Mapping Brain Networks

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Background[1]

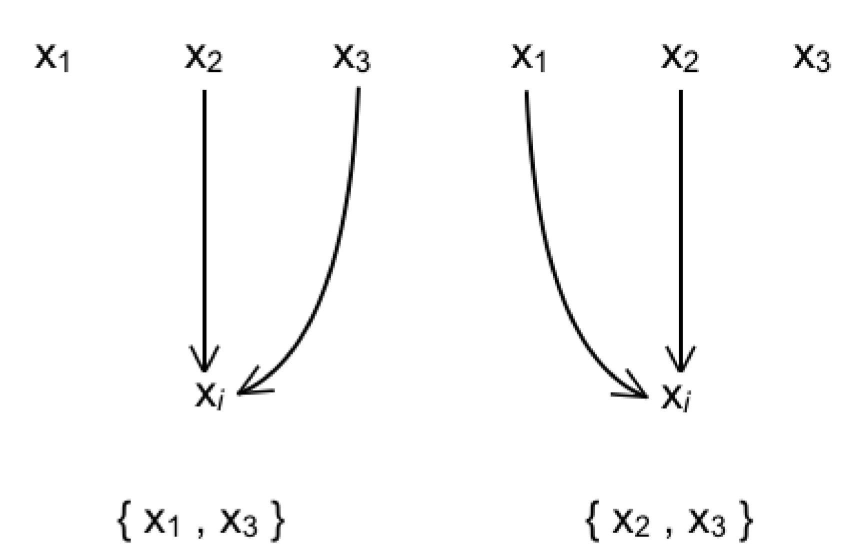
C. elegans is a widely studied nematode of which recent focus has been directed towards understanding its locomotion. We looked at how one models the brain networks of *C. elegans* using boolean functions. Observe the following input-output pairs for the function $f_i : \mathbb{F}_2^3 \to \mathbb{F}_2$:

$$f_i(1,1,1) = 0, f_i(0,0,0) = 0, f_i(1,1,0) = 1$$

- Each input represents a set of neurons which are turned on (1) or off (0)
- Each output says whether a certain neuron is turned on or off as a result of the input

The goal is to determine which set of variables f_i depends on; in essence, determine which neurons directly influence the neuron in question. We calculate this by:

1. Finding the monomials produced by inputs with different outputs:
$$m((1,1,1),(1,1,0)) = x_3, m((0,0,0),(1,1,0)) = x_1x_2$$
2. Creating a monomial ideal:
$$I_{\Delta_D^c} = \langle x_3, x_1x_2 \rangle$$
3. Determining ideal's primary decomposition:
$$I_{\Delta_D^c} = \langle x_1, x_3 \rangle \cap \langle x_2, x_3 \rangle$$
4. Using Stanley-Reisner Theory, draw the resulting minimal wiring diagrams:



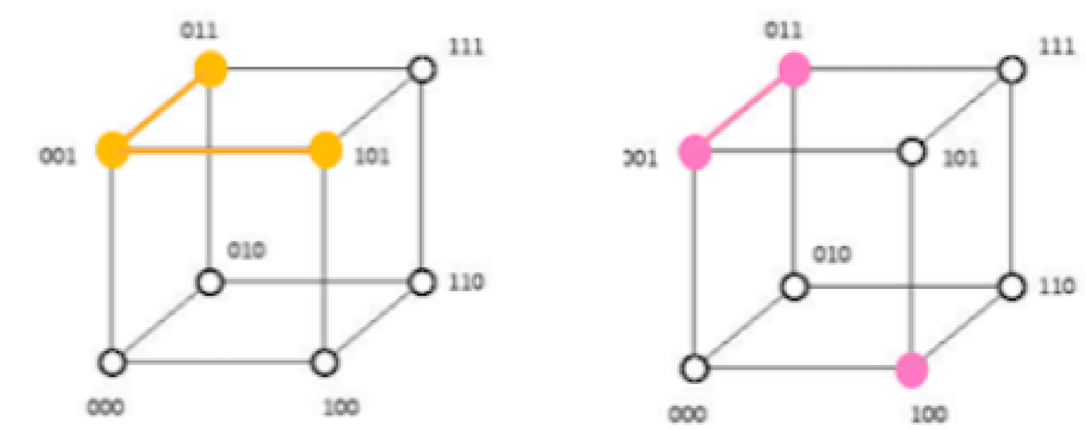
Each wiring diagram we call a min-set. In this example, multiple min-sets tells us the neuron in question either depends on x_1 and x_3, or x_2 and x_3 Ideally, the data produces a unique min-set, which means there is only one possible explanation.

