formal power series, let $\Phi_f : \Lambda \to \mathbb{R}$ denote the ring homomorphism defined by the rule $h_i \mapsto a_i$.
- If $A \subseteq \mathbb{R}[[t]]$, let $A(\mathbb{Z}) \subseteq \mathbb{Z}[[t]]$ denote the set of series in A with integer coefficients.
- Let $\mathcal{P} \subset \mathbb{R}[[t]]$ denote the set of Pólya Frequency series. By the Jacobi-Trudi identity and the Aiseen-Edrei-Schoenberg-Whitney theorem, this is precisely the set of series $f(t)$ satisfying satisfying $\Phi_f(1) = 1$ and $\Phi_f(s_\lambda) \geq 0$ for all $\lambda \in \mathrm{Par}$.
- Let $\star$ denote the Hadamard product in $\mathbb{R}[[t]]$, i.e. the product defined by

$$\left(\sum_{k\geq 0} a_k t^k\right) \star \left(\sum_{k\geq 0} b_k t^k\right) = \sum_{k\geq 0} a_k b_k t^k.$$

- If $A \subseteq \mathbb{R}[[t]]$, let $\overline{A}$ denote the closure of A under finite Hadamard products, that is

$$\overline{A} = \{g_1(t) \star \cdots \star g_\ell(t) : g_1(t), \ldots, g_\ell(t) \in A\}.$$

Hilbert Series of Koszul Algebras

Let $\mathcal{K}$ denote the set of series $f(t)$ with integer coefficients so that there exists a Koszul algebra $A = \bigoplus_{k\geq 0} A_i$ satisfying $\mathrm{Hilb}(A, t) = f(t)$.

Observation 1. (Sam-VandeBogert [2, Theorem 1.1]) Every Pólya Frequency series with integer coefficients is the Hilbert series of a Koszul algebra. That is, we have $\mathcal{P}(\mathbb{Z}) \subset \mathcal{K}$. Note that this containment is strict.

Let $\mathcal{R}$ denote the set of real formal power series $f(t) = \sum_{k\geq 0} a_k t^k$ satisfying $a_0 = 1$ and $\Phi_f(s_R) \geq 0$ for all ribbon skew shapes R. We call all such series *ribbon-positive*.

Observation 2. Every Hilbert series of a Koszul algebra is ribbon-positive. That is, we have $\mathcal{K} \subseteq \mathcal{R}(\mathbb{Z})$.

Proof sketch. I believe this follows from Lemma 4.23 in [3] and Remark 4.24 in [2]. □

Question 1. Is it true that $\mathcal{R}(\mathbb{Z}) \subseteq \mathcal{K}$, and thus that $\mathcal{K} = \mathcal{R}(\mathbb{Z})$?

Evidence. On the algebraic level, Lemma 4.23 in [3] is a biconditional statement. Thus it is plausible that Observation 2 is a biconditional statement at the numerical level, and thus that $\mathcal{K} = \mathcal{R}(\mathbb{Z})$. □

How would you prove it? Given a ribbon-positive series, we must find a Koszul algebra with that Hilbert series. Given a sequence $(a_0, a_1, \ldots)$ which 'numerically' satisfies all the relations to be the Hilbert series of a Koszul algebra, can we impose a Koszul algebra structure on, say, $\bigoplus_{k\geq 0} \mathbb{Q}^{a_k}$? □

Note that $\mathcal{P} \subset \overline{\mathcal{P}}$ is a strict containment. That is, the set of Pólya Frequency series is not closed under Hadamard products. However, the Segre product of graded algebras preserves the Koszul property. It followsw that $\overline{\mathcal{K}} = \mathcal{K}$. This leads to the following.

Observation 3. We have $\overline{\mathcal{P}(\mathbb{Z})} \subseteq \mathcal{K}$.

Proof. We have $\mathcal{P}(\mathbb{Z}) \subseteq \mathcal{K}$ and $\overline{\mathcal{K}} = \mathcal{K}$. Since $\mathcal{K}$ is a Hadamard-closed set containing $\mathcal{P}(\mathbb{Z})$, it must contain the Hadamard-closure $\overline{\mathcal{P}(\mathbb{Z})}$ of $\mathcal{P}(\mathbb{Z})$. □

Note that if $f(t) \in \overline{\mathcal{P}(\mathbb{Z})}$ has a prime coefficient in a nontrivial way, then in fact $f(t) \in \mathcal{P}(\mathbb{Z})$ (need to add details). Without having checked for examples, it does seem plausible that there are elements of $\mathcal{K}$ with prime coefficients. Thus it does *not* seem plausible that $\overline{\mathcal{P}(\mathbb{Z})} = \mathcal{K}$.

