# Assignment #1
# Math 584A

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Due September 10th at 1 am (via Gradescope)

For uploading to Gradescope, it will be easiest to put each solution on a different page. The code for this is commented out in the tex file.

**Problem 1.** *Fix any two points $x, y \in \mathbb{R}^n$. Show that*

$$\lim_{p\to\infty} d_p(x,y) = d_\infty(x,y).$$

*Solution.* Let $i \le n$ be defined such that $|x_i - y_i| \ge |x_m - y_m|$ for all $m \le n$. By definition, $|x_i - y_i| = d_\infty(x,y)$. Notice that

$$|x_i - y_i| \le |x_1 - y_1| + \ldots + |x_i - y_i| + \ldots + |x_n - y_n| \le n|x_i - y_i|.$$

Thus,

$$|x_i - y_i|^p \le |x_1 - y_1|^p + \ldots + |x_i - y_i|^p + \ldots + |x_n - y_n|^p \le n|x_i - y_i|^p.$$

By taking each expression in the inequality to the $1/p$ power, we get

$$(|x_i - y_i|^p)^{\frac{1}{p}} \le (|x_1 - y_1|^p + \ldots + |x_i - y_i|^p + \ldots + |x_n - y_n|^p)^{\frac{1}{p}} \le (n|x_i - y_i|^p)^{\frac{1}{p}},$$

and

$$|x_i - y_i| \le d_p(x,y) \le n^{\frac{1}{p}}|x_i - y_i|. \tag{1}$$

Taking the limit as p approaches infinity gives

$$\lim_{p\to\infty} |x_i - y_i| \le \lim_{p\to\infty} d_p(x,y) \le \lim_{p\to\infty} n^{\frac{1}{p}}|x_i - y_i|.$$

Thus,

$$d_\infty(x,y) \le \lim_{p\to\infty} d_p(x,y) \le d_\infty(x,y).$$

Therefore, by the Squeeze Theorem,

$$\lim_{p\to\infty} d_p(x,y) = d_\infty(x,y).$$

**Problem 2.** *Fix $p_1, p_2 \in [1,\infty]$ such that $p_1 < p_2$ (we allow the case $p_2 = \infty$). Find constants $\underline{C}, \overline{C} > 0$ such that*

$$\underline{C} d_{p_1}(\bar{x},\bar{y}) \le d_{p_2}(\bar{x},\bar{y}) \le \overline{C} d_{p_1}(\bar{x},\bar{y}),$$

*for all $x, y \in \mathbb{R}^N$. Hint: you may find it helpful to do the case $p_1 = 1$ and $p_2 = \infty$ first.*

*Solution.* The corresponding $d_{p_1}(\bar{x},\bar{y})$ for $p_1$. Let $d_\infty(\bar{x},\bar{y}) = |x_i - y_i|$ for some $i \le N$. By (1), we see that

$$n^{-\frac{1}{p_1}} d_{p_1}(x,y) \le |x_i - y_i|.$$

By setting $\underline{C} = n^{-\frac{1}{p_1}}$, we get $\underline{C} d_{p_1}(\bar{x},\bar{y}) \le d_\infty(\bar{x},\bar{y})$. Since the choice of $p_1 \ge 1$ was arbitrary, this is valid for any $d_p(\bar{x},\bar{y})$.

For the same $d_\infty(\bar{x},\bar{y})$, factoring out $|x_i - y_i|^{p_1}$ from the expression for $d_{p_1}(\bar{x},\bar{y})$ gives

$$\begin{aligned} d_{p_1}(\bar{x},\bar{y}) &= \left[|x_i - y_i|^{p_1}\left(\frac{|x_1-y_1|^{p_1}}{|x_i-y_i|^{p_1}} + \ldots + \frac{|x_i-y_i|^{p_1}}{|x_i-y_i|^{p_1}} + \ldots \frac{|x_N-y_N|^{p_1}}{|x_i-y_i|^{p_1}}\right)\right]^{1/p_1} \\ &= |x_i - y_i|\left[\left(\frac{|x_1-y_1|}{|x_i-y_i|}\right)^{p_1} + \ldots + 1 + \ldots \left(\frac{|x_N-y_N|}{|x_i-y_i|}\right)^{p_1}\right]^{1/p_1}. \end{aligned}$$

Let $a_m = \frac{|x_m - y_m|}{|x_i - y_i|}$ for each $m \le N$. Since $|x_i - y_i| \ge |x_m - y_m|$, $a_m < 1$ for all $m \le N$. Thus,

$$d_{p_1}(\bar{x},\bar{y}) = |x_i - y_i|(a_1^{p_1} + \ldots + 1 + \ldots + a_N^{p_1})^{1/p_1}. \tag{2}$$

Since we set $d_\infty(\bar{x},\bar{y}) = |x_i - y_i|$,

$$d_{p_1}(\bar{x},\bar{y}) = d_\infty(\bar{x},\bar{y})(a_1^{p_1} + \ldots + 1 + \ldots + a_N^{p_1})^{1/p_1}.$$

Because $a_1^{p_1} + \ldots + 1 + \ldots + a_N^{p_1} > 1$, $(a_1^{p_1} + \ldots + 1 + \ldots + a_N^{p_1})^{1/p_1} > 1$. Therefore,

$$d_\infty(\bar{x},\bar{y}) \le d_{p_1}(\bar{x},\bar{y}).$$

Thus,

$$\underline{C} d_{p_1}(\bar{x},\bar{y}) \le d_\infty(\bar{x},\bar{y}) \le \overline{C} d_{p_1}(\bar{x},\bar{y}) \tag{3}$$

where $\underline{C} = n^{-\frac{1}{p_1}}$ and $\overline{C} = 1$.

