Representation Theory I
Bonn, summer term 2019

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Sheet 8

Solutions to be handed in before class on Wednesday May 29

Problem 37. Let R be a fixed root system of rank r in the space V. Let α, β be non-zero elements of V, and $\theta = \theta_{\alpha,\beta}$ the corresponding angle. Then $(\alpha, \beta^\vee) = \|\alpha\|\|\beta\| \cos\theta$. Denoting $\langle \beta, \alpha^\vee \rangle = 2(\beta, \alpha^\vee)/(\alpha, \alpha^\vee)$ we get $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = 4\cos^2\theta$.

1. Using the axioms of a root system, make a table of the possible values of $\langle \alpha, \beta^\vee \rangle$, $\langle \beta, \alpha^\vee \rangle$, θ and $\|\beta\|^2/\|\alpha\|^2$. (1 point)

   Notice that one can assume that $\alpha \neq \pm\beta$, and $\|\beta\| \geq \|\alpha\|$.

2. Using the pictures from problem 35, explain how the maximal $\langle \beta, \alpha^\vee \rangle$ is realised by drawing the root system and explaining which choice of α and β realises this scenario. (1 point)

3. Let α and β be non-proportional roots. If $(\alpha, \beta^\vee) > 0$ then $\alpha - \beta$ is again a root. (2 points)

Problem 38. We continue the setup of the previous problem. Let α and β be non-proportional roots. The *α-string through β* is the set of roots of the form $\beta + n\alpha$ for $n \in \mathbb{Z}$.

1. Denote $r, q \geq 0$ the largest integers for which $\beta - r\alpha$ (resp. $\beta + q\alpha$) is still a root. Show that the string is unbroken, i.e. that it contains all elements between $\beta - r\alpha$ and $\beta + q\alpha$. (2 points)

2. Show that $r - q = \langle \beta, \alpha^\vee \rangle$. (1 point)

3. What is the maximal length of a root string? (1 point)

Problem 39. An element of a basis π for a root system R is called a *simple* root.

1. Let α be such a simple root. Show that the reflection s_α permutes the positive roots other than α. (2 points)

   Hint Consider the coefficient of α versus the coefficients of the other simple roots before and after applying s_α.

2. Let $\rho = \frac{1}{2}\sum_{\beta \in R^+} \beta$ be the half-sum of positive roots. Show that $\mathrm{s}_\alpha(\rho) = \rho - \alpha$ for α a simple root. (1 point)

Problem 40. Let k be of characteristic zero. Assuming the existence of bases for root systems, show that there exists no semisimple Lie algebras of dimension 4 or 5. (2 points)

Problem 41. Assume $R \subset V$ is a root system of rank 2, with basis α, β. Consider the space $V_{\mathbb{R}}$, with the hyperplanes H_{s_α} given by the reflections s_α, with $\alpha \in R$. Draw a picture.

Let $\mathcal{A}$ be the set of connected components of $V_{\mathbb{R}} \setminus \{H_{s_\alpha} \mid \alpha \in R\}$. Show that the Weyl group $W = W(R, V)$ acts transitively on $\mathcal{A}$. Deduce from this a bijection between W and $\mathcal{A}$ by picking an arbitrary element in $\mathcal{A}$ corresponding to $e \in W$. Illustrate this in a picture! (3 points)

We will give occasionally some optional problems for additional credit. These can be solved using computer algebra, in particular using Sage. You can use the online version at `https://cocalc.com`.

The documentation for root systems can be found at `http://doc.sagemath.org/html/en/reference/combinat/sage/combinat/root_system/__init__.html`.

Optional problem 1. Fill in the following table. We use $\alpha_1, \ldots, \alpha_n$ the simple roots in the usual construction of the root system as implemented in Sage, so that $\beta \in R$ is $\sum_{i=1}^{n} k_i \alpha_i$, and the height $\mathrm{ht}(\beta)$ of a root β is the sum of the coefficients in the simple roots. (1 point)

Dynkin type	$\#\{\beta \in R^+ \mid k_1 \geq 1\}$	$\#\{\beta \in R^+ \mid \mathrm{ht}(\beta) \geq 10\}$
A_{10}		
F_4		
E_6		
E_7		
E_8		