Homework 2

Joshua Cragun *

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*Prof. Savin, Math 3210

Question 1

A complex number z is called *algebraic* if there exists integers $a_0, a_1, ..., a_n$, such that $a_n z^n + \cdots + a_1 z + a_0 = 0$. Prove that the algebraic numbers are countable.

Proof. To begin, first consider all polynomials in the form

$$a_n z^n + \cdots + a_1 z + a_0$$

Let $\mathbb{F}$ be the set of all polynomials with integer coefficients and let F_i be a set of all polynomials with integer coefficients of degree i. Then clearly,

$$\mathbb{F} = \bigcup_{i=0}^{\infty} F_i$$

And clearly this is a countable union, since the ith term can be mapped to $i + 1$ in the natural numbers. Next, consider the mapping $f : F_i \mapsto \mathbb{Z}^{i+1}$:

$$a_i z^i + \cdots + a_1 z + a_0 \mapsto (a_0, ..., a_i)$$

Evidently this defines an bijection on F_i since if any outputs in $\mathbb{Z}^{i+1}$ were the same it would indicate the exact same polynomial (injective) and any $(a_0, ..., a_i) \in \mathbb{Z}^{i+1}$ will corresponds unqiuely to a polynomial in F_i (surjective). Further, this shows that F_i is countable since this is a bijection with $\mathbb{Z}^{i+1}$, which is countable as a consequence of theorem 2.13 in Rutin's *Principles of Mathematical Analysis*. Since $\mathbb{Z}^{i+1}$ is countable, there exists an injection $g : \mathbb{Z}^{i+1} \mapsto \mathbb{N}$. Then by taking the composition $g \circ f$ we get an injection from each F_i to $\mathbb{N}$. Hence, F_i is countable. In addition, $\mathbb{F}$ is countable since it is a countable union of countable sets.

Now, since each F_i is countable it is possible for each F_i to put all of their polynomials $f_n, n \in \mathbb{N}$ into a sequence $f_1, f_2, ...$ and so on. By the fundamental theorem of algebra, each f_n has at most i roots. Let $r_j, j \in \mathbb{N}$ correspond to the set of roots of the jth polynomial in $\{f_n\}$. Then we also have the sequence $r_1, r_2, ...$ for each $F_i \subseteq \mathbb{F}$. Define R_i as the union of each r_j derived from an F_i. Each R_i is therefore countable since it is a countable union of countable (or more spectifically, finite) sets. But then clearly the algebraic numbers are the union of all R_i, hence then the algebraic numbers must be countable since they can be expressed as a countable union of countable sets.

□

Question 2

Prove that the following two (X, d) are metric spaces:

- $X = \mathbb{R}^2$ and $d((x_1, x_2), (y_1, y_2)) = \max(|x_1 - y_1|, |x_2 - y_2|)$
- $X = \mathbb{Z}$ and $d(x, x) = 0$ or $d(x, y) = \frac{1}{2^n}$, if $x \neq y$, where 2^n is the largest power of 2 dividing $x - y$.

To show if any (X, d) is a metric space, one needs to show three things:

- $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$, $\forall x, y \in X$
- $d(x, y) = d(y, x)$, $\forall x, y \in X$
- $d(x, z) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in X$

Part 1: To prove the first property, one must consider two cases:

Case 1: $x = y$
If $x = y$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ then $x_1 = y_1$ and $x_2 = y_2$. Then $d(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|) = \max(|x_1 - x_1|, |x_2 - x_2|) = \max(|0|, |0|) = 0$. Thus, when $x = y$, $d(x, y) = 0$.

Case 2: $x \neq y$
Since $x \neq y$, $x_1 \neq y_1$ or $x_2 \neq y_2$. Then $d(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)$. Without loss of generality, assume that either $x_1 = y_1$ or $x_2 = y_2$. Then one of the arguments in $\max(|x_1 - y_1|, |x_2 - y_2|)$ is zero, however the other argument must be > 0 since the two points are not equal. Therefore then, the output must be that difference which is > 0. Conversely, if neither $x_k = y_k, k = 1, 2$ then both differnces will be greater than zero, and therefore the distance will be greater than 0 regardless of which one had the greater difference. Hence, $d(x, y) \geq 0, d(x, y) = 0 \iff x = y$.

For the next property, let x and y be as above.

Then $d(x, y) = \max(|x_1 - y_1|, |x_2 - y_2|)$. Suppose $x_1 - y_1 = n$ and $x_2 - y_2 = m$. Then $y_1 - x_1 = -n$ and $y_2 - x_2 = -m$. However

$$|y_1 - x_1| = |-n| = |n| = |x_1 - y_1|$$

And

$$|y_2 - x_2| = |-m| = |m| = |x_2 - y_2|$$

Therefore $\max(|x_1 - y_1|, |x_2 - y_2|) = \max(|y_1 - x_1|, |y_2 - x_2|) = d(y, x)$. Thus proving the second property.

Lastly, let x, y, and z be in the same form as in the previous parts. Then $d(x, z) = |x_k - z_k|$ where k can be 1 or 2 exclusively. Since $d(x, y) = |x - y|$ is knwon to form a metic space with $\mathbb{R}$, it is true that

$$|x_k - z_k| \leq |x_k - y_k| + |y_k - z_k|$$

Further, it is also true that

$$|x_k - y_k| \leq max(|x_1 - y_1|, |x_2 - y_2|)$$

$$|y_k - z_k| \leq max(|y_1 - z_1|, |y_2 - z_2|)$$

Hence, $d(x, z) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in \mathbb{R}^2$. Thus this is a metric space, as was to be shown.

Part 2: