Assignment #2
Math 584A

Due September 17th, 1 am (via Gradescope)

For uploading to Gradescope, it will be easiest to put each solution on a different page. The code for this is commented out in the tex file.

Problem 1. *Let (X, d_X) and (Y, d_Y) be metric spaces that are both subsets of some larger linear space Z; that is, $X, Y \subset Z$. Consider the following potential metrics on $X \cap Y$:*

(i) $d_p(v_1, v_2) = (d_X(v_1, v_2)^p + d_Y(v_1, v_2)^p)^{\frac{1}{p}}$ *for a fixed* $p \in [1, \infty)$,

(ii) $d_{\max}(v_1, v_2) = \max\{d_X(v_1, v_2), d_Y(v_1, v_2)\}$, *and*

(iii) $d_{\min}(v_1, v_2) = \min\{d_X(v_1, v_2), d_Y(v_1, v_2)\}$.

Each of these is well-defined in that it is finite for any $v_1, v_2 \in X \cap Y$. Which are metrics? Give a proof or counterexample for each.

Solution. What follows is a solution to part (i). By construction, $d_p(v_1, v_2)$ can only be zero if both $d_X(v_1, v_2)$ and $d_Y(v_1, v_2)$ are zero, and will be positive if both $d_X(v_1, v_2)$ and $d_Y(v_1, v_2)$. Since both d_X and d_Y are metrics, they satisfy positive definiteness property. Thus, d_p also satisfies positive definiteness.

Since both d_X and d_Y are metrics,

$$d_X(v_1, v_2) = d_X(v_2, v_1) \quad \text{and} \quad d_Y(v_1, v_2) = d_Y(v_2, v_1).$$

Therefore,

$$\begin{aligned} d_p(v_2, v_1) &= (d_X(v_2, v_1)^p + d_Y(v_2, v_1)^p)^{\frac{1}{p}} \\ &= (d_X(v_1, v_2)^p + d_Y(v_1, v_2)^p)^{\frac{1}{p}} \\ &= d_p(v_1, v_2). \end{aligned}$$

Thus, d_p satisfies the symmetry property.

Using the triangle inequality on the metrics d_X and d_Y yields

$$d_p(v_1, v_2) \leq ([d_X(v_1, v_3) + d_X(v_3, v_2)]^p + [d_Y(v_1, v_3) + d_Y(v_3, v_2)]^p)^({}^1\!/\!_p). \tag{1}$$

Using the definition

$$|x|_p = (|x_1|^p + \cdots + |x_N|^p)^{1/p},$$

notice that for $\bar{v} = (d_X(v_1, v_2), d_Y(v_1, v_2))$,

$$d_p(v_1, v_2) = |\bar{z}|_p.$$

Let vectors $\bar{x}$ and $\bar{y}$ be two vectors defined by

$$\bar{x} = (d_X(v_1, v_3), d_Y(v_1, v_3)) \quad \text{and} \quad \bar{y} = (d_X(v_3, v_2), d_Y(v_3, v_2)).$$

Applying our definition for $|x|_p$ gives

$$|\bar{x} + \bar{y}|_p = (|x_1 + y_1|^p + |x_2 + y_2|^p)^{1/p}.$$

Substituting in the values for each vector gives

$$|\bar{x} + \bar{y}|_p = (|d_X(v_1, v_3) + d_X(v_3, v_2)|^p + |d_Y(v_1, v_3) + d_Y(v_3, v_2)|^p)^{1/p}.$$

Substituting in with (1) gives

$$d_p(v_1, v_2) \leq |\bar{x} + \bar{y}|_p. \tag{2}$$

Since we proved in a previous homework that $|\cdot|_p$ satisfies the triangle inequality,

$$|\bar{x} + \bar{y}|_p \leq |\bar{x}|_p + |\bar{y}|_p.$$

Substituting in the values for each vector gives

$$\begin{aligned}|\bar{x} + \bar{y}|_p &\leq |d_X(v_1, v_3) + d_Y(v_1, v_3)|_p + |d_X(v_3, v_2) + d_Y(v_3, v_2)|_p \\ &= d_p(v_1, v_3)^p + d_p(v_3, v_2)^p\end{aligned}$$

Combining with (2) gives

$$d_p(v_1, v_2) \leq d_p(v_1, v_3)^p + d_p(v_3, v_2)^p,$$

thus satisfying the triangle inequality. Therefore, d_p is a metric.

