APPLIED MATH BOOT CAMP 2024

1. The natural numbers and induction

What are the natural numbers anyways?

Axiom 1.1 (Peano's Postulates)**.** *The natural numbers are defined as a set $\mathbb{N}$ together with a unary "successor" function $S : \mathbb{N} \to \mathbb{N}$ and a special element $1 \in \mathbb{N}$ satisfying the following postulates:*

- *I.* $1 \in \mathbb{N}$.
- *II.* *If $n \in \mathbb{N}$, then $S(n) \in \mathbb{N}$.*
- *III.* *There is no $n \in \mathbb{N}$ such that $S(n) = 1$.*
- *IV.* *If $n, m \in \mathbb{N}$ and $S(n) = S(m)$, then $n = m$.*
- *V.* *If $A \subset \mathbb{N}$ is a subset satisfying the two properties:*
  - $1 \in A$
  - *if $n \in A$, then $S(n) \in A$,*

  then $A = \mathbb{N}$.

Question 1.2. *Intuitively, what is $S(n)$?*

We take the following theorem for granted (we do not want to get into too much logic...).

Theorem 1.3 (Mathematical Induction)**.** *For each $n \in \mathbb{N}$, let $P(n)$ be a proposition. Suppose the following two results:*

(A) *$P(1)$ is true.*
(B) *If $P(n)$ is true, then $P(S(n))$ is true.*

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Statement (A) is called the base case and statement (B) is called the inductive step. The assumption that P is true of n is called the inductive hypothesis.

Remark 1.4. *From now on we assume that all the usual properties of $\mathbb{N}$ and $\mathbb{Z}$ hold. A list of properties will be posted.*

When you write a proof, you need to clearly declare what the inductive hypothesis is (if it is not obvious from context) and clearly delineate the parts of the proof relating to the base case and inductive step. Here is an example.

Proposition 1.5. *For every $n \in \mathbb{N}$, we have*

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1. \tag{1.1}$$

Proof. We prove this by induction. For the base case, take $n = 0$. Notice that the left hand side of (1.1) is

$$2^0 = 1,$$

while the right hand side is

$$2^{0+1} - 1 = 2 - 1 = 1.$$

These are equal. Thus, the base case is established.

We now prove the inductive step. Assume that (1.1) holds for n, and we prove it for $n+1$. By the inductive hypothesis, we have

$$\sum_{i=0}^{n+1} 2^i = 2^{n+1} + \sum_{i=0}^{n} 2^i = 2^{n+1} + \left(2^{n+1} - 1\right) = 2 \cdot 2^{n+1} - 1 = 2^{n+2} - 1.$$

The proof is complete. □

Exercise 1.6. *Show that*

$$\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}.$$

2. Sets and functions

Sets and functions are among the most fundamental objects in mathematics. A formal treatment of set theory was first undertaken at the end of the 19th Century and was finally codified in the form of the Zermelo-Fraenkel axioms. This goes well beyond our purposes here. Thus, we present a simplified version.'

Definition 2.1. *A* set *is an object S with the property that, given any x, we have the dichotomy that precisely one of the following two conditions is true: $x \in S$ or $x \notin S$. In the former case, we say that x is an* element *of S, and in the latter, we say that x is not an element of S.*

A set is often presented in one of the following forms:

- A complete listing of its elements.
  Example: the set $S = \{1, 2, 3, 4, 5\}$ contains precisely the five smallest positive integers.
- A listing of some of its elements with ellipses to indicate unnamed elements.
  Example 1: the set $S = \{3, 4, 5, \ldots, 100\}$ contains the positive integers from 3 to 100, including 6 through 99, even though these latter are not explicitly named.
  Example 2: the set $S = \{2, 4, 6, \ldots, 2n, \ldots\}$ is the set of all positive even integers.
- A two-part indication of the elements of the set by first identifying the source of all elements and then giving additional conditions for membership in the set.
  Example 1: $S = \{x \in \mathbb{N} \mid x \text{ is prime}\}$ is the set of primes.
  Example 2: $S = \{x \in \mathbb{Z} \mid x^2 < 3\}$ is the set of integers whose squares are less than 3.

Definition 2.2. *Two sets A and B are equal if they contain precisely the same elements, that is, $x \in A$ if and only if $x \in B$. When A and B are equal, we denote this by $A = B$.*

Definition 2.3. *A set A is a* subset *of a set B if every element of A is also an element of B, that is, if $x \in A$, then $x \in B$. When A is a subset of B, we denote this by $A \subset B$. If $A \subset B$ but $A \neq B$ we say that A is a* proper *subset of B.*

Exercise 2.4. *Let $A = \{1, \{2\}\}$. Is $1 \in A$? Is $2 \in A$? Is $\{1\} \subset A$? Is $\{2\} \subset A$? Is $1 \subset A$? Is $\{1\} \in A$? Is $\{2\} \in A$? Is $\{\{2\}\} \subset A$? Explain.*

Proof. Yes, $1 \in A$. No, $2 \notin A$. Yes, $\{1\} \subset A$. No, $\{2\} \not\subset A$. No, $1 \not\subset A$. No, $\{1\} \notin A$. Yes, $\{2\} \in A$. Yes, $\{\{2\}\} \subset A$ because $\{2\}$ is the only element of $\{\{2\}\}$ and $\{2\} \in A$. □

Definition 2.5. *Let A and B be two sets. The* union *of A and B is the set*

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Definition 2.6. *Let A and B be two sets. The* intersection *of A and B is the set*

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Theorem 2.7 (No proof required)**.** *Let A and B be two sets. Then:*

a) $A = B$ *if and only if* $A \subset B$ *and* $B \subset A$.
b) $A \subset A \cup B$.
c) $A \cap B \subset A$.

A special example of the intersection of two sets is when the two sets have no elements in common. This motivates the following definition.

Definition 2.8. *The* empty set *is the set with no elements, and it is denoted* $\emptyset$. *That is, no matter what* x *is, we have* $x \notin \emptyset$.

Definition 2.9. *Two sets* A *and* B *are* disjoint *if* $A \cap B = \emptyset$.

Exercise 2.10. *Show that if* A *is any set, then* $\emptyset \subset A$.

Proof. Since there exists no $x \in \emptyset$ where $x \notin A$, $\emptyset \subset A$. □

Definition 2.11. *Let* A *and* B *be two sets. The* difference *of* B *from* A *is the set*

$$A \setminus B = \{x \in A \mid x \notin B\}.$$

The set $A \setminus B$ is also called the *complement* of B relative to A. When the set A is clear from the context, this set is sometimes denoted B^c, but we will try to avoid this imprecise formulation and use it only with warning.

Theorem 2.12. *Let* X *be a set, and let* $A, B \subset X$. *Then:*
a) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$
b) $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$

Proof. Start with a proof statement a). Let $x \in X \setminus (A \cup B)$. Then $x \notin A \cup B$, which means that

$$x \in (X \setminus A) \text{ and } x \in (X \setminus B).$$

This is the definition of intersection, so

$$x \in (X \setminus A) \cap (X \setminus B).$$

Therefore,

$$X \setminus (A \cup B) \subset (X \setminus A) \cap (X \setminus B).$$

Now let $x \in (X \setminus A) \cap (X \setminus B)$. Then

$$x \in (X \setminus A) \text{ and } x \in (X \setminus B).$$

This means that $x \notin A$ and $x \notin B$. Therefore, $x \notin (A \cup B)$, and $x \in (X \setminus (A \cup B))$. As a result

$$(X \setminus A) \cap (X \setminus B) \subset (X \setminus (A \cup B)).$$

Combining the forward and reverse statements gives $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$.

What follows is a proof of statement b). Let $x \in X \setminus (A \cap B)$. Then,

$$x \notin (A \cap B),$$

and by definition of intersection,

$$x \notin A \text{ or } x \notin B.$$

Therefore,

$$x \in (X \setminus A) \cup (X \setminus B),$$

and

$$(X \setminus A \cap B) \subset (X \setminus A) \cup (X \setminus B).$$

Now I show that the right hand side is a subset of the left hand side. Let $x \in (X \setminus A) \cup (X \setminus B)$. Then,

$$x \notin A \text{ or } x \notin B.$$

Therefore,

$$x \notin (A \cap B) \text{ and } x \in X \setminus (A \cap B).$$

This means that

$$(X \setminus A) \cup (X \setminus B) \subset X \setminus (A \cap B).$$

Combining the forward and reverse relations gives $X \setminus A \cap B = (X \setminus A) \cup (X \setminus B)$. $\square$

Sometimes we will encounter families of sets. The definitions of intersection/union can be extended to infinitely many sets.

Definition 2.13. *Let $\mathcal{A} = \{A_\lambda \mid \lambda \in I\}$ be a collection of sets indexed by a nonempty set I. Then the intersection and union of $\mathcal{A}$ are the sets*

$$\bigcap_{\lambda \in I} A_\lambda = \{x \mid x \in A_\lambda, \text{ for all } \lambda \in I\},$$

and

$$\bigcup_{\lambda \in I} A_\lambda = \{x \mid x \in A_\lambda, \text{ for some } \lambda \in I\}.$$

Note: unions and intersections should almost never be written in-line because doing so causes spacing issues and is difficult to read. Please use display environments.

Exercise 2.14. *Give an example of an indexed family of sets.*

Example. Let $A_\lambda = \{\lambda\}$, and let $\mathcal{A} = \{A_\lambda \mid \lambda \text{ is prime}\}$ $\square$

Theorem 2.15. *Let X be a set, and let $\mathcal{A} = \{A_\lambda \mid \lambda \in I\}$ be a collection of subsets of X. Then:*

$$(1) \quad X \setminus \left(\bigcup_{\lambda \in I} A_\lambda \right) = \bigcap_{\lambda \in I} (X \setminus A_\lambda) \quad \textit{and} \quad (2) \quad X \setminus \left(\bigcap_{\lambda \in I} A_\lambda \right) = \bigcup_{\lambda \in I} (X \setminus A_\lambda).$$

Proof. Begin with a proof of statement (1). Let

$$x \in X \setminus (\bigcup_{\lambda \in I} A_\lambda).$$

This means that $x \in (X \setminus A_\lambda)$ for all $\lambda \in I$. This is the definition of the intersection, so

$$x \in \bigcap_{\lambda \in I} (X \setminus A_\lambda),$$

and

$$X \setminus \left(\bigcup_{\lambda \in I} A_\lambda \right) \subset \bigcap_{\lambda \in I} (X \setminus A_\lambda).$$

To show that the right hand side is a subset of the left hand side, let

$$x \in \bigcap_{\lambda \in I} (X \setminus A_\lambda).$$

This means that $x \notin A_\lambda$ for all $\lambda \in I$. So,

$$x \in (X \setminus (\bigcup_{\lambda \in I} A_\lambda)).$$

Thus,

$$\bigcap_{\lambda \in I} (X \setminus A_\lambda) \subset X \setminus (\bigcup_{\lambda \in I} A_\lambda).$$

Combining the forward and reverse statements gives

$$X \setminus \left(\bigcup_{\lambda \in I} A_\lambda \right) = \bigcap_{\lambda \in I} (X \setminus A_\lambda).$$

The proof for statement (2) follows. Let

$$x \in (X \setminus (\bigcap_{\lambda \in I} A_\lambda)).$$

There must then exist $\lambda' \in I$ where $x \notin A_{\lambda'}$ and $x \in X \setminus A'_\lambda$. Therefore,

$$x \in \bigcup_{\lambda \in I} (X \setminus A_\lambda),$$

and

$$X \setminus (\bigcap_{\lambda \in I} A_\lambda) \subset \bigcup_{\lambda \in I} (X \setminus A_\lambda).$$

To show that the right hand side is a subset of the left hand side, let

$$x \in \bigcup_{\lambda \in I} (X \setminus A_\lambda).$$

By definition of union, there must exist $\lambda' \in I$ such that $x \in X \setminus A_{\lambda'}$ or $x \notin A_{\lambda'}$. This means that

$$x \notin \bigcap_{\lambda \in I} A_\lambda,$$

and

$$x \in (X \setminus \bigcap_{\lambda \in I} A_\lambda).$$

Therefore,

$$\bigcup_{\lambda \in I} (X \setminus A_\lambda) \subset (X \setminus \bigcap_{\lambda \in I} A_\lambda).$$

Combining the forward and reverse statements gives

$$X \setminus (\bigcap_{\lambda \in I} A_\lambda) \subset \bigcup_{\lambda \in I} (X \setminus A_\lambda).$$

□

Definition 2.16. *Let A and B be two nonempty sets. The* Cartesian product *of A and B is the set of ordered pairs*

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

If (a, b) and $(a', b') \in A \times B$, we say that (a, b) and (a', b') are equal *if and only if $a = a'$ and $b = b'$. In this case, we write $(a, b) = (a', b')$.*

Definition 2.17. *Let A and B be two nonempty sets. A* function *f from A to B is a subset $f \subset A \times B$ such that for all $a \in A$ there exists a unique $b \in B$ satisfying $(a, b) \in f$. To express the idea that $(a, b) \in f$, we most often write $f(a) = b$. To express that f is a function from A to B in symbols we write $f \colon A \to B$.*

Exercise 2.18. *Let the function $f \colon \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = 2n$. Describe f as a subset of $\mathbb{N} \times \mathbb{N}$ in two ways (see the discussion below Definition 2.1).*

Solution. The function $f(n) = 2n$ can be represented as a set either $f = \{(1, 2), (2, 4), (3, 6), ...\}$ or $f = \{(n, 2n) \mid \text{for all } n \in \mathbb{N}\}$.

