Prof. Dr. Catharina Stroppel
Dr. Pieter Belmans

Sheet 7

Solutions to be handed in before class on Wednesday May 22

There is the following remark about last week's lecture:

> In the argument that the coroot is unique one needs that in the last step the field has characteristic zero, since n should not be zero.

Problem 34. Let V be a finite-dimensional vector space over a field of characteristic 0. Let $R \subseteq V$ be an irreducible root system. Let W be the associated Weyl group. Show that V is an irreducible representation of W. (3 points)

Problem 35. Recall that the *Cartan matrix* of a root system is the matrix $(a_{j,i}) = (2(\alpha_i, \alpha_j^\vee)/(\alpha_j, \alpha_j^\vee))_{i,j}$, where $\alpha_1, \ldots, \alpha_n$ are the simple roots.

Recall that the *Dynkin diagram* of a root system is a visual representation of the Cartan matrix. It is a graph whose vertices correspond to the simple roots. An undirected single edge is drawn if the angle of the roots is 60° or 120°. A directed double edge is drawn if the angle is 45° or 135°, oriented from the longer to the shorter root, and likewise a directed triple edge is drawn if the angle is 30° or 150°.

1. Show that up to isomorphism there are 4 rank 2 root systems, and draw their pictures. (4 points)
2. Compute their Cartan matrices, and draw their associated Dynkin diagrams. (4 points)
3. Which of these are Langlands dual or self-dual? (2 points)

Problem 36. The root system of type B_2 is constructed in the previous exercise, and is the one involving double (and not single or triple) edges. If you haven't constructed the root system in the previous exercise, take a look at `https://en.wikipedia.org/wiki/Root_system#/media/File:Root_system_B2.svg`.

Compute the order of the Weyl group of type B_2, and show which familiar group it is isomorphic to. (3 points)

Representation Theory I
Bonn, summer term 2019

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Sheet 8

Solutions to be handed in before class on Wednesday May 29

Problem 37. Let R be a fixed root system of rank r in the space V. Let α, β be non-zero elements of V, and $\theta = \theta_{\alpha,\beta}$ the corresponding angle. Then $(\alpha, \beta^\vee) = \|\alpha\|\|\beta\| \cos\theta$. Denoting $\langle \beta, \alpha^\vee \rangle = 2(\beta, \alpha^\vee)/(\alpha, \alpha^\vee)$ we get $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = 4\cos^2\theta$.

1. Using the axioms of a root system, make a table of the possible values of $\langle \alpha, \beta^\vee \rangle$, $\langle \beta, \alpha^\vee \rangle$, θ and $\|\beta\|^2/\|\alpha\|^2$. (1 point)

   Notice that one can assume that $\alpha \neq \pm\beta$, and $\|\beta\| \geq \|\alpha\|$.

2. Using the pictures from problem 35, explain how the maximal $\langle \beta, \alpha^\vee \rangle$ is realised by drawing the root system and explaining which choice of α and β realises this scenario. (1 point)

3. Let α and β be non-proportional roots. If $(\alpha, \beta^\vee) > 0$ then $\alpha - \beta$ is again a root. (2 points)

Problem 38. We continue the setup of the previous problem. Let α and β be non-proportional roots. The *α-string through β* is the set of roots of the form $\beta + n\alpha$ for $n \in \mathbb{Z}$.

1. Denote $r, q \geq 0$ the largest integers for which $\beta - r\alpha$ (resp. $\beta + q\alpha$) is still a root. Show that the string is unbroken, i.e. that it contains all elements between $\beta - r\alpha$ and $\beta + q\alpha$. (2 points)

2. Show that $r - q = \langle \beta, \alpha^\vee \rangle$. (1 point)

3. What is the maximal length of a root string? (1 point)

Problem 39. An element of a basis π for a root system R is called a *simple* root.

1. Let α be such a simple root. Show that the reflection s_α permutes the positive roots other than α. (2 points)

   Hint Consider the coefficient of α versus the coefficients of the other simple roots before and after applying s_α.

2. Let $\rho = \frac{1}{2}\sum_{\beta \in R^+} \beta$ be the half-sum of positive roots. Show that $\mathrm{s}_\alpha(\rho) = \rho - \alpha$ for α a simple root. (1 point)

Problem 40. Let k be of characteristic zero. Assuming the existence of bases for root systems, show that there exists no semisimple Lie algebras of dimension 4 or 5. (2 points)