Homework Assignment 3

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Question 1

Let x_n be a sequence of positive real numbers such that $\lim_n x_n = x > 0$. Prove that

- $\lim_n x_n^2 = x^2$.
- $\lim_n \sqrt{x_n} = \sqrt{x}$.

Part 1: Using the identity

$$x_n^2 - x^2 = (x_n - x)^2 + 2x(x_n - x)$$

Where for any given $\varepsilon > 0$ there exists an integer N such that for any $m \geq N$, $|x_m - x| < \sqrt{\varepsilon}$. This implies that $|(x_n - x)^2| < \varepsilon$ and therefore $\lim_{n\to\infty}(x_n - x)^2 = 0$. And clearly

$$\lim_{n\to\infty} 2x(x_n - x) = 2x \cdot 0 = 0$$

Thus $\lim_n x_n^2 - x^2 = \lim_n (x_n - x)^2 + 2x(x_n - x) = 0$ and therefore

$$\lim_n x_n^2 = x^2$$

$\square$

Part 2:
By what was given, one need not consider the cases where $x \leq 0$. If $x > 0$ then there exists an N such that if $m \geq N$, $|x_m - x| < \varepsilon\sqrt{x}$. Then, because this is over positive real numbers,

$$|\sqrt{x_m} - \sqrt{x}| = \frac{|x_m - x|}{|\sqrt{x_m} + \sqrt{x}|} < \frac{|x_m - x|}{\sqrt{x}} < \frac{\varepsilon\sqrt{x}}{\sqrt{x}} = \varepsilon$$

Thus $\lim_n \sqrt{x_n} = \sqrt{x}$. $\square$

Question 2

Define a sequence by $s_1 = 1$ and $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$. Prove that $s_n < 2$ for all n, and that s_n is an increasing sequence. Find the limit.

To begin, one sees immediately that s_n increases between $s_1 = 1$ and $s_2 = \sqrt{3}$. Suppose that s_n is increasing from $n = 1, ..., m$. Then,

$$\begin{aligned} s_m = \sqrt{2 + \sqrt{s_{m-1}}} &> s_{m-1} \\ \sqrt[4]{2 + \sqrt{s_{m-1}}} &> \sqrt{s_{m-1}} \\ 2 + \sqrt[4]{2 + \sqrt{s_{m-1}}} &> 2 + \sqrt{s_{m-1}} \\ \sqrt{2 + \sqrt[4]{2 + \sqrt{s_{m-1}}}} = s_{m+1} &> \sqrt{2 + \sqrt{s_{m-1}}} = s_m \end{aligned}$$

Hence $s_{m+1} > s_m$. Thus the sequence is increasing for all $n \geq 1$. Next, suppose that some $s_m \geq 2$. Then,

$$\begin{aligned} 2 &\leq \sqrt{2 + \sqrt{s_{m-1}}} \\ 4 &\leq 2 + \sqrt{s_{m-1}} \\ 2 &\leq \sqrt{s_{m-1}} \\ 4 &\leq s_{m-1} \end{aligned}$$

Thus, if $s_m \geq 2$ then $s_{m-1} \geq 4$. But this contradicts the fact that the sequence is strictly increasing. Thus $s_n < 2$ for all $n \geq 1$. To find the limit of the sequence, let $\lim_n \sqrt{2 + \sqrt{s_n}} = L$. Then

$$L = \sqrt{\lim_n (2 + \sqrt{s_n})} = \sqrt{2 + \lim_n \sqrt{s_n}} = \sqrt{2 + \sqrt{\lim_n s_n}} = \sqrt{2 + \sqrt{L}}$$

Thus the limit will be a solution to the expression above, or when rearranged:

$$L^4 - 4L^2 - L + 4 = 0$$

Which is approximately 1.83118, or in exact form:

$$\frac{1}{3}\Big(-1 + \sqrt[3]{\frac{1}{2}(79 - 3\sqrt{249})} + \sqrt[3]{\frac{1}{2}(79 + 3\sqrt{249})}\Big)$$

Question 3

Let $X = \mathbb{Z}$ and $d(x,x) = 0$ or $d(x,y) = \frac{1}{2^n}$, if $x \neq y$, where 2^n is the largest power of 2 dividing $x - y$. Prove that the following two series are Cauchy. One of them is convergent (find its sum) while the other not (explain).

