## **Supplementary Model Derivations**

### **Anonymous ACL submission**

**A.3** 

equation

**Uniqueness theorem for Poisson's** 

 $\nabla^2 \phi_1 = -\frac{\rho_f}{\varepsilon_0}$ 

 $\nabla^2 \phi_2 = -\frac{\rho_f}{\varepsilon_0}$ 

 $\nabla^2(\phi_2 - \phi_1) = \nabla^2\phi_2 - \nabla^2\phi_1$ 

 $\nabla^2(\phi_2 - \phi_1) = -\frac{\rho_f}{\varepsilon_0} - (-\frac{\rho_f}{\varepsilon_0})$ 

 $\nabla^2(\phi_2 - \phi_1) = 0$ 

 $\int_{S} \phi \nabla \phi \cdot d\mathbf{S} = \int_{V} (\nabla \phi)^{2} dV$ 

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**Abstract** 

This document contains derivations generated

by GPT-4 following the few-shot prompting

scheme discussed in the corresponding paper.

A S=0 (no premises removed)

0

A.1 Gauss' law: equivalence between

differential and integral forms

 $\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}$ 

 $\iint_{S} \mathbf{E} \cdot d\mathbf{A} = \iiint_{V} \nabla \cdot \mathbf{E} dV$ 

 $\rho = -\nabla \cdot \mathbf{P} + \nabla \cdot \mathbf{D}$ 

 $\rho = \nabla \cdot (\mathbf{D} - \mathbf{P})$ 

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009	$\frac{\mathcal{L}}{\varepsilon_0} = \iiint_V \nabla \cdot \mathbf{E} dV$	(3)			
010	$Q = \iiint_V \rho dV$	(4)	$\phi = \phi_2 - \phi_1$	(18)	028
	333 V		$ abla^2\phi=0$	(19)	029
011	$\frac{\iiint_V \rho dV}{\varepsilon_0} = \iiint_V \nabla \cdot \mathbf{E} dV$	(5)	A.4 Uniqueness theorem for Poisson's equation 2		030 031
012	$\iiint_V \nabla \cdot \mathbf{E} dV = \iiint_V \frac{\rho}{\varepsilon_0} dV$	(6)	Equation 2 $\nabla \cdot (\phi \nabla \phi) = (\nabla \phi)^2 + \phi \nabla^2 \phi$	(20)	031
013 014	A.2 Gauss' law: Equivalence of total an charge statements	d free	$\nabla^2 \phi = 0$	(21)	033
015	$ ho_b = - abla \cdot {f P}$	(7)	$\nabla \cdot (\phi \nabla \phi) = (\nabla \phi)^2$	(22)	034
016	$- ho_b =  abla \cdot \mathbf{P}$	(8)	$\mathbf{v} \cdot (\psi \mathbf{v} \psi) = (\mathbf{v} \psi)$	(22)	034
017	$ ho_f =  abla \cdot \mathbf{D}$	(9)	$\int_{V} \nabla \cdot (\phi \nabla \phi) dV = \int_{V} (\nabla \phi)^{2} dV$	(23)	035
018	$ ho =  ho_b +  ho_f$	(10)	ſ		
	7 D 7 D	(11)	$\int_{V}  abla \cdot (\phi  abla \phi) dV = \int_{S} \phi  abla \phi \cdot d\mathbf{S}$	(24)	036

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## A.5 Uniqueness theorem for Poisson's equation 6

$$\frac{\partial \phi}{\partial r} = 0 \tag{26}$$

$$\int \frac{\partial \phi}{\partial r} dr = \int 0 dr \tag{27}$$

$$\int 0dr = C_1 \tag{28}$$

$$\int \frac{\partial \phi}{\partial r} dr = \phi + C_2 \tag{29}$$

$$\phi + C_2 = C_1 \tag{30}$$

$$\phi = C_1 - C_2 \tag{31}$$

## **A.6** Uniqueness theorem for Poisson's equation 7

$$\phi = C_1 - C_2 \tag{32}$$

$$C = C_1 - C_2 (33)$$

$$\phi = C \tag{34}$$

$$\phi = \phi_2 - \phi_1 \tag{35}$$

$$\phi_2 - \phi_1 = C \tag{36}$$

### A.7 Poisson's equation: Newtonian gravity

$$\nabla \cdot \mathbf{g} = -4\pi G \rho \tag{37}$$

$$\mathbf{g} = -\nabla \phi \tag{38}$$

$$\nabla \cdot (-\nabla \phi) = -4\pi G \rho \tag{39}$$

$$-\nabla^2 \phi = -4\pi G \rho \tag{40}$$

$$\nabla^2 \phi = 4\pi G \rho \tag{41}$$

## A.8 Poisson's equation: Gravitational potential from Poisson's equation 2

$$\int_{V} \nabla \cdot \nabla \phi dV = \int_{V} 4\pi G \rho dV \qquad (42)$$

$$m = \int_{V} \rho dV \tag{43}$$

$$4\pi Gm = \int_{V} 4\pi G\rho dV \tag{44}$$

$$\int_{V} \nabla \cdot \nabla \phi dV = \int_{S} \nabla \phi \cdot d\mathbf{S} \tag{45}$$

$$\int_{S} \nabla \phi \cdot d\mathbf{S} = 4\pi Gm \tag{46}$$

## A.9 Poisson's equation: Gravitational potential from Poisson's equation 6

$$\int_{S} \frac{\partial \phi}{\partial r} dS = \int_{0}^{2\pi} \int_{0}^{\pi} \frac{\partial \phi}{\partial r} r^{2} \sin \theta d\theta d\varphi \qquad (47)$$

$$\int_0^{\pi} \sin \theta d\theta = 2 \tag{48}$$

$$\int_{0}^{2\pi} d\varphi = 2\pi \tag{49}$$

$$\int_{0}^{2\pi} \int_{0}^{\pi} \frac{\partial \phi}{\partial r} r^{2} \sin \theta d\theta d\varphi = \frac{\partial \phi}{\partial r} r^{2} \int_{0}^{2\pi} \int_{0}^{\pi} \sin \theta d\theta d\varphi \tag{50}$$

$$\frac{\partial \phi}{\partial r} r^2 \int_0^{2\pi} \int_0^{\pi} \sin \theta d\theta d\varphi = \frac{\partial \phi}{\partial r} r^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin \theta d\theta$$
(51)

$$\frac{\partial \phi}{\partial r} r^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta = \frac{\partial \phi}{\partial r} r^2 (2\pi \cdot 2) \quad (52)$$

$$\frac{\partial \phi}{\partial r}r^2(2\pi \cdot 2) = 4\pi \frac{\partial \phi}{\partial r}r^2 \tag{53}$$

$$\int_{S} \frac{\partial \phi}{\partial r} dS = 4\pi \frac{\partial \phi}{\partial r} r^2 \tag{54}$$

#### Poisson's equation: Gravitational Poisson's equation: Electrostatic 076 095 potential from Poisson's equation 8 potential from Poisson's equation 077 096 $\nabla^2 \phi = -\frac{\rho_f}{\epsilon}$ (71) $\frac{\partial \phi}{\partial r} = \frac{Gm}{r^2}$ 097 (55)078 $\nabla^2 \phi = \nabla \cdot \nabla \phi$ (72)098 $\int_{-\infty}^{\infty} \frac{\partial \phi}{\partial r} dr = \phi(r) - \phi(\infty)$ (56) $\int_{\mathcal{M}} \nabla^2 \phi dV = \int_{\mathcal{M}} \nabla \cdot \nabla \phi dV$ (73)099 $\int_{-r}^{r} \frac{Gm}{r^2} dr = \frac{-Gm}{r}$ $\int_{\mathcal{M}} \nabla^2 \phi dV = -\frac{1}{\varepsilon} \int_{\mathcal{M}} \rho_f dV$ (57)(74)100 $\int_{\mathcal{U}} \nabla^2 \phi dV = -\frac{Q}{\varepsilon}$ (75) $\phi(\infty) = 0$ (58) $\int_{V} \nabla \cdot \nabla \phi dV = -\frac{Q}{\varepsilon}$ (76)102 $\phi(r) - \phi(\infty) = \frac{-Gm}{r}$ (59)A.13 Poisson's equation: Electrostatic 103 potential from Poisson's equation 2 104 $\phi(r) = \frac{-Gm}{r}$ $\int_{\mathcal{U}} \nabla \cdot \nabla \phi dV = -\frac{Q}{\varepsilon}$ (60)(77)105 $\int_{V} \nabla \cdot \nabla \phi dV = \int_{S} \nabla \phi \cdot d\mathbf{S}$ **A.11** Poisson's equation: Electrostatics 084 (78)106 $\nabla \cdot \mathbf{D} = \rho_f$ (61) $-\frac{Q}{\varepsilon} = \int_{\mathcal{C}} \nabla \phi \cdot d\mathbf{S}$ (79)107 $\mathbf{D} = \varepsilon \mathbf{E}$ $\nabla \phi \cdot d\mathbf{S} = \frac{\partial \phi}{\partial \mathbf{n}} dS$ 086 (62)(80)108 $\int_{S} \nabla \phi \cdot d\mathbf{S} = \int_{S} \frac{\partial \phi}{\partial r} dS$ $\nabla \cdot \varepsilon \mathbf{E} = \rho_f$ (81)109 (63)087 $\int_{\mathcal{S}} \frac{\partial \phi}{\partial r} dS = -\frac{Q}{\varepsilon}$ (82)110 $\mathbf{E} = -\nabla \phi$ (64)088 A.14 Poisson's equation: Electrostatic 111 potential from Poisson's equation 4 112 $\varepsilon(-\nabla\phi) = \rho_f$ (65) $\int_{-\tau}^{\tau} \frac{\partial \phi}{\partial r} dr = \int_{-\tau}^{\tau} \frac{Q}{4\pi \varepsilon r^2} dr$ (83)113 $\int_{-\infty}^{r} \frac{\partial \phi}{\partial r} dr = \phi(r) - \phi(\infty)$ $-\varepsilon\nabla\phi=\rho_f$ (66)090 (84)114 $\int_{-\frac{Q}{4\pi\varepsilon r^2}} dr = \frac{Q}{4\pi\varepsilon r}$ (85)115 $-\nabla \cdot \varepsilon \nabla \phi = \rho_f$ (67)091 $\phi(\infty) = 0$ (86)116 $\nabla \cdot \nabla \phi = \nabla^2 \phi$ (68) $\phi(r) - \phi(\infty) = \frac{Q}{4\pi cr}$ (87)117 $-\varepsilon \nabla^2 \phi = \rho_f$ (69) $\phi(r) - 0 = \frac{Q}{4\pi\varepsilon r}$ (88)118

(70)

 $\phi(r) = \frac{Q}{4\pi\varepsilon r}$ 

(89)

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 $\nabla^2 \phi = -\frac{\rho_f}{\varepsilon}$ 

122	$\frac{d\mathbf{F}}{dV} = \frac{dq}{dV}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$	(90)	$U = q \int_{\infty}^{r} \nabla \phi \cdot d\mathbf{r} \tag{109}$	147
123	$\mathbf{f} = \frac{d\mathbf{F}}{dV}$	(91)	$\int_{\infty}^{r} \nabla \phi \cdot d\mathbf{r} = \int_{\infty}^{r} \frac{\partial \phi}{\partial r} dr \tag{110}$	148
124	$\rho = \frac{dq}{dV}$	(92)	$\int_{\infty}^{r} \frac{\partial \phi}{\partial r} dr = \phi(r) - \phi(\infty) \tag{111}$	149
125	$\mathbf{f} = \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B})$	(93)	$\phi(\infty) = 0 \tag{112}$	150
126	$\mathbf{f} = \rho \mathbf{E} + \rho \mathbf{v} \times \mathbf{B}$	(94)	$\phi(r) - \phi(\infty) = \phi(r) \tag{113}$	151
127 128	A.16 Lorentz force: continuous charge distribution 2	<b>;</b>		
129	$\mathbf{f} =  ho \mathbf{E} +  ho \mathbf{v}  imes \mathbf{B}$	(95)	$q \int_{\infty}^{\prime} \nabla \phi \cdot d\mathbf{r} = q\phi(r) \tag{114}$	152
130	$\mathbf{J}= ho\mathbf{v}$	(96)	$U = q\phi(r) \tag{115}$	153
131	$\mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}$	(97)	A.19 Laplace equation: Analytic functions (u)	154 155
132	$\mathbf{f} = \frac{d\mathbf{F}}{dV}$	(98)	$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{116}$	156
133	$ ho \mathbf{E} + \mathbf{J}  imes \mathbf{B} = rac{d\mathbf{F}}{dV}$	(99)	$\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial x^2} \tag{117}$	157
134	$\mathbf{F}=\iiint rac{d\mathbf{F}}{dV}dV$	(100)	$\frac{\partial}{\partial x}(\frac{\partial v}{\partial y}) = \frac{\partial^2 v}{\partial x \partial y} \tag{118}$	158
135	$\mathbf{F} = \iiint (\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}) dV$	(101)	$\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial v}{\partial y}) \tag{119}$	159
136 137	A.17 Lorentz force: Lorentz force in te potentials	erms of	$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{120}$	160
138	$\mathbf{E} = - abla \phi - rac{\partial \mathbf{A}}{\partial t}$	(102)	A.20 Laplace equation: Analytic functions (u) 2	161 162
139	$\mathbf{B} = \nabla \times \mathbf{A}$	(103)	$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{121}$	163
140	$\mathbf{v} \times \mathbf{B} = \mathbf{v} \times (\nabla \times \mathbf{A})$	(104)	$\frac{\partial}{\partial y}(\frac{\partial v}{\partial x}) = \frac{\partial}{\partial y}(-\frac{\partial u}{\partial y}) \tag{122}$	164
141	$\mathbf{v} \times (\nabla \times \mathbf{A}) = \nabla (\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla) \mathbf{A}$	(105)	$\frac{\partial}{\partial y}(-\frac{\partial u}{\partial y}) = -\frac{\partial}{\partial y}(\frac{\partial u}{\partial y}) \tag{123}$	165
142	$\mathbf{E} + \mathbf{v} \times \mathbf{B} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla (\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \mathbf{A})$	(106)	$-\frac{\partial}{\partial y}(\frac{\partial u}{\partial y}) = -\frac{\partial^2 u}{\partial y^2} \tag{124}$	166
143	$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$	(107)	$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y} (\frac{\partial v}{\partial x}) \tag{125}$	167

