Homework 1

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1 Question 1

For $i = 0, ..., 10$, construct the greatest rational number in the cut of the square root of 3 in the form of a (non reduced) fraction $a_i = \frac{x_i}{2^i}$, and the least rational number not in the cut in the same form, $b_i = \frac{y_i}{2^i}$.

Solution:

To be begin, we shall start with a_0 and b_0. Recall the definition of a cut bounded by $\sqrt{3}$.

$$(\sqrt{3})^* \coloneqq \{r \in \mathbb{Q}^+ \mid r^2 < 3\}$$

This will make it quite easy to determine which rational numbers are in and not in $(\sqrt{3})^*$. From the definition of a_0 and b_0, it is clear that they will be integers (specifcally, the two closest integers above and below $\sqrt{3}$) since the denominator is simply 2^0, or one.

Let us then describe an algorithm for defining what the next a_i and b_i will be. We know that there exists an integer n such that $(\frac{n}{m})^2 < 3$ and $(\frac{n+1}{m})^2 > 3$ where $m \in \mathbb{N}$ as a consequence of the Archimedean Principle. Because the binary operation of averaging two numbers naturally multiplies the denominator by 2 and finds the midpoint between two bounds (which will be tending towards $\sqrt{3}$), it is optimal for producing the following the following a_i and b_i in the series.

A "brute force" recurrence relation will then be described as follows:

1. Define possible greatest/smallest rational number in/out of the cut of a given denominator $m \in \mathbb{N}$ as $c_i = \frac{n}{m}$ as the average of a_{i-1} and b_{i-1} with $c_0 = \frac{0}{1}$
2. If $n^2 < 3 \cdot m^2$ increment n by one until $n^2 > 3 \cdot m^2$, then $\frac{n}{m} = b_i$ and $\frac{n-1}{m} = a_i$
3. Otherwise $n^2 > 3 \cdot m^2$, then decrement n by one until $n^2 < 3 \cdot m^2$, then $\frac{n}{m} = a_i$ and $\frac{n+1}{m} = b_i$

Given the exponential nature of the denominator, this could prove a tedious problem by hand, however computers can execute this algorithm very swiftly. But, for demonstrative purposes, I shall show how this algorithm finds a_0 and b_0.

Start with $c_0 = \frac{0}{1}$. Then $c_0^2 = 0 < 3$, so increment the numerator by 1 until $c_0^2 > 3$, which happens after two increments $(c_0 = \frac{2}{1})$ as $c_0^2 = 2^2 = 4 > 3$.

Then we have:

$$\begin{aligned} a_0 &= \frac{n-1}{m} = \frac{2-1}{1} = 1 \\ b_0 &= c_0 = 2 \end{aligned} \tag{1}$$

Continuing this process for $i = 1, 2, ..., 10$ we get:

1. $a_1 = \frac{3}{2}, b_1 = \frac{4}{2}$
2. $a_2 = \frac{6}{4}, b_2 = \frac{7}{4}$
3. $a_3 = \frac{13}{8}, b_3 = \frac{14}{8}$
4. $a_4 = \frac{27}{16}, b_4 = \frac{28}{16}$
5. $a_5 = \frac{55}{32}, b_5 = \frac{56}{32}$
6. $a_6 = \frac{110}{64}, b_6 = \frac{111}{64}$
7. $a_7 = \frac{221}{128}, b_7 = \frac{222}{128}$
8. $a_8 = \frac{443}{256}, b_8 = \frac{444}{256}$
9. $a_9 = \frac{886}{512}, b_9 = \frac{887}{512}$
10. $a_{10} = \frac{1773}{1024}, b_{10} = \frac{1774}{1024}$

2 Question 2

Let α and β be two cuts of $\mathbb{Q}^+$. Let $\alpha \cdot \beta = \{rs \mid r \in \alpha, s \in \beta\}$. Prove that $\alpha \cdot \beta$ is a cut of $\mathbb{Q}^+$.

Proof. To show that any given set is a cut, one must show that for a given set S:

1. $\forall x \in S$, if $y < x$, then $y \in S$.
2. S has no maximal element.

1. Let $c = a \cdot b \in \alpha \cdot \beta$ where $a \in \alpha$ and $b \in \beta$ and α and β are cuts of $\mathbb{Q}^+$. Let $d < c$. Thus $d < a \cdot b$. Then $\frac{d}{b} < a$ (the inequality will not flip since this is over $\mathbb{Q}^+$). Since α defines a cut and $\frac{d}{b} < a$, $\frac{d}{b} \in \alpha$. Then $d = \frac{d}{b} \cdot b$ where $\frac{d}{b} \in \alpha, b \in \beta$, so $d \in \alpha \cdot \beta$.

2. Let $c = a \cdot b \in \alpha \cdot \beta$. Since α is a cut, $\exists a_2 \in \alpha : a_2 > a$, which then means $a_2 \cdot b \in \alpha \cdot \beta$ and $a_2 \cdot b > a \cdot b = c$

And since properties 1 and 2 have been shown, $\alpha \cdot \beta$ must be a Dedekind cut. $\square$

3 Question 3

Let $1^* = \{r \in \mathbb{Q}^+ \mid r < 1\}$. Prove that $\alpha \cdot 1^* = \alpha$ for any cut of $\mathbb{Q}^+$.

Proof. Take any $a \cdot b \in \alpha \cdot 1^*$, where $a \in \alpha$ and $b \in 1^*$. Then by definition, $b < 1$. Then we have $a \cdot b < a \cdot 1 = a$, and therefore since α defines a Dedekind cut, $a \cdot b \in \alpha$. So then any member of $\alpha \cdot 1^*$ must also be a member of α, thus $\alpha \cdot 1^* \subseteq \alpha$.

Conversely,

$\square$