- $\sum_{n=0}^{\infty} 2^n$
- $\sum_{n=0}^{\infty} (-2)^n$

To begin, let s denote the first series, and let r denote the second. Consider two nonequal partial sums s_n, and s_m. Then,

$$s_n = 2^0 + 2^1 + \cdots + 2^n$$
$$s_m = 2^0 + 2^1 + \cdots + 2^m$$

Without loss of generality, assume $m > n$. Then,

$$\begin{aligned} s_m - s_n &= 2^{n+1} + \cdots + 2^m \\ &= 2^{n+1}(1 + 2^1 + \cdots + 2^{m-n-1}) \end{aligned}$$

Thus the differences between any two nonequal partial sums of degrees m and n with $m > n$ is $\frac{1}{2^{n+1}}$. If $n = m$ then the distance is zero, by definition. One can easily see that the same is true for r by simply replacing any 2 in the steps above with (-2) and then factoring out 2^{n+1} at the very end.

With this is it very straightforward to show that both series are Cauchy. Given any $\varepsilon > 0$ one can find a j such that $0 < \frac{1}{2^j} < \varepsilon$ and $\varepsilon \leq \frac{1}{2^{j+1}}$. Without loss of generality, take any partial sum of degree $k > j$ of s, then

$$d(s_k, s_j) = \frac{1}{2^{j+1}} < \varepsilon$$

It is very easy to see that the statement above is equally true for r. Hence, both series are Cauchy. □

Although both are Cauchy, only s is convergent, and it converges to -1. The reason is through considerations of the limit of the similar sequence $\lim_{n\to\infty} 2^n$. One sees that the distance between 0 and any 2^n is $\frac{1}{2^n}$ since 0 is divisible by any integer. Hence,

$$\lim_{n\to\infty} d(0, 2^n) = \lim_{n\to\infty} \frac{1}{2^n} = 0$$

Thus $\lim_{n\to\infty} 2^n = 0$ and since any partial sum of s, $s_n = 2^{n+1} - 1$,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 2^{n+1} - 1 = 0 - 1 = -1$$

The reason the other does not converge is slightly more nuanced, because in a sense, the series does converge. Consider the sequence of partial sums of r, $\{r_n\}$. We have

$$r_{n\geq 0} = \frac{1}{3}((-1)^n(2^{2n} + 1)$$

So Then

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} \frac{1}{3}((-1)^n 2^{2n} + 1)$$

Where $(-1)^n 2^{2n}$ will either be $-(2^{2n})$ or 2^{2n}. Hence $d((-1)^n 2^{2n}, 0)$ is $\frac{1}{2^{2n}}$, which as n increases converges to 0. Thus,

$$\lim_{n\to\infty} \frac{1}{3}((-1)^n 2^{2n} + 1) = \frac{1}{3}(0 + 1) = \frac{1}{3}$$

So if this metric space was over $\mathbb{Q}$ or above, the series would converge, however since $\frac{1}{3} \notin \mathbb{Z}$, the series can not converge, by definition. $\square$

Question 4

Let c_n be a sequence of positive numbers. Prove that

$$\text{liminf}_n \frac{c_{n+1}}{c_n} \leq \text{liminf}_n \sqrt[n]{c_n}.$$

Proof. Let $\alpha' = \text{liminf}_n \sqrt[n]{c_n}$ and $\alpha = \text{liminf}_n \frac{c_{n+1}}{c_n}$. Also let B be an arbitrary positive number such that $B < \alpha$ and let $z_m = \text{liminf}_{m \geq n} \frac{c_{m+1}}{c_m}$. Then $\exists N$ such that $z_N \geq B$ for all $m \geq N$. In other words we have,

$$B \leq \frac{c_{N+1}}{c_N}, B \leq \frac{c_{N+2}}{c_{N+1}}, \cdots, B \leq \frac{c_{m+1}}{c_m}$$

Or,

$$c_{N+1} \geq Bc_N, c_{N+2} \geq B^2 c_N, \cdots, c_m \geq B^{m-N} c_N$$

So then in another form, $c_m \geq \frac{c_N}{B^N} B^m$. Then we also have that

$$\sqrt[m]{c_m} \geq \sqrt[m]{\frac{c_N}{B^N} B^m}$$

However note that $\frac{c_N}{B^N}$ is constant, so as m increases, the value of the m-th root of the quotient approaches 1, so we then have

$$\sqrt[m]{c_m} \geq \sqrt[m]{\frac{c_N}{B^N} B^m} \to \sqrt[m]{B^m} = B$$

Thus for all $B < \alpha$, $\alpha' \geq B$. This implies that $\alpha' \geq \alpha$. In other words,

$$\text{liminf}_n \sqrt[n]{c_n} \geq \text{liminf}_n \frac{c_{n+1}}{c_n}.$$

$\square$

Question 5

Let (X, d) be a metric space. Let $x = \{x_n\}$ and $y = \{y_n\}$ be two Cauchy sequences. Prove that the sequence of distances $d(x_n, y_n)$ is a Cauchy sequence of real numbers.