A.18 Lorentz force: Potential energy

derivation from scalar potential 3

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A.15 Lorentz force: continuous charge

distribution

 $d\mathbf{F}$ 

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 $\mathbf{F} = q(-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} + \nabla(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\mathbf{A}) \quad (108)$ 

176		$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x^2}$	(132)		$\nabla^2 v = 0$	(149)	199
				A.25	Laplace equation: Electrostatics	<b>;</b>	200
177		$\nabla^2 u = 0$	(133)		$\mathbf{E} = (u, v)$	(150)	201
178	A.22	Laplace equation: Analytic fun	ections				
179		(v)			$ abla \cdot \mathbf{E} =  ho$	(151)	202
180		$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$	(134)		$ abla = (rac{\partial}{\partial x}, rac{\partial}{\partial u})$	(152)	202
181		$\frac{\partial}{\partial x}(\frac{\partial v}{\partial x}) = \frac{\partial^2 v}{\partial x^2}$	(135)				203
182		$\frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) = \frac{\partial^2 u}{\partial x \partial y}$	(136)		$\nabla \cdot \mathbf{E} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \cdot (u, v)$	(153)	204
183		$\frac{\partial}{\partial x}(-\frac{\partial u}{\partial y}) = -\frac{\partial^2 u}{\partial x \partial y}$	(137)		$\nabla \cdot \mathbf{E} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \rho$	(154)	205
184		$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$	(138)	A.26	Laplace equation: Electrostatics		207
185	A.23	Laplace equation: Analytic fun	ctions		$\frac{\partial \phi}{\partial x} = -u$	(156)	208
186		(v) 2			$\partial x$	,	
187		$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$	(139)		$\frac{\partial \phi}{\partial y} = -v$	(157)	209
188		$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y} (\frac{\partial v}{\partial y})$	(140)		$\frac{\partial u}{\partial x} = \rho - \frac{\partial v}{\partial y}$	(158)	210
189		$\frac{\partial}{\partial y}(\frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial y \partial x}$	(141)		$\frac{\partial^2 \phi}{\partial x^2} = -\frac{\partial u}{\partial x}$	(159)	211
190		$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y} (\frac{\partial u}{\partial x})$	(142)		$\frac{\partial^2 \phi}{\partial y^2} = -\frac{\partial v}{\partial y}$	(160)	212

**Laplace equation: Analytic functions** 

 $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$ 

 $\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x}$ 

 $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ 

 $\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial u^2}$ 

 $\nabla^2 v = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x}$ 

 $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\rho$ 

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**Laplace equation: Analytic functions** 

 $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ 

 $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$ 

 $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ 

 $\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2}$ 

 $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ 

(u) 3

(143)

#### **Lorentz force: Potential energy** $U = -q\mathbf{v} \cdot \mathbf{A}(r)$ (182)241 derivation from vector potential 221 **Lorentz force: Derivation of classical** 242 $\mathbf{F} = q(\nabla(\mathbf{v} \cdot \mathbf{A}) - \frac{d\mathbf{A}}{dt})$ (167)222 Lagrangian of EM field 243 $V = q\phi - q\dot{\mathbf{r}} \cdot \mathbf{A}$ (183)244 $\frac{d\mathbf{A}}{dt} = 0$ 223 (168) $T = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}$ (184)245 $\mathbf{F} = q\nabla(\mathbf{v} \cdot \mathbf{A})$ L = T - V224 (169)(185) $L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} - (q\phi - q\dot{\mathbf{r}}\cdot\mathbf{A})$ (186)247 $U = -\int_{-r}^{r} \mathbf{F} \cdot d\mathbf{r}$ 225 (170) $L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} - q\phi + q\dot{\mathbf{r}}\cdot\mathbf{A}$ (187)248 $U = -q \int^r \nabla(\mathbf{v} \cdot \mathbf{A}) \cdot d\mathbf{r}$ (171) $L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\dot{\mathbf{r}}\cdot\mathbf{A} - q\phi$ (188)249 A.29 **Lorentz force: Potential energy** A.32 Lorentz force: Derivation of classical 250 derivation from vector potential 3 Lagrangian of EM field 2 251 $\nabla (\mathbf{v} \cdot \mathbf{A}) \cdot \hat{\mathbf{r}} = \frac{\partial (\mathbf{v} \cdot \mathbf{A})}{\partial r}$ $L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\dot{\mathbf{r}}\cdot\mathbf{A} - q\phi$ (189)(172)252 229 $\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$ (190)253 $d\mathbf{r} = \hat{\mathbf{r}}dr$ (173)230 $\mathbf{A} = (A_x, A_y, A_z)$ (191)254 $\nabla (\mathbf{v} \cdot \mathbf{A}) \cdot d\mathbf{r} = \frac{\partial (\mathbf{v} \cdot \mathbf{A})}{\partial r} dr$ (174)231 $L = \frac{m}{2}(\dot{x}, \dot{y}, \dot{z}) \cdot (\dot{x}, \dot{y}, \dot{z}) + q(\dot{x}, \dot{y}, \dot{z}) \cdot \mathbf{A} - q\phi$ $U = -q \int_{0.0}^{r} \nabla (\mathbf{v} \cdot \mathbf{A}) \cdot d\mathbf{r}$ (192)255 (175)232

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**Lorentz force: Potential energy** 

derivation from vector potential 4

 $U = -q \int_{-r}^{r} \frac{\partial (\mathbf{v} \cdot \mathbf{A})}{\partial r} dr$ 

 $\int^{r} \frac{\partial (\mathbf{v} \cdot \mathbf{A})}{\partial r} dr = \mathbf{v} \cdot \mathbf{A}(r) - \mathbf{v} \cdot \mathbf{A}(\infty) \quad (178)$ 

 $\mathbf{A}(\infty) = 0$ 

 $U = -q(\mathbf{v} \cdot \mathbf{A}(r) - 0)$ 

 $L = \frac{m}{2}(\dot{x}, \dot{y}, \dot{z}) \cdot (\dot{x}, \dot{y}, \dot{z}) + q(\dot{x}, \dot{y}, \dot{z}) \cdot (A_x, A_y, A_z) - q\phi$ 

 $U = -a(\mathbf{v} \cdot \mathbf{A}(r) - \mathbf{v} \cdot \mathbf{A}(\infty))$ 

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Laplace equation: Electrostatics 3

 $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial u^2}$ 

 $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial u^2} = -\rho$ 

 $-\rho = 0$ 

 $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial u^2} = 0$ 

 $\nabla^2 \phi = 0$ 

 $U = -q \int^r \frac{\partial (\mathbf{v} \cdot \mathbf{A})}{\partial r} dr$ 

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### **Lorentz force: Derivation of Lorentz** force from classical Lagrangian (LHS) 4

force from classical Lagrangian (LHS) 4
$$dA_x = \frac{\partial A_x}{\partial t}dt + \frac{\partial A_x}{\partial x}dx + \frac{\partial A_x}{\partial y}dy + \frac{\partial A_x}{\partial z}dz$$
(194)

$$\frac{dx}{dt} = \dot{x} \tag{195}$$

$$\frac{dy}{dt} = \dot{y} \tag{196}$$

$$\frac{dz}{dt} = \dot{z} \tag{197}$$

$$dt = 1 (198)$$

$$dx = \dot{x}dt \tag{199}$$

$$dy = \dot{y}dt \tag{200}$$

$$dz = \dot{z}dt \tag{201}$$

$$dA_{x} = \frac{\partial A_{x}}{\partial t} + \frac{\partial A_{x}}{\partial x}\dot{x}dt + \frac{\partial A_{x}}{\partial y}\dot{y}dt + \frac{\partial A_{x}}{\partial z}\dot{z}dt$$
(202)

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} \quad (203)$$

### **A.34** Lorentz force: Derivation of Lorentz force from classical Lagrangian (LHS) 5

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\frac{d}{dt}\dot{x} + q\frac{d}{dt}A_x \tag{204}$$

$$\frac{d}{dt}\dot{x} = \ddot{x} \tag{205}$$

$$m\frac{d}{dt}\dot{x} = m\ddot{x} \tag{206}$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} \quad (207)$$

$$q\frac{dA_x}{dt} = q(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z})$$
(208)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + q(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}) \tag{209}$$

### **Lorentz force: Derivation of Lorentz** force from classical Lagrangian (RHS)

$$\frac{\partial L}{\partial x} = q \frac{\partial}{\partial x} (\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) - q \frac{\partial}{\partial x} \phi \quad (210)$$
280

$$I = \frac{\partial}{\partial x}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) \tag{211}$$

$$\frac{\partial}{\partial x}(\dot{x}A_x) + \frac{\partial}{\partial x}(\dot{y}A_y) + \frac{\partial}{\partial x}(\dot{z}A_z) = \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}$$
282

$$q\frac{\partial}{\partial x}(\dot{x}A_x+\dot{y}A_y+\dot{z}A_z)=q(\frac{\partial A_x}{\partial x}\dot{x}+\frac{\partial A_y}{\partial x}\dot{y}+\frac{\partial A_z}{\partial x}\dot{z}) \tag{213}$$

$$\frac{\partial L}{\partial x} = q(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}) - q\frac{\partial}{\partial x}\phi \quad (214)$$

## A.36 Lorentz force: Derivation of x component of electric field

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \tag{215}$$

$$\nabla \phi = (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \tag{216}$$

$$\mathbf{A} = (A_x, A_y, A_z) \tag{217}$$

$$\mathbf{E} = -((\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) - \frac{\partial}{\partial t}(A_x, A_y, A_z)) \tag{218}$$

$$\mathbf{E} \cdot (1,0,0) = -((\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) - \frac{\partial}{\partial t} (A_x, A_y, A_z)) \cdot (1,0,0)$$
(219)

### **Lorentz force: Derivation of Lorentz** force from classical Lagrangian 4

$$(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_x = \dot{y}(\nabla \times \mathbf{A})_z - \dot{z}(\nabla \times \mathbf{A})_y$$
 (220)

$$(\nabla \times \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \tag{221}$$

$$(\nabla \times \mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \tag{222}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + q(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}) \qquad \dot{y}(\nabla \times \mathbf{A})_z - \dot{z}(\nabla \times \mathbf{A})_y = \dot{y}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) - \dot{z}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) \tag{223}$$

$$F_x = qF_{xx} + q(\hat{y}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) + \hat{z}(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z})$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (222)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (225)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (225)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (225)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (225)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (225)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (226)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (226)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (227)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (227)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (228)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (229)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (229)$$

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$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (229)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (229)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (229)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (229)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (239)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (239)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (249)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (239)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (249)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (239)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (239)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (249)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (239)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (239)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (239)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (239)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (239)$$

$$F_x = qF_x + q(\hat{y}(\nabla \times \mathbf{A}))_x \quad (239)$$

$$F_x = qF_x + q($$

**Electromagnetic wave equation: The** 

320

(256)

(240)

345 
$$\nabla \times (\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$
 (257)

$$-\nabla^2 \mathbf{B} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$
 (258)

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0 \tag{259}$$

## A.43 Ampere's circuital law: Proof of equivalence 2

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$
 (260)

$$\nabla \times \frac{1}{\mu_0} \mathbf{B} = \nabla \times \mathbf{H} + \mathbf{J}_M \qquad (261)$$

$$\nabla \times \frac{1}{\mu_0} \mathbf{B} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_M \tag{262}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{263}$$

$$\frac{\partial \mathbf{D}}{\partial t} = \frac{\partial}{\partial t} \varepsilon_0 \mathbf{E} + \frac{\partial}{\partial t} \mathbf{P}$$
 (264)

$$\nabla \times \frac{1}{\mu_0} \mathbf{B} = \mathbf{J}_f + \frac{\partial}{\partial t} \varepsilon_0 \mathbf{E} + \frac{\partial}{\partial t} \mathbf{P} + \mathbf{J}_M \quad (265)$$

## A.44 Ampere's circuital law: Proof of equivalence 4

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J}_f + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_P + \mathbf{J}_M \quad (266)$$

$$\mathbf{J}_b = \mathbf{J}_P + \mathbf{J}_M \tag{267}$$

$$\mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + \mathbf{J}_P + \mathbf{J}_M \tag{268}$$

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b \tag{269}$$

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J}_f + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_b \qquad (270)$$

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$
 (271)

## A.45 Uncertainty principle: Kennard inequality proof part 1.1

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 \tag{272}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \cdot p(x) dx$$
 (273)

$$p(x) = |\psi(x)|^2 (274)$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \cdot p(x) dx$$
 (275)

$$\langle x \rangle = \int_{-\infty}^{\infty} x \cdot |\psi(x)|^2 dx$$
 (276)

$$\left\langle x^{2}\right\rangle =\int_{-\infty}^{\infty}x^{2}\cdot|\psi(x)|^{2}dx \tag{277}$$

