Mapping Brain Networks

Frost Research

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1 Necessary Definitions

Let S be a set of inputs, where $s_i, s_j \in S$

We call $\beta = \{s_i, s_j\}$ an **edge** if s_i, s_j differ at only one entry. Each edge **corresponds** to a square-free single-variable monomial, equivalent to $m(s_i, s_j)$

If two inputs s_i, s_j differ at one or more entries, then there exists a **path** between them, where each path α is a union of edges.

A given path α between s_i and s_j is **contained in S** if $\alpha \subseteq S$. An edge β is **contained in** α if $\beta \subseteq \alpha$
Theorem Let S be a set of input vectors. The following three statements are equivalent:

1. Regardless of their outputs, $I_{\Delta_D^c}$ corresponds to a unique min-set.
2. For all $s_i, s_j \in S$, the length of the shortest path contained in S between s_i and s_j is equal to the number of variables in $m(s_i, s_j)$.
3. For all $s_i, s_j \in S$, there exists a path α from s_i to s_j contained in S with the property that any edge contained in α corresponds to a variable that divides $m(s_i, s_j)$.

Proof. **Step One.** We will first demonstrate that statement 1 is equivalent to statement 2.

$$\Rightarrow$$

Suppose that S is a set of input vectors such that, regardless of their outputs, $I_{\Delta_D^c}$ will correspond to a unique min-set.

In the trivial case where for all $s_i, s_j \in S$ there exists an edge between s_i and s_j, then all inputs are connected by a path of length 1 (an edge) and the proof is complete.

Let $s_i, s_j \in S$ such that $m(s_i, s_j)$ is a multivariable square-free monomial of m variables. First, we will show that S contains a path between s_i and s_j. Seeking a contradiction, assume there exists an s_i and s_j with no path contained in S.

Define $U \subseteq S$ such that for all $s_k \in U$, S contains a path between s_i and s_k. Let $V = S - U$. Clearly, U and V form a partition of S. It is important to note that for all $s_k \in U$ and for all $s_r \in V$, $m(s_k, s_r)$ must be multivariate. Consider the following possible outputs: for all $s_k \in U$, $t_k = 0$ and for all $s_r \in V$, $t_r = 1$. Then, $I_{\Delta_D^c}$ must only have multivariate generators, and will not correspond to a unique min-set. Thus, these inputs do not produce unique min-sets regardless of outputs. This is a contradiction.

Having now shown that S must contain a path between s_i, s_j we wish to show that the shortest path between s_i and s_j that is connected in S has length m. Let α be the shortest path between s_i and s_j that is contained in S. We will say that α has length n. We denote the $n+1$ inputs contained in α as follows:

$$s_i = s_0, s_1, s_2, ..., s_m = s_j$$

Clearly, n cannot be less than m (we proved that there must be a path between any two points in S, and each path must be at least the length of the number of variables in the monomial produced by the two points). Seeking a contradiction, suppose $m > n$.

Case 1. *alpha* contains an edge that corresponds to a variable which is not expressed in $m(s_i, s_j)$. In other words, there exists $k \in \{1, ..., n\}$ such that $m(s_{k-1}, s_k)$ does not divide $m(s_i, s_j)$.

Consider then, the situation where $0 = t_0 = t_1 = ... = t_{k-1}$ and $1 = t_k = t_{k+1} = ... = t_n$.

So it is possible that $I_{\Delta_D^c}$ has a multivariariate generator, hence it is possible that $I_{\Delta_D^c}$ corresponds to multiple min-sets. This is a contradiction.

Case 2. α contains two or more edges which correspond to the same variable. Suppose that α contains exactly two such edges. More specifically, there must exist a $j, k \in \{1, 2, ..., n\}$ with $j \neq k$ such that $m(s_{j-1}, s_j) = m(s_{k-1}, s_k) = x_a$.

Consider the inputs s_{j-1} and s_k. It follows that the shortest path between them, β, is the portion of α which starts at s_{j-1} and ends at s_k. Notice that *beta* contains exactly two edges which correspond to a x_a and thus x_a does not divide $m(s_{j-1}, s_k)$. This means that β contains an edge that corresponds to a variable, x_a, that is not expressed in $m(s_{j-1}, s_k)$. This then falls onto case 1, and again reaches a contradiction.

Now suppose that α contains more than two edges which correspond to the same variable. That is, there exists $a_1, a_2, ..., a_k \in \{1, ..., n\}$ with $a_1 < a_2 < ... < a_k$ such that $m(s_{a_1-1}, s_{a_1}) = m(s_{a_2-1}, s_{a_2}) = ... = m(s_{a_k-1}, s_{a_k}) = x_a$. Again, we consider a portion of the path α between s_{a_1-1} and s_{a_2}, which we will call *beta*. Since x_a does not divide $m(s_{a_1-1}, s_{a_2})$ we see that once again β contains an edge that corresponds to a variable which is not expressed in $m(s_{a_1-1}, s_{a_2})$. So this also collapses into case 1.

Therefore, it must be the case that $m = n$. That is, for all $s_i, s_j \in S$ the shortest path between s_i and s_j has length equal to the number of variables in $m(s_i, s_j)$.

