Homework 2

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Question 1

A complex number z is called *algebraic* if there exists integers $a_0, a_1, ..., a_n$, such that $a_n z^n + \cdots + a_1 z + a_0$. Prove that the algebraic numbers are countable.

Proof. To begin, first consider all polynomials in the form

$$a_n z^n + \cdots + a_1 z + a_0$$

From the fundamental theorem of algebra, it is known that there are $n + 1$ complex solutions to the given expression (all of whom are therefore algebraic numbers). Let $\mathbb{F}$ be the set of all polynomials with integer coefficients and let F_i be a set of all polynomials with integer coefficients of degree i. Then clearly,

$$\mathbb{F} = \bigcup_{i=0}^{\infty} F_i$$

And clearly this is a countable union, since the ith term can be mapped to i in the integers. Next, consider the mapping $f : F_i \mapsto \mathbb{Z}^{i+1}$:

$$a_i z^i + \cdots + a_1 z + a_0 \mapsto (a_0, ..., a_i)$$

Evidently this defines an bijection on F_i since if any outputs in $\mathbb{Z}^{i+1}$ were the same it would indicate the exact same polynomial and any $(a_0, ..., a_i) \in \mathbb{Z}^{i+1}$ will corresponds unqiuely to a polynomial in F_i since there are no restraints on range of acceptable coefficients to any term of a given polynomial in $F - i$ or $\mathbb{F}$.

$\square$