Homework Assignment 1

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Question 1

a.) Let $f_n(x) = x^n \in \mathcal{C}([0,1])$. Prove that $\{f_n\}$ has no convergent sub sequence in the norm of $\mathcal{C}([0,1])$

Proof. First we must note that $\{f_n\}$ and any of its subsequences converge pointwise to the following function $f(x)$:

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

Proof. First note that $\forall n \in \mathbb{N}, f_n(0) = 0^n = 0$, and $f_n(1) = 1^n = 1$, hence $\{f_n\}$ and all of its subsequences converge to f at 0 and 1.

Next, consider all $0 < x < 1$. Then for any infinite and monotone increasing subsequence $s(n) : \mathbb{N} \to \mathbb{N}$ (including the identity function of the natural numbers, or in other words, $\{f_n\}$ itself.)

$$\lim_{n\to\infty} f_{s(n)}(x) \leq \sum_{n=1}^{\infty} f(x) = \sum_{n=1}^{\infty} x^n$$

But since $0 < x < 1$, $\sum_{n=1}^{\infty} x^n$ must be convergent; therefore implying that $\lim_{n\to\infty} f_{s(n)}(x) = 0$, thus showing that $f(x)$ is the pointwise limit of $\{f_n\}$. $\square$

It is immediately clear however that f is not continuous, since if $0 < \epsilon < 1$, then one cannot find a δ such that $|1 - x| < \delta \to |f(1) - f(x)| < \epsilon$ since $|f(1) - f(x)| = 1$ for all $0 \leq x < 1$. As a result then, $f(x) \notin \mathcal{C}([0,1])$, and hence, $\{f_n\}$ nor any of its subsequences converge in $\mathcal{C}([0,1])$. $\square$

b.) Use this to prove that the unit ball $B = \{f \in \mathcal{C}([0,1]) : ||f|| \leq 1\}$ is not compact.

Proof. Using the norm $||f(x)|| = sup_{x\in[0,1]}(|f(x)|)$, one can see that for all $f_i \in \{f_n\}, ||f_i|| = 1$ since each f_i is monotone increasing with a maximum on $[0,1]$ of $f_i(1) = 1$. Hence $\{f_n\} \in B$.

However, according to theorem 3.6 in Walter Rudin's *Priciples of Mathematical Analysis*, if $\{f_n\}$ is a sequence in a compact metric space, then some sub-sequence of $\{f_n\}$ converges to a point in $\mathcal{C}([0,1])$. However it has been shown that no subsequence of $\{f_n\}$ converges in $\mathcal{C}([0,1])$ (and by extention any of its subsets), and that all f_i lie in B. Hence one must conclude that B is not compact. $\square$

Question 2

a.) Let (X, d) be a metric space and let $K \subset X$ be a compact subset. Prove that for all $\epsilon > 0$ there are finitely many points $x_1, \ldots, x_n \in K$ so that, for every $x \in K$ there exists an $i, i = 1, \ldots, n$, such that $d(x, x_i) < \epsilon$

Proof. The set of all open balls centered at all points of K with radius ϵ creates an open cover of K. Call this set B. Because K is compact, then B has a finite sub-covering of open balls of radius ϵ, $B_1, B_2, \cdots, B_N$. Therefore since $K \subset \bigcup_{i=1}^{N} B_i$, it must be the case that for every $x \in K$, the distance between x and at least one of the ball's centers $\{x_1, x_2, \cdots x_n\}$ has a distance less than ϵ. Therefore, there exists an $i \in \mathbb{N} : 1 \leq i \leq n$ such that $\forall x \in K, d(x, x_i) < \epsilon$. $\square$

b.) Use this to prove that, if $K \subset \mathcal{C}([0, 1])$ is compact, then K is equicontinuous.

Proof. Since K is compact, given any $\epsilon > 0$ select $f_1, f_2, \cdots, f_n$ such that $\forall f \in K, \exists i \in \mathbb{N}([1, n])$ such that $|f(x) - f_i(x)| < \frac{\epsilon}{3}$. Further, since K is compact it is also known that each $f \in K$ are uniformly continuous. Therefore, for each f_i, $\exists \delta_i > 0$ such that if $|x - y| < \delta$, then $|f_i(x) - f_i(y)| < \frac{\epsilon}{3}$. Let $\delta = \inf(\delta_i)$. Then we have, if $|x - y| < \delta$

$$\begin{aligned}
|f(x) - f(y)| &= |(f(x) - f_i(x)) + (f_i(x) - f_i(y)) + (f_i(y) - f(y))| \\
&\leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