What follows is a solution to part (ii). Since d_X and d_Y are metrics

$$\max\{d_X(v_1, v_2), d_Y(v_1, v_2)\} \geq 0.$$

And since $d_X(v_1, v_2), d_Y(v_1, v_2) = 0$ if and only if $v_1 = v_2$, the same must be true for $d_{max}(v_1, v_2)$. Thus, d_{max} satisfies positive definiteness.

Consider $d_{\max}(v_2, v_1)$. But the symmetry property of d_X and d_Y

$$\begin{aligned}d_{\max}(v_2, v_1) &= \max\{d_X(v_2, v_1), d_Y(v_2, v_1)\} \\ &= \max\{d_X(v_1, v_2), d_Y(v_1, v_2)\} \\ &= d_{\max}(v_1, v_2).\end{aligned}$$

Therefore, $d_{\max}$ satisfies the symmetry property.

Using the triangle property on d_X and d_Y gives

$$d_{\max}(v_1, v_2) \leq \max\{(d_X(v_1, v_3) + d_X(v_3, v_2)), (d_Y(v_1, v_3) + d_Y(v_3, v_2))\}.$$

Notice that

$$
\begin{aligned}
d_X(v_1, v_3) + d_X(v_3, v_2) &\leq \max\{d_X(v_1, v_3), d_Y(v_1, v_3)\} + d_X(v_3, v_2) \\
&\leq \max\{d_X(v_1, v_3), d_Y(v_1, v_3)\} + \max\{d_X(v_3, v_2), d_Y(v_3, v_2)\} \\
&= d_{\max}(v_1, v_3) + d_{\max}(v_3, v_2)
\end{aligned}
$$

The same is also true for the sum $d_Y(v_1, v_3) + d_Y(v_3, v_2)$. Thus,

$$\max\left\{\left(d_X(v_1, v_3) + d_X(v_3, v_2)\right), \left(d_Y(v_1, v_3) + d_Y(v_3, v_2)\right)\right\} \leq d_{\max}(v_1, v_3) + d_{\max}(v_3, v_2),$$

and

$$d_{\max}(v_1, v_2) \leq d_{\max}(v_1, v_3) + d_{\max}(v_3, v_2).$$

Therefore, $d_{\max}$ satisfies the triangle property and is a metric.

What follows is a counterexample to the proposed metric in part (iii). Let $v_1 = (1, 1)$ and $v_2 =$

Problem 2. *Fix a metric space (X, d). Show that if $U_1, U_2, \ldots$ are open subsets of X then so is*

$$\bigcup_{i=1}^{\infty} U_i = \{u \in X : \text{ there exists } i \in \mathbb{N} \text{ such that } u \in U_i\}.$$

Solution. Fix any

$$x \in \bigcup_{i=1}^{\infty} U_i.$$

By the definition of union, there exists some i_x such that $x \in U_{i_x}$. Since each U_i is open, there exists $B_{r_x}(x)$ where $B_{r_x}(x) \subset U_{i_x}$. Because

$$U_{i_x} \subset \bigcup_{i=1}^{\infty} U_i,$$

it must be true that

$$B_{r_x}(x) \subset \bigcup_{i=1}^{\infty} U_i.$$

Since the choice of x was arbitrary, the intersection of an arbitrary number of open sets is open.

Problem 3. *Let (X, d) be a metric space where X is nonempty and d is the discrete metric.*

(i) Classify all continuous functions $f : \mathbb{R} \to X$ (you can take $\mathbb{R}$ to have the Euclidean metric).

(ii) Classify all continuous functions $f : X \to \mathbb{R}$.

Solution. Let $f : \mathbb{R} \to X$ be continuous and let $\varepsilon = 1$. Fix $x_0 \in \mathbb{R}$. By definition there exists some $\delta > 0$ such that whenever $|x - x_0| < \delta$ for some other $x \in \mathbb{R}$, $d_{disc}(f(x), f(x_0)) < 1$. However, this implies that $f(x) = f(x_0)$. Since the choice of x_0 was arbitrary, f must be the constant function.

