Assignment #1
Math 584A

John Cohen

Due September 10th at 1 am (via Gradescope)

For uploading to Gradescope, it will be easiest to put each solution on a different page. The code for this is commented out in the tex file.

Problem 1. *Fix any two points $x, y \in \mathbb{R}^n$. Show that*

$$\lim_{p\to\infty} d_p(x,y) = d_\infty(x,y).$$

Solution. Let $i \leq n$ be defined such that $|x_i - y_i| \geq |x_m - y_m|$ for all $m \leq n$. By definition, $|x_i - y_i| = d_\infty(x,y)$. Notice that

$$|x_i - y_i| \leq |x_1 - y_1| + \ldots + |x_i - y_i| + \ldots + |x_n - y_n| \leq n|x_i - y_i|.$$

Thus,

$$|x_i - y_i|^p \leq |x_1 - y_1|^p + \ldots + |x_i - y_i|^p + \ldots + |x_n - y_n|^p \leq n|x_i - y_i|^p.$$

By taking each expression in the inequality to the $1/p$ power, we get

$$(|x_i - y_i|^p)^{\frac{1}{p}} \leq (|x_1 - y_1|^p + \ldots + |x_i - y_i|^p + \ldots + |x_n - y_n|^p)^{\frac{1}{p}} \leq (n|x_i - y_i|^p)^{\frac{1}{p}},$$

and

$$|x_i - y_i| \leq d_p(x,y) \leq n^{\frac{1}{p}}|x_i - y_i|. \tag{1}$$

Taking the limit as p approaches infinity gives

$$\lim_{p\to\infty} |x_i - y_i| \leq \lim_{p\to\infty} d_p(x,y) \leq \lim_{p\to\infty} n^{\frac{1}{p}}|x_i - y_i|.$$

Thus,

$$d_\infty(x,y) \leq \lim_{p\to\infty} d_p(x,y) \leq d_\infty(x,y).$$

Therefore, by the trichotomy,

$$\lim_{p\to\infty} d_p(x,y) = d_\infty(x,y).$$

Problem 2. *Fix $p_1, p_2 \in [1, \infty]$ such that $p_1 < p_2$ (we allow the case $p_2 = \infty$). Find constants $\underline{C}, \overline{C} > 0$ such that*

$$\underline{C} d_{p_1}(\bar{x}, \bar{y}) \leq d_{p_2}(\bar{x}, \bar{y}) \leq \overline{C} d_{p_1}(\bar{x}, \bar{y}),$$

for all $x, y \in \mathbb{R}^N$. Hint: you may find it helpful to do the case $p_1 = 1$ and $p_2 = \infty$ first.

Solution. The corresponding $d_{p_1}(\bar{x}, \bar{y})$ for p_1. Let $d_\infty(\bar{x}, \bar{y}) = |x_i - y_i|$ for some $i \leq N$. By (1), we see that

$$n^{-\frac{1}{p_1}} d_{p_1}(x, y) \leq |x_i - y_i|.$$

By setting $\underline{C} = n^{-\frac{1}{p_1}}$, we get $\underline{C} d_{p_1}(\bar{x}, \bar{y}) \leq d_\infty(\bar{x}, \bar{y})$. Since the choice of $p_1 \geq 1$ was arbitrary, this is valid for any $d_p(\bar{x}, \bar{y})$.

For the same $d_\infty(\bar{x}, \bar{y})$, factoring out $|x_i - y_i|^{p_1}$ from the expression for $d_{p_1}(\bar{x}, \bar{y})$ gives

$$\begin{aligned} d_{p_1}(\bar{x}, \bar{y}) &= \left[|x_i - y_i|^{p_1} \left(\frac{|x_1 - y_1|^{p_1}}{|x_i - y_i|^{p_1}} + \ldots + \frac{|x_i - y_i|^{p_1}}{|x_i - y_i|^{p_1}} + \ldots \frac{|x_N - y_N|^{p_1}}{|x_i - y_i|^{p_1}} \right) \right]^{1/p_1} \\ &= |x_i - y_i| \left[\left(\frac{|x_1 - y_1|}{|x_i - y_i|} \right)^{p_1} + \ldots + 1 + \ldots \left(\frac{|x_N - y_N|}{|x_i - y_i|} \right)^{p_1} \right]^{1/p_1}. \end{aligned}$$

Let $a_m = \frac{|x_m - y_m|}{|x_i - y_i|}$ for each $m \leq N$. Since $|x_i - y_i| \geq |x_m - y_m|$, $a_m < 1$ for all $m \leq N$. Thus,

$$d_{p_1}(\bar{x}, \bar{y}) = |x_i - y_i| (a_1^{p_1} + \ldots + 1 + \ldots + a_N^{p_1})^{1/p_1}. \tag{2}$$