Thus proving K to be equicontinuous. $\square$

Question 3

Recall the norms $||f||_1 = \int_0^1 |f(x)|\ dx$ and $||f||_\infty = \sup_{x\in X}\{|f(x|)\}$ on $\mathcal{C}([0,1])$.

a.) Prove that $||f||_1 \leq ||f||_\infty$ for all $f \in \mathcal{C}([0,1])$

Proof. Note that

$$\int_a^b f(x)\ dx \leq (b-a)\cdot sup_{x\in[a,b]}(|f(x)|)$$

Then in this case,

$$||f||_1 = \int_0^1 |f(x)|\ dx \leq (1-0)\cdot sup_{x\in[0,1]}(|f(x)|) = ||f||_\infty$$

Hence $\forall f \in \mathcal{C}([0,1])$, $||f||_1 \leq ||f||_\infty$. $\square$

b.) Prove that there is no constant $C > 0$ such that $||f||_\infty < C||f||_1$ holds for all $f \in \mathcal{C}([0,1])$ by producing a sequence $f_n \in \mathcal{C}([0,1])$ with $||f_n||_\infty \to \infty$ and $||f_n||_1 = 1$

Proof. Consider the following sequence of functions, $\{f_n\}$:

$$f_n(x) = \begin{cases} 2n - 2n^2x & 0 \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \end{cases}$$

Observe that for any fixed n,

$$\begin{aligned} ||f_n||_1 &= \int_0^1 |f(x)|\ dx \\ &= \int_0^{\frac{1}{n}} |2n - 2n^2x|\ dx + \int_{\frac{1}{n}}^1 |0|\ dx \\ &= 2n\left(\frac{1}{n}\right) - n^2\left(\frac{1}{n}\right)^2 \\ &= 1 \end{aligned}$$

Also notice that since $||f_n||_\infty = 2n$, $\lim_{n\to\infty} ||f_n|| = \infty$. Fix any $n \in \mathbb{N}$, $C \in \mathbb{R}$ such that $||f_n||_\infty < C||f_n||_1$. Then for any $m \geq \lceil \frac{C}{2} \rceil$, we will have

$$||f_m||_\infty \;\geq\; C \;=\; C||f_m||_1$$

Hence there does not exist any such $C > 0$ such that $||f_i||_\infty < C||f_i||_1$ over all $f_i \in \{f_n\}$ since one can find an $m > n$ such that the inequality is no longer true. By extention it is the case that $\nexists\ C > 0$ such that $||f||_\infty < C||f||_1$ holds for all $f \in \mathcal{C}([0,1])$. □

c.) Prove that $\mathcal{C}([0,1])$ with norm $||f||_1$ is not a complete metric space. Observe that this gives another proof of (b).

Proof. To do this, one should consider a different sequence of functions in $\mathcal{C}([0,1])$, $\{g_n\}$ defined like so:

$$g_n(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ n\left(x - \frac{1}{2}\right) & \frac{1}{2} < x \leq \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} < x \leq 1 \end{cases}$$

It is easy to see that each g_n is continuous. For any $\epsilon > 0$ choose $0 < \delta < \frac{\epsilon}{n}$. Then there are three cases to consider:

Case 1: $x \leq \frac{1}{2}$ and $y \leq \frac{1}{2}$ or $x > \frac{1}{2} + \frac{1}{n}$ and $y > \frac{1}{2} + \frac{1}{n}$.
Given any $\epsilon > 0$, and if $|x - y| < \delta$, then $|g_n(x) - g_n(y)| = 0 < \epsilon$

Case 2: $\frac{1}{2} < x \leq \frac{1}{2} + \frac{1}{n}$ and $\frac{1}{2} < y \leq \frac{1}{2} + \frac{1}{n}$
Given any $\epsilon > 0$, and if $|x - y| < \delta$, then

$$\begin{aligned} |g_n(x) - g_n(y)| &= \left|\left(nx - \frac{n}{2}\right) - \left(ny - \frac{n}{2}\right)\right| \\ &= |nx - ny| = |n(x - y)| \\ &= |n| \cdot |x - y| < n \cdot \frac{\epsilon}{n} = \epsilon \end{aligned}$$

Case 3: Either $\min(x,y) \leq \frac{1}{2}$ and $\frac{1}{2} < \max(x,y) \leq \frac{1}{2} + \frac{1}{n}$ (which will be referred to as the "lower case") or $\frac{1}{2} < \min(x,y) \leq \frac{1}{2} + \frac{1}{n}$ and $\frac{1}{2} + \frac{1}{n} < \max(x,y)$ (which will be referred to as the "upper case").