□

Definition 2.19. *Let $f\colon A \to B$ be a function. The* domain *of f is A and the* codomain *of f is B.*
If $X \subset A$, then the image of X under f *is the set*
$$f(X) = \{f(x) \in B \mid x \in X\}.$$
If $Y \subset B$, then the preimage of Y under f *is the set*
$$f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}.$$

Exercise 2.20. *Must $f(f^{-1}(Y)) = Y$ and $f^{-1}(f(X)) = X$? For each, either prove that it always holds or give a counterexample.*

Counterexample. No, $f(f^{-1}(Y)) = Y$ is not always true. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined as $f(n) = 2n$ and let $Y = \{2, 3\}$. Then
$$f^{-1}(Y) = \{1\}$$
as there is no $n \in \mathbb{N}$ for which $2n = 3$. Taking the image gives
$$f(\{1\}) = \{2\},$$
which is different from Y, so the statement $f(f^{-1}(Y)) = Y$ does not hold.

No, $f^{-1}(f(X)) = X$ is not always true. Let $f : \mathbb{N} \to \mathbb{N}$ be a function defined as $f(n) = 1$ and let $X = \{2, 3\}$. Then
$$f(X) = \{1\}.$$
However, as $f(n) = 1$ for all $n \in \mathbb{N}$,
$$f^{-1}(\{1\}) = \mathbb{N}.$$
Since $X \neq \mathbb{N}$, $f^{-1}(f(X)) = X$ does not hold.

□

Definition 2.21. *A function $f\colon A \to B$ is* surjective *(also known as 'onto') if, for every $b \in B$, there is some $a \in A$ such that $f(a) = b$. The function f is* injective *(also known as 'one-to-one') if for all $a, a' \in A$, if $f(a) = f(a')$, then $a = a'$. The function f is* bijective*, (also known as a bijection or a 'one-to-one' correspondence) if it is surjective and injective.*

Exercise 2.22. *Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n^2$. Is f injective? Is f surjective?*

Solution. The function f is injective. Because f is strictly increasing, if
$$n_1 < n_2 \text{ for } n_1, n_2 \in \mathbb{N},$$
then
$$(n_1)^2 < (n_2)^2.$$
Therefore, if
$$(n_1)^2 = (n_2)^2,$$
it must be true that
$$n_1 = n_2.$$

The function f is not surjective because there exists no $n \in \mathbb{N}$ such that $n^2 = 2$. □

Exercise 2.23. *Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n + 2$. Is f injective? Is f surjective?*

Solution. Yes, f is injective. Because f is strictly increasing, if
$$n_1 < n_2 \text{ for } n_1, n_2 \in \mathbb{N},$$
it follows that
$$n_1 + 2 < n_2 + 2.$$
Therefore, if
$$n_1 + 2 = n_2 + 2,$$

it must be true that

$$n_1 = n_2.$$

No, f is not surjective because there exists no $n \in \mathbb{N}$ such that $n + 2 = 1$. □

Exercise 2.24. *Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^2$. Is f injective? Is f surjective?*

Solution. No, f is not injective because $(-1)^2 = 1$ and $1^2 = 1$. No, f is not surjective because there exists no $z \in \mathbb{Z}$ such that $z^2 = -1$. □

Exercise 2.25. *Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 2$. Is f injective? Is f surjective?*

Solution. Yes, f is injective. f is strictly increasing, so if

$$z_1 < z_2 \text{ for } z_1, z_2 \in \mathbb{Z},$$

then

$$z_1 + 2 < z_2 + 2.$$

Therefore, if

$$z_1 + 2 = z_2 + 2,$$

it follows that

$$z_1 = z_2.$$

Yes, f is surjective because for all $z_2 \in \mathbb{Z}$ there exists $z_1 \in \mathbb{Z}$ such that $z_1 + 2 = z_2$. □

Definition 2.26. *Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$. Then the* composition *$g \circ f : A \longrightarrow C$ is defined by $(g \circ f)(x) = g(f(x))$, for all $x \in A$.*

Proposition 2.27. *Let A, B, and C be sets and suppose that $f : A \longrightarrow B$ and $g : B \longrightarrow C$. Then $g \circ f : A \longrightarrow C$ and*

a) if f and g are both injections, so is $g \circ f$.
b) if f and g are both surjections, so is $g \circ f$.
c) if f and g are both bijections, so is $g \circ f$.
d) if f is an injection and g is not, $g \circ f$. is not injective
e) if f is not an injection and g is, $g \circ f$. is not injective
f) if f is a surjection and g is not, $g \circ f$. is not surjective
g) if f is not a surjection and g is, $g \circ f$. is not surjective

Proof. What follows is a proof of statement a). Fix any $a_1, a_2 \in A$ such that $g(f(a_1)) = g(f(a_2))$. Because g is an injection, $f(a_1) = f(a_2)$, and because f is also an injection, $a_1 = a_2$. Therefore, $g(f(a_1)) = g(f(a_2))$ gives $a_1 = a_2$, and $g \circ f$ is an injection.

What follows is a proof of statement b).Given that g is a surjection, for any $c \in C$, there must exist $b \in B$, such that $g(b) = c$. Similarly, because f is a surjection, for all $b \in B$, there must $a \in A$, such that $f(a) = b$. Therefore, for all $c \in C$, $g(b) = c$ as $g(f(a)) = c$. Therefore, $g \circ f$ must also be a surjection.

What follows is a proof of statement c). If f and g are both bijections, they are both injections and surjections. By parts (a) and (b), $g \circ f$ must also be both an injection and surjection, and is therefore a bijection.

To show d) is true, let $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) = x$ for all $x \in \mathbb{N}$, and let $g : \mathbb{N} \to \mathbb{N}$ such that $g(y) = 1$ for all $y \in \mathbb{N}$. Then

$$\begin{aligned} g(f(1)) &= g(1) \\ &= 1. \end{aligned}$$

Also,

$$\begin{aligned} g(f(2)) &= g(2) \\ &= 1. \end{aligned}$$

Thus $g(f(1)) = g(f(2))$, and $g \circ f$ is not an injection.

To show e) is true, let $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) = 1$ for all $x \in \mathbb{N}$, and let $g : \mathbb{N} \to \mathbb{N}$ such that $g(y) = y$ for all $y \in \mathbb{N}$. Then

$$\begin{aligned} g(f(1)) &= g(1) \\ &= 1. \end{aligned}$$

Also,

$$\begin{aligned} g(f(2)) &= g(1) \\ &= 1. \end{aligned}$$

Thus $g(f(1)) = g(f(2))$, and $g \circ f$ is not an injection.

To show that f) is true, let $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) = 2x$ for all $x \in \mathbb{N}$, and let $g : \mathbb{N} \to \mathbb{N}$ such that $g(y) = y$ for all $y \in \mathbb{N}$. 3 is in the codomain of g, and has representation $g(3) = 3$. 3 is also in the codomain of f, but there exists no $x \in \mathbb{N}$ such that $2x = 3$, and by extension there exists no $x \in \mathbb{N}$ such that $(g \circ f)(x) = 3$. Therefore, $g \circ f$ is not a surjection.

To show that g) is true, let $g : \mathbb{N} \to \mathbb{N}$ such that $g(y) = 2y$ for all $y \in \mathbb{N}$. 3 is in the codomain of g, but there exists no $y \in \mathbb{N}$ such that $2y = 3$. Therefore, no matter what $f : A \to \mathbb{N}$ is, $g \circ f$ is not a surjection. □

Proposition 2.28. *Suppose that $f\colon A \to B$ is bijective. Then there exists a bijection $g\colon B \to A$ that satisfies $(g \circ f)(a) = a$ for all $a \in A$, and $(f \circ g)(b) = b$, for all $b \in B$. The function g is often called the* inverse *of f and denoted f^{-1}. It should not be confused with the preimage.*

Proof. Define $g : B \to A$ as follows. Fix $b \in B$. By the bijectivity of f, there exists a unique $a \in A$ such that $f(a_b) = b$. Let $g(b) = a_b$. Applying f to both sides of the definition for g gives

$$\begin{aligned} f(g(b)) &= f(a_b) \\ &= b. \end{aligned}$$

Similarly, applying our definition of g to the equation of $f(a_b) = b$ gives

$$\begin{aligned} g(f(a_b)) &= g(b) \\ &= a_b. \end{aligned}$$

To show that g is an injection, fix $b_1, b_2 \in B$, and set $g(b_1) = g(b_2)$. Applying f gives

$$\begin{aligned} f(g(b_1)) &= f(g(b_2)) \\ b_1 &= b_2. \end{aligned}$$

This satisfies the definition of an injection. To show that g must be a surjection, assume for contradiction that g is not a surjection. That would mean that there exists some $b \in B$, such that $g(b) \neq a$ for all $a \in A$. Applying f to both sides gives

$$\begin{aligned} f(g(b)) &\neq f(a) \\ b &\neq f(a). \end{aligned}$$

This contradicts f as a bijection, so g must be a surjection, and therefore a bijection as well.

□

Definition 2.29. *We say that two sets A and B are in* bijective correspondence *when there exists a bijection from A to B or, equivalently, from B to A.*

3. Building the real numbers

This sheet introduces a continuum C through a series of axioms. We will construct such an object in a bit.

Axiom 3.1. *A continuum is a nonempty set C.*

We often refer to elements of C as *points*.

Definition 3.2. *Let X be a set. An* ordering *on the set X is a subset $<$ of $X \times X$, with elements $(x, y) \in <$ written as $x < y$, satisfying the following properties:*

(a) *(Trichotomy) For all $x, y \in X$ exactly one of the following holds: $x < y$, $y < x$ or $x = y$.*
(b) *(Transitivity) For all $x, y, z \in X$, if $x < y$ and $y < z$ then $x < z$.*

Remark 3.3. *a) In mathematics "or" is understood to be inclusive unless stated otherwise. So in a) above, the word "exactly" is needed.*
b) $x < y$ may also be written as $y > x$.
c) By $x \leq y$, we mean $x < y$ or $x = y$; similarly for $x \geq y$.