Since we set $d_\infty(\bar{x}, \bar{y}) = |x_i - y_i|$,

$$d_{p_1}(\bar{x}, \bar{y}) = d_\infty(\bar{x}, \bar{y}) (a_1^{p_1} + \ldots + 1 + \ldots + a_N^{p_1})^{1/p_1}.$$

Because $a_1^{p_1} + \ldots + 1 + \ldots + a_N^{p_1} > 1$, $(a_1^{p_1} + \ldots + 1 + \ldots + a_N^{p_1})^{1/p_1} > 1$. Therefore,

$$d_\infty(\bar{x}, \bar{y}) \leq d_{p_1}(\bar{x}, \bar{y}).$$

Thus,

$$\underline{C} d_{p_1}(\bar{x}, \bar{y}) \leq d_\infty(\bar{x}, \bar{y}) \leq \overline{C} d_{p_1}(\bar{x}, \bar{y}) \tag{3}$$

where $\underline{C} = n^{-\frac{1}{p_1}}$ and $\overline{C} = 1$.

Consider the p_2 metric on the same vectors $\bar{x}$ and $\bar{y}$. Equation (2) for p_2 gives

$$d_{p_2}(\bar{x}, \bar{y}) = |x_i - y_i| (a_1^{p_2} + \ldots + 1 + \ldots + a_N^{p_2})^{1/p_2}.$$

Since $p_2 > p_1$ and $a_m < 1$, $a_m^{p_2} \leq a_m^{p_1}$ and

$$(a_1^{p_2} + \ldots + 1 + \ldots + a_N^{p_2})^{1/p_2} \leq a_1^{p_1} + \ldots + 1 + \ldots + a_N^{p_1})^{1/p_1}.$$

Therefore,

$$d_{p_2}(\bar{x}, \bar{y}) \leq d_{p_1}(\bar{x}, \bar{y}).$$

Combining this with (3) gives

$$n^{-\frac{1}{p_1}} d_{p_1}(\bar{x}, \bar{y}) \leq d_\infty(\bar{x}, \bar{y}) \leq d_{p_2}(\bar{x}, \bar{y}) \leq d_{p_1}(\bar{x}, \bar{y}).$$

Thus,

$$\underline{C} d_{p_1}(\bar{x}, \bar{y}) \leq d_{p_2}(\bar{x}, \bar{y}) \leq \overline{C} d_{p_1}(\bar{x}, \bar{y})$$

where $\underline{C} = n^{-\frac{1}{p_1}}$ and $\overline{C} = 1$.

Problem 3. *Consider the metric space (X, d_{disc}), where X is a nonempty set. Suppose that $(x_n)_n$ is a sequence in X. Show that*

$$\lim_{n\to\infty} x_n = x$$

if and only if there exists N such that $x_n = x$ for all $n \geq N$.

Solution. Fix any ϵ such that $0 < \epsilon < 1$. Since $\lim_{n\to\infty} x_n = x$, there must exist $N \in \mathbb{N}$ such that $d_{disc}(x_n, x) < \epsilon$ whenever $n \geq N$. Since we are in the discrete metric, $d_{disc}(x_n, x) < 1$ only when $x_n = x$. Therefore, $x_n = x$ for all $n \geq N$.

Assume there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n = n$. Thus, for all $n \geq N$, $d_{disc}(x,x) = 0$. Therefore, $d_{disc}(x,x) < \epsilon$ for any $\epsilon > 0$. Thus, $\lim_{n\to\infty} x_n = x$.

Problem 4. *Prove Young's inequality: for any $p, q > 1$ such that*

$$1 = \frac{1}{p} + \frac{1}{q}$$

for any $x, y \in \mathbb{R}$,

$$xy \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}.$$

Hint: for any fixed $y \geq 0$, consider the function $f : [0, \infty) \to \mathbb{R}$, defined by

$$f(x) = \frac{x^p}{p} + \frac{y^q}{q} - xy,$$

and show that the minimum of f is zero.

Solution. Consider first the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{x^p}{p} + \frac{y^q}{q} - xy.$$

For a fixed $y \geq 0$, we will show that the minimum of this function is zero, and thus

$$\frac{x^p}{p} + \frac{y^q}{q} > xy.$$

Checking first the bounds at $x = 0$ and the limit as $x \to \infty$ gives

$$f(0) = \frac{y^q}{q} \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty.$$

Since $y \geq 0$, $f(0) \geq 0$.