Nevertheless, it could still be true that $\overline{\mathcal{P}}(\mathbb{Z}) = \mathcal{K}$. In other words, perhaps every Hilbert series of a Koszul algebra can be realized as a finite Hadamard product of Pólya Frequency series. The Pólya Frequency series factors are not required to have integer coefficients individually to lie in $\overline{\mathcal{P}}(\mathbb{Z})$. We only require that their Hadamard product has integer coefficients.

Motivated by this possibility, we discuss the possible relationship between $\overline{\mathcal{P}}$ and $\mathcal{R}$.

Observation 4. We (probably) have $\overline{\mathcal{P}} \subseteq \mathcal{R}$.

Proof sketch. The proof techniques in [1] should readily generalize to show that the property $\Phi_{f\star g}(s_R) \geq 0$ holds when $f, g \in \mathcal{P}$. Thus the ribbon-positive property is preserved under Hadamard prodcuts of Pólya Frequency series, and it follows that $\overline{\mathcal{P}} \subseteq \mathcal{R}$. □

Question 2. Is the set $\mathcal{R}$ Hadamard-closed? That is, do we have $\overline{\mathcal{R}} = \mathcal{R}$?

Evidence. Given my suspicion that $\mathcal{R}(\mathbb{Z}) = \mathcal{K}$, and the fact that $\mathcal{K}$ is Hadamard-closed, I am inclined to believe this.

Additionally, I think we may be able to generalize the techniques in [1] to show $\Phi_{f\star g}(s_R) \geq 0$ for any $f, g \in \mathcal{R}$, not just $f, g \in \mathcal{P}$. However, it would require more work. □

Observation 5. A series $f(t) = \sum_{k\geq 0} a_k t^k \in \mathcal{R}$ if and only if every Temperley–Lieb immanant of the matrix

$$\begin{bmatrix} a_1 & a_2 & & & a_n \\ 1 & a_1 & a_2 & & \\ 0 & 1 & a_1 & a_2 & \\ 0 & 0 & 1 & a_1 & a_2 \\ 0 & 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is nonnegative for all $n \geq 1$.

Sketch of proof. Use [1, Corollary 3.16] to show that the Temperley–Lieb immanants of this matrix are precisely the images of ribbons under Φ_f. □

Question 3. Is it true that $\mathcal{R} \subseteq \overline{\mathcal{P}}$, and thus that $\mathcal{R} = \overline{\mathcal{P}}$?

How would you prove it? Given a ribbon-positive sequence, find a way to express it as a Hadamard product of Pólya Frequency series.

Here is one way that could potentially work:

- Use a result of Rhoades and Skandera to find a planar network N, possibly with some negative edge weights, which realizes the matrix from Observation 5.
- Somehow use the hypothesis that all Temperley–Lieb immanants are positive to create a bijection between path families on N and path families on a tuple $N_1, \ldots, N_k$ of networks with *positive* edge weights.
- Observe that taking tuples of path families corresponds to the Hadamard product of path matrices.
- Observe that the positive edge weight requirement implies that the path matrices for $N_1, \ldots, N_k$ come from Pólya Frequency series.

□

Upshot

If it is indeed true that $\mathcal{R}(\mathbb{Z}) = \mathcal{K}$, then we have a totally *combinatorial* way of thinking about $\mathcal{K}$, for example using the concatenation/near-concatenation identity. If $\mathcal{R}(\mathbb{Z}) = \mathcal{K}$ *and* it is also true that $\mathcal{R} = \overline{\mathcal{P}}$, then we have $\overline{\mathcal{P}}(\mathbb{Z}) = \mathcal{K}$. This gives a totally *analytic* way of thinking about $\mathcal{K}$. Of course, by default, we have a way of thinking about $\mathcal{K}$ algebraically. Perhaps these three persectives in tandem would allow us to prove new results about $\mathcal{K}$.

References

[1] Robert Angarone et al. *Hadamard Products of dual Jacobi-Trudi matrices.* 2025. arXiv: `2511.08969 [math.CO]`. URL: `https://arxiv.org/abs/2511.08969`.

[2] Steven V Sam and Keller VandeBogert. *From total positivity to pure free resolutions.* 2025. arXiv: `2408.10408 [math.AC]`.

[3] Keller VandeBogert. "Ribbon Schur functors". In: *Algebra Number Theory* 19.4 (2025), pp. 771–834. ISSN: 1937-0652,1944-7833. DOI: `10.2140/ant.2025.19.771`.

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