In this case, notice that, given any $\epsilon > 0$, and x, y such that $|x - y| < \delta$ we have

$$|g_n(x) - g_n(y)| \leq \left|\left(nx - \frac{n}{2}\right) - \left(ny - \frac{n}{2}\right)\right|$$

For the lower case suppose that, without loss of generality, $\min(x, y) = x$. This imples that $x \leq \frac{1}{2} < y$ and thus

$$nx - \frac{n}{2} \leq \frac{n}{2} - \frac{n}{2} = 0$$

and

$$ny - \frac{n}{2} > \frac{n}{2} - \frac{n}{2} = 0$$

Then since $g_n(y) = ny - \frac{n}{2} > 0$ and $f(x) = 0$ but $nx - \frac{n}{2} \leq 0$, it follows that the absolute difference between $g_n(y)$ and $nx - \frac{n}{2}$ must be greater than or equal to the absolute difference between $g_n(x)$ and $g_n(y)$, as asserted above.

For the upper case suppose that, without loss of generality, $\max(x, y) = y$. This means that $\frac{1}{2} < x \leq \frac{1}{2} + \frac{1}{n}$ and $\frac{1}{2} + \frac{1}{n} < y$. It then follows that

$$ny - \frac{n}{2} > n\left(\frac{1}{2} + \frac{1}{n}\right) - \frac{n}{2} = \frac{n}{2} + 1 - \frac{n}{2} = 1$$

Hence $ny - \frac{n}{2} > 1$ and $g_n(y) = 1$. Since expressions in the previous case show that $nx - \frac{n}{2} = g_n(x) > 0$, it follows that $\left|g_n(x) - \left(ny - \frac{n}{2}\right)\right| \geq |g_n(x) - g_n(y)|$ since $ny - \frac{n}{2} \geq g_n(y)$.

Then it is very straightforward since in either the lower or upper case we have:

$$\begin{aligned} |g_n(x) - g_n(y)| &\leq \left|\left(nx - \frac{n}{2}\right) - \left(ny - \frac{n}{2}\right)\right| \\ &= |nx - ny| = |n(x - y)| \\ &= |n| \cdot |x - y| < n \cdot \frac{\epsilon}{n} = \epsilon \end{aligned}$$

Hence $\{g_n\}$ is a sequence of continuous functions in $\mathcal{C}([0, 1])$. Next, it is plain to see that $\{g_n\}$ converges pointwise to the following function g:

$$g(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} < x \leq 1 \end{cases}$$

Since for any n, if $0 \leq x \leq \frac{1}{2}$, $g_n(x) = 0$. For any $\frac{1}{2} < x < 1$, choose any $N \in \mathbb{N}$ such that $N > \left(x - \frac{1}{2}\right)^{-1}$. It follows that $x > \frac{1}{N} + \frac{1}{2}$ and thus $g_N(x) = 1$, proving $g(x)$ to be the pointwise limit of $\{g_n\}$.

It is also clear that $g(x)$ is not continuous since $\forall \delta > 0$, $\exists x \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta]$ such that $|g\left(\frac{1}{2}\right) - g(x)| = 1$ since $\frac{1}{2} + \delta > \frac{1}{2}$. It follows then that for all

$\epsilon > 0$, $|x - y| < \delta$ does not imply that $|g(x) - g(y)| < \epsilon$. Hence $g(x)$ is not continuous and $g \notin \mathcal{C}([0,1])$. This also implies that $\{g_n\}$ does not converge in $\mathcal{C}([0,1])$.

It then suffices to prove that $\mathcal{C}([0,1])$ is not a complete metric space by showing that that $\{g_n\}$ is Cauchy, since it has already been shown to not converge in $\mathcal{C}([0,1])$.

Proof. Given some $\epsilon > 0$, select any $m, n, N \in \mathbb{N}$ such that $n \neq m$; $m, n \geq N$ and $N > \frac{1}{\epsilon}$. Without loss of generality, suppose that $n > m$. Then

$$||g_n(x) - g_m(x)||_1 = \int_0^1 |g_n(x) - g_m(x)| dx$$

Since $g_n(x) - g_m(x) = 0$ for all $0 \leq x \leq \frac{1}{2}$ and $\frac{1}{2} + \frac{1}{m} < x \leq 1$, one need only consider two intervals of x:

$$\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} |(n-m)\Big(x - \frac{1}{2}\Big)| \ dx + \int_{\frac{1}{2}+\frac{1}{n}}^{\frac{1}{2}+\frac{1}{m}} |1 - m\Big(x - \frac{1}{2}\Big)| \ dx$$