$\Leftarrow$

Suppose that S is a set of input vectors with the property that for all $s_i, s_j \in S$ the shortest path between s_i and s_j has length equal to the number of variables in $m(s_i, s_j)$. Further suppose that there exists $s_i, s_j \in S$ with $t_i \neq t_j$, where $m(s_i, s_j)$ is a multivariable square-free monomial of m variables. Since $t_i \neq t_j$, $m(s_i, s_j) \in I_{\Delta_D^c}$. We know that S contains a path α between s_i and s_j of length m. We will denote the $m + 1$ inputs contained in α as follows:

$$s_i = s_0, s_1, s_2, ..., s_m = s_j$$

so that for all $k \in \{1, 2, ..., m\}$ there is an edge between s_{k-1} and s_k. That is $m(s_{k-1}, s_k)$ is an unvariate square-free monomial (a single variable) which divides $m(s_i, s_j)$. Notice that because $t_i \neq t_j$ there must exists $k \in \{1, ..., m\}$ such that $t_{k-1} \neq t_k$. Hence, $m(s_{k-1}, s_k) \in I_{\Delta_D^c}$. Therefore, for any multivariable monomial in $I_{\Delta_D^c}$, there exists a variable also in $I_{\Delta_D^c}$ by which it is divisible. We have then shown that $I_{\Delta_D^c}$ can only have univariable square-free monomials as generators and thus must correspond to a unique min-set.

Step Two. We will show that that statement 3 is equivalent to statement 1.

$\Rightarrow$

Let S be a set of input vectors such that for all $s_i, s_j \in S$, S contains a path α between s_i and s_j with the following property: any edge contained in α corresponds to a variable that divides $m(s_i, s_j)$.

Suppose that there exists $s_i, s_j \in S$ where $t_i \neq t_j$ and $m(s_i, s_j)$ is a multivariate square-free monomial. We then know that S contains a path α between s_i and s_j where each edge in α corresponds to a variable that divides $m(s_i, s_j)$. We will denote the inputs which lie on α as follows:

$$s_i = s_0, s_1, s_2, ..., s_m = s_j$$

So that for all $k \in \{1, ..., m\}$ there exists an edge between s_{k-1} and s_k. Because $t_i \neq t_j$ ($t_0 \neq t_m$), there must exist $k \in \{1, ..., m\}$ such that $t_{k-1} \neq t_k$. Thus, $m(s_{k-1}, s_k) \in I_{\Delta_D^c}$. Since $m(s_{k-1}, s_k)$ is a single variable that divides $m(s_i, s_j)$ we see that $I_{\Delta_D^c}$ can not have any multivariate generators.

Therefore, $I_{\Delta_D^c}$ corresponds to a unique min-set.

$\Leftarrow$

Now let S be a set of input vectors such that regardless of their outputs $I_{\Delta_D^c}$ will correspond to a unique min-set. First, we must consider the simple case where for all $s_i, s_j \in S$ there exists an edge between the two inputs. Then any path $\alpha \subseteq S$ is only comprised of a single edge. So S satisfies the criteria of the proposition and the proof is complete. Otherwise, there exists s_i, s_j such that $m(s_i, s_j)$ is a multivariate square free monomial. Suppose that $t_i \neq t_j$ so that $m(s_i, s_j) \in I_{\Delta_D^c}$. First, we will show that S contains a path between s_i and s_j. Seeking a contradiction, assume there exists an s_i and s_j with no path contained in S.

Define $U \subseteq S$ such that for all $s_k \in U$, S contains a path between s_i and s_k. Let $V = S - U$. Clearly, U and V form a partition of S. It is important to note that for all $s_k \in U$ and for all $s_r \in V$, $m(s_k, s_r)$ must be multivariate. Consider the following possible outputs: for all $s_k \in U$, $t_k = 0$ and for all $s_r \in V$, $t_r = 1$. Then, $I_{\Delta_D^c}$ must only have multivariate generators, and will not correspond to a unique min-set. Thus, these inputs do not produce unique min-sets regardless of outputs. This is a contradiction. Having now shown that S must contain a path between s_i, s_j we wish to show that S must contain such a path with the further property that every edge $\beta \subseteq \alpha$ must correspond to a variable which divides $m(s_i, s_j)$. Seeking a contradiction, assume that for all paths $\alpha \subseteq S$ between s_i and s_j, there exists an edge $\beta \subseteq \alpha$ such that β corresponds to a variable that does not divide $m(s_i, s_j)$. We will once again describe the inputs contained in α as follows:

$$s_i = s_0, s_1, s_2, ..., s_m = s_j$$

Let $k \in \{1, ..., m\}$ be such that $\beta = \{s_{k-1}, s_k\}$. Consider the case where $0 = t_o = t_i = ... = t_{k-1}$ and $1 = t_k = t_{k+1} = ... = t_m$. Then the only single variable monomial that is an element of $I_{\Delta_D^c}$ is $m(s_{k-1}, s_k)$. Since $m(s_{k-1}, s_k)$ does not divide $m(s_i, s_j)$, $I_{\Delta_D^c}$ has a multivariate generator and thus cannot correspond to a unique min-set. This is a contradiction. Thus, we have shown that for all $s_i, s_j \in S$, S must contain a path α between s_i and s_j with the property that for all edges $\beta \subseteq \alpha$, β corresponds to a variable that divides $m(s_i, s_j)$.

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