Axiom 3.4. *A continuum C has an ordering $<$.*

Definition 3.5. *If $A \subset C$ is a subset of C, then a point $a \in A$ is a* first *point of A if, for every element $x \in A$, either $a < x$ or $a = x$. Similarly, a point $b \in A$ is called a* last *point of A if, for every $x \in A$, either $x < b$ or $x = b$.*

Definition 3.6. *A set A is finite if $A = \emptyset$ or if there exists $n \in \mathbb{N}$ and a bijection $f : A \to \{1, 2, \ldots, n\}$. In the former case, we say that A has no elements, and in the latter case, we say that A has n elements.*

Lemma 3.7. *Fix $n \in \mathbb{N}$. Suppose that A has $n + 1$ elements and that $a \in A$. Then $A \setminus \{a\}$ has n elements.*

Proof. Let $f : \{2, 3, \ldots\} \to \mathbb{N}$ be a function defined as $f(n) = n - 1$. This function is strictly increasing, so if $n_1 < n_2$, it must be true that $n_1 - 1 < n_2 - 1$. Therefore, if $f(n_1) = f(n_2)$, $n_1 = n_2$, and f must be an injection.

The function f must also be a surjection because for all $m \in \mathbb{N}$, there exists $n \in \{2, 3, \ldots\}$ such that $n - 1 = m$.

This subtraction by one function must be a bijection, and therefore for any $n \in \mathbb{N}$ and set A with $n + 1$ elements, $A \setminus \{a\}$ with $a \in A$, has n elements. □

Lemma 3.8. *If A is a nonempty, finite subset of a continuum C, then A has a first and last point.*

Proof. What follows is a proof by induction on a set with n elements. The base case is the set which only has 1 element. This element is both the first and last point of the set.

The induction hypothesis is that any set with $n \in \mathbb{N}$ elements has a first point and last point. Consider any set A with $n + 1$ elements. Now define a new set

$$A' = A \setminus \{a\}, \text{ where } a \in A.$$

By Lemma 3.7, A' has n elements, and by the induction hypothesis has a first point a_f and last point a_f. If $a < a_f$, then a is the first point of A. Similarly, if $a > a_l$, then a is the last point of A. Otherwise, the a_f and a_l values for $A \setminus \{a\}$ are the a_f and a_l values for A.

Therefore, by induction, every set with $n + 1$ elements has a first and last point. □

Theorem 3.9. *Suppose that A is a set of n distinct points in a continuum C, or, in other words, $A \subset C$ has cardinality n. Then symbols $a_1, \ldots, a_n$ may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$, i.e. $a_i < a_{i+1}$ for $1 \leq i \leq n - 1$.*

Proof. What follows is a proof by induction on a set with n elements. The base case is the set that only has 1 element. This one element is ordered and can be designated a_1.

The induction hypothesis is that any set with n elements, where $n \in \mathbb{N}$, has symbols $a_1, \dots, a_n$ that may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$. Consider a set A with $n+1$ elements. By Lemma 3.8, A has a last element $a_l \in A$. Define

$$A' = A \setminus \{a_l\}.$$

Since A' has n elements, by the induction hypothesis, its elements have an ordering from least to greatest, terminating at a_n. Because each element is distinct, for all $a \in A$, $a_l > a$. Therefore, for all $a' \in A'$, $a_l > a'$. Tus, the set A has the same ordering as A' but with $a_l = a_{n+1}$.

Therefore, by induction, any set with n elements can have its elements labeled and ordered from least to greatest. $\square$

Definition 3.10. *If $x, y, z \in C$ and either (i) both $x < y$ and $y < z$ or (ii) both $z < y$ and $y < x$, then we say that y is* between *x and z.*

Corollary 3.11. *Of three distinct points in a continuum, one must be between the other two.*

Proof. Let $A = \{x, y, z\}$ where $x, y, z \in C$ and are distinct. Then, A is a finite subset of C and has a first and last element. Because of the choice of x, y, and z is arbitrary, we can set x as the first point and z as the last point without the loss of generality. By definition 3.10, this must mean that $x < y$ and $y < z$. Therefore, y is between x and z. $\square$

Axiom 3.12. *A continuum C has no first or last point.*

Definition 3.13. *If $a, b \in C$ and $a < b$, then the set of points between a and b is called an* interval*, denoted by (a, b).*

Theorem 3.14. *If x is a point of a continuum C, then there exists an interval (a, b) such that $x \in (a, b)$.*

Proof. Because $x \in C$ and C is a continuum, by Axiom 3.12, it is always possible to find $a, b \in C$ such that $a < x$ and $x < b$. By definition 3.10, x is between a and b, and by definition 3.13, (a, b) constitutes an interval. Therefore, $x \in (a, b)$. $\square$

Definition 3.15. *Let A be a subset of a continuum C. A point p of C is called a* limit point *of A if every interval I containing p has nonempty intersection with $A \setminus \{p\}$. Explicitly, this means:*

$$\text{for every interval } I \text{ with } p \in I, \text{ we have } I \cap (A \setminus \{p\}) \neq \emptyset.$$

Notice that we do not require that a limit point p of A be an element of A. We will use the notation $LP(A)$ to denote the set of limit points of A.

Theorem 3.16. *If p is a limit point of A and $A \subset B$, then p is a limit point of B.*

Proof. Let I be any interval such that $p \in I$. Since p is a limit point of set A, $I \cap (A \setminus \{p\}) \neq \emptyset$. Let $x \in I \cap (A \setminus \{p\})$, so $x \in I$ and $x \in A \setminus \{p\}$. Because $A \subset B$, $A \setminus \{p\} \subset B \setminus \{p\}$. Thus, $x \in B \setminus \{p\}$. Therefore $x \in I \cap (B \setminus \{p\})$ and x must be a limit point of B. $\square$

Definition 3.17. *If (a, b) is an interval in a continuum C, then $C \setminus (\{a\} \cup (a, b) \cup \{b\})$ is called the* exterior *of (a, b) and is denoted by $\operatorname{ext}(a, b)$.*

Lemma 3.18. *If (a, b) is an interval in a continuum C, then*

$$\operatorname{ext}(a, b) = \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}.$$

Proof. Let $x \in \operatorname{ext}(a, b)$. By definition 3.17, $x \notin (a, b)$, and by definition 3.13, is not between a and b. Therefore, $x \leq a$ or $x \geq b$. However $a, b \notin \operatorname{ext}(a, b)$, so $x < a$ or $x > b$. This as a set gives $\operatorname{ext}(a, b) = \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$. $\square$

Lemma 3.19. *No point in the exterior of an interval is a limit point of that interval. No point of an interval is a limit point of the exterior of that interval.*

Proof. Let $p \in \operatorname{ext}(a, b)$. By lemma 3.18, either $p < a$ or $p > b$.

Consider just the case that $p < a$. Because $\operatorname{ext}(a, b)$ is a subset of a continuum C, by axiom 3.12, there must be $x \in C$ such that $x < p$. By definition 3.10, p is between x and a, and $p \in (x, a)$ by definition 3.13. However, the intersection $(x, a) \cap (a, b) = \emptyset$, so by definition 3.15, p cannot be a limit point of (a, b).

The justification is very similar for the case where $p > b$, as there must now exist $x \in C$ such that $x > p$. Thus $p \in (b, x)$, and $(b, x) \cap (a, b) = \emptyset$. Therefore, no point of the exterior of an interval can be a limit point of the interval.

To show that no point of an interval can be a limit point of the exterior of that interval, let $p \in (a, b)$, and by definition of the exterior, $\operatorname{ext}(a, b) \cup (a, b) = \emptyset$. This proves the statement. □

Theorem 3.20. *If two intervals have a point x in common, their intersection is an interval containing x.*

Proof. Let $(a, b), (c, d) \subset C$ with $x \in (a, b)$ and $x \in (c, d)$. This is the definition of the intersection and $x \in (a, b) \cap (c, d)$. By definition 3.10, $x > a$ and $x < b$. Also, $x > c$ and $x < d$. Let the function $\max(x, y)$ for $x, y \in C$ be defined as

$$\max(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{if } x < y, \end{cases}$$

Similarly, let the function $\min(x, y)$ for $x, y \in C$ be defined as

$$\min(x, y) = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } x > y, \end{cases}$$

Since $x > a$ and $x > c$, it must be true that $x > \max(a, c)$. Similarly, as $x < b$ and $x < d$, $x < \min(b, d)$. By definition 3.10, x is between $\max(a, c)$ and $\min(b, d)$, and by definition 3.13, the set of all possible values for x describe the interval $(\max(a, c), \min(b, d))$. □

Corollary 3.21. *If n intervals $I_1, \ldots, I_n$ have a point x in common, then their intersection $I_1 \cap \cdots \cap I_n$ is an interval containing x.*

Proof. This will be proven by induction on the set of intervals I_n all containing x. Let I_1 be one of those intervals. Because every interval contains x, $x \in I_1$.

The induction hypothesis is that the intersection of n intervals which all contain x, is itself an interval that contains x. Let $I_1, \ldots, I_n$ where $x \in I_n$ for all n. Then the intersection,

$$I_{n+1} \cap \bigcap_{i=1}^{n} I_i = \bigcap_{i=1}^{n+1} I_i$$

is the intersection of two intervals, both of which contain x. By theorem 3.20, the intersection

$$\bigcap_{i=1}^{n+1} I_i$$

is also an interval that contains x.

Therefore, by induction, the intersection of any number of intervals containing a point x must also be an interval that contains x. □

Exercise 3.22. *Is Corollary 3.21 true for* infinite *intersections of intervals?*

Counterexample. Consider the set of nested intervals $I_n = (x - 1/n, x + 1/n)$ where $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Every set I_n contains x. Consider any point $x + a$, where $a \in \mathbb{R}$ is arbitrarily small. Because $\mathbb{N}$ is unbounded, for any choice of a, it must be true that there exists $n \in \mathbb{N}$ such that $1/n < |a|$. Therefore, for any choice of a, there must exist I_n such that $x + a \notin I_n$. Thus, the infinite intersection of these intervals must not contain anything other than x and cannot be an interval. $\square$

Theorem 3.23. *Let A, B be subsets of a continuum C. Then p is a limit point of $A \cup B$ if, and only if, p is a limit point of at least one of A or B.*

Proof. Let p be a limit point of A. By theorem 2.7, $A \subset A \cup B$, and by theorem 3.16, p is a limit point of $A \cup B$. By the same reasoning, if p is limit point of B, then p is a limit point of $A \cup B$.

The reverse implication will be proven using the contrapositive, which states that if p is not a limit point of both B and A, then it is not a limit point of $A \cup B$. Since p is not a limit point of A and B, there must exist intervals $I_1, I_2 \in C$ such that

$$p \in I_1, I_2, \quad I_1 \cap (A \setminus \{p\}) = \emptyset, \quad \text{and} \quad I_2 \cap (B \setminus \{p\}) = \emptyset.$$

Since I_1 and I_2 both contain p, by theorem 3.20, there must exist an interval I_3 such that $I_3 = I_1 \cap I_2$ and $p \in I_3$. By theorem 2.7, $I_3 \subset I_1$ and $I_3 \subset I_2$, so

$$I_3 \cap (A \setminus \{p\}) = \emptyset \quad \text{and} \quad I_3 \cap (B \setminus \{p\}) = \emptyset.$$

Therefore, it must be true that $I_3 \cap ((A \cup B) \setminus \{p\}) = \emptyset$. $\square$

Corollary 3.24. *Let $A_1, \ldots, A_n$ be n subsets of a continuum C. Then p is a limit point of $A_1 \cup \cdots \cup A_n$ if, and only if, p is a limit point of at least one of the sets A_k.*

Proof. Let p be a limit point of some set $A_m \in \{A_1, \ldots, A_n\}$. Since

$$A_m \subset \bigcup_{i=1}^{n} A_i,$$

by theorem 3.16,

$$p \text{ is a limit point of } of \bigcup_{i=1}^{n} A_i.$$

The reverse implication will be proven by the contrapositive and induction on the union of the sets A_n. The contrapositive of this statement is that if p is not a limit point of every set $A_m \in \{A_1, \ldots, A_n\}$, then p is not a limit point of

$$\bigcup_{i=1}^{n} A_i.$$

The base case is for when there is one set A_1 and p is not the limit point of A_1.