Examining the left hand expression, note that since $n > m$, $n - m > 0$ and since $x \geq \frac{1}{2}$, we have $x - \frac{1}{2} \geq 0$. As a result $(n-m)(x - \frac{1}{2}) \geq 0$ and $|(n-m)(x-\frac{1}{2})| = (n-m)(x - \frac{1}{2})$. So then we have

$$\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} (n-m)\Big(x - \frac{1}{2}\Big) = \frac{n-m}{2}\Big(x(x-1)\Big|_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}}\Big)$$

Which by plugging in the integral's bounds gives us

$$\frac{n-m}{2}\Big(\Big(\frac{1}{n} + \frac{1}{2}\Big)\Big(\frac{1}{n} - \frac{1}{2}\Big) - \Big(\frac{1}{2}\Big)\Big(-\frac{1}{2}\Big)\Big) = \frac{1}{2n} - \frac{m}{2n^2}$$

Next consider the right hand expression. Because $x \leq \frac{1}{2} + \frac{1}{m}$, $m(x - \frac{1}{2}) \leq m((\frac{1}{m} + \frac{1}{2} - \frac{1}{2})) = 1$. So since $m(x - \frac{1}{2}) \leq 1$, it follows that $1 - m(x - \frac{1}{2}) \geq 0$ and hence $|1 - m(x - \frac{1}{2})| = 1 - m(x - \frac{1}{2})$. Then we have

$$\int_{\frac{1}{2}+\frac{1}{n}}^{\frac{1}{2}+\frac{1}{m}} 1 - m\Big(x - \frac{1}{2}\Big) \ dx = \frac{1}{m} - \frac{1}{n} - \frac{m}{2}\Big(x(x-1)\Big|_{\frac{1}{2}+\frac{1}{n}}^{\frac{1}{2}+\frac{1}{m}}\Big)$$

Evaluating the rightmost term yields

$$\Big(\frac{1}{m} - \frac{1}{n}\Big) - \Big(\frac{1}{2m} - \frac{m}{2n^2}\Big) = \frac{m}{2n^2} + \frac{1}{2m} - \frac{1}{n}$$

Combining the two integrals then gives us

$$\left(\frac{1}{2n} - \frac{m}{2n^2}\right) + \left(\frac{m}{2n^2} + \frac{1}{2m} - \frac{1}{n}\right) = \frac{1}{2m} - \frac{1}{2n}$$

However since $n > m$ it follows that $\frac{1}{2m} > \frac{1}{2n}$ and hence $\frac{1}{2m} - \frac{1}{2n} > 0$. It is also then the case that

$$\frac{1}{2m} - \frac{1}{2n} < \frac{1}{2m} < \frac{1}{m} \leq \frac{1}{N} < \epsilon$$

Thereby proving that $\{g_n\}$ is a Cauchy sequence. $\square$

Thus $\mathcal{C}([0,1])$ is not complete since $\{g_n\} \subset \mathcal{C}([0,1])$ is Cauchy and $\{g_n\}$ does not converge in $\mathcal{C}([0,1])$. $\square$

Question 4

Given the following function $f(x)$,

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$$

a.) For which $x \in \mathbb{R}$ does the series converge (absolutely)?

It is clear to see that the series does not converge when $x = 0$ since the series devolves into $1 + 1 + 1 + \cdots = \infty$. It also does not converge when $x = \frac{-1}{k^2}$, where $k = 1, 2, 3, ...$ since the k-th element of the sequence will be undefined. Then consider any other $xin\mathbb{R}$. If $x < -1$ or $x > 0$, then

$$\left|\frac{1}{1+n^2x}\right| \leq \left|\frac{1}{n^2x}\right|$$

And is therefore absolutely convergent by direct comparison to the $p = 2$ series. For $-1 > x > 0$, the problem is a bit more complex. Given any $1 > \delta > 0$, where $x \in (-1, -\delta]$ is not of an illegal form $(\frac{-1}{k^2})$, then when $n \geq \sqrt{(\frac{2}{\delta})}$ we have

$$\left|\frac{1}{1+n^2x}\right| \leq \frac{1}{n^2} \cdot \frac{1}{\delta - \frac{1}{n^2}} \leq \frac{2}{n^2\delta}$$

Which implies that $f(x)$ converges absolutely between $(-1, 0)$ as well. This answers both part a.) and part b.) of this question.

c.) Is f bounded?

It appears to be that f is not bounded since for $x < 0$ we have

$$f(x) = \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{\frac{1}{x} + n^2}$$

Where $0 < \frac{1}{\frac{1}{x}+n^2} < \frac{1}{n^2}$, and is therefore convergent. But then

$$\lim_{x \to 0^+} \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{\frac{1}{x} + n^2} = \infty$$

Question 5