Research Questions

1. Given input-output pairs that correspond to multiple min-sets, what other experiments can be performed so that there is a unique min-set?
2. What sets of inputs always correspond to a unique min-set regardless of the output?

Three Dimensions

Observe the following sets of inputs and their representations on the cube:

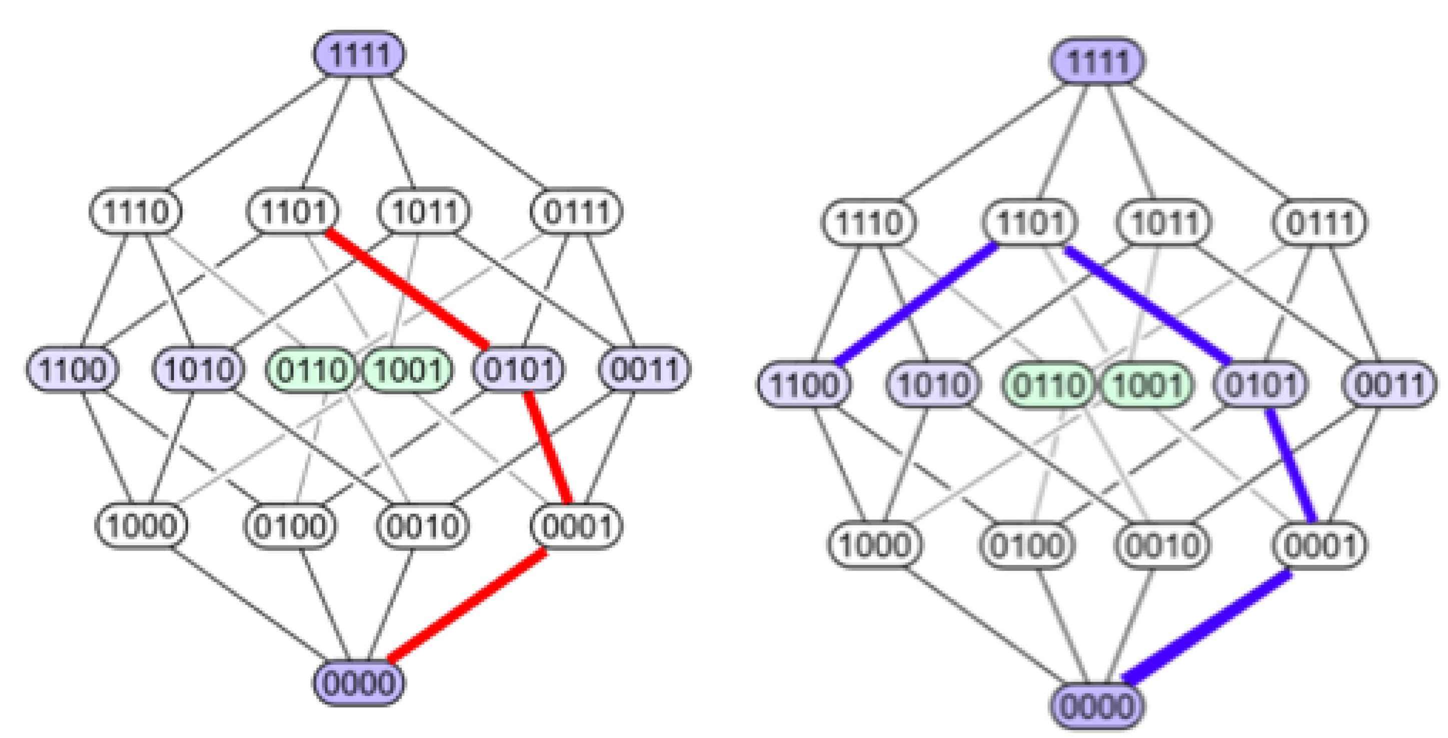


- (left) Will always produce unique min-set: $\{(0,1,1),(0,0,1),(1,0,1)\}$
- (right) Does not always produce unique min-set: $\{(0,1,1),(0,0,1),(1,0,0)\}$

One example where $\{(0,1,1),(0,0,1),(1,0,0)\}$ fails to produce a unique min-set is when $(0,1,1)$ has an output of 1, $(0,0,1)$ has an output of 1, and $(1,0,0)$ has an output of 0. The resulting monomials are $x_1x_2x_3$ and x_1x_3. Thus, the ideal is $\langle x_1x_2x_3, x_1x_3 \rangle$ and its primary decomposition is $\langle x_1 \rangle \cap \langle x_3 \rangle$.

Four Dimensions

Observe the following sets of inputs and their representation on the hypercube:



- (left) Will always produce unique min-set: $\{(0,0,0,0),(0,0,0,1),(0,1,0,1),(1,1,0,1)\}$
- (right) Does not always produce unique min-set: $\{(0,0,0,0),(0,0,0,1),(0,1,0,1),(1,1,0,1),(1,1,0,0)\}$

One might notice that a set of input points will always produce a unique min-set when **each input is connected** and **is connected in the shortest route possible**

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Results

Theorem Let S be a set of input vectors. The following three statements are equivalent:

1. Regardless of their outputs, $I_{\Delta_D^c}$ corresponds to a unique min-set.
2. For all $s_i, s_j \in S$, the length of the shortest path between s_i and s_j contained in S is equal to the number of variables in $m(s_i, s_j)$.