The induction hypothesis is that the union of $n - 1$ sets for which p is not a limit point of any individual set, does not have p as its limit point. Let

$$\bigcup_{i=1}^{n-1} A_i$$

be the union of $n - 1$ sets A_i, where p is not a limit point for each A_i. Let A_n be a set for which p is not a limit point. By the induction hypothesis, p is not a limit point of

$$\bigcup_{i=1}^{n-1} A_i.$$

Since p is not a limit point of A_n,
$$\bigcup_{i=1}^{n-1} A_i \cup A_n = \bigcup_{i=1}^{n} A_i$$
does not have p as its limit point by Theorem 3.23. Therefore, by induction, if p is not a limit point of each set $A_m \in \{A_1, \ldots, A_n\}$ it is not a limit point of
$$\bigcup_{i=1}^{n} A_i.$$
□

Theorem 3.25. *If p and q are distinct points of a continuum C, then there exist disjoint intervals I and J containing p and q, respectively.*

Proof. There are two cases that need to be proven separately: the case in which there exists a distinct element $x \in C$ that is between p and q, and the case in which there does not.

Let $p < q$ and thereby the interval (p, q) exists. Consider first the case where there exists $x \in (p, q)$. Since $p, q \in C$, by axiom 3.12, there must exist $a, b \in C$ such that $a < p$ and $q < b$. Since $x \in (p, q)$, $p < x$ and $x < q$. We can therefore construct interval (a, x) with $p \in (a, x)$, and interval (x, b) with $q \in (x, b)$. As a result, $(a, x) \cap (x, b) = \emptyset$.

Now consider the case where there exists no $x \in C$ such that $x \in (p, q)$. Define $a, b \in C$ the same as in the first case. Since $a < p$ and $p < q$, there exists interval (a, q) with $p \in (a, q)$. Similarly, since $p < q$ and $q < b$, there exists interval (p, b) with $q \in (p, b)$. The intersection $(a, q) \cap (p, b) = (p, q)$, but we defined this set to be empty, so $(a, q) \cap (p, b) = \emptyset$. □

Corollary 3.26. *A subset of a continuum C consisting of one point has no limit points.*

Proof. Let A be a subset of continuum C with only one element. Consider the case that the single element $p \in A$ is the limit point. Then, no matter the choice of I with $p \in I$, $I \cap (A \setminus \{p\}) = \emptyset$. This contradicts p as a limit point of A.

Now consider any point $p \notin A$ as a potential limit point of A. Let $a \in A$. Since a and p are distinct, by Theorem 3.25, disjoint intervals I_a and I_p can be formed where $a \in I_a$ and $p \in I_p$. Since $a \neq p$ is the only element of A, $A \setminus \{p\} \subset A$, and $A \subset I_a$. Therefore, $A \setminus \{p\} \cap I_p = \emptyset$, and p cannot be a limit point of A. □

Theorem 3.27. *A finite subset A of a continuum C has no limit points.*

Proof. This will be proven by induction on a set A with n elements. The base case is the case in which A only has one element. Corollary 3.26 ensures that this set does not have limit points.

The induction hypothesis is that any set with n elements does not have any limit points. Let A have $n+1$ elements. By Theorem 3.9, each element of A can be assigned a symbol $a_1, \ldots, a_{n+1}$. Rewrite A as $A \setminus \{a_{n+1}\} \cup \{a_{n+1}\}$. By Lemma 3.7, $A \setminus \{a_{n+1}\}$ has n elements. By the induction hypothesis $A \setminus \{a_{n+1}\}$ cannot have any limit points. Since $\{a_{n+1}\}$ cannot have any limit points by Corollary 3.26, $A = A \setminus \{a_{n+1}\} \cup \{a_{n+1}\}$ cannot have any limit points by Theorem 3.23.

Therefore, by induction, no finite set can have limit points. □

Corollary 3.28. *If A is a finite subset of a continuum C and $x \in A$, then there exists an interval R, containing x, such that $A \cap R = \{x\}$.*

Proof. Consider any point x in some finite set A. By Theorem 3.28, A has no limit points. By Definition 3.15, there exists some interval I where $x \in I$, and $I \cap A \setminus \{x\} = \emptyset$. Thus, there must be no element $(y \neq x) \in C$ such that $y \in I$ and $y \in A$. Therefore, $I \cap A = \{x\}$. □

Theorem 3.29. *If p is a limit point of A and I is an interval containing p, then the set $I \cap A$ is infinite.*

Proof. Rewrite A as $A = (A \cap I) \cup (A \setminus I)$. Then, $I \cap (A \setminus I) = \emptyset$. Since $p \in I$, p cannot be a limit point of $A \setminus I$. By Theorem 3.23, p must be a limit point of $(A \cap I)$. Therefore, $(A \cap I)$ cannot be finite by Theorem 3.27. $\square$

3.1. **Building a continuum: defining $\mathbb{R}$.** We take it for granted that we have the rational numbers:

$$\mathbb{Q} = \{{}^{a}/_{b} : a \in \mathbb{Z}, b \in \mathbb{N}\}.$$

This inherits an ordering from the one on $\mathbb{Z}$:

$$\frac{a}{b} < \frac{c}{d} \qquad \text{if and only if} \qquad ad < bc.$$

Definition 3.30. *A (Dedekind) cut is a set $A \subset \mathbb{Q}$ such that*

(i) $A \neq \mathbb{Q}, \emptyset$*;*
(ii) there is no largest point: if $a \in A$, there is a point $a' \in A$ such that $a < a'$;
(iii) if $a \in A$ and $b \in \mathbb{Q}$ satisfies $b < a$, then $b \in A$.

Let us denote $\mathbb{R}$ to be the set of cuts.

It may seem strange that we are defining $\mathbb{R}$ here. Surely, we already know what it is!? Actually, the more you think about it, the more you will have trouble pinning down an actual definition of the real numbers. This is one approach to it (the other one is related to taking the "completion" of a metric space, which we discuss during the semester). As you go through these exercises, you may find it interesting to think about how real numbers that you "know," like $\sqrt{2}$ and π, could be defined using cuts.

Theorem 3.31. *For any $A, B \in \mathbb{R}$, we define $<$ by*

$$A \prec B \qquad \textit{if and only if } A \subsetneq B.$$

Show that $\prec$ is an ordering on $\mathbb{R}$.

Proof. Let $A, B \in \mathbb{R}$ be any cuts and fix any $a \in A$. Consider first the case where there exists $b \in B$ such that $b \notin A$. By the ordering on $\mathbb{Q}$, one must be true: $a < b$, $a > b$, or $a = b$.

If $a > b$, by Definition 3.30 (iii), $b \in A$. This contradicts our choice of b so is not possible.

If $a = b$, then $b \in A$, which again contradicts our choice of b.

Therefore, $a < b$, and by Definition 3.30 (iii), $a \in B$. Since the choice of $a \in A$ was arbitrary, $a < b$ for any choice of a. Therefore, $A \subset B$.

Now consider the other case where for any choice of b, $b \in A$. Therefore, $B \subset A$. In this case, there are two possible subcases: $B = A$ and $B \subsetneq A$.

Thus the only possibilities are that $A \subsetneq B$, $A = B$, or $B \subsetneq A$, and $\prec$ satisfies the trichotomy condition of an ordering.

Let $A, B, C \in \mathbb{R}$, with $A \prec B$ and $B \prec C$. Fix any $a \in A$. Since $A \prec B$, $a \in B$. And since $B \prec C$, $a \in C$. This means that $A \prec C$, thus satisfying the transitivity condition of an ordering. $\square$

Definition 3.32. *If a cut C_a has the form*

$$C_a = \{b \in \mathbb{Q} : b < a\}$$

for some $a \in \mathbb{Q}$, we call C_a a rational cut*. Let $C_{\mathbb{Q}}$ be the set of all rational cuts.*

Lemma 3.33. *Given $a \in \mathbb{Q}$, show that C_a is a cut; that is, $C_a \in \mathbb{R}$.*

Proof. Let $b, c \in \mathbb{Q}$ with $b < a$ and $c > a$. Then by Definition 3.32, $b \in C_a$ while $c \notin C_a$, so C_a is nonempty and not all of $\mathbb{Q}$. This satisfies Definition 3.30 part (i) of a cut.

Using the same b defined above, define $a' = (b + a)/2$. Then, $a' \in \mathbb{Q}$ and must be between b and a. Since the choice of b was arbitrary, it is always possible to find a larger value in C_a, thus C_a satisfies Definition 3.30 (ii).

Again using the same b defined above, define $d \in \mathbb{Q}$ such that $d < b$. Since $d < b$ and $b < a$, $d < a$ and by definition of C_a, $d \in C_a$. Therefore, C_a satisfies definition 3.30 (iii).

Since C_a satisfies all properties of a cut, C_a must be a cut. □

Theorem 3.34. *Show that $\mathbb{R}$ is nonempty and has no first or last point.*

Proof. Fix $a \in \mathbb{Q}$. By Lemma 3.32, for any $q \in \mathbb{Q}$, C_q must exist. Therefore so C_a must exist and $\mathbb{R}$ is nonempty.

Assume for contradiction that $\mathbb{R}$ has a first point. Let this cut be defined as $F \in \mathbb{R}$. Since F must be a nonempty cut, there must exist some point $p \in F$. Since $p \in \mathbb{Q}$, by Lemma 3.33, we can form a rational cut C_p. Seeing as every $x \in C_p$ must be less than p, and $p \in F$, $x \in F$. Since F has no largest point, there must also exist $q \in F$ where $p < q$. Therefore $q \notin C_p$, and $C_p \prec F$. Therefore, by contradiction $\mathbb{R}$ cannot have a first point.

To show that $\mathbb{R}$ has no last point, assume for contradiction that it does have a last point or last cut. Let this cut be defined as $L \in \mathbb{R}$. Since no cut can be all of $\mathbb{Q}$, $\mathbb{Q} \setminus L$ is nonempty. Let $d \in \mathbb{Q} \setminus L$ and fix any $a \in L$. Due to the trichotomy property of the ordering on $\mathbb{Q}$, a and d must fit one of three relations: $a < d$, $a = d$, or $a > d$.

It cannot be true that $d > a$, as that would mean $d \in L$, and could not be in $\mathbb{Q} \setminus L$.

It cannot be true that $d = a$. Since a is not the largest element of L, there exists $b \in L$ such that $a < b$. Therefore, $d < b$ and $d \in L$. This contradicts our choice of $d \in \mathbb{Q} \setminus L$.

By elimination, $d > a$.

Since $d \in \mathbb{Q}$, $d + 1 \in \mathbb{Q}$, and the rational cut C_{d+1} can be formed. Because $d < d + 1$, $d \in C_{d+1}$. The choice of $a \in C_l$ was arbitrary, so every element $a \in C_l$ satisfies $a < d$, which means $a < d + 1$, and therefore $a \in C_{d+1}$. Therefore, $C_l \prec C_{d+1}$, contradicting C_l as the last cut. Thus, $\mathbb{R}$ cannot have a last cut. □

Exercise 3.35. *Show that*

$$\{C \in \mathbb{R} : C \text{ is a rational cut } \} \subsetneq \mathbb{R}.$$

Think about how the set on the left hand side is a "copy" of $\mathbb{Q}$. More formally, find $\varphi : \mathbb{Q} \to C_{\mathbb{Q}}$ that is bijective, increasing (that is, $\varphi(a_1) \prec \varphi(a_2)$ if $a_1 < a_2$), and that $C_{\mathbb{Q}} \neq \mathbb{R}$.