Let $f : X \to \mathbb{R}$ be continuous. Fix $x_0 \in \mathbb{R}$ and consider when $|f(x) - f(x_0)| < \varepsilon$ for some $\varepsilon > 0$. By the definition of continuity, there must be some $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $d_{disc}(x, x_0) < \delta$. This is clearly true for the case when $x = x_0$, but when $x \neq x_0$, $d_{disc}(x, x_0) = 1$. Therefore, whenever $|f(x) - f(x_0)| < \varepsilon$ and $x \neq x_0$, $d_{disc}(x, x_0) = 1$. Thus $\delta > 1$ in order to satisfy the definition of continuity. But since $d_{disc}(x, x_0) = 1$ for all $x \neq x_0$, $|f(x) - f(x_0)| < \varepsilon$ for all $x \in X$.

Problem 4. *Fix any non-negative function $K \in L^1_{\text{prel}}(\mathbb{R})$. Define a function*

$$T : L^1_{\text{prel}}([0,1]) \to L^1_{\text{prel}}([0,1])$$

by, for every $f \in L^1_{\text{prel}}([0,1])$,

$$(Tf)(x) = \int_0^1 K(x-y)f(y)dy.$$

(i) Show that T is well-defined (i.e. $Tf \in L^1_{\text{prel}}([0,1])$ whenever $f \in L^1_{\text{prel}}([0,1])$). Note: you can exchange the order of integration freely in this problem. We will justify this later in the course.

Also, you may find it helpful to show that if $f \in L^1_{\text{prel}}$, there are nonnegative $f_+, f_- \in L^1_{\text{prel}}$ such that $f = f_+ - f_-$.

(ii) Show that T is continuous.

(iii) Is T uniformly continuous?

Solution. We begin proving $Tf \in L^1_{\text{prel}}([0,1])$ by first showing that Tf is bounded. Let $f \in L^1_{\text{prel}}([0,1])$. By definition of the $L^1_{\text{prel}}([0,1])$ norm,

$$\begin{aligned}
||(Tf)(x)||_{L^1_{\text{prel}}([0,1])} &= \int_0^1 \left| \int_0^1 K(x-y)f(y)dy \right| dx \\
&= \int_0^1 \int_0^1 |K(x-y)f(y)|dydx \\
&\le \int_0^1 \int_0^1 |K(x-y)||f(y)|dydx \\
&= \int_0^1 |f(y)| \int_0^1 |K(x-y)|dxdy \\
&\le \int_0^1 |f(y)| \int_{-\infty}^{\infty} |K(x-y)|dxdy.
\end{aligned}$$

Since $K \in L^1_{\text{prel}}(\mathbb{R})$, there exists some $K \in R$ such that

$$\int_{-\infty}^{\infty} |K(x)|dx \le K.$$

Thus,

$$\int_0^1 |f(y)| \int_{-\infty}^{\infty} |K(x-y)|dxdy \le K \int_0^1 |f(y)|dy.$$

And since, $f \in L^1_{\text{prel}}([0,1])$, there exists some $F \in R$ such that

$$\int_0^1 |f(y)|dy \le F.$$

Thereby,

$$||(Tf)(x)||_{L^1_{\text{prel}}([0,1])} \le KF,$$

and Tf must be bounded.

To show that Tf is continuous, fix any $\varepsilon > 0$ and $x_0 \in \mathbb{R}$. For some other $x \in \mathbb{R}$,

$$\begin{aligned} |Tf(x) - Tf(x_0)| &= \left| \int_0^1 K(x-y)f(y)dy - \int_0^1 K(x_0-y)f(y)dy \right| \\ &= \left| \int_0^1 \left(K(x-y) - K(x_0-y)\right) f(y)dy \right| \\ &\leq \int_0^1 |(K(x-y) - K(x_0-y))| \, |f(y)|dy. \end{aligned}$$

Since $K \in L^1_{\text{prel}}(\mathbb{R})$, K must be continuous. Thus for F as defined earlier, there must exist some $\delta > 0$ such that $|x - x_0| < \delta$ implies $|K(x) - K(x_0)| < \varepsilon/F$. Notice that

$$\begin{aligned} |x - x_0| &= |x - y + y - x_0| \\ &= |(x-y) - (x_0 - y)|, \end{aligned}$$

and let $|x - x_0| < \delta$. Then,

$$\begin{aligned} |Tf(x) - Tf(x_0)| &\leq \frac{\varepsilon}{F} \int_0^1 |f(y)|dy \\ &\leq \frac{\varepsilon}{F}(F) \\ &= \varepsilon. \end{aligned}$$

Because the choice of f was arbitrary, Tf is continuous. Consequently, $Tf \in L^1_{\text{prel}}([0,1])$.

