Homework Assignment 4

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Question 1

Let $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| \leq (x-y)^2$. Prove that f is constant.

Proof. As given, we have two cases to consider since $f(x) - f(y) \leq (x-y)^2$ and $f(x) - f(y) \geq -(x-y)^2$.

In the former, we have

$$f'(x) = \lim_{c \to x} \frac{f(c) - f(x)}{c - x} \leq \frac{(c-x)^2}{c-x} = c - x = 0$$

In the latter we have

$$f'(x) = \lim_{c \to x} \frac{f(c) - f(x)}{c - x} \geq \frac{-(c-x)^2}{c-x} = x - c = 0$$

So then we have that $f'(x) \geq 0$ and $f'(x) \leq 0$. Hence $f'(x) = 0$, showing that f is constant. □

Question 2

Let f be a differentiable function defined in a neighborhood of x. Assume that $f''(x)$ exists. Prove that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Proof. To begin, first consider the derivative for some artbitrary differentiable function g.

$$g'(x) = \lim_{c \to x} \frac{g(c) - g(x)}{c - x} = \lim_{c \to x} \frac{g(x + (c-x)) - g(x)}{c - x}$$

Hence if one examines the limit intead with respect to $h = (c - x)$ we get

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

Then, using this notation instead we have

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

Where

$$f'(x+h) = \lim_{k\to 0} \frac{f(x+h+k) - f(x+h)}{k}$$
$$f'(x) = \lim_{l\to 0} \frac{f(x+l) - f(x)}{l}$$

So then if we choose $k = l = -h$, we get

$$\begin{aligned} f''(x) &= \lim_{h\to 0} \frac{\frac{f(x)-f(x+h)}{-h} - \frac{f(x-h)-f(x)}{-h}}{h} \\ &= \lim_{h\to 0} \frac{\frac{f(x+h)-f(x)}{h} + \frac{f(x-h)-f(x)}{h}}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} \end{aligned}$$

As was to be shown. $\square$

Question 3 (Fixed Point Theorem)

Let $f : \mathbb{R} \to \mathbb{R}$ such that $|f'(x)| \leq C$ for some $0 \leq C < 1$ and all x. A number x is a fixed point for f if $f(x) = x$. Prove that f cannot have two fixed points. Let x_1 be any real number, and define a sequence by $x_{n+1} = f(x_n)$. Prove that the sequence $\{x_n\}$ is Cauchy. Prove that the limit is a fixed point of f.

Part 1:

Proof. Let f be as above and suppose f had any number n of fixed points $x_1, ..., x_n$ with $n \geq 2$. Take x_1 and any fixed point $x_m, 1 < m \leq n$. Then

$$\frac{f(x_m) - f(x_1)}{x_m - x_1} = \frac{x_m - x_1}{x_m - x_1} = 1$$

Which by the Mean Value Theorem implies that $\exists c \in (x_1, x_m)$ such that $f'(c) = 1$. But by definition, $\forall x \in \mathbb{R}, f'(x) < 1$. This is a contradiction, which implies that f can at most have one fixed point, and thus cannot have two fixed points. $\square$

Part 2:

Proof. Let f be as above and let $\epsilon > 0$. By the triangle inequality we have for any fixed $n > 1$ (w.l.o.g. assume $m > n$):

$$|x_m - x_n| \leq |x_m - x_{m-1}| + \cdots + |x_{n+1} - x_n| \tag{1}$$

And since $|x_{n+1} - x_n| \leq C|x_n - x_{n-1}|$, it is also the case that

$$\begin{aligned} |x_{n+1} - x_n| &\leq C|x_n - x_{n-1}| \\ |x_{n+2} - x_{n+1}| &\leq C^2|x_n - x_{n-1}| \\ |x_{n+3} - x_{n+2}| &\leq C^3|x_n - x_{n-1}| \\ &\cdots \end{aligned}$$

Hence we have $|x_m - x_{m-1}| \leq C^{m-n}|x_n - x_{n-1}|$, which when combined with (1), we get

$$\begin{aligned} |x_m - x_n| &\leq C^{m-n}|x_n - x_{n-1}| + C^{m-n-1}|x_n - x_{n-1}| + \cdots + C|x_n - x_{n-1}| \\ &= \left(C^{m-n} + C^{m-n-1} + \cdots\right)|x_n - x_{n-1}| \\ &= \left(\sum_{i=1}^{m-n} C^i\right)|x_n - x_{n-1}| \\ &\leq \left(\sum_{i=m-n}^{\infty} C^i\right)|x_n - x_{n-1}| \end{aligned}$$