Taking the derivative and second derivative of $f(x)$ with respect to x gives

$$f'(x) = x^{p-1} - y \text{ and } f'' = (p-1)x^{p-2}.$$

Solving for $f'(x) = 0$ gives a critical point at $x = y^{\frac{q}{p}}$. Since $p > 1$ and $x \geq 0$, $f'' \geq 0$ for all $x \in [0, \infty)$. This critical point is therefore, a local minimum. Evaluating f at this value of x gives

$$\begin{aligned}
f(x = y^{\frac{q}{p}}) &= \frac{(y^{\frac{q}{p}})^p}{p} + \frac{y}{q} - y^{\frac{q}{p}} y \\
&= \frac{y^q}{p} + \frac{y^q}{q} - y^{\frac{q+p}{p}} \\
&= y^q \left(\frac{1}{p} + \frac{1}{q} \right) - y^{\frac{q+p}{p}} \\
&= y^q - y^{\frac{q+p}{p}}
\end{aligned}$$

To continue, we need a rearrangement of Young's Inequality.

$$\begin{aligned}
1 &= \frac{1}{p} + \frac{1}{q} \\
&= \frac{q}{pq} + \frac{p}{pq} \\
&= \frac{p+q}{pq}
\end{aligned}$$

Thus,

$$\begin{aligned} f(x = y^{\frac{q}{p}}) &= y^q - y^{q(\frac{q+p}{pq})} \\ &= y^q - y^{q(1)} \\ &= 0. \end{aligned}$$

Therefore, zero is the absolute minimum of this function. Since an identical argument can be made for y, $f(x) \geq 0$ for all $x, y \geq 0$. In this case, $x = |x|$ and $y = |y|$, and

$$|x||y| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}.$$

Finally, by Cauchy-Schwarz inequality,

$$xy \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}.$$

for any $x, y \in \mathbb{R}$.

Problem 5. *Fix $N \geq 1$ and $p > 1$. Let q be as in problem 4. Show the following:*

(i) Show that, for any $\bar{x}, \bar{y} \in \mathbb{R}^N$, we have

$$x \cdot y \leq \frac{|x|_p^p}{p} + \frac{|y|_q^q}{q}.$$

Recall that

$$|x|_p = \left(|x_1|^p + \cdots + |x_N|^p\right)^{1/p},$$

and similarly for $|\cdot|_q$.

(ii) Show that d_p is a metric on $\mathbb{R}^N$. You may find it helpful to write, for any $i = 1, \ldots, N$,

$$|x_i - y_i|^p = (x_i - y_i) z_i$$

for a well-chosen z_i.

Solution. [Part (i)] The dot product of the vectors $\bar{x}$ and $\bar{y}$ is

$$\bar{x} \cdot \bar{y} = x_1 y_1 + \ldots + x_N y_N.$$

Since p and q satisfy Young's Inequality,

$$x_n y_n \leq \frac{|x_n|^p}{p} + \frac{|y_n|^q}{q}.$$

for $n \leq N$. Thus,

$$\bar{x} \cdot \bar{y} \leq \left(\frac{|x_1|^p}{p} + \frac{|y_1|^q}{q}\right) + \ldots + \left(\frac{|x_N|^p}{p} + \frac{|y_N|^q}{q}\right).$$

Combining the terms containing x and the terms containing y gives

$$\begin{aligned}\bar{x} \cdot \bar{y} &\leq \left(\frac{|x_1|^p + \ldots + |x_N|^p}{p}\right) + \left(\frac{|y_1|^q + \ldots + |y_N|^q}{q}\right) \\ &\leq \frac{|\bar{x}|_p^p}{p} + \frac{|\bar{y}|_q^q}{q}.\end{aligned}$$

Solution. [Part (ii)] To show that d_p is a metric on $\mathbb{R}^N$, recall the definition of d_p for any $p > 1$:

$$d_p(\bar{x}, \bar{y}) = \left(|x_1 - y_1|^p + \cdots + |x_N - y_n|^p\right)^{1/p}.$$

Fix any $n \leq N$. Since, $|x_n - y_n| = 0$ only when $x_n = y_n$, $|x_n - y_n|^p = 0$ only when $x_n = y_n$. Since the choice of n was arbitrary $|x_1 - y_1|^p + \cdots + |x_N - y_n|^p = 0$ only if $x_n = y_n$ for all $n \leq N$. The same can also be said about that sum raised to the $1/p$ power, thus satisfying positive definiteness.