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 \tag{278}$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} x^2 \cdot |\psi(x)|^2 dx - (\int_{-\infty}^{\infty} x \cdot |\psi(x)|^2 dx)^2$$
(279)

## A.46 Uncertainty principle: Kennard inequality proof part 1.4

$$f^*(x) \cdot f(x) = x^2 \cdot (\psi^*(x) \cdot \psi(x))$$
 (280)

$$\psi^*(x) \cdot \psi(x) = |\psi(x)|^2$$
 (281)

$$x^2 \cdot |\psi(x)|^2 = x^2 \cdot f^*(x) \cdot f(x)$$
 (282)

$$\sigma_x^2 = \int_{-\infty}^{\infty} x^2 \cdot |\psi(x)|^2 dx \qquad (283)$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} x^2 \cdot f^*(x) \cdot f(x) dx \qquad (284)$$

$$\langle f|f\rangle = \int_{-\infty}^{\infty} f^*(x) \cdot f(x) dx$$
 (285)

$$\sigma_x^2 = \langle f | f \rangle \tag{286}$$

### **Uncertainty principle: Kennard** inequality proof part 2.2

$$\frac{dv}{dx} = e^{\frac{-ip\chi}{\hbar}} \tag{287}$$

$$uv(-\infty) = 0 \frac{\hbar}{-ip} e^{\frac{-ip(-\infty)}{\hbar}}$$
 (304)

$$uv(\infty) = 0 \tag{305}$$

(306)

$$v = \int \frac{dv}{d\chi} d\chi \tag{288}$$

$$\int d\chi \, d\chi \qquad (200)$$

$$= inv$$

$$b = \frac{-ip\chi}{\hbar} \tag{289}$$

$$(uv) \Big|^{\infty} = 0 \tag{307}$$

$$\frac{dv}{d\chi}=e^b$$
 (290) A.50 Uncertainty principle: Kennard inequality proof part 2.5

$$v = \int e^b d\chi \qquad (291) \qquad I = (uv) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v \frac{du}{d\chi} d\chi \qquad (308)$$

$$v = \frac{\hbar}{-ip}e^b + C$$
 (292)  $(uv)\Big|_{-\infty}^{\infty} = 0$  (309)

A.48 Uncertainty principle: Kennard inequality proof part 2.3 
$$I = -\int_{-\infty}^{\infty} v \frac{du}{d\chi} d\chi \qquad (310)$$

$$\frac{du}{d\chi} = \frac{d\psi(\chi)}{d\chi} \tag{311}$$

$$v = \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}}$$

$$v = \frac{\sqrt{h}}{-ip} e^{\frac{-ip\chi}{\hbar}}$$
(312)

$$I = -\int_{-\infty}^{\infty} \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \frac{d\psi(\chi)}{d\chi} d\chi \qquad (313)$$

$$I = \frac{\hbar}{ip} \int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi} e^{\frac{-ip\chi}{\hbar}} d\chi \tag{314}$$

## Uncertainty principle: Kennard

$$u = \psi(\chi) \tag{293}$$

$$v = \frac{\hbar}{-ip}e^b \tag{294}$$

$$b = \frac{-ip\chi}{\hbar} \tag{295}$$

$$v = \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \tag{296}$$

$$uv = \psi(\chi) \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}}$$
 (297)

#### A.49 **Uncertainty principle: Kennard** inequality proof part 2.4

$$uv = \psi(\chi) \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}}$$
 (298)

$$uv(\infty) = \psi(\infty) \frac{\hbar}{-ip} e^{\frac{-ip\infty}{\hbar}}$$
 (299)

$$uv(-\infty) = \psi(-\infty) \frac{\hbar}{-in} e^{\frac{-ip(-\infty)}{\hbar}}$$
 (300)

$$\psi(\infty) = 0 \tag{301}$$

$$\psi(-\infty) = 0 \tag{302}$$

$$uv(\infty) = 0 \frac{\hbar}{-ip} e^{\frac{-ip\infty}{\hbar}}$$
 (303)

421	$g(x) = \frac{\hbar}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi} e^{i(x-\chi)b} d\chi$	(db) (315)	$\sigma_x^2 = \langle f   f \rangle$	(332)	444
422	$\int_{-\infty}^{\infty} e^{i(x-\chi)b} db = 2\pi\delta(x-\chi)$	(316)	$\sigma_p^2 = \langle g   g  angle$	(333)	445
	$\frac{\hbar}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi} 2\pi \delta(x-\chi) d\chi db = \frac{\hbar}{i}$	$\int^{\infty} \frac{dy}{y}$	$\frac{\psi(\chi)}{dx}\delta(x-\chi)d\chi$ $\langle f f\rangle\cdot\langle g g\rangle=\sigma_x^2\sigma_p^2$	(334)	446
423	$2\pi i J_{-\infty} J_{-\infty}  a\chi$	(317)		(335)	447
424	$\int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi} 2\pi \delta(x - \chi) d\chi = 2\pi \frac{d\psi(x)}{dx}$	(318)	A.55 Uncertainty principle: Kennard inequality proof part 4.3		448
425	$\frac{\hbar}{i} \int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi} \delta(x - \chi) d\chi = \frac{\hbar}{i} 2\pi \frac{d\psi(x)}{dx}$	(319)	$ z ^2 = (\text{Re}(z))^2 + (\text{Im}(z))^2$	(336)	450
426	$g(x) = \frac{\hbar}{i} \left(\frac{d\psi(x)}{dx}\right)$	(320)	$(\text{Re}(z))^2 + (\text{Im}(z))^2 \ge (\text{Im}(z))^2$	(337)	451
427 428	A.52 Uncertainty principle: Kennard inequality proof part 3.2				
429	$\tilde{g}^*(p) \cdot \tilde{g}(p) = p^2 \varphi^*(p) \cdot \varphi(p)$	(321)	$\langle f a\rangle - \langle a f\rangle$		
430	$\varphi^*(p) \cdot \varphi(p) =  \varphi(p) ^2$	(322)	$(\operatorname{Im}(z))^2 = \left(\frac{\langle f g\rangle - \langle g f\rangle}{2i}\right)^2$	(338)	452
431	$p^2 \varphi^*(p) \cdot \varphi(p) = p^2  \varphi(p) ^2$	(323)	$ z ^2 \ge (\operatorname{Im}(z))^2$	(339)	453
432	$\tilde{g}^*(p) \cdot \tilde{g}(p) =  \tilde{g}(p) ^2$	(324)	/ f   a \		
433	$ \tilde{g}(p) ^2 = p^2  \varphi(p) ^2$	(325)	$ z ^2 \ge \left(\frac{\langle f g\rangle - \langle g f\rangle}{2i}\right)^2$	(340)	454
434 435	A.53 Uncertainty principle: Kennard inequality proof part 3.3		A.56 Uncertainty principle: Kennard inequality proof part 4.4		455 456
436	$\sigma_p^2 = \int_{-\infty}^{\infty} p^2  \varphi(p) ^2 dp$	(326)	$\sigma_x^2\sigma_p^2 \geq  raket{f g} ^2$	(341)	457
437	$ \tilde{g}(p) ^2 = p^2  \varphi(p) ^2$	(327)	$ z ^2 \ge \left(\frac{\langle f g\rangle - \langle g f\rangle}{2i}\right)^2$	(342)	458
438	$\int_{-\infty}^{\infty}  \tilde{g}(p) ^2 dp = \int_{-\infty}^{\infty} p^2  \varphi(p) ^2 dp$	(328)	$z=\langle f g angle$	(343)	459
439	$\int_{-\infty}^{\infty}  \tilde{g}(p) ^2 dp = \int_{-\infty}^{\infty}  g(x) ^2 dx$	(329)	$ \langle f g\rangle ^2 = (\frac{\langle f g\rangle - \langle g f\rangle}{2i})^2$	(344)	460
440	$\langle g g\rangle = \int_{-\infty}^{\infty}  g(x) ^2 dx$	(330)	$1\sqrt{3}\sqrt{3}$ $2i$	()	
441	$\sigma_p^2 = \langle g g \rangle$	(331)	$\sigma_x^2 \sigma_p^2 \ge (\frac{\langle f g \rangle - \langle g f \rangle}{2i})^2$	(345)	461

A.54 Uncertainty principle: Kennard

inequality proof part 4.1

442

443

461

A.51 Uncertainty principle: Kennard

inequality proof part 2.9

419

420

## A.57 Uncertainty principle: Kennard inequality proof part 5.1

$$f(x) = x \cdot \psi(x) \tag{346}$$

$$g(x) = (-i\hbar \frac{d}{dx}) \cdot \psi(x) \tag{347}$$

$$\langle f|g\rangle = \int_{-\infty}^{\infty} f^*(x) \cdot g(x) dx$$
 (348)

$$\langle f|g\rangle = \int_{-\infty}^{\infty} (x \cdot \psi^*(x)) \cdot ((-i\hbar \frac{d}{dx}) \cdot \psi(x)) dx$$
(349)

$$\langle f|g\rangle = -i\hbar \int_{-\infty}^{\infty} x\psi^*(x) \frac{d\psi(x)}{dx} dx$$
 (350)

## A.58 Uncertainty principle: Kennard inequality proof part 5.2

$$f(x) = x \cdot \psi(x) \tag{351}$$

$$g^*(x) = \psi^*(x) \cdot \left(-i\hbar \frac{d}{dx}\right) \tag{352}$$

$$\langle g|f\rangle = \int_{-\infty}^{\infty} g^*(x) \cdot f(x) dx$$
 (353)

$$\langle g|f\rangle = \int_{-\infty}^{\infty} \psi^*(x) \cdot (-i\hbar \frac{d}{dx}) \cdot (x \cdot \psi(x)) dx$$
(354)

$$\langle g|f\rangle = -i\hbar \int_{-\infty}^{\infty} \psi^*(x) \frac{d}{dx} (x\psi(x)) dx$$
 (355)

## A.59 Uncertainty principle: Kennard inequality proof part 5.6

$$\langle f|g\rangle - \langle g|f\rangle = i\hbar \int_{-\infty}^{\infty} |\psi(x)|^2 dx$$
 (356)

$$p(x) = |\psi(x)|^2 (357)$$

$$\int_{-\infty}^{\infty} p(x)dx = \int_{-\infty}^{\infty} |\psi(x)|^2 dx \qquad (358)$$

$$i\hbar \int_{-\infty}^{\infty} |\psi(x)|^2 dx = i\hbar \int_{-\infty}^{\infty} p(x) dx \qquad (359)$$

$$\int_{-\infty}^{\infty} p(x)dx = 1 \tag{360}$$

# A.60 Particle in a box: Wavefunction angular velocity as a function of particle mass from Schrödinger's equation 7

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t)$$
(361)

$$V(x) = 0 (362)$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)+V(x)\psi(x,t)=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{363}$$

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \hbar \omega \psi(x,t)$$
 (364)

$$\hbar\omega\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t}$$
 (365)

$$\hbar\omega\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{366}$$

$$\hbar\omega\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{367}$$

# A.61 Particle in a box: Wavefunction angular velocity as a function of particle mass from Schrödinger's equation 8

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{\hbar^2 k^2}{2m}\psi(x,t)$$
 (368)

$$\frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m\hbar}$$
 (369)

$$\hbar\omega\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{370}$$

$$\hbar\omega = \frac{p^2}{2m} \tag{371}$$

$$\omega = \frac{p^2}{2m\hbar} \tag{372}$$

## A.62 Quantum harmonic oscillator: Ladder operator method 4

$$aa^{\dagger} - a^{\dagger}a = \frac{i}{\hbar}(\hat{p}\hat{x} - \hat{x}\hat{p}) \tag{373}$$