3. For all $s_i, s_j \in S$, there exists a path α from s_i to s_j contained in S with the property that any edge contained in α corresponds to a variable that divides $m(s_i, s_j)$.

Using the following link to see the proof and accompanying code.

Implementation

To algebraically determine if a set of input vectors will correspond to a unique min-set, perform the following algorithm.
Input: Set of input vectors
Output: Whether the set of input vectors is guaranteed to correspond to a unique min-set or not

1. Find all the monomial differences between input pairs.
2. For each multivariate monomial, do the following:
a. With the two inputs, s_i, s_j, that produce the multivariate monomial, assume their outputs, t_i, t_j, differ.
b. Then for every input pair that produces a univariate monomial that is part of the mulitvariate monomial, set the output for both of the inputs in the input pair to be the same.
c. If t_i and t_j differ, then let $t_i + t_j = 1$, otherwise $t_i + t_j = 0$
d. Put these equations into an augmented matrix and perform Gaussian Elimination. Record if it is consistent or inconsistent.
3. After doing this for all multivariate monomials, if any of them were consistent, then we say that the set of inputs S, does not correspond to a unique min-set. Otherwise, S does correspond to a unique min-set.

Future Research Questions

- Can this result be generalized to fields other than $\mathbb{F}_2$?
- For what input-output data sets do signed min-sets exist?
- Are there input data sets which are guaranteed to have signed min-sets?
- Which input-output data sets correspond to simplexes?

[1]A. Jarrah, R. Laubenbacher, B. Stigler, and M. Stillman. Reverse-engineering of polynomial dynamical systems. Advances inApplied Mathematics, 39:477–489, 2007.