Consider

$$d_p(\bar{y}, \bar{x}) = \left(|y_1 - x_1|^p + \cdots + |y_N - x_n|^p\right)^{1/p}.$$

Since $|a-b| = |b-a|$ for all $a, b \in \mathbb{R}$, $d_p(\bar{y}, \bar{x}) = d_p(\bar{x}, \bar{y})$. Therefore, $d_p(\bar{y}, \bar{x})$ satisfies the symmetry property of metric spaces.

To prove the Triangle inequality, we first prove the Cauchy-Schwarz inequality for this system. Fix $\epsilon \geq 0$ to be chosen and consider the product $\epsilon\bar{x} \cdot \bar{y}/\epsilon$. By part (i) of this problem we have

$$\epsilon\bar{x} \cdot \frac{\bar{y}}{\epsilon} \leq \frac{\epsilon^p |\bar{x}|_p^p}{p} + \frac{|\bar{y}|_q^q}{\epsilon^q q}.$$

We wish to find the value of epsilon for which this upper bound is minimized. Taking the derivative with respect to epsilon and setting that value equal to zero gives

$$\begin{aligned} 0 &= \frac{p\epsilon^{p-1}|\bar{x}|_p^p}{p} + \frac{-q|\bar{y}|_q^q}{\epsilon^{q+1} q} \\ &= \epsilon^{p-1}|\bar{x}|_p^p - |\bar{y}|_q^q \epsilon^{-q-1}. \end{aligned}$$

Rearranging and solving for epsilon gives

$$\begin{aligned} |\bar{y}|_q^q \epsilon^{-q-1} &= \epsilon^{p-1}|\bar{x}|_p^p, \\ \frac{|\bar{y}|_q^q}{|\bar{x}|_p^p} &= \epsilon^{p-1}\epsilon^{q+1}, \\ \left(\frac{|\bar{y}|_q^q}{|\bar{x}|_p^p}\right)^{\frac{1}{p+q}} &= \epsilon. \end{aligned}$$

Using eq 3, this becomes

$$\left(\frac{|\bar{y}|_q^q}{|\bar{x}|_p^p}\right)^{\frac{1}{pq}} = \epsilon.$$

Using this value for epsilon in our original inequality gives and simplifying gives

$$\begin{aligned} \epsilon\bar{x} \cdot \frac{\bar{y}}{\epsilon} &\leq \left[\left(\frac{|\bar{y}|_q^q}{|\bar{x}|_p^p}\right)^{\frac{1}{pq}}\right]^p \frac{|\bar{x}|_p^p}{p} + \left[\left(\frac{|\bar{y}|_q^q}{|\bar{x}|_p^p}\right)^{\frac{1}{pq}}\right]^{-q} \frac{|\bar{y}|_q^q}{q} \\ &= \left(\frac{|\bar{y}|_q^q}{|\bar{x}|_p^p}\right)^{\frac{1}{q}} \frac{|\bar{x}|_p^p}{p} + \left(\frac{|\bar{y}|_q^q}{|\bar{x}|_p^p}\right)^{\frac{-1}{p}} \frac{|\bar{y}|_q^q}{q} \\ &= \left(\frac{|\bar{y}|_q^{q/q}}{|\bar{x}|_p^{p/q}}\right) \frac{|\bar{x}|_p^p}{p} + \left(\frac{|\bar{y}|_q^{-q/p}}{|\bar{x}|_p^{-p/p}}\right) \frac{|\bar{y}|_q^q}{q} \\ &= \frac{|\bar{y}|_q |\bar{x}|_p^{p-p/q}}{p} + \frac{|\bar{x}|_p |\bar{y}|_q^{q-q/p}}{q} \end{aligned}$$

To simplify further consider a rearrangement of Young's Inequality:

$$\begin{aligned} 1 &= \frac{1}{p} + \frac{1}{q} \\ 1 - \frac{1}{p} &= \frac{1}{q} \\ q - \frac{q}{p} &= 1. \end{aligned}$$

By similar steps, the same inequality holds if the positions of p and q are swapped. Continuing with the simplification gives

$$\begin{aligned}
\epsilon\bar{x}\cdot\frac{\bar{y}}{\epsilon} &\leq \frac{|\bar{y}|_q|\bar{x}|_p^1}{p} + \frac{|\bar{x}|_p|\bar{y}|_q^1}{q} \\
&= |\bar{y}|_q|\bar{x}|_p\left(\frac{1}{p}+\frac{1}{q}\right) \\
&= |\bar{y}|_q|\bar{x}|_p.
\end{aligned}$$