$$[\hat{p}, \hat{x}] = \hat{p}\hat{x} - \hat{x}\hat{p} \tag{374}$$

$$[\hat{p}, \hat{x}] = -i\hbar \tag{375}$$

506	$-i \cdot i = 1$	(376)		Ladder operators for the quant harmonic oscillator part 3.1	um	531 532
507	$\left[a,a^{\dagger}\right]=aa^{\dagger}-a^{\dagger}a$	(377)		$a = \frac{1}{\sqrt{2}}(\frac{d}{dq} + q)$	(394)	533
508	$\left[a,a^{\dagger}\right]=\frac{i}{\hbar}[\hat{p},\hat{x}]$	(378)		$p = -i\frac{d}{dq}$	(395)	534
509	$\left[a,a^{\dagger}\right]=\frac{i}{\hbar}(-i\hbar)$	(379)		$-i \cdot i = 1$	(396)	535
510	$\left[a,a^{\dagger} ight]=1$	(380)		$\frac{d}{da} = -ip$	(397)	536
511	A.63 Creation and annihilation operator	ors:		aq		
512	Ladder operators for the quantum	n		1 ,	(200)	
513	harmonic oscillator part 1.6			$a = \frac{1}{\sqrt{2}}(-ip + q)$	(398)	537
514	$\hbar\omega(\frac{1}{2} + \frac{1}{\sqrt{2}}(-\frac{d^2}{dq^2} + q^2)\frac{1}{\sqrt{2}}(\frac{d^2}{dq^2} + q^2))\psi(q)$	$(381) = E\psi(q)$		$a = \frac{1}{\sqrt{2}}(ip + q)$	(399)	538
314			<b>A.66</b>	Creation and annihilation opera	ators•	539
	$1 \cdot d^2$		1.00	Ladder operators for the quant		540
515	$a = \frac{1}{\sqrt{2}} (\frac{d^2}{da^2} + q^2)$	(382)		harmonic oscillator part 3.2	um	
	V 2 · uq					541
516	$a^{\dagger} = \frac{1}{\sqrt{2}}(-\frac{d^2}{dq^2} + q^2)$	(383)		$a^{\dagger} = \frac{1}{\sqrt{2}}(-\frac{d}{dq} + q)$	(400)	542
517	$\hbar\omega(\frac{1}{2} + a^{\dagger}a)\psi(q) = E\psi(q)$	(384)		$p = -i\frac{d}{dq}$	(401)	543
518	$E = \hbar\omega(a^{\dagger}a + \frac{1}{2})$	(385)		$-i \cdot i = 1$	(402)	544
519	A.64 Creation and annihilation operato	ors:		d		
520	Ladder operators for the quantum			$-\frac{d}{da} = ip$	(403)	545
521	harmonic oscillator part 2	-		uq		
522	[q,p] = qp - pq	(386)		$a^{\dagger} = \frac{1}{\sqrt{2}}(-ip + q)$	(404)	546
	d	A	<b>A.67</b>	Creation and annihilation opera	ators:	547
523	$p = -i\frac{d}{da}$	(387)	1.07	Ladder operators for the quant		548
	aq			harmonic oscillator part 3.6	uiii	549
	$d \cdot d \cdot d \cdot d$	(200)		<u>-</u>	(40.5)	
524	$[q, p] = q(-i\frac{d}{dq}) - (-i\frac{d}{dq})q$	(388)		$aa^{\dagger} - a^{\dagger}a = i(pq - qp)$	(405)	550
	, ,					
525	$[q,p] = -iq\frac{d}{da} + i\frac{d}{da}q$	(389)		[p,q] = pq - qp	(406)	551
	$dq dq^{T}$	()			( )	
526	$(\frac{d}{dq}q - q\frac{d}{dq}) = 1$	(390)		$aa^{\dagger} - a^{\dagger}a = i[p, q]$	(407)	552
527	$-iq\frac{d}{dq} + i\frac{d}{dq}q = i$	(391)		[p,q] = -i	(408)	553
528	[q,p]=i	(392)		$aa^{\dagger} - a^{\dagger}a = -i \cdot i$	(409)	554
529	[q,p]f(q) = if(q)	(393)		$-i \cdot i = 1$	(410)	555

A.65 Creation and annihilation operators:

(427)

 $\left[\hat{H}, \hat{x}(t)\right] = \hat{H}\hat{x}(t) - \hat{x}(t)\hat{H}$ 

595	$[\hat{x}_0, \hat{p}_0] = i\hbar$	(439)		$P_e(t) =  \langle e, 0   \Psi(t) \rangle ^2$	(454)	614
	$\frac{1}{\omega m}[\hat{x}_0, \hat{p}_0]\sin(\omega t_2 - \omega t_1) = \frac{i\hbar}{\omega m}\sin(\omega t_2)$	$-\omega t_1$		$P_e(t) = \cos^2(\frac{\Omega t}{2})$	(455)	615
596		(440)	A.75	Vacuum Rabi Oscillations: group probability 2	nd state	616 617
597	$[\hat{x}(t_1), \hat{x}(t_2)] = \frac{i\hbar}{\omega m} \sin(\omega t_2 - \omega t_1)$	(441)	$P_g(t)$	• •		
598	A.73 Heisenberg picture: momentum commutator 3				(456)	618
599 600	$[\hat{p}(t_1), \hat{p}(t_2)] = m\omega \hat{p}_0 \hat{x}_0 \sin(\omega t_2 - \omega t_1) + r$	$m\omega\hat{x}_0\hat{p}_0$ si (442)	$\operatorname{in}(\omega t_1$	$-\omega t_2) \qquad \langle g, 1 e, 0\rangle = 0$	(457)	619
601	$\sin(\omega t_1 - \omega t_2) = -\sin(\omega t_2 - \omega t_1)$	(443)	$P_g(t)$	$=  \cos\left(\frac{\Omega t}{2}\right)*0 - i\sin\left(\frac{\Omega t}{2}\right)\langle g, 1\rangle$	$ g,1\rangle ^2$ (458)	620
602	$m\omega\hat{x}_0\hat{p}_0\sin(\omega t_1 - \omega t_2) = -m\omega\hat{x}_0\hat{p}_0\sin(\omega t_1 - \omega t_2)$	$\omega t_2 - \omega t_1 \tag{444}$	$P_{s}$	$g(t) =  -i\sin\left(\frac{\Omega t}{2}\right)\langle g, 1 g, 1\rangle ^2$	(459)	621
	$[\hat{p}(t_1), \hat{p}(t_2)] = m\omega \hat{p}_0 \hat{x}_0 \sin(\omega t_2 - \omega t_1) - r$	ກ/ປາຂົດກິດ ຮ່	in(wto	$\langle g, 1 g, 1\rangle = 1$	(460)	622
603		(445)	$m(\omega v_2)$	$P_g(t) =  -i\sin\left(\frac{\Omega t}{2}\right) * 1 ^2$	(461)	623
604	$\hat{x}_0\hat{p}_0 - \hat{p}_0\hat{x}_0 = i\hbar$	(446)		$ -i ^2 = 1$	(462)	624
605	$m\omega(\hat{x}_0\hat{p}_0-\hat{p}_0\hat{x}_0)\sin(\omega t_2-\omega t_1)=i\hbar m\omega s$	$\sin(\omega t_2 - (447))$	$-\omega t_1)$	$P_g(t) =  \sin\left(\frac{\Omega t}{2}\right) ^2$	(463)	625
606	$[\hat{p}(t_1), \hat{p}(t_2)] = i\hbar m\omega \sin(\omega t_2 - \omega t_1)$	(448)		$P_g(t) = \sin^2(\frac{\Omega t}{2})$	(464)	626
607	A.74 Vacuum Rabi Oscillations: excited	l state	A.76	Expectation value: integral expre		627
608	probability (O4)			$\left\langle \hat{X} ight angle _{\Psi}=\left\langle \Psi ightert \hat{X}\leftert \Psi ight angle$	(465)	628
609	$ \Psi(t)\rangle = \cos\left(\frac{\Omega t}{2}\right) e,0\rangle - i\sin\left(\frac{\Omega t}{2}\right) _{\Delta t}$	$g,1\rangle$ (449)		$\hat{X} = \mathbb{I}\hat{X}\mathbb{I}$	(466)	629
	$\langle e,0 \Psi(t)\rangle=\cos\!\left(\frac{\Omega t}{2}\right)\langle e,0 e,0\rangle-i\sin\!\left(\frac{\Omega t}{2}\right)$	$\frac{\Omega t}{2}$	$ a,1\rangle$	$\left\langle \hat{X}\right\rangle _{\Psi}=\left\langle \Psi\right \left\  \hat{X}\right\ \left \Psi\right\rangle$	(467)	630
610	$(0,0)^{2}(0)^{2} \qquad (2)^{2}(0,0)^{2}(0,0)$	(450)	IJ; ±/	$\mathbb{I} = \int \ket{x}ra{x}dx$	(468)	631

(438)

 $\langle e, 0|e, 0\rangle = 1$ 

 $\langle e,0|g,1\rangle=0$ 

 $\langle e, 0 | \Psi(t) \rangle = \cos\left(\frac{\Omega t}{2}\right)$ 

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A.72 Heisenberg picture: position

commutator 4

commutator -  $[\hat{x}(t_1), \hat{x}(t_2)] = \frac{1}{\omega m} (\hat{x}_0 \hat{p}_0 - \hat{p}_0 \hat{x}_0) \sin(\omega t_2 - \omega t_1)$  (437)

 $\hat{x}_0 \hat{p}_0 - \hat{p}_0 \hat{x}_0 = [\hat{x}_0, \hat{p}_0]$ 

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632	$\left\langle \hat{X} \right\rangle_{\Psi} = \left\langle \Psi \right  \left( \int \left  x \right\rangle \left\langle x \right  dx \right) \hat{X} \left( \int \left  x' \right\rangle \left\langle x \right  dx \right)$		79 Euler-lagrange equation: Full derivative of the perturbation Lagrangian with respect to $\varepsilon$ 2 $\frac{dg_{\varepsilon}}{d\varepsilon} = \frac{d}{d\varepsilon}(\varepsilon \eta(x))$	(484)	649 650 651
633	$\left\langle \hat{X} \right\rangle_{\Psi} = \int \int \left\langle \Psi   x \right\rangle \left\langle x   \hat{X}   x' \right\rangle \left\langle x'   \Psi \right\rangle dx$	xdx' (470)	$\frac{d}{d\varepsilon}(\varepsilon\eta(x)) = \eta(x) + \varepsilon \frac{d\eta(x)}{d\varepsilon}$	(485)	653
634	A.77 Expectation value: integral expres $\left\langle \hat{X} \right\rangle_{\Psi} = \int \int \left\langle \Psi   x \right\rangle \left\langle x   \hat{X}   x' \right\rangle \left\langle x'   \Psi \right\rangle dx$		$\frac{d\eta(x)}{d\varepsilon} = 0$	(486)	654
635	`	(471)	$\eta(x) + \varepsilon \frac{d\eta(x)}{d\varepsilon} = \eta(x)$	(487)	655
636	$\hat{X}   x' \rangle = x'   x' \rangle$ $\langle \Psi   x \rangle = \langle x   \Psi \rangle^{\dagger}$	(472) (473)	$\frac{dg_{\varepsilon}}{d\varepsilon} = \eta(x)$	(488)	656
637	$\langle \Psi   x \rangle = \langle x   \Psi \rangle$	(4/3) A.	80 Euler-Lagrange equation: Deriv	vation	657
638	$\langle x x'\rangle=\delta(x-x')$	(474)	$J = \int_{a}^{b} L(x, f(x), f'(x))$	(489)	658
639	$\int \int \left\langle \Psi   x \right\rangle \left\langle x   x' \left  x' \right\rangle \left\langle x' \middle  \Psi \right\rangle dx dx' = \int \int dx dx' dx' dx' dx' dx' dx' dx' dx' dx' $	$\int \langle x   \Psi \rangle^{\dagger} x'   $ (475)	$g_{\varepsilon}(x) = f(x) + \varepsilon \eta(x)$ $x' \rangle \langle x'   \Psi \rangle dx dx'$	(490)	659
	$\int \int \langle x \Psi\rangle^{\dagger} x' \delta(x-x') \langle x' \Psi\rangle dx dx' = \langle$	$\langle \hat{X} \rangle_{_{ m M}}$	$J_{\varepsilon} = \int_{a}^{b} L(x, g_{\varepsilon}(x), g'_{\varepsilon}(x)) dx$	(491)	660
640	J	(476)	$L_{\varepsilon} = L(x, g_{\varepsilon}(x), g'_{\varepsilon}(x))$	(492)	661
641	A.78 Expectation value: integral expres $\left\langle \hat{X} \right\rangle_{\Psi} = \int \int \left\langle x   \Psi \right\rangle^{\dagger} x' \delta(x-x') \left\langle x'   \Psi \right\rangle dx'$	dxdx'	$\frac{dJ_{\varepsilon}}{d\varepsilon} = \frac{d}{d\varepsilon} \int_{a}^{b} L_{\varepsilon} dx$	(493)	662
642		(477)	$\frac{dJ_{\varepsilon}}{d\varepsilon} = \int_{a}^{b} \frac{dL_{\varepsilon}}{d\varepsilon} dx$	(494)	663
	$\int \langle x \Psi\rangle^{\dagger} x' \delta(x-x') \langle x' \Psi\rangle dx' = \langle x \Psi\rangle^{\dagger} x'$	$x\langle x \Psi angle$ <b>A.</b>	81 Euler-Lagrange equation: Deriv	vation 4	664
643	J	(478)	$\frac{dv}{dx} = \eta'(x)$	(495)	665
644	$\langle x \Psi\rangle=\Psi(x)$	(479)	$v = \int \frac{dv}{dx} dx$	(496)	666
645	$\langle x \Psi\rangle^{\dagger} x \langle x \Psi\rangle = \Psi^{\dagger}(x)x\Psi(x)$	(480)	$\eta'(x) = \frac{d\eta(x)}{dx}$	(497)	667
646	$\Psi^{\dagger}(x)\Psi(x) =  \Psi(x) ^2$	(481)	$\frac{dv}{dx} = \frac{d\eta(x)}{dx}$	(498)	668
647	$\Psi^{\dagger}(x)x\Psi(x) = x \Psi(x) ^2$	(482)	$v = \int \frac{d\eta(x)}{dx} dx$	(499)	669
648	$\left\langle \hat{X} \right\rangle_{\Psi} = \int x  \Psi(x) ^2 dx$	(483)	$v = \eta(x)$	(500)	670

