Assignment #1
Math 584A

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Due September 10th at 1 am (via Gradescope)

For uploading to Gradescope, it will be easiest to put each solution on a different page. The code for this is commented out in the tex file.

Problem 1. *Fix any two points* $x, y \in \mathbb{R}^n$. *Show that*

$$\lim_{p\to\infty} d_p(x,y) = d_\infty(x,y).$$

Solution. Let $i \leq n$ be defined such that $|x_i - y_i| \geq |x_m - y_m|$ for all $m \leq n$. By definition, $|x_i - y_i| = d_\infty(x,y)$. Notice that

$$|x_i - y_i| \leq |x_1 - y_1| + \ldots + |x_i - y_i| + \ldots + |x_n - y_n| \leq n|x_i - y_i|.$$

Thus,

$$|x_i - y_i|^p \leq |x_1 - y_1|^p + \ldots + |x_i - y_i|^p + \ldots + |x_n - y_n|^p \leq n|x_i - y_i|^p.$$

By taking the each expression in the inequality to the $1/p$ power, we get

$$(|x_i - y_i|^p)^{\frac{1}{p}} \leq (|x_1 - y_1|^p + \ldots + |x_i - y_i|^p + \ldots + |x_n - y_n|^p)^{\frac{1}{p}} \leq (n|x_i - y_i|^p)^{\frac{1}{p}},$$

and

$$|x_i - y_i| \leq d_p(x,y) \leq n^{\frac{1}{p}}|x_i - y_i|.$$

Taking the limit as p approaches infinity gives

$$\lim_{p\to\infty} |x_i - y_i| \leq \lim_{p\to\infty} d_p(x,y) \leq \lim_{p\to\infty} n^{\frac{1}{p}}|x_i - y_i|.$$

Thus,

$$d_\infty(x,y) \leq \lim_{p\to\infty} d_p(x,y) \leq d_\infty(x,y).$$

Therefore, by the Squeeze Theorem,

$$\lim_{p\to\infty} d_p(x,y) = d_\infty(x,y).$$

Problem 2. *Fix $p_1, p_2 \in [1, \infty]$ such that $p_1 < p_2$ (we allow the case $p_2 = \infty$). Find constants $\underline{C}, \overline{C} > 0$ such that*

$$\underline{C} d_{p_1}(\bar{x}, \bar{y}) \leq d_{p_2}(\bar{x}, \bar{y}) \leq \overline{C} d_{p_1}(\bar{x}, \bar{y}),$$

for all $x, y \in \mathbb{R}^N$. Hint: you may find it helpful to do the case $p_1 = 1$ and $p_2 = \infty$ first.

Problem 3. *Consider the metric space (X, d_{disc}), where X is a nonempty set. Suppose that $(x_n)_n$ is a sequence in X. Show that*

$$\lim_{n\to\infty} x_n = x$$

if and only if there exists N such that $x_n = x$ for all $n \geq N$.

Problem 4. *Prove Young's inequality: for any $p, q > 1$ such that*

$$1 = \frac{1}{p} + \frac{1}{q}$$

for any $x, y \in \mathbb{R}$,

$$xy \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}.$$

Hint: for any fixed $y \geq 0$, consider the function $f : [0, \infty) \to \mathbb{R}$, defined by

$$f(x) = \frac{x^p}{p} + \frac{y^q}{q} - xy,$$

and show that the minimum of f is zero.

Problem 5. *Fix $N \geq 1$ and $p > 1$. Let q be as in problem 4. Show the following:*

(i) Show that, for any $\bar{x}, \bar{y} \in \mathbb{R}^N$, we have

$$x \cdot y \leq \frac{|x|_p^p}{p} + \frac{|y|_q^q}{q}.$$

Recall that

$$|x|_p = \left(|x_1|^p + \cdots + |x_N|^p\right)^{1/p},$$

and similarly for $|\cdot|_q$.

(ii) Show that d_p is a metric on $\mathbb{R}^N$. You may find it helpful to write, for any $i = 1, \ldots, N$,

$$|x_i - y_i|^p = (x_i - y_i) z_i$$

for a well-chosen z_i.

Problem 6. *Suppose that (X, d) is a metric space and $x, y, z \in X$. Show that*

$$d(x, z) \geq |d(x, y) - d(y, z)|.$$

Problem 7. *Show that the following functions are continuous:*

(i) $\psi : \mathbb{R} \to \mathbb{R}$ defined by $\psi(x) = 2x^3 + 1$.

(ii) $E : C(\mathbb{R}) \to \mathbb{R}$ *defined by* $E(f) = f(0)$

(iii) $\mathcal{S} : C(\mathbb{R}) \to C(\mathbb{R})$ *defined by* $\mathcal{S}(f) = f^2$

You may find it helpful to note that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. *Also, for (iii), please show that* $\mathcal{S}$ *is well-defined; that is, show that* $f^2 \in C(\mathbb{R})$.

Problem 8. *Suppose that* d_1, d_2 *are equivalent metrics on a set* X. *Suppose that* (Y, d_Y) *is a metric space and* $f : (X, d_1) \to (Y, d_Y)$ *is continuous. Show that* $\tilde{f} : (X, d_2) \to (Y, d_Y)$, *defined by* $\tilde{f}(x) = f(x)$ *for all* $x \in X$, *is continuous.*