Representation Theory II
Bonn, winter term 2019–2020

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Sheet 10

Solutions to be handed in before class on Friday December 20.

Problem 43. In this problem we consider the Chow ring of $\mathbb{P}^2$ and $\mathbb{P}^3$.

1. Compute explicitly $\mathrm{A}(\mathbb{P}^2)$ (resp. $\mathrm{A}(\mathbb{P}^3)$) by expressing the products of Schubert classes $[\Omega_w]$ for $w \in \{e, s_1, s_2s_1\}$ (resp. $w \in \{e, s_1, s_2s_1, s_3s_2s_1\}$) in terms of Schubert classes.
2. Use the labelling of Schubert varieties Ω_λ via partitions λ for
$$\lambda \in \left\{\emptyset, \square, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right\}, \text{resp. } \lambda \in \left\{\emptyset, \square, \begin{smallmatrix}\square\\\square\end{smallmatrix}, \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right\} \tag{24}$$
and consider the Littlewood–Richardson coefficients
$$[\Omega_\lambda][\Omega_\mu] = \sum_\nu \mathrm{c}_{\lambda,\mu}^\nu [\Omega_\nu]. \tag{25}$$
Show that
$$\mathrm{c}_{\lambda,\mu}^\nu = \#\{\text{semistandard (skew) tableaux of shape } \nu/\lambda \text{ of weight } \mu\}. \tag{26}$$
3. Show that if $|\lambda| + |\mu| = |\nu|$ that
$$\mathrm{c}_{\lambda,\mu}^\nu = [S(\nu) : S(\lambda) \otimes S(\mu)] \tag{27}$$
where $S(\lambda)$ for λ a partition of n denotes the corresponding irreducible S_n-module.

(5 points)

Problem 44. Calculate or describe via a picture the geometric representation for the Coxeter group (W, S) attached to

$$\bullet\!-\!\!-\!\!-\!\!-\!\bullet$$

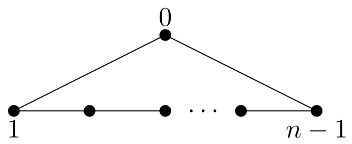
and

$$\bullet\overset{3}{-\!\!-\!\!-\!\!-}\bullet$$

(2 points)

Problem 45. Consider the following set of periodic permutations of the integers
$$\left\{ f\colon \mathbb{Z} \to \mathbb{Z} \mid f(i+n) = f(i) + n, \sum_{i=1}^n f(i) = \sum_{i=1}^n i, f \text{ is bijective} \right\}. \tag{28}$$

Show that this defines (via composition of maps) a group which is isomorphic to the Coxeter group with Coxeter diagram



(3 points)

Problem 46. Consider the Coxeter group (W, S) associated to

with $S = \{s, t\}$.

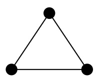
1. Describe all the elements of W.
2. Show that $W \cong \mathrm{S}_2 \ltimes \mathbb{Z}$ as abstract groups.
3. Realise W as an affine reflection group. For this, consider the subspace $U = \mathbb{R}\alpha_s$ of the geometric representation and the affine hyperplanes

   $$H_{\alpha,r} \coloneqq \{\lambda \in U \mid 2(\lambda, \alpha) = r\}. \tag{29}$$

   Show that W is isomorphic to the affine reflection group generated by $s_\alpha = s_{\alpha,0}$ and $s_{\alpha,1}$, where

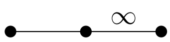
   $$s_{\alpha,r} \colon \lambda \mapsto \lambda - (2(\lambda, \alpha) - r)\alpha \tag{30}$$

   is the affine reflection in the hyperplane $H_{\alpha,r}$.
4. Give an analogous description for the Coxeter group (W, S) associated to



(6 points)

Optional problem 3. Show that the Coxeter group attached to the Coxeter diagram



is isomorphic to $\mathrm{PGL}_2(\mathbb{Z}) \coloneqq \mathrm{GL}_2(\mathbb{Z})/\{\pm 1\}$.

Here you can use that

1. $\mathrm{PGL}_2(\mathbb{Z})$ is generated by the order two elements

   $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{31}$$

   which gives a morphism $\varphi \colon W \to \mathrm{PGL}_2(\mathbb{Z})$;
2. the subgroup $\mathrm{PSL}_2(\mathbb{Z})$ of index 2 is the free product of the groups of order 2 and 3 generated by $\varphi(s_1 s_2)$ and $\varphi(s_2 s_3)$.

(4 points)