#### **Euler-Lagrange equation: Derivation 5 Euler-Lagrange equation: Straight line** 671 690 $u = \frac{\partial L}{\partial \, f'}$ $S = \int_{-b}^{b} ds$ (501)672 (518)691 $ds = \sqrt{dx^2 + dy^2}$ (519)692 $v = \eta(x)$ (502)673 dy = y'dx(520)693 n(a) = 0(503)674 $ds = \sqrt{dx^2 + (u'dx)^2}$ (521)694 $\eta(b) = 0$ (504)675 $ds = dx\sqrt{1 + u'^2}$ (522)695 $uv = \frac{\partial L}{\partial f'} \eta(x)$ (505) $S = \int^b \sqrt{1 + y'^2} dx$ (523) $uv\Big|^b = \frac{\partial L}{\partial f'}\eta(x)\Big|^b$ (506)677 A.85 **Euler-Lagrange equation: Straight line** 697 698 $\left. \frac{\partial L}{\partial f'} \eta(x) \right|^b = \frac{\partial L}{\partial f'} \left. \eta(x) \right|^b$ $\frac{dL}{du} - \frac{d}{dx}\frac{dL}{du'} = 0$ (507)(524)678 699 $\frac{dL}{du} = 0$ (525)700 $\left. \frac{\partial L}{\partial f'} \eta(x) \right|^b = \frac{\partial L}{\partial f'} (\eta(b) - \eta(a))$ (508)679 $\frac{dL}{du'} = y'(1+y'^2)^{-\frac{1}{2}}$ (526)701 $\frac{dL}{du} - \frac{d}{dx}y'(1+y'^2)^{-\frac{1}{2}} = 0$ $\frac{\partial L}{\partial f'}(\eta(b) - \eta(a)) = \frac{\partial L}{\partial f'}(0 - 0)$ (527)702 (509)680 $-\frac{d}{dx}y'(1+y'^2)^{-\frac{1}{2}}=0$ (528)703 $\frac{\partial L}{\partial f'}(0-0) = 0$ (510)681 $\int -\frac{d}{dx} (y'(1+y'^2)^{-\frac{1}{2}}) dx = \int 0 dx$ (529) $(uv)\Big|^b = 0$ 704 (511) $\int \frac{d}{dx} (y'(1+y'^2)^{-\frac{1}{2}}) dx = C$ **Euler-Lagrange equation: Derivation 6** (530)705 $I = \int_{-\infty}^{b} \frac{\partial L}{\partial f'} \eta'(x) dx$ (512)684 A.86 Euler-Lagrange equation: Straight line 706 $I = (uv) \Big|_{0}^{b} - \int_{0}^{b} v \frac{du}{dx} dx$ $\frac{dy}{dx} = C(1 - C^2)^{-1/2}$ (531)708 (513)685 $C(1 - C^2)^{-1/2} = A$ (532)709 $(uv)\Big|^b = 0$ (514)686 $\frac{dy}{dx} = A$ (533)710 $\frac{du}{dx} = \frac{d}{dx} \frac{\partial L}{\partial f'}$ (515)687 $\int \frac{dy}{dx} dx = \int A dx$ (534)711 $I = -\int^b v \frac{du}{dx} dx$ (516) $\int Adx = Ax + C$ (535)712

y = Ax + C

(536)

713

(517)

 $I = -\int^{b} \eta(x) \frac{d}{dx} \frac{\partial L}{\partial f'} dx$ 

714 A.87 Escape velocity 
$$F = \frac{GMm}{r^2}$$
 (537)  $\frac{dT}{dx} = \frac{x}{v_1(x^2 + a^2)^{\frac{1}{2}}} + \frac{x - t}{v_2((x-t)^2 + b^2)^{\frac{1}{2}}}$  (555) 726

716  $dW = Fdx$  (538)  $\frac{x}{(x^2 + a^2)^{\frac{1}{2}}} = \sin \theta_1$  (556) 727

718  $W = \int_{r_0}^{\infty} dW$  (540)  $\frac{dT}{dx} = \frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2}$  (557) 738

719  $W = \int_{r_0}^{\infty} \frac{GMm}{r^2} dr$  (541)  $\frac{dT}{dx} = \frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2}$  (558) 740

720 A.88 Escape velocity 2  $\frac{\sin \theta_1}{r^2} - \frac{\sin \theta_2}{r^2}$  (542)  $\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0$  (560) 741

721  $W = \int_{r_0}^{\infty} \frac{GMm}{r^2} dr$  (542)  $\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0$  (561) 743

722  $F = \frac{GMm}{r^2}$  (543)  $\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0$  (561) 743

723  $F = mg$  (544)  $\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0$  (562) 744

724  $\frac{GMm}{r^2} - mg$  (545)  $\frac{1}{v_1} - \frac{v_1}{v_2} - \frac{v_2}{v_2}$  (562) 745

725  $GMm = mgr^2$  (546)  $\frac{1}{v_1} - \frac{v_2}{v_2} - \frac{v_2}{v_2}$  (565) 747

726  $M = mgr^2$  (547)  $\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0$  (561) 743

727  $W = mgr_0$  (548)  $c\sin \theta_1 = n_1 \sin \theta_1 = c\sin \theta_2$  (565) 747

728 A.89 Escape velocity 3

729  $W = mgr_0$  (549)  $n_1 \sin \theta_1 = n_2 \sin \theta_2$  (567) 749

750  $M = \frac{1}{2}mv^2_{ssc}$  (550)  $\frac{\partial^2 v_1(x,t)}{\partial t^2} - (-i\omega)\frac{\partial}{\partial t}(e^{-i\omega t}f(x))$  (568) 722

751  $W = mgr_0$  (549)  $n_1 \sin \theta_1 = n_2 \sin \theta_2$  (567) 749

752  $\frac{\partial^2 v_2(x,t)}{\partial t^2} - (-i\omega)\frac{\partial}{\partial t}(e^{-i\omega t}f(x))$  (568) 722

753  $M = W = E$  (551)

754  $M = E$  (551)

755  $M = \frac{1}{2}mv^2_{ssc}$  (552)  $\frac{\partial^2 v_1(x,t)}{\partial t^2} - (-i\omega)\frac{\partial}{\partial t}(e^{-i\omega t}f(x)) - i\omega e^{-i\omega t}f(x)$  (569) 753

(554)

 $i \cdot i = -1$ 

(571)

755

 $v_{esc} = \sqrt{2gr_0}$ 

$$-\omega^{2}e^{-i\omega t}f(x) = e^{2\frac{\partial^{2}u(x,t)}{\partial x^{2}}} \qquad (574) \qquad u(x,t) = F(x-ct) + G(x+ct) \qquad (588) \qquad 776$$

$$759 \qquad A.93 \qquad Wave equation: plane wave eigenmodes \\ 4 \qquad u(x,t) = Ae^{-i(kx-\omega t)} + Be^{i(kx-\omega t)} \qquad (575) \qquad F_{H} = F_{x+2h} - F_{x} \qquad (589) \qquad 778$$

$$761 \qquad u(x,t) = Ae^{-i(kx-\omega t)} + Be^{i(kx-\omega t)} \qquad (575) \qquad F_{N} = ma(t) \qquad (590) \qquad 779$$

$$762 \qquad u(x,t) = \int_{-\infty}^{\infty} s(\omega)u(x,t)d\omega \qquad (576) \qquad F_{N} = F_{H} \qquad (591) \qquad 780$$

$$763 \qquad s_{+}(\omega) = As(\omega) \qquad (577) \qquad a(t) = \frac{\partial^{2}}{\partial t^{2}}u(x+h,t) \qquad (592) \qquad 781$$

$$764 \qquad s_{-}(\omega) = Bs(\omega) \qquad (578) \qquad ma(t) = F_{H} \qquad (593) \qquad 782$$

$$765 \qquad Ae^{-i(kx-\omega t)} = \int_{-\infty}^{\infty} s_{+}(\omega)e^{-i(kx-\omega t)}d\omega \qquad (579) \qquad m\frac{\partial^{2}}{\partial t^{2}}u(x+h,t) = F_{H} \qquad (594) \qquad 783$$

$$766 \qquad Be^{i(kx-\omega t)} = \int_{-\infty}^{\infty} s_{-}(\omega)e^{i(kx-\omega t)}d\omega \qquad (580) \qquad n\frac{\partial^{2}}{\partial t^{2}}u(x+h,t) = F_{x+2h} - F_{x} \qquad (595) \qquad 784$$

$$767 \qquad A.94 \qquad Wave equation: plane wave eigenmodes \qquad 5 \qquad m\frac{\partial^{2}}{\partial t^{2}}u(x+h,t) = F_{x+2h} - F_{x} \qquad (596) \qquad 786$$

$$768 \qquad A.94 \qquad Wave equation: plane wave eigenmodes \qquad 5 \qquad F_{x+2h} = ku(x+2h,t) - ku(x+h,t) \qquad (597) \qquad 787$$

$$768 \qquad A.94 \qquad Wave equation: plane wave eigenmodes \qquad 5 \qquad F_{x+2h} = ku(x+2h,t) - ku(x+h,t) + ku($$

(572)

(573)

 $-\omega^2 e^{-i\omega t} f(x) = -\omega^2 e^{-i\omega t} f(x)$ 

 $\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$ 

756

757

 $\int_{-\infty}^{\infty} s_{-}(\omega)e^{i(kx-\omega t)}d\omega = \int_{-\infty}^{\infty} s_{-}(\omega)e^{ik(x-ct)}d\omega$ 

 $\int_{-\infty}^{\infty} s_{-}(\omega)e^{ik(x-ct)}d\omega = G(x+ct)$ 

(586)

(587)

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791

 $\int_{-\infty}^{\infty} s_{+}(\omega)e^{-ik(x-ct)}d\omega = F(x-ct) \quad (585) \qquad \frac{\partial^{2}}{\partial t^{2}}u(x+h,t) = \frac{k}{m}\left(u(x+2h,t)-2u(x+h,t)+u(x,t)\right)$ 

792	A.97 Wave equation: Hooke's law 3	<b>B</b> .5	S=1 (one premise removed)		811
793	$\frac{\partial^2}{\partial t^2}u(x+h,t) = \frac{k}{m} (u(x+2h,t) - 2u(x+h,t) + \frac{k}{m} (u(x+h,t) - 2u(x+h,t) + \frac{k}{m} (u($		Gauss' law: equivalence between differential and integral forms		812 813
794	$N = \frac{L}{h} \tag{60}$	,	$\iint_{S} \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0}$	(619)	814
795	$m = \frac{M}{N} \tag{66}$	04)	$\iint_{S} \mathbf{E} \cdot d\mathbf{A} = \iiint_{V} \nabla \cdot \mathbf{E} dV$	(620)	815
796	$k = KN \tag{60}$	05)	$\frac{Q}{\varepsilon_0} = \iiint_V \nabla \cdot \mathbf{E} dV$	(621)	816
797	$\frac{k}{m} = \frac{KN}{\frac{M}{N}} \tag{60}$	06)	$Q=\iiint_V \rho dV$	(622)	817
798	$\frac{k}{m} = \frac{KN^2}{M} \tag{60}$	07)	$\frac{\iiint_V \rho dV}{\varepsilon_0} = \iiint_V \nabla \cdot \mathbf{E} dV$	(623)	818
799	$\frac{k}{m} = \frac{K\frac{L^2}{h^2}}{M} \tag{60}$	08)	$\iiint_{V} \nabla \cdot \mathbf{E} dV = \iiint_{V} \frac{\rho}{\varepsilon_{0}} dV$	(624)	819
	-2	<b>B.2</b>	Gauss' law: Equivalence of total and charge statements	nd free	820 821
800	$\frac{\partial^2}{\partial t^2}u(x+h,t) = \frac{K\frac{L^2}{h^2}}{M}(u(x+2h,t)-2u(x+h,t))$ (60)	,	$ ho_b = - abla \cdot \mathbf{P}$	(625)	822
000		,	$- ho_b =  abla \cdot \mathbf{P}$	(626)	823
801	$\frac{\partial^2}{\partial t^2}u(x+h,t) = \frac{KL^2}{Mh^2} (u(x+2h,t) - 2u(x+h,t)) $ (61)		$ ho_f =  abla \cdot \mathbf{D}$	(627)	824
802	A.98 Wave equation: stress pulse in a bar 2	2		(620)	
803	$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{KL^2}{M} \frac{\partial^2 u(x,t)}{\partial x^2} \tag{61}$	11)	$ ho =  ho_f -  ho_b$	(628)	825
804	$K = \frac{EA}{L} \tag{61}$	12)	$\rho = \nabla \cdot \mathbf{D} - \nabla \cdot \mathbf{P}$	(629)	826
	M		$\rho = \nabla \cdot (\mathbf{D} - \mathbf{P})$	(630)	827
805	$\rho = \frac{M}{V} \tag{61}$	B.3	Uniqueness theorem for Poisson's equation		828 829
806	V = AL   (61	14)	$\nabla^2 \phi_1 = -\frac{\rho_f}{\varepsilon_0}$	(631)	830
807	$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{EA}{L} \frac{L^2}{M} \frac{\partial^2 u(x,t)}{\partial x^2} \tag{61}$	15)	$\nabla^2 \phi_2 = -\frac{\rho_f}{\varepsilon_0}$	(632)	831
808	$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{EAL}{M} \frac{\partial^2 u(x,t)}{\partial x^2} \tag{61}$	16)	$\nabla^2 \phi_1 + \nabla^2 \phi_2 = -\frac{\rho_f}{\varepsilon_0} + -\frac{\rho_f}{\varepsilon_0}$	(633)	832
809	$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{E}{\frac{M}{AL}} \frac{\partial^2 u(x,t)}{\partial x^2} $ (61)	17)	$\nabla^2 \phi_1 + \nabla^2 \phi_2 = -2 \frac{\rho_f}{\varepsilon_0}$	(634)	833
810	$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u(x,t)}{\partial x^2} \tag{61}$	18)	$\nabla^2(\phi_1 + \phi_2) = -2\frac{\rho_f}{\varepsilon_0}$	(635)	834

$$\nabla^2 \phi = -\frac{2^{Pf}}{c_0} \qquad (637) \qquad C - C_1 - C_2 \qquad (652) \qquad 857$$

$$\nabla^2 \phi = 0 \qquad (638) \qquad \phi_2 - C_1 - C_2 \qquad (653) \qquad 858$$

$$8.4 \qquad \text{Uniqueness theorem for Poisson's equation 2} \qquad \phi_1 - C_1 - C_2 \qquad (654) \qquad 859$$

$$\nabla \cdot (\phi \nabla \phi) = (\nabla \phi)^2 + \phi \nabla^2 \phi \qquad (639)$$

$$\nabla^2 \phi = 0 \qquad (640) \qquad \phi_2 - \phi_1 = C_1 - C_2 - (C_1 - C_2) \qquad (655) \qquad 860$$

$$\nabla^2 \phi = 0 \qquad (640) \qquad \phi_2 - \phi_1 = C \qquad (656) \qquad 861$$

$$\nabla \nabla \cdot (\phi \nabla \phi) = (\nabla \phi)^2 \qquad (642) \qquad \nabla \nabla \cdot (\phi - \nabla \phi) = (\nabla \phi)^2 \qquad (642) \qquad \nabla \cdot (\phi - \nabla \phi) = (\nabla \phi)^2 \qquad (643)$$

$$\nabla \cdot (\phi \nabla \phi) dV = \int_V (\nabla \phi)^2 dV \qquad (643) \qquad \nabla \cdot (-\nabla \phi) = -4\pi G\rho \qquad (657) \qquad 863$$

$$\nabla \cdot (\phi \nabla \phi) dV = \int_V (\nabla \phi)^2 dV \qquad (643) \qquad \nabla^2 \phi = 4\pi G\rho \qquad (660) \qquad 866$$

$$\nabla^2 \phi = 4\pi G\rho \qquad (661) \qquad 867$$

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**B.6** 

(636)

