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Sheet 12

Solutions to be handed in before class on Friday January 17.

Problem 52. Let $\pi\colon E \to X$ and $\pi'\colon F \to X$ be vector bundles. Construct

1. the direct sum bundle $E \oplus F$, whose fibres are the direct sum of the fibres;
2. the tensor product bundle $E \otimes F$, whose fibres are the tensor product of the fibres;
3. the Hom-bundle $\mathcal{H}\mathrm{om}(E, F)$, whose fibres are the Hom of the fibres.

(2 points)

Problem 53. Let G be an algebraic group, and B a Borel subgroup. Let V be a finite-dimensional representation of B. Show that $G \times_B V$ is a G-equivariant vector bundle on G/B.

(3 points)

Problem 54. Consider the algebraic Peter–Weyl theorem from the lecture for $\mathbb{C}[\mathrm{SL}_2(\mathbb{C})] = \mathbb{C}[a, b, c, d]/(ad - bc - 1)$.

1. Equip this algebra with a filtration by assigning degree 1 to the generators, and prove that the left and right actions of $\mathrm{SL}_2(\mathbb{C})$ preserve the filtration, and therefore pass to the associated graded.
2. Determine the Hilbert series of the associated graded.
3. Using the Peter–Weyl theorem, give a complete classification (up to isomorphism) of finite-dimensional irreducible representations of $\mathrm{SL}_2(\mathbb{C})$, by mimicking the construction used in Problem 51.

(4 points)

Problem 55. Consider $\mathbb{P}^1$ as $\mathrm{SL}_2(\mathbb{C})/B$.

1. Prove that the Borel subgroup B is abelian.
2. Classify explicitly all B-equivariant vector bundles up to isomorphism by giving a complete list.

(4 points)

Problem 56.

1. Show that a rank n vector bundle $E \to X$ is trivial if and only if $\mathrm{H}^0(X, E)$ is n-dimensional, spanned by sections $s_1, \ldots, s_n$ which are linearly independent in every fibre.
2. Compute $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$. How many linearly independent sections are there?
3. Explain why the condition of part 1 does not apply.

(3 points)