However as m increases $\sum_{i=m-n}^{\infty} C^i$ approaches zero, implying that there exists an m such that $\sum_{i=m-n}^{\infty} C^i < \frac{\epsilon}{|x_n - x_{n-1}|}$. Hence,

$$|x_m - x_n| \leq \left(\sum_{i=m-n}^{\infty} C^i\right)|x_n - x_{n-1}| < |x_n - x_{n-1}|\frac{\epsilon}{|x_n - x_{n-1}|} = \epsilon$$

Thus proving the sequence to be Cauchy. □

Part 3:

Proof. It is clear that a limit to $\{x_n\}$ exists, since the sequence is Cauchy over a complete metric space. It is also true that

$$\lim_{n\to\infty} x_n = L = \lim_{n\to\infty} x_{n-1}$$

And since $x_n = f(x_{n-1})$ and f is continuous we have

$$\lim_{n\to\infty} f(x_{n-1}) = f(\lim_{n\to\infty} x_{n-1}) = f(L)$$

Thus we have $\lim_{n\to\infty} x_n = L = f(L)$. Hence the limit is a fixed point. $\square$

Question 4 (Concavity)

Let $f : (\alpha, \beta) \to \mathbb{R}$ be twice differentiable function such that $f'' \geq 0$ on the interval. Let $c \in (\alpha, \beta)$ and let $g(x)$ be the linear function whose graph is the tangent line of the graph of f at c i.e. $g(x) = f(c) + f'(c)(x - c)$. Prove that $f(x) \geq g(x)$ for $x \in (\alpha, \beta)$.

Proof. Let f, g be as above. Since it is only guranteed that f is twice differentiable, one can represent f using a Taylor function like so:

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(b)(x - c)^2}{2}$$

With $b \in (\alpha, x)$. Using $g(x)$ as given we then have

$$\begin{aligned} f(x) - g(x) &= \left(f(c) + f'(c)(x - c) + \frac{f''(b)(x - c)^2}{2}\right) - f(c) + f'(c)(x - c) \\ &= \frac{f''(b)(x - c)^2}{2} \end{aligned}$$

However $\forall x \in (\alpha, \beta), f''(x) \geq 0$ as is $\frac{(x-c)^2}{2}$. Hence $f(x) - g(x) \geq 0$ and thus $f(x) \geq g(x)$ $\square$

Question 5 (Newton Method)

Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable function. Let $[a, b]$ be a closed interval such that $f(a) < 0$ and $f(b) > 0$, $f'(x) \geq \delta > 0$, and $f''(x) \geq 0$ for $x \in [a, b]$. Prove that there is unique $c \in (a, b)$ such that $f(c) = 0$. Define a sequence by $x_1 = b$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Prove that the sequence is decreasing and bounded from below by c, it has a limit. Prove that the limit is c. Check that the conditions are satisfied for $f(x) = x^2 - 2$ and the interval $[1, 2]$. What is the limit of the sequence $\{x_n\}$? Compute x_n for $n = 1, 2, 3, 4$.

Part 1:

Proof. Since $f(a) < 0$ and $f(b) > 0$, $0 \in [f(a), f(b)]$ because segments in $\mathbb{R}$ are connected. And since the function is continuous (since it is differentiable) $f([a, b]) = [f(a), f(b)]$, and since both are closed and bounded, the segments must be compact. Hence there exists at least one $c \in [a, b]$ such that $f(c) = 0$. Suppose then that there were many unique $c_j, j > 1$ such that $f(c_j) = 0$. Then we have for $1 \leq k \neq l \leq j$

$$\frac{f(c_k) - f(c_l)}{k - l} = 0$$

Which implies, by the mean value theorem, that $\exists d \in [a, b]$ such that $f'(d) = 0$. However $f'(x) \geq \delta > 0$. This is a condraction, hence there is only one unique point c such that $f(c) = 0$. $\square$

Part 2:

Proof. First consider the base case,

$$x_2 = b - \frac{f(b)}{f'(b)}$$

Since $f(b) > 0$ and $f'(x) > 0$, $x_1 > x_2$. Assume $x_1 \geq x_2 \geq \cdots \geq x_n$. Then we have

$$x_n = f(x_{n-1}) - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

Rearranging we get

$$0 = f(x_{n-1}) + f'(x_{n-1})(x_n - x_{n-1})$$

Which is also the tangent line of x_{n-1} at the point x_n. Since $f(x_n)$ is greater than or equal to the tangent line at x_{n-1}, (shown in question 4) we can conclude then that $f(x_n) \geq 0$. Since

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We see that the rightmost term is either positive or zero, thus $x_{n+1} \leq x_n$. Thus the sequence is decreasing. □

Part 3:

Proof.