Consider the $p_2$ metric on the same vectors $\bar{x}$ and $\bar{y}$. Equation (2) for $p_2$ gives

$$d_{p_2}(\bar{x},\bar{y}) = |x_i - y_i|(a_1^{p_2} + \ldots + 1 + \ldots + a_N^{p_2})^{1/p_2}.$$

Since $p_2 > p_1$ and $a_m < 1$, $a_m^{p_2} \le a_m^{p_1}$ and

$$(a_1^{p_2} + \ldots + 1 + \ldots + a_N^{p_2})^{1/p_2} \le a_1^{p_1} + \ldots + 1 + \ldots + a_N^{p_1})^{1/p_1}.$$

Therefore,

$$d_{p_2}(\bar{x},\bar{y}) \le d_{p_1}(\bar{x},\bar{y}).$$

Combining this with (3) gives

$$n^{-\frac{1}{p_1}} d_{p_1}(\bar{x},\bar{y}) \le d_\infty(\bar{x},\bar{y}) \le d_{p_2}(\bar{x},\bar{y}) \le d_{p_1}(\bar{x},\bar{y}).$$

Thus,

$$\underline{C} d_{p_1}(\bar{x},\bar{y}) \le d_{p_2}(\bar{x},\bar{y}) \le \overline{C} d_{p_1}(\bar{x},\bar{y})$$

where $\underline{C} = n^{-\frac{1}{p_1}}$ and $\overline{C} = 1$.

**Problem 3.** *Consider the metric space* $(X, d_{\text{disc}})$, *where* $X$ *is a nonempty set. Suppose that* $(x_n)_n$ *is a sequence in* $X$. *Show that*

$$\lim_{n\to\infty} x_n = x$$

*if and only if there exists* $N$ *such that* $x_n = x$ *for all* $n \geq N$.

*Solution.* Fix any $\epsilon$ such that $0 < \epsilon < 1$. Since $\lim_{n\to\infty} x_n = x$, there must exist $N \in \mathbb{N}$ such that $d_{disc}(x_n, x) < \epsilon$ whenever $n \geq N$. Since we are in the discrete metric, $d_{disc}(x_n, x) < 1$ only when $x_n = x$. Therefore, $x_n = x$ for all $n \geq N$.

Assume there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n = n$. Thus, for all $n \geq N$, $d_{disc}(x,x) = 0$. Therefore, $d_{disc}(x,x) < \epsilon$ for any $\epsilon > 0$. Thus, $\lim_{n\to\infty} x_n = x$.

**Problem 4.** *Prove Young's inequality: for any $p, q > 1$ such that*

$$1 = \frac{1}{p} + \frac{1}{q}$$

*for any $x, y \in \mathbb{R}$,*

$$xy \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}.$$

*Hint: for any fixed $y \geq 0$, consider the function $f : [0, \infty) \to \mathbb{R}$, defined by*

$$f(x) = \frac{x^p}{p} + \frac{y^q}{q} - xy,$$

*and show that the minimum of $f$ is zero.*

**Problem 5.** *Fix $N \geq 1$ and $p > 1$. Let $q$ be as in problem 4. Show the following:*

*(i) Show that, for any $\bar{x}, \bar{y} \in \mathbb{R}^N$, we have*

$$x \cdot y \leq \frac{|x|_p^p}{p} + \frac{|y|_q^q}{q}.$$

*Recall that*

$$|x|_p = \left(|x_1|^p + \cdots + |x_N|^p\right)^{1/p},$$

*and similarly for $|\cdot|_q$.*

*(ii) Show that $d_p$ is a metric on $\mathbb{R}^N$. You may find it helpful to write, for any $i = 1, \ldots, N$,*

$$|x_i - y_i|^p = (x_i - y_i)z_i$$

*for a well-chosen $z_i$.*

**Problem 6.** *Suppose that $(X, d)$ is a metric space and $x, y, z \in X$. Show that*

$$d(x, z) \geq |d(x, y) - d(y, z)|.$$

**Problem 7.** *Show that the following functions are continuous:*

*(i) $\psi : \mathbb{R} \to \mathbb{R}$ defined by $\psi(x) = 2x^3 + 1$.*

*(ii) $E : C(\mathbb{R}) \to \mathbb{R}$ defined by $E(f) = f(0)$*

*(iii) $\mathcal{S} : C(\mathbb{R}) \to C(\mathbb{R})$ defined by $\mathcal{S}(f) = f^2$*

*You may find it helpful to note that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. Also, for (iii), please show that $\mathcal{S}$ is well-defined; that is, show that $f^2 \in C(\mathbb{R})$.*

**Problem 8.** *Suppose that $d_1, d_2$ are equivalent metrics on a set $X$. Suppose that $(Y, d_Y)$ is a metric space and $f : (X, d_1) \to (Y, d_Y)$ is continuous. Show that $\tilde{f} : (X, d_2) \to (Y, d_Y)$, defined by $\tilde{f}(x) = f(x)$ for all $x \in X$, is continuous.*