Solution. Let $\varphi : \mathbb{Q} - > C_{\mathbb{Q}}$ be defined as $\varphi(C_{\mathbb{Q}}) = C_{\mathbb{Q}}$. This function preserves the ordering on $\mathbb{Q}$ so if $a, b \in \mathbb{Q}$, where $a < b$, it must be true that $C_a \prec C_b$. Therefore, if $C_a = C_b$, it must be true that $a = b$. Thus, φ is an injection. By the way C_a is defined, it is not possible for $C_a \, inC_{\mathbb{Q}}$ and $a \notin \mathbb{Q}$. Thus, φ is a surjection. Therefore, φ is a bijection and $\{C \in \mathbb{R} : C \text{ is a rational cut}\} \subset \mathbb{R}$

To show that $\{C \in \mathbb{R} : C \text{ is a rational cut}\} \subsetneq \mathbb{R}$, I claim that $C_{\sqrt{2}} = \{x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2\}$ is a cut which is distinct from any rational cut.

Consider $-1, 2 \in \mathbb{Q}$. Since $-1 < 0$, $-1 \in C_{\sqrt{2}}$. Also, $2^2 > 2$ so $2 \notin C_{\sqrt{2}}$. Therefore, $C_{\sqrt{2}}$ is nonempty and not all of $\mathbb{Q}$, satisfying Definition 3.30 (i).

Consider any $a \in \mathbb{Q}$ and $a < 0$ or $a^2 < 2$. From our definition of $C_{\sqrt{2}}$, $a \in C_{\sqrt{2}}$. Consider $a - 2 \in \mathbb{Q}$. If $a < 0$ it must be true that $a - 2 < 0$ and therefore in $C_{\sqrt{2}}$. If $a > 0$ it must be true that $a^2 < 2$. Therefore $a < 2$ and it follows that $a - 2 < 0$. Since the choice of a was arbitrary, for any $a \in C_{\sqrt{2}}$, $a - 2 \in C_{\sqrt{2}}$, satisfying Definition 3.30 (iii).

To show that there is no largest element of $C_{\sqrt{2}}$, let $a \in C_{\sqrt{2}}$. First consider the case where $a < 0$. For any $a < 0$, $a/2 > a$ and $a/2 < 0$.

For the case where $a > 0$, consider $(a + 1/n)$ where n is some $n \in \mathbb{N}$. Squaring gives

$$\begin{aligned}(a + 1/n)^2 &= a^2 + 2a/n + 1/n^2 \\ &< a^2 + 2a/n + 1/n \\ &= a^2 + (2a + 1)/n\end{aligned}$$

I claim that there must exist $n_0 \in \mathbb{N}$ such that $a^2 + (2a+1)/n_0 < 2$. Rearranging this equality gives

$$(2a+1)/(2-a^2) < n_0.$$

This n_0 must exist as $2 - a^2 > 0$ and $\mathbb{N}$ is unbounded. So,

$$a^2 + (2a+1)/n_0 < 2,$$

and

$$\begin{aligned}(a + 1/n_0)^2 &< a^2 + (2a+1)/n_0 \\ (a + 1/n_0)^2 &< 2,\end{aligned}$$

Therefore, $(a + 1/n_0) \in C_{\sqrt{2}}$. Since the choice of a was arbitrary there can be no largest element of $C_{\sqrt{2}}$, which satisfies Definition 3.30 (ii).

All three criteria have been satisfied, and $C_{\sqrt{2}}$ is a cut.

What remains to be shown is that $C_{\sqrt{2}}$ is distinct from any rational cut. Let $a \in \mathbb{Q}$ be any element such that $2 < a^2$. Consider some smaller element $(a - 1/n)$ where $n \in \mathbb{N}$. Squaring this smaller element gives

$$\begin{aligned}(a - 1/n)^2 &= a^2 - 2a/n + 1/n^2 \\ &> a^2 - 2a/n.\end{aligned}$$

I claim there must exist $n_0 \in \mathbb{N}$ such that $a^2 - 2a/n_0 > 2$. Rearranging this equality gives

$$n_0 > 2a/(a^2 - 2).$$

Since $a^2 - 2$ must be positive and $\mathbb{N}$ is unbounded, n_0 must exist. Therefore,

$$a^2 - 2a/n_0 > 2.$$

Since

$$(a - 1/n_0)^2 > a^2 - 2a/n_0,$$

it must be true that

$$(a - 1/n_0)^2 > 2.$$

Thus, for any choice of $a \in \mathbb{Q}$ where $a > \sqrt{2}$, there exists $b = a - 1/n \in \mathbb{Q}$ such that $b < a$ and $b > \sqrt{2}$. The rational cut C_b exists, with $b \in C_a$ and $b \notin C_{\sqrt{2}}$. Thus, $C_{\sqrt{2}} \prec C_a$. Therefore, $C_{\sqrt{2}} \notin \{C \in \mathbb{R} : C \text{ is a rational cut}\}$, and

$$\{C \in \mathbb{R} : C \text{ is a rational cut}\} \subsetneq \mathbb{R}.$$

□

4. Adding a topology

In this sheet we give a continuum C a topology. Roughly speaking, this is a way to describe how the points of C are 'glued together'.

Definition 4.1. *A subset of a continuum is* closed *if it contains all of its limit points.*

Theorem 4.2. *The sets $\emptyset$ and C are closed.*

Proof. Since the empty set has no points, it cannot have any limit points. Therefore, there exists no limit point p where $p \notin \emptyset$. The empty set is therefore closed.

Let p be any limit point of a continuum C. By Definition 3.15, $p \in C$. Therefore, C is closed. □

Theorem 4.3. *A subset of C containing a finite number of points is closed.*

Proof. Let A be any finite subset of C. By Theorem 3.27, A does not have any limit points. Therefore, there exists no limit point p where $p \notin A$. The empty set is therefore closed. $\square$

Definition 4.4. *Let X be a subset of C. The* closure *of X is the subset $\overline{X}$ of C defined by:*

$$\overline{X} = X \cup LP(X).$$

Theorem 4.5. *For any $X \subset C$, X is closed if and only if $X = \overline{X}$.*

Proof. Let $X = \overline{X}$. Then by the definition of closure, $LP(X) \subset X$, and X is closed.

Let X be closed. For any point $p \in LP(X)$ it must be true that $p \in X$. Therefore, $LP(X) \subset X$ and $X \cup LP(X) = X$. By the definition of closure, $X = \overline{X}$. $\square$

Theorem 4.6. *The closure of $X \subset C$ satisfies $\overline{X} = \overline{\overline{X}}$.*

Proof. By the definition of closure, $\overline{\overline{X}} = \overline{X} \cup LP(\overline{X})$. To arrive at the desired conclusion, it needs to be shown that $LP(\overline{X}) \subset \overline{X}$, and thereby $\overline{X} \cup LP(\overline{X}) = \overline{X}$.

Since $\overline{X} = X \cup LP(X)$, if $p \in LP(\overline{X})$, p must be a limit point of X or a limit point of $LP(X)$ by Theorem 3.23. Consider any limit point p of $LP(X)$. Let $p \in I_1$. By Definition 2.15,

$$I_1 \cap (LP(X) \setminus \{p\}) \neq \emptyset.$$

There must then exist some $y \in I_1$ and $y \in LP(X)$. Therefore, y is a limit point of X. Construct interval I_2, where $y \in I_2$, as follows. Since $y \neq p$, by Theorem 3.25, disjoint intervals I_y and I_p exist with $y \in I_y$ and $p \in I_p$. Since $y, p \in I_1$, by Theorem 3.20, we can define the interval $I_2 = I_1 \cap I_y$ such that $y \in I_2$ and $p \notin I_2$. Since $y \in LP(X)$ and $y \in I_2$, by definition 2.25,

$$I_2 \cap (X \setminus \{y\}) \neq \emptyset.$$

Since $p \notin I_2$ the above intersection is equivalent to

$$I_2 \cap (X \setminus \{y, p\}) \neq \emptyset.$$

Since $I_2 \subset I_1$ and $X \setminus \{y, p\} \subset X \setminus \{p\}$, it must bet true that

$$I_1 \cap (X \setminus \{p\}) \neq \emptyset.$$

Seeing as I_2 can be constructed from any I_1, and the choice of I_1 was arbitrary for any limit point p of $LP(X)$, $LP(LP(X)) \subset LP(X)$.

$\square$

Corollary 4.7. *Given any subset $X \subset C$, the closure $\overline{X}$ is closed.*

Proof. By Theorem 4.6, $\overline{X} = \overline{\overline{X}}$, so by Theorem 4.5, $\overline{X}$ is closed. $\square$

Definition 4.8. *A subset G of a continuum C is* open *if its complement $C \setminus G$ is closed.*

Theorem 4.9. *The sets $\emptyset$ and C are open.*

Proof. The complement of C is the empty set. Since C is closed by Theorem 4.2, $\emptyset$ is open by Definition 4.8.

The complement of the empty set is C. Since by Theorem 4.2, $\emptyset$ is closed, by Definition 4.8, C is open. $\square$

The following is a very useful criterion to determine whether a set of points is open.

Theorem 4.10. *Let $G \subset C$. Then G is open if and only if for all $x \in G$, there exists an interval I such that $x \in I \subset G$.*

Proof. Since G is open, by Definition 4.8, $C \setminus G$ is closed. Let $x \in G$. Since $x \notin C \setminus G$, there must exist an interval I such that $x \in I$ and $I \cap (C \setminus G) \setminus \{x\}) = \emptyset$. Therefore, $I \subset G$, proving the forward relation.

To prove the reverse implication, consider any $x \in G$. Let $x \in I$ where $I \subset G$. It follows that $I \cap (C \setminus G) = \emptyset$. Then, $I \cap ((C \setminus G) \setminus \{x\}) = \emptyset$. Therefore, x cannot be a limit point of $C \setminus G$. Since the choice of $x \in G$ was arbitrary, no element of G can be a limit point of $C \setminus G$. Therefore, for any limit point p of $C \setminus G$, $p \in C \setminus G$. Thus, $C \setminus G$ is closed. By Definition 4.8, G must be open. □

Corollary 4.11. *Every interval I is open. Every complement of an interval, $C \setminus I$, is closed.*

Proof. Let interval I exist with $x \in I$. Seeing as $I \subset I$, I satisfies Definition 4.10, and is therefore open.

Since I is open, by Definition 4.8, $C \setminus I$ is closed. □

Corollary 4.12. *Let $G \subset C$. Then G is open if and only if for all $x \in G$, there exists a subset $V \subset G$ such that $x \in V$ and V is open.*

Proof. Let G be open and let $x \in G$. By Theorem 3.10, there must exist I_x such that $x \in I_x \subset G$. Since I_x is an open set by Corollary 4.11, this satisfies the reverse implication.

To prove the forward implication, pick any $x \in G$. Let there exist a subset $V \subset G$ such that $x \in V$ and V is open. Since V is open and $x \in V$, by Definition 4.10, there must exist an interval I_x where $x \in I_x$ and $I_x \subset V$. Since $V \subset G$, it follows that $I_x \subset G$. Since the choice of x was arbitrary, for every $x \in G$, there exists an interval I_x where $x \in I_x$ and $I_x \subset G$. Therefore, G is open by Definition 4.10. □

Corollary 4.13. *Let $a \in C$. Then the sets $\{x \mid x < a\}$ and $\{x \mid a < x\}$ are open.*

Proof. Consider any $b \in \{x \mid x < a\}$. Since C has no first point, $b - 1 \in \{x \mid x < a\}$. Therefore, the interval $(b-1, a)$ exists, with $b \in (b-1, a)$ and $(b-1, a) \subset \{x \mid x < a\}$. Since the choice of b can be any element of $\{x \mid x < a\}$, the set is therefore open by Theorem 4.10.