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n-1} - \frac{\lim_{n\to\infty} f(x_{n-1})}{\lim_{n\to\infty} f'(x_{n-1})}$$

However $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n-1}$. Thus

$$\frac{\lim_{n\to\infty} f(x_{n-1})}{\lim_{n\to\infty} f'(x_{n-1})} = 0$$

Which implies that $\lim_{n\to\infty} f(x_{n-1}) = f(\lim_{n\to\infty} x_{n-1}) = 0$ since f is continuous. This implies that the limit is c. □

Part 4:

- $f(1) = -1 < 0$
- $f(2) = 2 > 0$
- $f'(1) = 2 > 0$
- $f'(2) = 4 > 0$
- $f''(x) = 2 \geq 0$

Thus $f(x) = x^2 - 2$ satisfies the conditions for this problem. Since the limit of the sequence is the solution of the formula, the limit is $\sqrt{2}$.

Part 5:

- $x_1 = 2$
- $x_2 = 2 - \frac{2}{4} = \frac{3}{2}$
- $x_3 = \frac{3}{2} - \frac{\frac{1}{4}}{3} = \frac{17}{12}$
- $x_4 = \frac{17}{12} - \frac{\frac{1}{144}}{\frac{17}{6}} = \frac{577}{408}$

Question 6

Consider the power series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$, i.e. the sequence whose n-th term is $(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$. Compute the radius of convergence of this series. Use the theorem of Taylor to prove that $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ for every x. Use this series to find a rational number that approximates $\sin(1/2)$ with an error less than $1/10^3$.

Part 1: The radius of convergence is $\frac{1}{\alpha}$ where $\alpha = limsup\sqrt[n]{|a_n|}$. However, it is also known that if the limit test converges, so too does the root test. Attempting to perform the root test of the series we get

$$\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1}(2n-1)!}{(-1^n)(2(n+1)-1)!}$$

$$= \frac{(-1)^{n+1}(2n-1)!}{(-1^n)(2n+1}!$$

$$= \frac{-1}{(2n+1)(2n)}$$

$$= \frac{-1}{4n^2+2n}$$

And

$$\lim_{n\to\infty} \frac{-1}{4n^2+2n} = 0$$

Hence the radius of convergence is $\frac{1}{0}$, or infinite.

Part 2:

Proof. To begin, we have the Taylor theorem, or

$$f(x) = f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2!}+\cdots\frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!}+\frac{f^{(n)}(c)(x-a)^{n-1}}{(n)!}$$

Assume $a = 0$. Then we have

$$sin(x) = sin(0) + cos(0)x - \frac{sin(0)x^2}{2!} - \frac{cos(0)x^3}{3!} + \cdots$$

Since $sin(0) = 0$, we can eliminate all the sin terms. And since $cos(0) = 1$, what is left is

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

As was to be shown □

Part 3: Knowing the equation above, the most important part is finding a suffieciently large n such that the error is less than $\frac{1}{1000}$. Examining the error term we have

$$\frac{sin^{(n)}(c)x^n}{n!} < \frac{1}{1000}$$

Knowing that $sin^{(n)}(c)$ is at most 1, it should be treated accordingly. Plugging $\frac{1}{2}$ into x gives us

$$\frac{(\frac{1}{2})^n}{n!} < \frac{1}{1000}$$

Which, as far as I'm aware, n is best determined by starting at one and incrementing n by two until the value is less than $\frac{1}{1000}$. Doing so gives us $n = 5$ with an error of

$$\frac{(\frac{1}{2})^5}{5!} = \frac{1}{3840} < \frac{1}{1000}$$

Our computed approximation is then

$$sin\left(\frac{1}{2}\right) \approx \frac{1}{2} - \frac{(\frac{1}{2})^8}{3!} + \frac{(\frac{1}{2})^5}{5!} = \frac{23}{48}$$