To prove the triangle inequality for d_p, we start by proving $|\bar{x}+\bar{y}|_p \leq |\bar{x}_p| + |\bar{y}_p|$ for any $\bar{x},\bar{y} \in \mathbb{R}^N$. Let $|x_i+y_i|^p = (x_i+y_i)z_i$. For this to hold $z_i = |x_i+y_i|^{p-1}sign(x_i+y_i)$ where

$$sign(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Thus,

$$\begin{aligned}
|\bar{x}+\bar{y}|_p^p &= |x_1+y_1|^p + \ldots + |x_1+y_1|^p \\
&= (x_1+y_1)z_1 + \ldots + (x_1+y_1)z_N \\
&= (x_1z_1 + \ldots + x_Nz_N) + (y_1z_1 + \ldots + y_Nz_N) \\
&= \bar{x}\cdot\bar{z} + \bar{y}\cdot\bar{z} \\
&\leq |\bar{x}|_p|\bar{z}|_q + |\bar{y}|_p|\bar{z}|_q \\
&= (|\bar{x}|_p| + |\bar{y}|_p)|\bar{z}|_q.
\end{aligned}$$

We now compute $|\bar{z}|_q$.

$$\begin{aligned}
|\bar{z}|_q &= \left(\sum_{i=1}^{N}\left[|x_i+y_i|^{p-1}sign(x_i+y_i)\right]^q\right)^{1/q} \\
&= \left(\sum_{i=1}^{N}|x_i+y_i|^{q(p-1)}\right)^{1/q}.
\end{aligned}$$

The $sign(x_i+y_i)$ terms can be removed within the absolute value as $|-1| = |1|$ and if $sign(x_i+y_i) = 0$, $|x_i+y_i|^{p-1} = 0$ as well. Thus, the values are unchanged. By eq 3,

$$\begin{aligned}
qp - q &= (p+q) - q \\
&= p.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\bar{z}|_q &= \left(\sum_{i=1}^{N}|x_i+y_i|^p\right)^{1/q} \\
&= |\bar{x}+\bar{y}|_p^{p/q}
\end{aligned}$$

Returning to our expression for $|\bar{x}+\bar{y}|_p^p$,

$$\begin{aligned}
|\bar{x}+\bar{y}|_p^p &\leq (|\bar{x}|_p| + |\bar{y}|_p)|\bar{z}|_q \\
&= (|\bar{x}|_p| + |\bar{y}|_p)|\bar{x}+\bar{y}|_p^{p/q}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\bar{x}|_p| + |\bar{y}|_p &\geq \frac{|\bar{x}+\bar{y}|_p^p}{|\bar{x}+\bar{y}|_p^{p/q}} \\
&= |\bar{x}+\bar{y}|_p^{p-p/q} \\
&= |\bar{x}+\bar{y}|_p.
\end{aligned}$$

Since $p - {}^p/_q = 1$ as shown in eq 3,

$$|\bar{x}|_p| + |\bar{y}|_p \geq |\bar{x}+\bar{y}|_p.$$

Consider again d_p.

$$\begin{aligned}
d_p(\bar{x},\bar{z}) &= |\bar{x}-\bar{z}|_p \\
&= |(\bar{x}-\bar{y}) + (\bar{y}-\bar{z})|_p.
\end{aligned}$$

By the above inequality,

$$\begin{aligned}
d_p(\bar{x},\bar{z}) &\leq |\bar{x}-\bar{y}|_p + |\bar{y}-\bar{z}|_p \\
&\leq d_p(\bar{x},\bar{y}) + d_p(\bar{y},\bar{z}).
\end{aligned}$$

Thus, d_p satisfies the triangle inequality, and is therefore a metric.

Problem 6. *Suppose that (X, d) is a metric space and $x, y, z \in X$. Show that*

$$d(x, z) \geq |d(x, y) - d(y, z)|.$$

Problem 7. *Show that the following functions are continuous:*

(i) $\psi : \mathbb{R} \to \mathbb{R}$ *defined by* $\psi(x) = 2x^3 + 1$.

(ii) $E : C(\mathbb{R}) \to \mathbb{R}$ *defined by* $E(f) = f(0)$

(iii) $\mathcal{S} : C(\mathbb{R}) \to C(\mathbb{R})$ *defined by* $\mathcal{S}(f) = f^2$

You may find it helpful to note that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. Also, for (iii), please show that $\mathcal{S}$ is well-defined; that is, show that $f^2 \in C(\mathbb{R})$.

Problem 8. *Suppose that d_1, d_2 are equivalent metrics on a set X. Suppose that (Y, d_Y) is a metric space and $f : (X, d_1) \to (Y, d_Y)$ is continuous. Show that $\tilde{f} : (X, d_2) \to (Y, d_Y)$, defined by $\tilde{f}(x) = f(x)$ for all $x \in X$, is continuous.*