In this part of exam, all steps must be shown clearly and thoroughly. Answers can either be directly typed or inserted photos of work as images. The first part of exam must be submitted as a single file and no extra files are not accepted.

1. [10 pts] Using the definition, $E[\boldsymbol{x}] = \mu$, show

$$E[\boldsymbol{x}\boldsymbol{x}^T] = \boldsymbol{\mu}\boldsymbol{\mu}^T + \boldsymbol{\Sigma}$$

and

$$E[\boldsymbol{x}_n\boldsymbol{x}_m^T] = \boldsymbol{\mu}\boldsymbol{\mu}^T + I_{nm}\boldsymbol{\Sigma}$$

where $\boldsymbol{x}_n$ denotes a data point sampled from a Gaussian distribution with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$, and I_{nm} denotes the (n, m) element of the identity matrix.

$$\begin{aligned}
E[xx^T] &= \mu\mu^T + \Sigma \\
&= \mu\mu^T + E[(x-\mu)(x^T - \mu^T)] \\
&= \mu\mu^T + E[xx^T - x\mu^T - \mu x^T + \mu\mu^T] \\
&= \mu\mu^T + E[xx^T] - E[x\mu^T] - E[\mu x^T] + E[\mu\mu^T] \\
&= \mu\mu^T + E[xx^T] - \mu^T E[x] - \mu E[x^T] + \mu\mu^T \\
&= \cancel{\mu\mu^T} + E[xx^T] - \cancel{\mu\mu^T} - \cancel{\mu\mu^T} + \cancel{\mu\mu^T} \\
&\boxed{= E[xx^T]}
\end{aligned}$$

$$\begin{aligned}
E[x_n x_m^T] &= \mu\mu^T + I_{nm}\Sigma \\
&= \mu\mu^T + I_{nm}(E[(x_n - \mu)(x_m - \mu^T)]) \\
&= \mu\mu^T + I_{nm}(E[x_n x_m^T] - \mu\mu^T) \\
&= \mu\mu^T + I_{nm}E[x_n x_m^T] - I_{nm}(\mu\mu^T)
\end{aligned}$$

$$\begin{cases} n == m \Rightarrow I_{nm} = 1 \\ n != m \Rightarrow I_{nm} = 0 \end{cases}$$

if $n = m$: $= \cancel{\mu\mu^T} + E[x_n x_m^T] - \cancel{\mu\mu^T}$

$E[x_n x_m^T] = E[x_n x_m^T]$ ✓

if $n \neq m$:

$E[x_n x_m^T] = \mu\mu^T$

2. [10 pts] Consider a data set in which each data point t_n is associated with a weighting factor $r_n > 0$, so that the sum-of-squares error function becomes

$$E_D(\boldsymbol{w}) = \frac{1}{2}\sum_{n=1}^{N} r_n\{t_n - \boldsymbol{w}^T\boldsymbol{x}_n\}^2.$$

Find an expression for the solution $\boldsymbol{w}^*$ that minimizes this error function.

r_n, t_n = Lagrange Multipliers

$$\frac{\partial E}{\partial w} = 2r_n(t_n - x_n)$$

$$\boxed{w^* = \frac{1}{2}\sum 2r_n(t_n - x_n)}$$