Consider any $b \in \{x \mid x > a\}$. Since C has no last point $b + 1 \in \{x \mid x > a\}$. Therefore, the interval $(a, b+1)$ exists, with $b \in (a, b+1)$ and $(a, b+1) \subset \{x \mid x > a\}$. Since the choice of b can be any element of $\{x \mid x > a\}$, the set is open by Theorem 4.10. □

Theorem 4.14. *Let G be a nonempty open set. Then G is the union of a collection of intervals.*

Proof. For each $x \in G$, let I_x be an interval containing x and $I_x \subset G$. These intervals must exist by Theorem 4.10. Let $\mathcal{I}$ be the family of sets defined as

$$\mathcal{I} = \{I_x \subset G \mid x \in G \text{ and } x \in I_x\}.$$

Let $m \in G$. Then m must be in some interval I_x. Therefore,

$$m \in \bigcup_x I_x,$$

and

$$G \subset \bigcup_x I_x.$$

Since each set I_x is a subset of G,

$$\bigcup_x I_x \subset G.$$

Therefore, G is the union of intervals. □

Exercise 4.15. *Is it true that, for any continuum C, there exist subsets $X \subset C$ that are neither open nor closed?*

Example. The answer depends on the continuum. The set $(0, 1]$ in the reals is neither open nor closed. It is not closed because 0 is a limit point of $(0, 1]$ and $0 \notin (0, 1]$. It is not open as the complement is $(-\infty, 0] \cup (1, \infty)$ is not closed as 1 is a limit point of $(1, \infty)$, and $1 \notin (1, \infty)$.

However there is no such set in $\mathbb{Z}$. Any set in $\mathbb{Z}$ must be closed. Consider any set $A \subset \mathbb{Z}$. For any $z \in A$, an interval $(z-1, z+1)$ can be constructed where $(z-1, z+1) \cap A \setminus \{z\} = \emptyset$. Therefore, A has no limit points, and as a result, A is closed. □

Theorem 4.16. *Let Λ be an indexing set and $\{X_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of closed subsets of a continuum C. Then the intersection*

$$\bigcap_{\lambda \in \Lambda} X_\lambda$$

is closed. Let us note that we sometimes (lazily) write

$$\bigcap_{\lambda} X_\lambda$$

when the indexing set is clear from context.

Proof. Let p be a limit point of

$$\bigcap_{\lambda} X_\lambda.$$

Fix any interval I_p such that $p \in I_p$. Thus,

$$I_p \cap \left(\bigcap_{\lambda} X_\lambda \right) \setminus \{p\} \neq \emptyset.$$

Let

$$y \in \left(\bigcap_{\lambda} X_\lambda \right) \setminus \{p\},$$

and fix any X_{λ_0}. By the definition of intersection, $y \in X_{\lambda_0} \setminus \{p\}$. Since $y \in I_p$, $y \in I_p \cap X_\lambda \setminus \{p\}$. Therefore, p is a limit point of X_{λ_0}.

Since X_{λ_0} is closed, $p \in X_{\lambda_0}$. Thus,

$$p \in \bigcap_{\lambda} X_\lambda.$$

Since p was any limit point of

$$\bigcap_{\lambda} X_\lambda,$$

it must be closed. □

Theorem 4.17. *Let $G_1, \ldots, G_n$ be a finite collection of open subsets of a continuum C.. Then the intersection $G_1 \cap \cdots \cap G_n$ is open.*

Proof. We proceed by induction on the number of open sets included in the intersection. The base case is the case in which there is only one set G_1, which is given as open.

For the induction hypothesis, assume that the intersection of n open sets is open. Therefore, the intersection $G_1 \cap \cdots \cap G_n$ is open. Take the intersection with a new open set G_{n+1}. If these sets are disjoint, the result is the empty set, which is open. For the case when the intersection is nonempty, let

$$x \in \bigcap_{i=1}^{n} G_i \cap G_{n+1}.$$

Since both are open sets, by Theorem 4.10, there exist intervals I_1 and I_2, where $x \in I_1, I_2$, and

$$I_1 \subset \bigcap_{i=1}^{n} G_i \quad \text{and} \quad I_2 \subset G_{n+1}.$$

By Theorem 3.20, there exists an interval I_3 where $I_3 = I_2 \cap I_1$ and $x \in I_3$. By definition, $I_3 \subset I_1, I_2$ so,

$$I_3 \subset \bigcap_{i=1}^{n} G_i \quad \text{and} \quad I_3 \subset G_{n+1}.$$

Thus,

$$\begin{aligned} I_3 &\subset \bigcap_{i=1}^{n} G_i \cap G_{n+1} \\ &\subset \bigcap_{i=1}^{n+1} G_i. \end{aligned}$$

Since the choice of x in the intersection was arbitrary, the intersection is an open set by Theorem 4.10. $\square$

Exercise 4.18. *Is it necessarily the case that the intersection of an infinite number of open sets is open? Is it possible to construct an infinite collection of open sets whose intersection is not open? Equivalently, is it possible to construct an infinite collection of closed sets whose union is not closed?*

Example. Consider the family of open sets defined $\mathcal{I} = \{(0, 1 + {}^1\!/\!{}_n) \mid \text{ for all } n \in \mathbb{N}\}$. Each set in $\mathcal{I}$ is open. Consider any arbitrarily small $x \in \mathbb{Q}$ where $1 + x > 1$. Since $\mathbb{N}$ is unbounded, there must exists $n_0 \in N$ where $1/n_0 < q$. Therefore $1 + x \notin (0, 1 + {}^1\!/\!{}_{n_0})$. Thus, the infinite intersection of this family of sets can include no value greater than 1 and is $(0, 1]$.

Consider the family of closed sets defined $\mathcal{J} = \{[0, 1 - {}^1\!/\!{}_n] \mid \text{ for all } n \in \mathbb{N}\}$. Consider any arbitrarily small $q \in \mathbb{Q}$ where $1 - q < 1$. Since $\mathbb{N}$ is unbounded, there must exists $n_0 \in N$ where $1/n_0 < q$. Therefore $1 - q \in [0, 1 + {}^1\!/\!{}_{n_0}]$. Thus, the infinite intersection of this family of sets includes every value less than one, but not 1 itself and equals $[0, 1)$. $\square$

Corollary 4.19. *Let Λ be an indexing set and $\{G_\lambda : \lambda \in \Lambda\}$ be an arbitrary collection of open subsets of a continuum C. Then the union*

$$\bigcup_{\lambda} G_\lambda$$

is open. Let $X_1, \dots, X_n$ be a finite collection of closed subsets of a continuum C. Then the union $X_1 \cup \dots \cup X_n$ is closed.

Proof. By Theorem 2.5,

$$C \setminus \bigcup_{\lambda} G_\lambda = \bigcap_{\lambda} (C \setminus G_\lambda).$$

By Theorem 4.8, each $C \setminus G_\lambda$ is closed. By Theorem 4.16, the arbitrary intersection of closed sets is closed. Therefore, the intersection of every set G_λ is an open set by Theorem 4.8.

By Theorem 2.5,

$$C \setminus \bigcup_{\lambda} X_\lambda = \bigcap_{\lambda} (C \setminus X_\lambda).$$

Each $C \setminus X_\lambda$ is an open set by Definition 4.8. By Theorem 4.17, the intersection of all the open sets $C \setminus X_\lambda$ is an open set. Therefore, the intersection of every set X_λ is a closed set by Theorem 4.8. $\square$

Theorem 4.14 says that every nonempty open set is the union of a collection of intervals. This necessary condition for open sets is also sufficient:

Corollary 4.20. *Let $G \subset C$ be nonempty. Then G is open if and only if G is the union of a collection of intervals.*

Proof. We begin by proving the forward implication. If G is the union of a collection of intervals, G is an interval by Theorem 4.19. Since an interval is an open set by Corollary 4.11, G is open.

To prove the reverse implication, assume G is open. By Theorem 4.14, G is the union of a collection of intervals. $\square$

Corollary 4.21. *If (a, b) is an interval in C, then $\text{ext}(a, b)$ is open.*

Proof. By Lemma 3.16, $\text{ext}\,(a, b) = \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$. By Corollary 4.13, both $\{x \in C \mid x < a\}$ and $\{x \in C \mid b < x\}$ are open. The union of open sets is open by Corollary 4.19, so $\text{ext}\,(a, b)$ is open. $\square$

Definition 4.22. *Let $X \subset C$. Then X is disconnected if it may be written as $X = A \cup B$, where A and B are disjoint, non-empty open sets in X. We say that X is connected if it is not disconnected.*

Exercise 4.23. *Let C be a continuum and $a \in C$. Prove that $C \setminus \{a\}$ is a disconnected continuum.*

Solution. If C is nonempty and has no first or last point, $C \setminus \{a\}$ will be nonempty and have no first or last point. Since every element of C satisfies the ordering property, every element of $C \setminus \{a\}$ satisfies the same ordering.

Observe that $C \setminus \{a\} = (-\infty, a) \cup (a, \infty)$, where $(-\infty, a)$ and (a, ∞) are disjoint. By Definition 4.22, $C \setminus \{a\}$ is disconnected. $\square$

5. Connectedness and boundedness

Axiom 5.1. *A continuum is connected.*

Theorem 5.2. *The only subsets of a continuum C that are both open and closed are $\emptyset$ and C.*

Proof. Let $A \subset C$ be a set that is both open and closed. By Definition 4.8, $C \setminus A$ is both open and closed. For any set, $A \cup C \setminus A = C$. Both A and $C \setminus A$ are open, and by definition of the complement A and $C \setminus A$ are disjoint. Therefore, C is the union of two disjoint open sets and is disconnected. This contradicts Axiom 5.1, so this set A cannot exist. $\square$

Theorem 5.3. *For all $x, y \in C$, if $x < y$, then there exists $z \in C$ such that z is in between x and y.*

Proof. Assume for contradiction that no such z exists. Form intervals $(-\infty, y)$ and (x, ∞). Since $x < y$, $(-\infty, y) \cup (x, \infty) = C$. These intervals are open and the intersection $(-\infty, y) \cap (x, \infty) = (x, y)$ is empty by assumption. Therefore, C is disconnected by Definition 4.22, producing a contradiction. $\square$

Corollary 5.4. *Every interval is infinite.*

Proof. Assume for contradiction that an interval $I = (a, b)$, where $a, b \in C$, is finite and nonempty. Then, I must have a first point $a_f \in I$ by Lemma 3.8. By Definition of an interval, $a < a_f$, thus interval $(a, a_f) \subset I$ can be formed. By Theorem 5.3, there must exist $b \in (a, a_f)$. Thus, $b \in I$ and $(b < a_f)$. This contradicts a_f as a first point of I. Therefore, by contradiction, no finite interval can exist. $\square$

Corollary 5.5. *Every point of C is a limit point of C.*

Proof. Assume there exists $p \in C$ where p is not a limit point of C. Then there exists an interval I, where $p \in I$, and $I \cap C \setminus \{p\} = \emptyset$. Thus $I \cap C = \{p\}$, which means I is not infinite, which contradicts Corollary 5.4. $\square$

Corollary 5.6. *Every point of an interval (a, b) is a limit point of (a, b).*

Proof. Let (a, b) be an interval and $p \in (a, b)$. This interval is open by Corollary 4.7, so let I_p be an interval where $p \in I_p$ and $I_p \subset (a, b)$. By Corollary 5.4, both intervals are infinite. Thus $(a, b) \setminus \{p\} \cap I_p \neq \emptyset$. Since the choice of p in (a, b) was arbitrary, any point of (a, b) is a limit point of (a, b). $\square$

We will now introduce boundedness. The first definition should be intuitively clear. The second is subtle and powerful.

Definition 5.7. *Let X be a subset of C. A point u is called an* upper bound *of X if for all $x \in X$, $x \leq u$. A point l is called a* lower bound *of X if for all $x \in X$, $l \leq x$. If there exists an upper bound of X, then we say that X is* bounded above*. If there exists a lower bound of X, then we say that X is* bounded below*. If X is bounded above and below, then we simply say that X is* bounded*.*

Definition 5.8. *Let X be a subset of C. We say that u is a* least upper bound *or* supremum *of X and write $u = \sup X$ if:*

(1) u is an upper bound of X, and
(2) if u' is an upper bound of X, then $u \leq u'$.