 $\nabla^2 \phi = \nabla^2 (\phi_1 + \phi_2)$ 

835

equation 7

Uniqueness theorem for Poisson's

 $\phi = C_1 - C_2$ 

854

855

856

(651)

## Poisson's equation: Gravitational potential from Poisson's equation 6

$$\nabla \cdot (\varepsilon(-\nabla \phi)) = \rho_f \tag{685}$$

$$\int_{S} \frac{\partial \phi}{\partial r} dS = \int_{0}^{2\pi} \int_{0}^{\pi} \frac{\partial \phi}{\partial r} r^{2} \sin \theta d\theta d\varphi \quad (668)$$

$$\varepsilon \nabla \cdot (-\nabla \phi) = \rho_f \tag{686}$$

$$\int_0^{\pi} \sin \theta d\theta = 2 \tag{669}$$

$$-\varepsilon \nabla^2 \phi = \rho_f \tag{687}$$

$$\int_0^{2\pi} d\varphi = 2\pi \tag{670}$$

$$\nabla^2 \phi = -\frac{\rho_f}{\varepsilon} \tag{688}$$

$$\int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta d\theta d\varphi = 2\pi \cdot 2 \cdot r^2 = 4\pi r^2$$
 (671)

### Poisson's equation: Electrostatic potential from Poisson's equation

$$\int_{S} \frac{\partial \phi}{\partial r} dS = \int_{0}^{2\pi} \int_{0}^{\pi} \frac{\partial \phi}{\partial r} r^{2} \sin \theta d\theta d\varphi = \frac{\partial \phi}{\partial r} \int_{0}^{2\pi} \int_{0}^{\pi} r^{2} \sin \theta d\theta d\varphi = \frac{\partial \phi}{\partial r} \cdot 4\pi r^{2}$$
(689)
$$\int_{S} \frac{\partial \phi}{\partial r} dS = \int_{0}^{2\pi} \int_{0}^{\pi} \frac{\partial \phi}{\partial r} r^{2} \sin \theta d\theta d\varphi = \frac{\partial \phi}{\partial r} \int_{0}^{2\pi} r^{2} \sin \theta d\theta d\varphi = \frac{\partial \phi}{\partial r} \cdot 4\pi r^{2}$$
(690)

$$\int_{S} \frac{\partial \phi}{\partial r} dS = 4\pi \frac{\partial \phi}{\partial r} r^2 \tag{673}$$

#### $\int_{\mathcal{U}} \nabla^2 \phi dV = \int_{\mathcal{U}} \nabla \cdot \nabla \phi dV$ (691)

#### **B.10** Poisson's equation: Gravitational potential from Poisson's equation 8

$$\frac{\partial \phi}{\partial r} = \frac{Gm}{r^2} \tag{674}$$

$$\int_{V} \nabla^{2} \phi dV = -\frac{1}{\varepsilon} \int_{V} \rho_{f} dV \qquad (692)$$

$$\int_{-\infty}^{r} \frac{\partial \phi}{\partial r} dr = \phi(r) - \phi(\infty) \tag{675}$$

$$\int_{V} \nabla \cdot \nabla \phi dV = -\frac{1}{\varepsilon} \int_{V} \rho_{f} dV \tag{693}$$

$$\int_{-\infty}^{r} \frac{Gm}{r^2} dr = \frac{-Gm}{r} \tag{676}$$

$$\int_{V} \nabla \cdot \nabla \phi dV = -\frac{Q}{\varepsilon} \tag{694}$$

$$\phi(r) - \phi(\infty) = \frac{-Gm}{r} \tag{677}$$

#### **B.13** Poisson's equation: Electrostatic potential from Poisson's equation 2

$$\phi(r) = \frac{-Gm}{r} + \phi(\infty) \tag{678}$$

$$\int_{V} \nabla \cdot \nabla \phi dV = -\frac{Q}{\varepsilon} \tag{695}$$

$$\phi(\infty) = 0 \tag{679}$$

$$\int_{V} \nabla \cdot \nabla \phi dV = \int_{S} \nabla \phi \cdot d\mathbf{S} \tag{696}$$

$$\phi(r) = \frac{-Gm}{r} \tag{680}$$

$$-\frac{Q}{\varepsilon} = \int_{C} \nabla \phi \cdot d\mathbf{S} \tag{697}$$

#### **B.11 Poisson's equation: Electrostatics**

 $\mathbf{D} = \varepsilon \mathbf{E}$ 

$$\nabla \phi = \frac{\partial \phi}{\partial x} \tag{698}$$

$$\nabla \cdot \mathbf{D} = \rho_f \tag{681}$$

$$\int_{G} \nabla \phi \cdot d\mathbf{S} = \int_{G} \frac{\partial \phi}{\partial r} dS \tag{699}$$

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{682}$$

$$J_S = J_S \partial r$$

$$\nabla \cdot (\varepsilon \mathbf{E}) = \rho_f \tag{683}$$

$$\int_{S} \frac{\partial \phi}{\partial r} dS = -\frac{Q}{\varepsilon} \tag{700}$$

$$\mathbf{E} = -\nabla\phi \tag{684}$$

$$\phi = -\mathbf{v}\,\psi$$
 (064)

#### $\int_{-\infty}^{\infty} \frac{\partial \phi}{\partial r} dr = \phi(r) - \phi(\infty)$ (702) $\mathbf{B} = \nabla \times \mathbf{A}$ 921 (720)945 $\int_{0}^{r} -\frac{Q}{4\pi\varepsilon r^2} dr = \frac{Q}{4\pi\varepsilon r}$ (703) $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ 922 (721)946 $\phi(r) - \phi(\infty) = \frac{Q}{4\pi\epsilon r}$ (704)923 $\mathbf{F} = q(-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\times\nabla\times\mathbf{A})$ (722)947 $\phi(r) = \frac{Q}{4\pi cr} + \phi(\infty)$ (705)924 $\mathbf{F} = q(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A})$ (723) 948 $assuming\phi(\infty) = 0, then\phi(r) = \frac{Q}{4\pi c r}$ (706) 925 **B.18 Lorentz force: Potential energy** 949 **B.15** Lorentz force: continuous charge 926 derivation from scalar potential 3 950 distribution 927 $U = q \int_{-\tau}^{\tau} \nabla \phi \cdot d\mathbf{r}$ $\frac{d\mathbf{F}}{dV} = \frac{dq}{dV}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ (724)951 (707) $\int_{-r}^{r} \nabla \phi \cdot d\mathbf{r} = \int_{-r}^{r} \frac{\partial \phi}{\partial r} dr$ $\mathbf{f} = \frac{d\mathbf{F}}{dV}$ (725)952 (708)929 $\mathbf{f} = \frac{dq}{dV} (\mathbf{E} + \mathbf{v} \times \mathbf{B})$ $\int_{-\infty}^{\infty} \frac{\partial \phi}{\partial r} dr = \phi(r) - \phi(\infty)$ (709)(726)930 953 $\rho = \frac{dq}{dV}$ $U = q(\phi(r) - \phi(\infty))$ (710)931 (727)954 $\mathbf{f} = \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ (711)932 $U = q\phi(r) - q\phi(\infty)$ (728)955 $\mathbf{f} = \rho \mathbf{E} + \rho \mathbf{v} \times \mathbf{B}$ (712)933 $U = a\phi(r)if\phi(\infty) = 0$ (729)956 Lorentz force: continuous charge **B.16** 934 **Laplace equation: Analytic functions** 957 935 distribution 2 (**u**) 958 $\mathbf{f} = \rho \mathbf{E} + \rho \mathbf{v} \times \mathbf{B}$ (713)936 $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial u}$ (730)959 $J = \rho v$ (714)937 $\frac{\partial}{\partial u}(\frac{\partial u}{\partial x}) = \frac{\partial}{\partial u}(\frac{\partial v}{\partial u})$ (731)960 $f = \rho E + J \times B$ (715)938 $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}$ (732)961 $\frac{d\mathbf{F}}{dV} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}$ 939 (716) $\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial^2 u}{\partial x \partial u})$ $\mathbf{F} = \iiint \frac{d\mathbf{F}}{dV} dV$ (733)962 (717)940 $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$ $\mathbf{F} = \iiint (\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}) dV$ (718)(734)941 963 23

(701)

**Lorentz force: Lorentz force in terms of** 

 $\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$ 

potentials

942

943

944

(719)

Poisson's equation: Electrostatic

 $\int_{-r}^{r} \frac{\partial \phi}{\partial r} dr = \int_{r}^{r} -\frac{Q}{4\pi \varepsilon r^{2}} dr$ 

potential from Poisson's equation 4

918

919

967		$\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial}{\partial y} (\frac{\partial u}{\partial y})$	(736)	$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x}$	(753)	992
968		$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y} (\frac{\partial u}{\partial y})$	(737)	$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$	(754)	993
969		$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$	(738)	$-\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$	(755)	994
970	<b>B.21</b>	Laplace equation: Analytic fur	nctions			
971		(u) 3 $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$	(739)	$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$	(756)	995
973		$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$	(740)	$ abla^2 v = 0$ B.25 Laplace equation: Electrostatic	(757)	996 997
						331
974		$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$	(741)	$\mathbf{E} = (u, v)$	(758)	998
975		$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$	(742)	$\nabla \cdot \mathbf{E} = \rho$	(759)	999
976		$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$	(743)	$\nabla \cdot \mathbf{E} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$	(760)	1000
977		$\nabla^2 u = 0$	(744)	$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \rho$	(761)	1001
				·		
978	<b>B.22</b>	Laplace equation: Analytic fur	nctions	B.26 Laplace equation: Electrostatic	s 2	1002
978 979	B.22	Laplace equation: Analytic fun (v)	nctions	<b>B.26</b> Laplace equation: Electrostatic	s 2	1002
	B.22		(745)	<b>B.26</b> Laplace equation: Electrostatic $\frac{\partial \phi}{\partial x} = -u$	s <b>2</b> (762)	1002 1003
979	B.22	(v)		• •		
979 980	B.22	$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$	(745)	$\frac{\partial \phi}{\partial x} = -u$	(762)	1003
979 980 981	B.22 B.23	(v) $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ $\frac{\partial}{\partial x} (\frac{\partial v}{\partial x}) = \frac{\partial}{\partial x} (-\frac{\partial u}{\partial y})$	(745) (746) (747)	$\frac{\partial \phi}{\partial x} = -u$ $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial (-u)}{\partial x}$ $\frac{\partial \phi}{\partial y} = -v$	(762) (763) (764)	1003
979 980 981 982		$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ $\frac{\partial}{\partial x} (\frac{\partial v}{\partial x}) = \frac{\partial}{\partial x} (-\frac{\partial u}{\partial y})$ $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$	(745) (746) (747)	$\frac{\partial \phi}{\partial x} = -u$ $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial (-u)}{\partial x}$ $\frac{\partial \phi}{\partial y} = -v$	(762) (763)	1003
<ul><li>979</li><li>980</li><li>981</li><li>982</li><li>983</li></ul>		$ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} $ $ \frac{\partial}{\partial x} (\frac{\partial v}{\partial x}) = \frac{\partial}{\partial x} (-\frac{\partial u}{\partial y}) $ $ \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} $ Laplace equation: Analytic fun	(745) (746) (747)	$\frac{\partial \phi}{\partial x} = -u$ $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial (-u)}{\partial x}$ $\frac{\partial \phi}{\partial y} = -v$ $\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial (-v)}{\partial y}$	(762) (763) (764)	1003 1004 1005
979 980 981 982 983 984		$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ $\frac{\partial}{\partial x} (\frac{\partial v}{\partial x}) = \frac{\partial}{\partial x} (-\frac{\partial u}{\partial y})$ $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$ Laplace equation: Analytic fun(v) 2	(745) (746) (747) <b>nctions</b>	$\frac{\partial \phi}{\partial x} = -u$ $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial (-u)}{\partial x}$ $\frac{\partial \phi}{\partial y} = -v$	(762) (763) (764)	1003 1004 1005
979 980 981 982 983 984 985		$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ $\frac{\partial}{\partial x} (\frac{\partial v}{\partial x}) = \frac{\partial}{\partial x} (-\frac{\partial u}{\partial y})$ $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$ Laplace equation: Analytic fun(v) 2 $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$	(745) (746) (747) nctions (748)	$\frac{\partial \phi}{\partial x} = -u$ $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial (-u)}{\partial x}$ $\frac{\partial \phi}{\partial y} = -v$ $\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial (-v)}{\partial y}$	<ul><li>(762)</li><li>(763)</li><li>(764)</li><li>(765)</li></ul>	1004 1005 1006
979 980 981 982 983 984 985		$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ $\frac{\partial}{\partial x} (\frac{\partial v}{\partial x}) = \frac{\partial}{\partial x} (-\frac{\partial u}{\partial y})$ $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$ Laplace equation: Analytic function: (v) 2 $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y} (\frac{\partial v}{\partial y})$	(745) (746) (747) nctions (748) (749)	$\frac{\partial \phi}{\partial x} = -u$ $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial (-u)}{\partial x}$ $\frac{\partial \phi}{\partial y} = -v$ $\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial (-v)}{\partial y}$ $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial (-u)}{\partial x} + \frac{\partial (-v)}{\partial y}$	<ul><li>(762)</li><li>(763)</li><li>(764)</li><li>(765)</li><li>(766)</li></ul>	1003 1004 1005 1006
979 980 981 982 983 984 985		$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ $\frac{\partial}{\partial x} (\frac{\partial v}{\partial x}) = \frac{\partial}{\partial x} (-\frac{\partial u}{\partial y})$ $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$ Laplace equation: Analytic function (v) 2 $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y} (\frac{\partial v}{\partial y})$ $\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y} (\frac{\partial u}{\partial x})$	(745) (746) (747) netions (748) (749) (750)	$\frac{\partial \phi}{\partial x} = -u$ $\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial (-u)}{\partial x}$ $\frac{\partial \phi}{\partial y} = -v$ $\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial (-v)}{\partial y}$ $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial (-u)}{\partial x} + \frac{\partial (-v)}{\partial y}$ $let \rho = \frac{\partial (-u)}{\partial x} + \frac{\partial (-v)}{\partial y}$ $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\rho$	<ul><li>(762)</li><li>(763)</li><li>(764)</li><li>(765)</li><li>(766)</li><li>(767)</li></ul>	1003 1004 1005 1006