To show that T is continuous, fix any $\varepsilon > 0$, and let $f \in L^1_{\text{prel}}([0,1])$. Let $g \in L^1_{\text{prel}}([0,1])$ such that $||f - g||_{L^1_{\text{prel}}([0,1])} < \varepsilon/K$ where K is the same value defined above. Then,

$$\begin{aligned} ||Tf(x) - Tg(x)||_{L^1_{\text{prel}}([0,1])} &= \int_0^1 \left| \int_0^1 K(x-y)f(y)dy - \int_0^1 K(x-y)g(y)dy \right| dx \\ &= \int_0^1 \left| \int_0^1 K(x-y)(f(y) - g(y))dy \right| dx \\ &\leq \int_0^1 \int_0^1 |K(x-y)||(f(y) - g(y))|dydx. \end{aligned}$$

Using the same value for K defined above,

$$\begin{aligned} ||Tf(x) - Tg(x)||_{L^1_{\text{prel}}([0,1])} &\leq K \int_0^1 |(f(y) - g(y))|dy \\ &= K||f - g||_{L^1_{\text{prel}}([0,1])}. \end{aligned}$$

But by assumption,

$$\begin{aligned} ||Tf(x) - Tg(x)||_{L^1_{\text{prel}}([0,1])} &\leq K\frac{\varepsilon}{K} \\ &= \varepsilon. \end{aligned}$$

Since the choice of f was arbitrary, T is continuous.

Since the value of δ only depends on K, T and not on the choice of f or g, T is uniformly continuous.

Problem 5. *Fix a metric space* (X, d)*. Suppose that* $x_0 \in X$ *and* $r > 0$*. Show that* $B_r(x_0)$ *is an open set.*

Solution. The first case is if $B_r(x_0)$ is empty. But the empty set is open.

The second case is when $B_r(x_0)$ is nonempty. Let $y \in B_r(x_0)$. Thus, $d(x_0, y) < r$. There must then exist $s \in \mathbb{R}$ such that

$$d(x_0, y) + s < r.$$

Let $B_s(y)$ be the open ball centered at y with radius s. Notice that $B_s(y)$ cannot be empty as $y \in B_s(y)$ by definition. Let $z \in B_s(y)$ be any element of $B_s(y)$. Thus, $d(y, z) < s$. Therefore, $d(x_0, y) + d(y, z) < r$, and by the triangle inequality

$$d(x_0, z) \leq d(x_0, y) + d(y, z) < r.$$

Therefore, $z \in B_r(x_0)$. Since the choice of z was arbitrary, $B_r(x_0)$ is open.

Problem 6. *Fix any constant $\theta \in \mathbb{R}$ and let x_n be a convergent sequence. Show that*

$$\lim_{n\to\infty} \theta x_n = \theta \lim_{n\to\infty} x_n.$$

Solution. Since the sequence (x_n) converges, let

$$\lim_{n\to\infty} x_n = x.$$

Then for any x_n, consider $|\theta x_n - \theta x| = |\theta||x_n - x|$. Since, (x_n) converges to x, there exists some $N \in \mathbb{N}$ such that whenever $n \geq N$,

$$|x_n - x| < \frac{\varepsilon}{|\theta|}.$$

Thus, for $n \geq N$,

$$\begin{aligned} |\theta x_n - \theta x| &< |\theta| \frac{\varepsilon}{|\theta|} \\ &= \varepsilon. \end{aligned}$$

Therefore, θx is the limit of the sequence (θx_n).

Problem 7. *Suppose that (X, d) is a metric space and $f, g : (X, d) \to \mathbb{R}$ are continuous functions.*

(i) Show that $f + g$ is continuous.

(ii) Show that $f \cdot g$ is continuous.

(iii) If f, g are uniformly continuous, show that $f + g$ is uniformly continuous.

(iv) If f, g are uniformly continuous, is $f \cdot g$ uniformly continuous?