We say that l is a greatest lower bound *or* infimum *and write $l = \inf X$ if:*

(1) l is a lower bound of X, and
(2) if l' is a lower bound of X, then $l' \leq l$.

Exercise 5.9. *If $\sup X$ exists, then it is unique, and similarly for $\inf X$.*

Solution. Let $s = \sup X$ and consider any least upper bound, s', for X. By the trichotomy property, three relations are possible: $s < s'$, $s > s'$, $s = s'$.

If $s = s'$, $s = \sup X$ and is unique.

If $s < s'$, then s' is not a supremum for X, and $s = \sup X$ and is unique.

If $s > s'$, then s is not a supremum for X, and $s' = \sup X$ and is unique.

Let $i = \inf X$ and consider any greatest lower bound, i', for X. By the trichotomy property, three relations are possible: $i < i'$, $i > i'$, $i = i'$.

If $i = i'$, $i = \inf X$ and is unique.

If $i < i'$, then i is not a infimum for X, and $i' = \inf X$ and is unique.

If $i > i'$, then i' is not a infimum for X, and $i = \inf X$ and is unique. $\square$

The following lemma is extremely useful when dealing with suprema; an analogous statement can be made for infima.

Lemma 5.10. *Suppose that $X \subset C$ and $s = \sup X$. If $p < s$, then there exists an $x \in X$ such that $p < x \leq s$.*

Proof. Let $p < s$, and assume for contradiction that there exists no $x \in X$ such that $p < x \leq s$. Then, for all $x \in X$, $x < p$. This means p is an upper bound for X, which contradicts s as the supremum. Therefore, there must exist $x \in X$ such that $p < x \leq s$. $\square$

Theorem 5.11. *Let $a < b$. The least upper bound and greatest lower bound of the interval (a, b) are:*

$$\sup(a, b) = b \quad \textit{and} \quad \inf(a, b) = a.$$

Proof. Let x be any element in (a, b). Since this interval is the set of values such that $x < a$ and $x > b$, a and b satisfy the definitions of upper and lower bounds respectively.

Consider any other $b' > a$. If $b > b'$, then there exists an interval $(b', b) \subset (a, b)$. This interval is nonempty by Corollary 5.4, so let $y \in (b', b)$. Thus, $y \in (a, b)$. Since, $b' < y$, b' is not an upper bound for (a,b). Therefore, if $b \geq b'$, b' is an upper bound for (a,b), which means b is the supremum of (a,b).

Consider any other $a' < b$. If $a < a'$, then there exists an interval $(a,a') \subset (a,b)$. This interval is nonempty by Corollary 5.4, so let $z \in (a,a')$. Thus, $z \in (a,b)$. Since, $a' > z$, a' is not a lower bound for (a,b). Therefore, if $a \geq a'$, a' is lower bound for (a,b), which means a is the infimum of (a,b). □

Lemma 5.12. *Let X be a subset of C. Suppose that $\sup X$ exists and $\sup X \notin X$. Then $\sup X$ is a limit point of X. The same holds for $\inf X$.*

Proof. Let $s = \sup X$. By Lemma 5.10, for any $p < s$, there exists $x \in X$ such that $p < x \leq s$. Since $s \notin X$, it must be true that $p < x < s$. Since C has no last point, let $b \in C$ where $b > s$. Then, the interval (p,b) exists with $x, s \in (p,b)$. Since, $x \in X$ as well, $(p,b) \cap X \setminus \{s\} \neq \emptyset$. Since the choice of $p < s$ and $b > s$ was arbitrary, s is a limit point of X.

Let $i = \inf X$. By Lemma 5.10 for infimum, for any $p > i$, there exists $x \in X$ such that $p > x \geq i$. Since $i \notin X$, it must be true that $p > x > i$. Since C has no first point, let $b \in C$ where $b < i$. Then, the interval (b,p) exists with $x, i \in (b,p)$. Since, $x \in X$ as well, $(b,p) \cap X \setminus \{i\} \neq \emptyset$. Since the choice of $p > i$ and $b < i$ was arbitrary, i is a limit point of X. □

Corollary 5.13. *Both a and b are limit points of the interval (a,b).*

Proof. By Theorem 5.11, a and b are infimum and supremum of (a,b). By Lemma 5.12, a and b are limit points of (a,b). □

Let $[a,b]$ denote the closure $\overline{(a,b)}$ of the interval (a,b).

Corollary 5.14. *The closed ineterval $[a,b] = \{x \in C \mid a \leq x \leq b\}$.*

Proof. By the definition of closure, $[a,b] = (a,b) \cup LP((a,b))$. By Corollary 5.6, every element of (a,b) is a limit point of (a,b). Thus,

$$(a,b) \subset LP((a,b)) \text{ and } [a,b] = LP((a,b)).$$

By Corollary 5.13, a and b are limit points of (a,b). Thus,

$$[\{a\} \cup (a,b) \cup \{b\}] \subset LP((a,b)).$$

Consider any point in $C \setminus [\{a\} \cup (a,b) \cup \{b\}]$. By Definition 3.15, this the external of (a,b). By Lemma 3.19, no point of $\text{ext}(a,b)$ can be a limit point of (a,b). Therefore,

$$[\{a\} \cup (a,b) \cup \{b\}] = LP((a,b)), \text{ and } [a,b] = \{x \in C \mid a \leq x \leq b\}.$$

□

Lemma 5.15. *Let $X \subset C$ and define:*

$$\Psi(X) = \{x \in C \mid x \textit{ is not an upper bound of } X\}.$$

Then $\Psi(X)$ is open. Define:

$$\Omega(X) = \{x \in C \mid x \textit{ is not a lower bound of } X\}.$$

Then $\Omega(X)$ is open.

Proof. Let $y \in \Psi(X)$. Since y is not an upper bound for X, there must exist $x \in X$ such that $y < x$. Therefore, by Theorem 5.3, there exists $z \in C$ such that $y < z$ and $z < x$. Since C is a continuum with no first point, let $a \in C$ such that $a < y$. Form the interval (a,z). Since $y \in \Psi(X)$, and $a < y$, $a \in \Psi(X)$. Because $z < x$, $z \in \Psi(X)$. Thus, $y \in (a,z)$ and $(a,z) \subset \Psi(X)$. Since the initial choice of y was arbitrary, $\Psi(X)$ is an open set.

Let $y \in \Omega(X)$. Since y is not a lower bound for X, there must exist $x \in X$ such that $y > x$. Therefore, by Theorem 5.3, there exists $z \in C$ such that $y > z$ and $z > x$. Since C is a continuum with no last point, let $b \in C$ such that $b > y$. Form the interval (z, b). Since $y < b$, $b \in \Omega(X)$. Since $z > x$, $z \in \Omega(X)$. Thus, $y \in (z, b)$ and $(z, b) \subset \Omega(X)$. Since the initial choice of y was arbitrary, $\Omega(X)$ is an open set. □

Theorem 5.16. *Suppose that X is nonempty and bounded above. Then $\sup X$ exists. Similarly, if X is nonempty and bounded below, then $\inf X$ exists.*

Proof. Since X is bounded above, $\Psi(X)$ exists, and is open by Lemma 5.15. The complement $C \setminus \Psi(X)$, which is the set of all upper bounds of X, is closed by Definition 4.8.

Assume for contradiction that for all $x \in C \setminus \Psi(X)$, there exists $y \in C \setminus \Psi(X)$ such that $y < x$. Since C is a continuum with no last point, there exists $a \in C$ such that $x < a$. Form interval (y, a). Thus, $x \in (y, a)$. Since $x < a$, $a \in C \setminus \Psi(X)$ as well. Therefore, $(y, a) \subset C \setminus \Psi(X)$. Since the choice of x was arbitrary, this holds for any $x \in C \setminus \Psi(X)$. Thus, $C \setminus \Psi(X)$ is open. Therefore, by contradiction, there must exist $s \in C \setminus \Psi(X)$ such that for all $u \in C \setminus \Psi(X)$, $s \leq u$. Therefore, s is the supremum of X.

Since X is bounded below, $\Omega(X)$ exists, and is open by Lemma 5.15. The complement $C \setminus \Omega(X)$, which is the set of all lower bounds of X, is closed by Definition 4.8.

Assume for contradiction that for all $x \in C \setminus \Omega(X)$, there exists $y \in C \setminus \Omega(X)$ such that $y > x$. Since C is a continuum with no first point, there exists $b \in C$ such that $x > b$. Form interval (b, y). Thus, $x \in (b, y)$. Since $x > b$, $b \in C \setminus \Omega(X)$ as well. Therefore, $(b, y) \subset C \setminus \Omega(X)$. Since the choice of x was arbitrary, this holds for any $x \in C \setminus \Omega(X)$. Thus, $C \setminus \Omega(X)$ is open. Therefore, by contradiction, there must exist $i \in C \setminus \Omega(X)$ such that for all $l \in C \setminus \Omega(X)$, $i \geq l$. Therefore, i is the supremum of X. □

Corollary 5.17. *Every nonempty closed and bounded set has a first point and a last point.*

Proof. Consider any nonempty, closed, and bounded set. By Theorem 5.16, $\sup(X)$ and $\inf(X)$ exist. Let $s = \sup(X)$ and $i = \inf(X)$. Assume for contradiction that $i, s \notin X$. By Lemma 5.12, i and s are limit points of X. Since X is closed, $i, s \in X$. Thus, for all $x \in X$, $i \leq x \leq s$. Since, $i, s \in X$, i and s are the first and last points of X by Definition 3.5.

□

Exercise 5.18. *Is this true for $\mathbb{Q}$?*

Counterexample. Consider the set $C = \{q \in \mathbb{Q} \mid q^2 < 2\}$ which represents the interval $(-\sqrt{2}, \sqrt{2}) \subset \mathbb{Q}$. This interval is bounded, and by Corollary 4.11, is open. By Definition 4.8, the complement $\mathbb{Q} \setminus C = \{q \in \mathbb{Q} \mid q^2 \geq 2\}$ is closed. Since there exists no $q \in \mathbb{Q}$ such that $q^2 = 2$, the complement $\mathbb{Q} \setminus C = \{q \in \mathbb{Q} \mid q^2 > 2\}$. Thus, $\mathbb{Q} \setminus C = (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$ and is also open. Therefore, by Definition 4.8, C is also closed. Thus, C is a closed set in $\mathbb{Q}$ with no first or last point. □

5.1. **Building a continuum: the real numbers are connected.**

Exercise 5.19. *Show that $\mathbb{Q}$ is not connected.*

Solution. Consider the set $C = \{q \in \mathbb{Q} \mid q^2 < 2 \text{ or } q < 0\}$. By Exercise 3.35, this set represents a cut. Since a cut has no smallest element and no last point, for any $b \in C$, there exists $a, c \in C$ such that $a < b$ and $b < c$. Thus, interval $(a, c) \subset C$ exists with $b \in (a, c)$. Since the choice of b was arbitrary, C is open.

Now consider $\mathbb{Q} \setminus C = \{q \in \mathbb{Q} \mid q^2 \geq 2 \text{ and } q > 0\}$. Since there exists no $q \in \mathbb{Q}$ such that $q^2 = 2$, $\mathbb{Q} \setminus C = \{q \in \mathbb{Q} \mid q^2 > 2 \text{ and } q > 0\}$. This has no last point by construction, and has no first point by Exercise 3.35. Therefore for any $e \in \mathbb{Q} \setminus C$, there exists $d, f \in \mathbb{Q} \setminus C$ such that $d < e$ and $e < f$. Thus, interval $(e, f) \subset \mathbb{Q} \setminus C$ exists with $e \in (d, f)$. Since the choice of d was arbitrary, $\mathbb{Q} \setminus C$ is open.

Since $C \cup \mathbb{Q} \setminus C = \mathbb{Q}$, and $C \cap \mathbb{Q} \setminus C = \emptyset$, with C and $\mathbb{Q} \setminus C$ both being open, $\mathbb{Q}$ is disconnected. □

Lemma 5.20. *Let Λ be an indexing set, and for all $\lambda \in \Lambda$, let C_λ be a cut. Then,*

$$\mathcal{C} = \bigcup_{\lambda \in \Lambda} C_\lambda$$

is a cut or all of $\mathbb{Q}$.