Proof. Since $\{x_n\}$ and $\{y_n\}$ are both Cauchy, it is true that for any $\varepsilon > 0$, $\exists N$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ for all $m \geq N$ (and likewise for $\{y_n\}$). Then because X is a metric space,

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \tag{1}$$

$$d(x_n, y_n) < \frac{\varepsilon}{2} + d(x_m, y_m) + \frac{\varepsilon}{2} \tag{2}$$

By rearranging (2) we get

$$d(x_n, y_n) - d(x_m, y_m) < \varepsilon$$

And by multiplying (1) by -1 and rearranging, we find that

$$d(x_m, y_m) - d(x_n, y_n) > -\varepsilon$$

Which implies that

$$|d(x_n, y_n) - d(x_m, y_m)| < \varepsilon$$

And since this is the standard distance formula for the real numbers, this shows that the sequence of distances $d(x_n, y_n)$ is a Cauchy sequence in $\mathbb{R}$. $\square$

Question 6

Let (X, d) be a metric space. Two Cauchy sequences $x = \{x_n\}$ and $y = \{y_n\}$ are equivalent if for every $\epsilon > 0$, there exists n such that $d(x_m, y_m) < \epsilon$ for all $m \geq n$. Prove that this is an equivalence relation.

Proof. Since X is a metric space, $d(x, x) = 0$ with any $x \in X$. Hence for any sequence $\{x_n\}$

$$d(x_m, x_m) = 0 < \varepsilon$$

Thus $\{x_n\}$ is equivalent to itself, and thus the relation is reflexive. Next, suppose a sequence $x = \{x_n\}$ was equivalent to another sequence $y = \{y_n\}$. Then since X is a metric space,

$$d(y_m, x_m) = d(x_m, y_m) < \epsilon$$

Hence y is equivalent to x and is therefore a symmetric relation. Now suppose a sequence $x = \{x_n\}$ was equivalent to another sequence $y = \{y_n\}$ and suppose that y was equivalent to another sequence $z = \{z_n\}$. Then there exists an N_1, N_2 such that $d(x_m, y_m) < \frac{\varepsilon}{2}$ and $d(y_m, z_m) < \frac{\varepsilon}{2}$ for $m \geq max(N_1, N_2)$. Then since X is a metric space,

$$\begin{aligned} d(x_m, z_m) &\leq d(x_m, y_m) + d(y_m, z_m) \\ d(x_m, z_m) &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

therefore x and z are also equivalent and the relation is transitive. Since the reltaion is reflexive, symmetric, and transitive, the relation defines an equivalence relation. $\square$

Question 7

Let Y be a non-empty set and $d : Y \times Y \to [0, \infty)$ a "distance" function such that

- $d(x, x) = 0$ for all $x \in Y$.
- $d(x, y) = d(y, x)$ for all $x, y \in Y$.
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in Y$.

In words, d is almost a distance function, however, $d(x, y) = 0$ is allowed for different x and y. We say that x and y are equivalent if $d(x, y) = 0$. (1) Prove that this is an equivalence relation. (2) Prove that, if x is equivalent to y then $d(x, z) = d(y, z)$ for all $z \in Y$.

Proof. Reflexivity and symmetry are given. All that is left to show is transitivity. Let x be equivalent to y and y be equivalent to z. Then,

$$d(x, z) \leq d(x, y) + d(y, z)$$

And since x and y are equivalent,

$$d(x, z) \leq 0 + d(y, z) = d(y, z)$$

So $d(x, z) \leq d(y, z)$. Now instead consider $d(y, z)$

$$\begin{aligned} d(y, z) &\leq d(y, x) + d(x, z) \\ &\leq d(x, y) + d(x, z) \\ &\leq 0 + d(x, z) \end{aligned}$$

So we have $d(x, z) \leq d(y, z)$ and $d(y, z) \leq d(x, z)$. Thus $d(x, z) = d(y, z)$, immediately proving (2). However if y and z are equivalent, then $d(y, z) = 0$. So then in the case above, $d(x, z) = d(y, z) = 0$, meaning that x must be equivalent to z, proving transitivity, and therefore that the relation is an equivalence relation. $\square$