Solution. To show that $f + g$ is continuous, fix some $\varepsilon > 0$ and some $x_0 \in X$. For some other $x \in X$,

$$\begin{aligned}|(f(x) + g(x)) - (f(x_0) + g(x_0))| &= |(f(x) - f(x_0)) + (g(x) - g(x_0))| \\ &\leq |f(x) - f(x_0)| + |g(x) - g(x_0)|.\end{aligned}$$

Since both f and g are continuous, there exists δ_f and δ_g such that $|f(x) - f(x_0)| < \varepsilon/2$ whenever $d(x, x_0) < \delta_f$ and $|g(x) - g(x_0)| < \varepsilon/2$ whenever $d(x, x_0) < \delta_g$. Let $\delta = \max\{\delta_f, \delta_g\}$. Then, $d(x, x_0) < \delta$ implies

$$\begin{aligned}|(f(x) + g(x)) - (f(x_0) + g(x_0))| &\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Since the choice of x_0 was arbitrary, $f + g$ is continuous.

For part (ii), fix some $\varepsilon > 0$ and some $x_0 \in X$. For some other $x \in X$,

$$\begin{aligned}|f(x)g(x) - f(x_0)g(x_0)| &= |f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) + f(x_0)g(x_0)| \\ &= |g(x)(f(x) - f(x_0)) + f(x_0)(g(x) - g(x_0))| \\ &\leq |g(x)(f(x) - f(x_0))| + |f(x_0)(g(x) - g(x_0))| \\ &\leq |g(x)||f(x) - f(x_0)| + |f(x_0)||g(x) - g(x_0)|.\end{aligned}$$

Since both f and g are continuous, there exists δ_f and δ_g such that $|f(x) - f(x_0)| < \varepsilon_f$ whenever $d(x, x_0) < \delta_f$, and $|g(x) - g(x_0)| < \varepsilon_f$ whenever $d(x, x_0) < \delta_g$. Let $\delta = \max\{\delta_f, \delta_g\}$ and let $d(x, x_0) < \delta$. To get an upper bound for $|g(x)|$, let $A = (x_0 - \delta, x_0 + \delta)$ be an interval and let

$$G = \sup_{x \in A} |g(x)|.$$

Let

$$\varepsilon_f = \frac{\varepsilon}{2G} \quad \text{and} \quad \varepsilon_g = \frac{\varepsilon}{2|f(x_0)|}.$$

Then $d(x, x_0) < \delta$ implies,

$$\begin{aligned}|f(x)g(x) - f(x_0)g(x_0)| &\leq G\frac{\varepsilon}{2G} + |f(x_0)|\frac{\varepsilon}{2|f(x_0)|} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Since the choice of x_0 was arbitrary, $f \cdot g$ is continuous.

For part (iii), fix any $\varepsilon > 0$ and any $x, y \in X$. Then,

$$\begin{aligned}|(f(x) + g(x)) - (f(y) + g(y))| &= |(f(x) - f(y)) + (g(x) - g(y))| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)|.\end{aligned}$$

Since both f and g are uniformly continuous, there exists δ_f and δ_g such that $|f(x) - f(y)| < \varepsilon/2$ whenever $d(x, y) < \delta_f$ and $|g(x) - g(y)| < \varepsilon/2$ whenever $d(x, y) < \delta_g$. Let $\delta = \max\{\delta_f, \delta_g\}$. Then, $d(x, x_0) < \delta$ implies

$$\begin{aligned}|(f(x) + g(x)) - (f(x_0) + g(x_0))| &\leq |f(x) - f(x_0)| + |g(x) - g(x_0)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Since the choice of x and y was arbitrary, and in neither case does the choice of δ depend on x or y, $f + g$ is uniformly continuous.

To show that $f \cdot g$ is not necessarily uniformly continuous, let $f, g = x$. We demonstrated in class that this function is uniformly continuous. Assume for contradiction that $f \cdot g = x^2$ is uniformly continuous. Then there exists δ such that for any $x, y \in X$ where $d(x, y) < \delta$, $|x^2 - y^2| < 1$. Let $y = x + \delta/2$. Then,

$$\begin{aligned}1 &> |x^2 - y^2| \\ &= |x^2 - (x + \delta/2)^2| \\ &= |x^2 - (x^2 + x\delta + \delta^2/4)| \\ &= |x\delta + \delta^2/4|.\end{aligned}$$

Since δ is positive, this value is maximized when x is positive. In this case, the absolute value can be dropped and rearranging for x yields

$$\frac{1 - \delta^2/4}{\delta} > x.$$

This inequality sets an upper bound for x, and therefore does not apply for all x and y in $\mathbb{R}$. Therefore, $f \cdot g$ is not uniformly continuous, despite f and g both being uniformly continuous.