Homework Assignment 4

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Question 1

Let $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| \le (x-y)^2$. Prove that f is constant.

Proof. As given, we have two cases to consider since $f(x) - f(y) \le (x-y)^2$ and $f(x) - f(y) \ge -(x-y)^2$.

In the former, we have

$$f'(x) = \lim_{c \to x} \frac{f(c) - f(x)}{c - x} \le \frac{(c-x)^2}{c-x} = c - x = 0$$

In the latter we have

$$f'(x) = \lim_{c \to x} \frac{f(c) - f(x)}{c - x} \ge \frac{-(c-x)^2}{c-x} = x - c = 0$$

So then we have that $f'(x) \ge 0$ and $f'(x) \le 0$. Hence $f'(x) = 0$, showing that f is constant. $\square$

Question 2

Let f be a differentiable function defined in a neighborhood of x. Assume that $f''(x)$ exists. Prove that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Proof. To begin, first consider the derivative for some artbitrary differentiable function g.

$$g'(x) = \lim_{c \to x} \frac{g(c) - g(x)}{c - x} = \lim_{c \to x} \frac{g(x + (c-x)) - g(x)}{c - x}$$

Hence if one examines the limit intead with respect to $h = (c - x)$ we get

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

Then, using this notation instead we have

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

Where

$$
\begin{aligned}
f'(x+h) &= \lim_{k\to 0} \frac{f(x+h+k) - f(x+h)}{k} \\
f'(x) &= \lim_{l\to 0} \frac{f(x+l) - f(x)}{l}
\end{aligned}
$$

So then if we choose $k = l = -h$, we get

$$
\begin{aligned}
f''(x) &= \lim_{h\to 0} \frac{\frac{f(x)-f(x+h)}{-h} - \frac{f(x-h)-f(x)}{-h}}{h} \\
&= \lim_{h\to 0} \frac{\frac{f(x+h)-f(x)}{h} + \frac{f(x-h)-f(x)}{h}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}
\end{aligned}
$$

As was to be shown. $\square$

Question 3 (Fixed Point Theorem)

Let $f : \mathbb{R} \to \mathbb{R}$ such that $|f'(x)| \leq C$ for some $0 \leq C < 1$ and all x. A number x is a fixed point for f if $f(x) = x$. Prove that f cannot have two fixed points. Let x_1 be any real number, and define a sequence by $x_{n+1} = f(x_n)$. Prove that the sequence $\{x_n\}$ is Cauchy.

Question 4 (Concavity)

Let $f : (\alpha, \beta) \to \mathbb{R}$ be twice differentiable function such that $f'' \geq 0$ on the interval. Let $c \in (\alpha, \beta)$ and let $g(x)$ be the linear function whose graph is the tangent line of the graph of f at c i.e. $g(x) = f(c) + f'(c)(x - c)$. Prove that $f(x) \geq g(x)$ for $x \in (\alpha, \beta)$.

Question 5 (Newton Method)

Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable function. Let $[a, b]$ be a closed interval such that $f(a) < 0$ and $f(b) > 0$, $f'(x) \geq \delta > 0$, and $f''(x) \geq 0$ for $x \in [a, b]$.

Prove that there is unique $c \in (a, b)$ such that $f(c) = 0$. Define a sequence by $x_1 = b$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Prove that the sequence is decreasing and bounded from below by c, it has a limit. Prove that the limit is c. Check that the conditions are satisfied for $f(x) = x^2 - 2$ and the interval $[1, 2]$. What is the limit of the sequence $\{x_n\}$? Compute x_n for $n = 1, 2, 3, 4$.

Question 6

Consider the power series $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$, i.e. the sequence whose n-th term is $(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$. Compute the radius of convergence of this series. Use the theorem of Taylor to prove that $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ for every x. Use this series to find a rational number that approximates $\sin(1/2)$ with an error less than $1/10^3$.