**B.24** Laplace equation: Analytic functions

(v) 3

(735)

989

990

991

(752)

**B.20** Laplace equation: Analytic functions

(u) 2

964

965

#### $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial u^2}$ (769)1011 $U = -q \int_{-r}^{r} \frac{\partial (\mathbf{v} \cdot \mathbf{A})}{\partial r} dr$ (784)1032 $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\rho$ (770)1012 $\int_{-r}^{r} \frac{\partial (\mathbf{v} \cdot \mathbf{A})}{\partial r} dr = \mathbf{v} \cdot \mathbf{A}(r) - \mathbf{v} \cdot \mathbf{A}(\infty) \quad (785)$ 1033 $-\rho = 0$ (771)1013 $U = -a(\mathbf{v} \cdot \mathbf{A}(r) - \mathbf{v} \cdot \mathbf{A}(\infty))$ (786)1034 $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ (772)1014 $U = -q\mathbf{v} \cdot \mathbf{A}(r) + q\mathbf{v} \cdot \mathbf{A}(\infty)$ (787)1035 $\nabla^2 \phi = 0$ (773)1015 1016 **B.28 Lorentz force: Potential energy** $assuming \mathbf{v} \cdot \mathbf{A}(\infty) = 0, then U = -q \mathbf{v} \cdot \mathbf{A}(r)$ derivation from vector potential (788)1036 $\mathbf{F} = q(\nabla(\mathbf{v} \cdot \mathbf{A}) - \frac{d\mathbf{A}}{dt})$ (774)1018 **Lorentz force: Derivation of classical** 1037 Lagrangian of EM field $U = -\int_{-r}^{r} \mathbf{F} \cdot d\mathbf{r}$ $V = q\phi - q\dot{\mathbf{r}} \cdot \mathbf{A}$ (789)1039 (775)1019 $q\phi = V + q\dot{\mathbf{r}} \cdot \mathbf{A}$ (790)1040 $U = -q \int_{-\infty}^{r} (\nabla (\mathbf{v} \cdot \mathbf{A}) - \frac{d\mathbf{A}}{dt}) \cdot d\mathbf{r}$ 1020 (776) $T = \frac{m}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$ (791)1041 L = T - V(792)1042 $U = -q \int_{-\pi}^{r} \nabla(\mathbf{v} \cdot \mathbf{A}) \cdot d\mathbf{r} + q \int_{-\pi}^{r} \frac{d\mathbf{A}}{dt} \cdot d\mathbf{r}$ (777) 1021 $L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} - (q\phi - q\dot{\mathbf{r}}\cdot\mathbf{A})$ (793)1043 $U = -q \int^r \nabla(\mathbf{v} \cdot \mathbf{A}) \cdot d\mathbf{r}$ (778)1022 $L = \frac{m}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + q\dot{\mathbf{r}} \cdot \mathbf{A} - q\phi$ (794)1044 **B.29 Lorentz force: Potential energy** 1023 **B.32 Lorentz force: Derivation of classical** 1045 derivation from vector potential 3 1024 Lagrangian of EM field 2 $\nabla (\mathbf{v} \cdot \mathbf{A}) \cdot \hat{\mathbf{r}} = \frac{\partial (\mathbf{v} \cdot \mathbf{A})}{\partial r}$ $L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\dot{\mathbf{r}}\cdot\mathbf{A} - q\phi$ (795)1047 (779)1025 $\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$ (796)1048 $d\mathbf{r} = \hat{\mathbf{r}}dr$ (780) $\frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} = \frac{m}{2}(\dot{x},\dot{y},\dot{z})\cdot(\dot{x},\dot{y},\dot{z})$ (797)1049 $\int_{-\infty}^{r} \frac{\partial (\mathbf{v} \cdot \mathbf{A})}{\partial r} dr = \int_{-\infty}^{r} \nabla (\mathbf{v} \cdot \mathbf{A}) \cdot d\mathbf{r}$ (781)1027 $q\dot{\mathbf{r}}\cdot\mathbf{A}=q(\dot{x},\dot{y},\dot{z})\cdot(A_x,A_y,A_z)$ (798)1050 $U = -q \int^r \nabla (\mathbf{v} \cdot \mathbf{A}) \cdot d\mathbf{r}$ (782)1028

**Lorentz force: Potential energy** 

derivation from vector potential 4

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Laplace equation: Electrostatics 3

1010

1029

 $U = -q \int_{-r}^{r} \frac{\partial (\mathbf{v} \cdot \mathbf{A})}{\partial r} dr$ 

 $L = \frac{m}{2}(\dot{x}, \dot{y}, \dot{z}) \cdot (\dot{x}, \dot{y}, \dot{z}) + q(\dot{x}, \dot{y}, \dot{z}) \cdot (A_x, A_y, A_z) - q\phi$ 

### **Lorentz force: Derivation of Lorentz** force from classical Lagrangian (LHS) 4

$$dA_{x} = \frac{\partial A_{x}}{\partial t}dt + \frac{\partial A_{x}}{\partial x}dx + \frac{\partial A_{x}}{\partial y}dy + \frac{\partial A_{x}}{\partial z}dz$$
(800)

$$\frac{dx}{dt} = \dot{x} \tag{801}$$

$$\frac{dy}{dt} = \dot{y} \tag{802}$$

$$dx = \dot{x}dt \tag{803}$$

$$dy = \dot{y}dt \tag{804}$$

$$dA_{x} = \frac{\partial A_{x}}{\partial t}dt + \frac{\partial A_{x}}{\partial x}\dot{x}dt + \frac{\partial A_{x}}{\partial y}\dot{y}dt + \frac{\partial A_{x}}{\partial z}\dot{z}dt$$
(805)

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}$$
 (806)

#### **Lorentz force: Derivation of Lorentz B.34** force from classical Lagrangian (LHS) 5

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\frac{d}{dt}\dot{x} + q\frac{d}{dt}A_x \tag{807}$$

$$\frac{d}{dt}\dot{x} = \ddot{x} \tag{808}$$

$$m\frac{d}{dt}\dot{x} = m\ddot{x} \tag{809}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + q\frac{d}{dt}A_x \tag{810}$$

$$\frac{d}{dt}A_x = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}$$
(811)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + q\left(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}\right) \qquad F_x = qE_x + q\left(\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right) \tag{827}$$

### **Lorentz force: Derivation of Lorentz** force from classical Lagrangian (RHS) 2

$$\frac{\partial L}{\partial x} = q \frac{\partial}{\partial x} (\dot{x} A_x + \dot{y} A_y + \dot{z} A_z) - q \frac{\partial}{\partial x} \phi$$
 (813)

$$I = \frac{\partial}{\partial x}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) \tag{814}$$

$$\frac{\partial}{\partial x}(\dot{x}A_x+\dot{y}A_y+\dot{z}A_z) = \frac{\partial A_x}{\partial x}\dot{x}+\frac{\partial A_y}{\partial x}\dot{y}+\frac{\partial A_z}{\partial x}\dot{z} \tag{815}$$

$$q\frac{\partial}{\partial x}(\dot{x}A_x+\dot{y}A_y+\dot{z}A_z)=q(\frac{\partial A_x}{\partial x}\dot{x}+\frac{\partial A_y}{\partial x}\dot{y}+\frac{\partial A_z}{\partial x}\dot{z}) \tag{816}$$

$$\frac{\partial L}{\partial x} = q(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}) - q\frac{\partial}{\partial x}\phi \quad (817)$$

## **Lorentz force: Derivation of x** component of electric field

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \tag{818}$$

$$\nabla \phi = (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \tag{819}$$

$$\mathbf{E} = -((\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) - \frac{\partial}{\partial t}(A_x, A_y, A_z)) \tag{820}$$

$$\mathbf{E} \cdot (1, 0, 0) = -((\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) - \frac{\partial}{\partial t} (A_x, A_y, A_z)) \cdot (1, 0, 0)$$
(821)

### **B.37** Lorentz force: Derivation of Lorentz force from classical Lagrangian 4

$$(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_x = \dot{y}(\nabla \times \mathbf{A})_z - \dot{z}(\nabla \times \mathbf{A})_y$$
 (822)

$$(\nabla \times \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$
 (823)

$$(\nabla \times \mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$
 (824)

$$\dot{y}(\nabla \times \mathbf{A})_z - \dot{z}(\nabla \times \mathbf{A})_y = \dot{y}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) - \dot{z}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})$$
(825)

$$F_x = qE_x + q(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_x \tag{826}$$

$$F_x = qE_x + q(\dot{y}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) - \dot{z}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})) \tag{827}$$

#### $\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E})$ (848)1121 $qE_x = q\mathbf{E} \cdot \hat{\mathbf{x}}$ (833)1097 $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ $q(\dot{\mathbf{r}} \times \mathbf{B})_x = q(\dot{\mathbf{r}} \times \mathbf{B}) \cdot \hat{\mathbf{x}}$ (834)1098 (849)1122 $\mathbf{F} \cdot \hat{\mathbf{x}} = q\mathbf{E} \cdot \hat{\mathbf{x}} + q(\dot{\mathbf{r}} \times \mathbf{B}) \cdot \hat{\mathbf{x}}$ (835)1099 $\mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E}) = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t} \frac{\partial \mathbf{B}}{\partial t}$ (850)1123 **B.39** 1100 **Electromagnetic wave equation: The** origin of the electromagnetic wave 1101 equation in 2 1102 $\mu_0 \varepsilon_0 \frac{\partial}{\partial t} \frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$ $\nabla\times(\nabla\times\mathbf{E}) = -\frac{\partial}{\partial t}(\nabla\times\mathbf{B})$ (851)1124 (836)1103 $\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ $\nabla \times (\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$ 1104 (837)(852)1125 **B.42** Electromagnetic wave equation: The 1126 $-\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\frac{\partial}{\partial t}(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t})$ origin of the electromagnetic wave 1127 (838)1105 equation in 3 1128 $\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$ (853)1129 $-\frac{\partial}{\partial t}(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$ (839)1106 $\nabla \cdot \mathbf{B} = 0$ (854)1130 $\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$ (840)1107 **Electromagnetic wave equation: The** 1108 $\nabla \times (\nabla \times \mathbf{B}) = \nabla(0) - \nabla^2 \mathbf{B}$ (855)1131 origin of the electromagnetic wave 1109 equation in 3 1110 $\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ (841)1111 $\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{B}$ (856)1132

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**B.41** 

 $\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla \times (\nabla \times \mathbf{E})$ 

 $\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\nabla^2 \mathbf{E}$ 

 $\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0$ 

Electromagnetic wave equation: The

origin of the electromagnetic wave

 $\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = \nabla \times (\nabla \times \mathbf{B})$ 

