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Sheet 11

Solutions to be handed in before class on Wednesday June 26

Problem 51. Consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with the usual diagonal Cartan subalgebra $\mathfrak{h}$ and associated root system $R = \{\pm\alpha\}$. Take $R^+ = \{\alpha\}$ as a system of positive roots, and $\rho = \alpha/2$. Let $z \in \mathbb{C}$.

1. Describe all $\mathrm{U}(\mathfrak{g})$-submodules of the Verma module $M(z\rho)$. (2 points)
2. Show that the simple quotient $L(z\rho)$ of $M(z\rho)$ has finite dimension (equal to $z+1$) if and only if $z \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. (2 points)

Problem 52. Consider the natural representation $V = \mathbb{C}^n$ of $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$.

1. Show that the exterior power $\bigwedge^d V$ is an irreducible representation of $\mathfrak{g}$, for all $d = 0, \ldots, \dim_{\mathbb{C}} V$. (2 points)
2. Determine the highest weight of $\bigwedge^d V$ with respect to the standard diagonal Cartan subalgebra $\mathfrak{h}$ and choose a basis of R such that $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$ is given by the upper triangular matrices in $\mathfrak{g}$. (2 points)

Problem 53. Let $\mathfrak{b} \subset \mathfrak{g}$ be an inclusion of Lie algebras over a field k. Let M be a representation of $\mathfrak{b}$ and E a representation of $\mathfrak{g}$. Show that there is a canonical isomorphism of $\mathfrak{g}$-representations

$$\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} (E \otimes_k M) \xrightarrow{\sim} E \otimes_k (\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{b})} M). \tag{40}$$

(4 points)

Hint Here E on the left is considered as a representation of $\mathfrak{b}$ by restriction. Either define an isomorphism explicitly or deduce it from the fact that the left adjoint of a composition is the composition of the left adjoints, provided they exist.

Problem 54. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with a Cartan subalgebra $\mathfrak{h}$ and $R^+ \subseteq R = R(\mathfrak{g}, \mathfrak{h})$ a system of positive roots. We say that a $\mathrm{U}(\mathfrak{g})$-module M *has a Verma flag* if there is a sequence

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M \tag{41}$$

of submodules such that each quotient M_i/M_{i-1} is isomorphic to a Verma module. It can be shown that the tensor product $E \otimes_{\mathbb{C}} \Delta(\lambda)$ of a Verma module $\Delta(\lambda)$ with a finite-dimensional $\mathrm{U}(\mathfrak{g})$-module E has a Verma flag.

Specialising $\mathfrak{g}$ to $\mathfrak{sl}_2(\mathbb{C})$ with the notation of problem 51, find Verma flags for $V \otimes_{\mathbb{C}} \Delta(0)$ and $V \otimes_{\mathbb{C}} \Delta(-\rho)$, where $\rho = \alpha/2$. Are these tensor products direct sums of Verma modules? (2 points)

Problem 55. Consider $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with the Cartan subalgebra and positive roots as in problem 51. Let $\mathcal{C}$ be the smallest subcategory of the category of all $\mathrm{U}(\mathfrak{g})$-modules which contains all Verma modules and is closed under tensor products with finite-dimensional modules. A short exact sequence in $\mathcal{C}$ is by definition a short exact sequence of $\mathrm{U}(\mathfrak{g})$-modules whose terms lie in $\mathcal{C}$.

Show that there is a non-split short exact sequence

$$0 \to M(0) \to P \to M(s_\alpha \cdot 0) \to 0 \tag{42}$$

in $\mathcal{C}$. (2 points)