Proof. Let $\mathcal{C} \neq \mathbb{Q}$. Since each C_λ is nonempty, $\mathcal{C}$ is nonempty.

Let $x \in \mathcal{C}$. By definition of union, there exists some λ_0 such that $x \in C_{\lambda_0}$. Since C_{λ_0} is a cut, there exists $x' \in C_{\lambda_0}$ where $x < x'$. And since $C_{\lambda_0} \subset \mathcal{C}$, $x' \in \mathcal{C}$. Since the choice of x was arbitrary, $\mathcal{C}$ has no last point.

Let $y \in \mathcal{C}$. By definition of union, there exists some λ_y such that $y \in C_{\lambda_y}$. Since C_{λ_y} is a cut, if there exists $q \in \mathbb{Q}$ such that $q < y$, $q \in C_{\lambda_y}$. And since $C_{\lambda_y} \subset \mathcal{C}$, $y \in \mathcal{C}$. Since the choice of y was arbitrary, $\mathcal{C}$ satisfies part (iii) of Definition 3.30. $\square$

Lemma 5.21. *(i) Suppose that $\mathcal{A} \subset \mathbb{R}$ is bounded from above. Show that $\mathcal{A}$ has a supremum.*
(ii) Suppose that $\mathcal{A} \subset \mathbb{R}$ is bounded from below. Show that $\mathcal{A}$ has a infimum.

Proof. What follows is a proof of part (i). Since $\mathcal{A}$ is bounded above, the intersection

$$\bigcup_{A \in \mathcal{A}} A \neq \mathbb{Q}.$$

By Lemma 5.21, this intersection is a cut. Let this intersection be C. Fix any $B \in \mathcal{A}$. By construction, $B \subseteq C$. By the arbitrariness of B, C is an upper bound for $\mathcal{A}$.

Fix $x \in C$. By the definition of the union, there exists $A_x \in \mathcal{A}$ such that $x \in A_x$. Let U be any upper bound for $\mathcal{A}$. Then, U is an upper bound for A_x and $A_x \preceq U$. Since the choice of $x \in C$ was arbitrary, $C \preceq U$. Thus, C is the supremum of $\mathcal{A}$.

A proof of statement (ii) follows. Consider the family of sets $\mathcal{L} = \{L \in \mathbb{R} \mid L \text{ is a lower bound of } \mathcal{A}\}$. Since $\mathcal{A}$ is bounded below, $\mathcal{L}$ is nonempty and bounded above. By Lemma 5.21, let the cut I be defined as

$$I = \bigcup_{L \in \mathcal{L}} L.$$

Let $i \in I$. By the definition of union, there must exist $L_i \in \mathcal{L}$ such that $i \in L_i$. Since, $L_i \in \mathcal{L}$, L_i is a lower bound of $\mathcal{A}$. By the arbitrariness of i, I is a lower bound for $\mathcal{A}$.

Let Y be any lower bound of $\mathcal{A}$. Thus, $Y \in \mathcal{L}$, and $Y \preceq I$. By the arbitrariness of Y, for all $L \in \mathcal{L}$, $L \preceq I$. Therefore, I is the infimum of $\mathcal{A}$. $\square$

Lemma 5.22. *If $\mathcal{A} \subset \mathbb{R}$ is an open set, and* $\inf A$ *and* $\sup A$ *exist,* $\inf A, \sup A \notin A$.

Proof. Assume for contradiction that $\inf A, \sup A \in A$. Since $\mathcal{A}$ is open, then there exist intervals I_i and I_s such that $\inf A \in I_i$ and $\sup A \in I_s$ and $I_i, I_s \subset \mathcal{A}$. Since every interval is infinite by Corollary 5.4, there must exist $j \in I_i$ and $t \in I_s$ such that $j < i$ and $s < t$. Since $I_i, I_s \subset \mathcal{A}$, $j, t \in \mathcal{A}$. This contradicts s, i as the supremeum and infimum, and is therefore impossible. $\square$

Lemma 5.23. *Suppose that $\mathcal{A}$ is an open set in $\mathbb{R}$, and let $A \in \mathcal{A}$. If there is $\underline{B}, \overline{B} \in \mathbb{R} \setminus \mathcal{A}$ such that $\underline{B} < A < \overline{B}$, then there are points $P_1, P_2 \in \mathbb{R} \setminus \mathcal{A}$ such that*

$$A \in (P_1, P_2) \quad \textit{and} \quad (P_1, P_2) \subset \mathcal{A}.$$

Proof. Since $\mathcal{A}$ is open, by Theorem 4.10, there exists an interval $I \subset \mathcal{A}$ such that $A \in I$. Define $\mathcal{I}$ to be the family of such intervals. Thus $\mathcal{I} = \{I \subset \mathcal{A} \mid A \in I\}$. Let

$$\widetilde{A} = \bigcup_{I \in \mathcal{I}} I.$$

By Corollary 4.19, $\widetilde{A}$ is open. Since $\widetilde{A}$ is the union intervals each sharing a point A in common, $\widetilde{A}$ is an interval containing A.

Let $\underline{B}, \overline{B} \in \mathbb{R} \setminus \mathcal{A}$ such that $\underline{B} < A < \overline{B}$. Thus, $(\underline{B}, \overline{B})$ is an interval and $\widetilde{A} \subset (\underline{B}, \overline{B})$. Since $(\underline{B}, \overline{B})$ is bounded, so is $\widetilde{A}$. Hence, $P_1 = \inf \widetilde{A}$ and $P_2 = \sup \widetilde{A}$ exist by Theorem 5.21. Since $\widetilde{A}$ is an interval, define $\widetilde{A} = (a, b)$ where $a, b \in \mathbb{R}$. By Theorem 5.11, $a = \inf \widetilde{A}$ and $b = \sup \widetilde{A}$. Thus $\widetilde{A} = (P_1, P_2) \subset \mathcal{A}$, and by Lemma 5.22, $P_1, P_2 \notin \widetilde{A}$.

□

Theorem 5.24. *Show that $\mathbb{R}$ is connected.*

Proof. Assume for contradiction that $\mathbb{R}$ is disconnected. Then, there exist open and disjoint sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}$ such that $\mathcal{A} \cup \mathcal{B} = \mathbb{R}$. Fix any $A \in \mathcal{A}$. When compared to elements in $\mathcal{B}$ there are only two cases to consider: either $A \prec B$ for all $B \in \mathcal{B}$ or there exists $\underline{B} \in \mathcal{B}$ such that $\underline{B} \prec A$.

Consider the first case. This means that $\mathcal{B}$ is bounded below. Therefore, $L = \inf \mathcal{B}$ exists by Lemma 5.21. For $\mathcal{A} \cup \mathcal{B} = \mathbb{R}$, $\mathcal{B} = \mathbb{R} \setminus \mathcal{A}$. Because $\mathcal{A}$ is open, $\mathcal{B}$ is closed by definition 4.8. Hence, $\inf \mathcal{B} \in \mathcal{B}$. But we assumed that $\mathcal{B}$ is open, and by Lemma 5.23, $\inf \mathcal{B} \notin \mathcal{B}$. This case is therefore impossible.

Consider the second case. Assuming that B is not the only element of $\mathcal{B}$, (in which case $\mathcal{B}$ is not open), let $\overline{B} \in \mathbb{R}$ such that $A \prec \overline{B}$. If $\overline{B}$ does not exist, then A is an upper bound for $\mathcal{B}$. Thus $S = \sup \mathcal{B}$ exists by Lemma 5.21. By the same argument above, $\mathcal{B}$ is closed and $\sup \mathcal{B} \in \mathcal{B}$. But we assumed that $\mathcal{B}$ is open. Thus $\sup \mathcal{B} \notin \mathcal{B}$, producing a contradiction.

If $\overline{B}$ exists, then $\underline{B} < A < \overline{B}$, and by Lemma 5.23, there exists $P_1, P_2 \in \mathbb{R} \setminus \mathcal{A}$ such that $A \in (P_1, P_2)$ and $(P_1, P_2) \subset \mathcal{A}$. Since (P_1, P_2) is bounded, $I = \inf(P_1, P_2)$ and $S = \sup(P_1, P_2)$ exist by Lemma 5.21. And since (P_1, P_2) is an interval, it is open. Thus $I, S \notin (P_1, P_2)$, by Lemma 5.22, and $I, S \notin \mathcal{A}$. However, $\mathcal{B}$ is open, so $\mathbb{R} \setminus \mathcal{B} = \mathcal{A}$ is closed. Thus, $I, S \in \mathcal{A}$ producing a contradiction. □

At this point, we see that $\mathbb{R}$ is a continuum. (Side note: can you think of any other sets that satisfy the axioms of a continuum?) All previous results, thus, apply to it. If you feel inclined, think about how our usual actions of addition, subtraction, multiplication, and division can be defined using the Dedekind cuts. From this point on, we can always imagine C to be $\mathbb{R}$. This is not necessary, but it may help with intuition.

6. Compactness

Definition 6.1. *Let X be a subset of C and let $\mathcal{G} = \{G_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of C. We say that $\mathcal{G}$ is a* cover *of X if every point of X is in some G_λ, or in other words:*

$$X \subset \bigcup_{\lambda \in \Lambda} G_\lambda.$$

We say that the collection $\mathcal{G}$ is an open cover *if each G_λ is open.*

Definition 6.2. *Let X be a subset of C. X is* compact *if for every open cover $\mathcal{G}$ of X, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover.*

A good summary of the definition of compactness is "every open cover contains a finite subcover".

Exercise 6.3. *Show that all finite subsets of C are compact.*

Lemma 6.4. *No finite collection of intervals covers C.*

Theorem 6.5. *C is not compact.*

Exercise 6.6. *Show that intervals are not compact.*

Theorem 6.7. *If X is compact, then X is bounded.*

Lemma 6.8. *Let $X \subset C$ and $p \in C \setminus X$. Then*

$$\mathcal{G} = \{\text{ext}(a, b) \mid (a, b) \text{ contains } p\}$$

is an open cover of X.

Theorem 6.9. *If X is compact, then X is closed.*

It will turn out that the two properties of compactness in Theorem 6.7 and Theorem 6.9 characterize compact sets completely (at least in C), meaning that every bounded closed set is compact. The rest of the sheet is concerned with proving this fact.

For the next three results, fix points $a, b \in C$ and suppose $\mathcal{G}$ is an open cover of $[a, b]$.

Lemma 6.10. *For all $s \in [a, b]$, there exist $G \in \mathcal{G}$ and $p, q \in C$ such that $p < s < q$ and $[p, q] \subset G$.*

Definition 6.11. *Fix $\bar{x} \in C$. We say that $\bar{x}$ is* reachable *from a if there exist $n \in \mathbb{N} \cup \{0\}$, $x_0, \ldots, x_n \in C$, and $G_1, \ldots, G_n \in \mathcal{G}$ such that*

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = \bar{x}$$

and

$$[x_{i-1}, x_i] \subset G_i \quad \text{for each } i \in \{1, 2, \ldots, n\}.$$

Theorem 6.12. *Let X be the set of all $\bar{x} \in C$ that are* reachable *from a. Then the point b is not an upper bound for X.*

Corollary 6.13. *There is a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is a cover of $[a, b]$.*

Corollary 6.14. *The set $[a, b]$ is compact.*

This last corollary is the main ingredient for the proof of the Heine-Borel theorem.

Lemma 6.15. *A closed subset Y of a compact set $X \subset C$ is compact.*

Theorem 6.16. *If $X \subset C$, then X is compact if and only if X is closed and bounded.*

Lemma 6.17. *A compact set $X \subset C$ with no limit points must be finite.*

Theorem 6.18. *Every bounded infinite subset of C has at least one limit point.*