5.13)

Proof Idea

We will show $USELESS_{TM}$ is undecidable by reducing E_{TM} to it. Recall that E_{TM} is the set of all Turing machines that accept no strings.

A Turing machine in E_{TM} (i.e. one that does not accept any strings) will always have an useless state, namely q_{accept}. It may also have some other useless states, but if we use the universal turing machine described on page 174 to simulate the machine, the construction ensures the only two potential useless states are q_{accept} and q_{reject}. We can alter the alphabet of the of the inputted Turing machine with a new symbol and have the simulation reject any string that contains this new symbol. Then the only possible useless state is q_{accept}, since there will always be strings the machine will reject. So if the resulting machine has a useless state, then no string ever forces it into the accept state. Likewise, if it doesn't have a useless state, then some string must cause it to reach an accept state. This is the key to the reduction

Proof

Assume $USELESS_{TM}$ is decidable and let T be some decider for it. Now consider the following decider for E_{TM}.

$S =$ On input $< M >$

1. Create the following TM

   $U =$ "On input x:

   Simulate M on input x using the universal Turing machine mentioned in Theorem 4.11 modified according to the proof idea above.
2. Run decider T for $USELESS_{TM}$ on U
3. *Accept* if U accepts, else *reject*.

This decider is a contradiction, so the decider T must be impossible. Hence $USELESS_{TM}$ is undecidable.