 $\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0$ 

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equation in 2

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**Lorentz force: Derivation of Lorentz** 

force from classical Lagrangian 5

 $\mathbf{B} = \nabla \times \mathbf{A}$ 

 $F_x = qE_x + q(\dot{\mathbf{r}} \times \mathbf{B})_x$ 

 $F_x = \mathbf{F} \cdot \hat{\mathbf{x}}$ 

 $E_x = \mathbf{E} \cdot \hat{\mathbf{x}}$ 

 $\nabla \cdot \mathbf{E} = 0$ 

 $\nabla(\nabla \cdot \mathbf{E}) = \nabla(0)$ 

 $\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E}$ 

 $F_x = qE_x + q(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_x$ 

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#### $\sigma_x^2 = \int_0^\infty x^2 \cdot |\psi(x)|^2 dx$ **B.44** Ampere's circuital law: Proof of (881)1165 equivalence 4 $\frac{1}{u_0}(\nabla \times \mathbf{B}) = \mathbf{J}_f + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_P + \mathbf{J}_M$ $\sigma_x^2 = \langle f | f \rangle$ (882)1166 **Uncertainty principle: Kennard B.47** 1167 $\mathbf{J}_b = \mathbf{J}_P + \mathbf{J}_M$ (866)inequality proof part 2.2 1168 $\frac{dv}{dy} = e^{\frac{-ip\chi}{\hbar}}$ (883)1169 $\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J}_f + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_b$ (867) $v = \int e^{\frac{-ip\chi}{\hbar}} d\chi$ (884)1170 $\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b$ (868) $v = \frac{\hbar}{-in} \int e^{\frac{-ip\chi}{\hbar}} d\chi$ 1148 (885)1171 $\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ $v = \frac{\hbar}{-in}e^{\frac{-ip\chi}{\hbar}} + C$ (869)1149 (886)1172 **Uncertainty principle: Kennard** $v = \frac{\hbar}{-ip}e^b + C$ inequality proof part 1.1 (887)1151 1173 $\sigma_r^2 = \langle x^2 \rangle - \langle x \rangle^2$ (870)1152 **B.48 Uncertainty principle: Kennard** 1174 inequality proof part 2.3 1175 $\langle x \rangle = \int_{-\infty}^{\infty} x \cdot p(x) dx$ (871)1153 $u = \psi(\chi)$ (888)1176 $p(x) = |\psi(x)|^2$ $v = -\frac{\hbar}{im}e^b$ (872)1154 (889)1177

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 $\langle x^2 \rangle = \int_0^\infty x^2 \cdot |\psi(x)|^2 dx$ 

 $\sigma_r^2 = \langle x^2 \rangle - \langle x \rangle^2$ 

 $\sigma_x^2 = \int_0^\infty x^2 \cdot |\psi(x)|^2 dx - (\int_0^\infty x \cdot |\psi(x)|^2 dx)^2$ 

**Uncertainty principle: Kennard** 

inequality proof part 1.4

 $f^*(x) \cdot f(x) = x^2 \cdot (\psi^*(x) \cdot \psi(x))$ 

 $x^2 \cdot (\psi^*(x) \cdot \psi(x)) = x^2 \cdot |\psi(x)|^2$ 

 $\int_{-\infty}^{\infty} f^*(x) \cdot f(x) dx = \int_{-\infty}^{\infty} x^2 \cdot |\psi(x)|^2 dx \quad (880)$ 

 $uv = \psi(\chi) \frac{\hbar}{-in} e^b$ 

 $uv = \psi(\chi) \frac{\hbar}{-in} e^{\frac{-ip\chi}{\hbar}}$ 

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Ampere's circuital law: Proof of

 $\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$ 

 $\frac{\partial \mathbf{D}}{\partial t} = \frac{\partial}{\partial t} (\varepsilon_0 \mathbf{E} + \mathbf{P})$ 

 $\nabla \times \frac{1}{\mu_0} \mathbf{B} = \nabla \times \mathbf{H} + \mathbf{J}_M$ 

 $\nabla \times \frac{1}{u_0} \mathbf{B} = \mathbf{J}_f + \frac{\partial}{\partial t} (\varepsilon_0 \mathbf{E} + \mathbf{P}) + \mathbf{J}_M$  (863)

 $\nabla \times \frac{1}{u_0} \mathbf{B} = \mathbf{J}_f + \frac{\partial}{\partial t} \varepsilon_0 \mathbf{E} + \frac{\partial}{\partial t} \mathbf{P} + \mathbf{J}_M \quad (864)$ 

 $\langle x \rangle = \int_{-\infty}^{\infty} x \cdot |\psi(x)|^2 dx$ 

 $\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} x^2 \cdot p(x) dx$ 

 $\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial}{\partial t} (\varepsilon_0 \mathbf{E} + \mathbf{P})$ 

equivalence 2

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#### **B.52 Uncertainty principle: Kennard** 1206 inequality proof part 3.2 1207 $(uv)\Big|^{\infty} = 0 - uv(-\infty)$ $\tilde{g}^*(p) \cdot \tilde{g}(p) = p^2 \varphi^*(p) \cdot \varphi(p)$ (898)1188 (912)1208 $\varphi^*(p) \cdot \varphi(p) = |\varphi(p)|^2$ (913)1209 $(uv)\Big|^{\infty} = 0$ (899) $p^2 \varphi^*(p) \cdot \varphi(p) = p^2 |\varphi(p)|^2$ (914)1210 **B.50 Uncertainty principle: Kennard** 1190 inequality proof part 2.5 $\tilde{q}^*(p) \cdot \tilde{q}(p) = p^2 |\varphi(p)|^2$ 1191 (915)1211 $I = (uv) \Big|_{\infty} - \int_{-\infty}^{\infty} v \frac{du}{dx} dx$ (900)1192 $|\tilde{a}(p)|^2 = p^2 |\varphi(p)|^2$ (916)1212 **B.53 Uncertainty principle: Kennard** 1213 inequality proof part 3.3 $(uv)\Big|^{\infty} = 0$ 1214 (901)1193 $\sigma_p^2 = \int_{-\infty}^{\infty} p^2 |\varphi(p)|^2 dp$ (917)1215 $I = -\int_{-\infty}^{\infty} v \frac{du}{dx} d\chi$ $|\tilde{q}(p)|^2 = p^2 |\varphi(p)|^2$ (902)(918)1194 1216 $\int_{-\infty}^{\infty} |\tilde{g}(p)|^2 dp = \int_{-\infty}^{\infty} |g(x)|^2 dx$ $\frac{du}{d\chi} = \frac{d\psi(\chi)}{d\gamma}$ (919)1217 (903)1195 $\sigma_p^2 = \int_0^\infty |\tilde{g}(p)|^2 dp$ (920)1218 $I = -\int_{-\infty}^{\infty} v \frac{d\psi(\chi)}{d\chi} d\chi$ (904)1196 $\sigma_p^2 = \int_{-\infty}^{\infty} |g(x)|^2 dx$ (921)1219 $I = \frac{\hbar}{in} \int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi} e^{\frac{-ip\chi}{\hbar}} d\chi$ (905)1197

**Uncertainty principle: Kennard** 

 $g(x) = \frac{\hbar}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\nu} e^{i(x-\chi)b} d\chi db$ 

 $g(x) = \frac{\hbar}{2\pi i} \int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi} \int_{-\infty}^{\infty} e^{i(x-\chi)b} db d\chi$ 

inequality proof part 2.9

 $\int_{-\infty}^{\infty} e^{i(x-\chi)b} db = 2\pi\delta(x-\chi)$ 

 $g(x) = \frac{\hbar}{2\pi i} \int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi} 2\pi \delta(x - \chi) d\chi$ 

 $g(x) = \frac{\hbar}{i} \int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi} \delta(x - \chi) d\chi$ 

 $g(x) = \frac{\hbar}{i} \left( \frac{d\psi(x)}{dx} \right)$ 

 $\sigma_n^2 = \langle g|g\rangle$ 

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**B.49** 

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**Uncertainty principle: Kennard** 

 $uv = \psi(\chi) \frac{\hbar}{-i\pi} e^{\frac{-ip\chi}{\hbar}}$ 

 $\psi(\infty) = 0$ 

 $uv(\infty) = \psi(\infty) \frac{\hbar}{-in} e^{\frac{-ip\infty}{\hbar}}$ 

 $uv(\infty) = 0 \frac{\hbar}{-in} e^{\frac{-ip\infty}{\hbar}}$ 

 $uv(\infty) = 0$ 

 $(uv)\Big|^{\infty} = uv(\infty) - uv(-\infty)$ 

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inequality proof part 2.4

1223	$\sigma_x^2 = \langle f   f  angle$	(923)	$f(x) = x \cdot \psi(x)$	(937)	1243
1224	$\sigma_p^2 = \langle g   g  angle$	(924)	$g(x) = (-i\hbar \frac{d}{dx}) \cdot \psi(x)$	(938)	1244
	p (SIS)	,	$\langle f g\rangle = \int_{-\infty}^{\infty} f^*(x)g(x)dx$	(939)	1245
1225	$\sigma_x^2 \sigma_p^2 = \langle f   f \rangle  \langle g   g \rangle$	(925)	$\int_{-\infty}^{\infty} (1+c)(1+c)$	<b>\\</b> 7	
1226	$\sigma_x^2 \sigma_p^2 \ge  raket{f g} ^2$	(926)	$\langle f g\rangle = \int_{-\infty}^{\infty} (x \cdot \psi^*(x))((-i\hbar \frac{d}{dx}) \cdot \psi(x))$	(940)	1246
1227 1228	B.55 Uncertainty principle: Kennard inequality proof part 4.3		$\langle f g\rangle = -i\hbar \int_{-\infty}^{\infty} x\psi^*(x) \frac{d\psi(x)}{dx} dx$	(941)	1247
1229	$ z ^2 = (\text{Re}(z))^2 + (\text{Im}(z))^2$	(927)	B.58 Uncertainty principle: Kennard inequality proof part 5.2		1248 1249
	(5. ( )) 2. (7. ( )) 2. (7. ( )) 2	(2.2.0)	$f(x) = x \cdot \psi(x)$	(942)	1250
1230	$(\text{Re}(z))^2 + (\text{Im}(z))^2 \ge (\text{Im}(z))^2$	(928)	$g^*(x) = \psi^*(x) \cdot (-i\hbar \frac{d}{dx})$	(943)	1251
1231	$ z ^2 \ge (\operatorname{Im}(z))^2$	(929)	$\langle g f\rangle = \int_{-\infty}^{\infty} g^*(x)f(x)dx$	(944)	1252
1232	$letz = rac{\langle f g angle - \langle g f angle}{2i}$	(930)	$\langle g f\rangle = \int_{-\infty}^{\infty} \psi^*(x)(-i\hbar \frac{d}{dx})(x\psi(x))dx$	(945)	1253
1233	$ z ^2 = \left(\frac{\langle f g\rangle - \langle g f\rangle}{2i}\right)^2$	(931)	$\langle g f\rangle = -i\hbar \int_{-\infty}^{\infty} \psi^*(x) \frac{d}{dx} (x\psi(x)) dx$	(946)	1254
1234	$ z ^2 \ge \left(\frac{\langle f g\rangle - \langle g f\rangle}{2i}\right)^2$	(932)	B.59 Uncertainty principle: Kennard inequality proof part 5.6		1255 1256
1235 1236	B.56 Uncertainty principle: Kennard inequality proof part 4.4		$\langle f g\rangle - \langle g f\rangle = i\hbar \int_{-\infty}^{\infty}  \psi(x) ^2 dx$	(947)	1257
1237	$\sigma_x^2 \sigma_p^2 \ge  \left< f   g \right> ^2$	(933)	$p(x) =  \psi(x) ^2$	(948)	1258

**B.57** 

Uncertainty principle: Kennard

 $f(x) = x \cdot \psi(x)$ 

inequality proof part 5.1

 $i\hbar \int_{-\infty}^{\infty} p(x)dx = i\hbar \int_{-\infty}^{\infty} |\psi(x)|^2 dx$ 

 $\int_{-\infty}^{\infty} p(x)dx = \frac{1}{i\hbar} (\langle f|g\rangle - \langle g|f\rangle)$ 

 $\int_{-\infty}^{\infty} p(x)dx = 1$ 

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**Uncertainty principle: Kennard** 

 $|z|^2 \ge \left(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i}\right)^2$ 

 $\sigma_x^2 \sigma_p^2 \ge |z|^2$ 

 $\sigma_x^2 \sigma_p^2 \ge (\frac{\langle f|g\rangle - \langle g|f\rangle}{2i})^2$ 

inequality proof part 4.1

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1264	from Schrodinger's equation 7		16		
1265	$i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t)$	$ \psi(x,t) \\ (952) $	$aa^{\dagger} - a^{\dagger}a = \frac{i}{\hbar}[\hat{p}, \hat{x}]$	(968)	1286
1266	V(x) = 0	(953)	$aa^{\dagger} - a^{\dagger}a = 1$	(969)	1287
1267	$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)$	(954)	$\left[a,a^{\dagger} ight]=1$	(970)	1288
1268	$\frac{\partial}{\partial t}\psi(x,t) = -\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)$	(955)	B.63 Creation and annihilation operator Ladder operators for the quantum harmonic oscillator part 1.6	l	1289 1290 1291
1269	$\hbar\omega\psi(x,t)=-\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)$	(956)	$\hbar\omega(\frac{1}{2} + \frac{1}{\sqrt{2}}(-\frac{d^2}{dq^2} + q^2)\frac{1}{\sqrt{2}}(\frac{d^2}{dq^2} + q^2))\psi(q)$	$= E\psi(q)$ (971)	1292
1270 1271	B.61 Particle in a box: Wavefunction at velocity as a function of particle m	_	$a = \frac{1}{\sqrt{2}} \left( \frac{d^2}{dq^2} + q^2 \right)$	(972)	1293
1272	from Schrödinger's equation 8 $-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{\hbar^2k^2}{2m}\psi(x,t)$	(957)	$a^{\dagger} = \frac{1}{\sqrt{2}}(-\frac{d^2}{dq^2} + q^2)$	(973)	1294
1274	$\frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m\hbar}$	(958)	$\hbar\omega(\frac{1}{2} + a^{\dagger}a)\psi(q) = E\psi(q)$	(974)	1295
1275	$\hbar\omega\psi(x,t) = -rac{\hbar}{2m}rac{\partial^2}{\partial x^2}\psi(x,t)$	(959)	$E = \hbar\omega(a^{\dagger}a + \frac{1}{2})$	(975)	1296
1276	$\hbar\omega = rac{p^2}{2m}$	(960)	B.64 Creation and annihilation operator Ladder operators for the quantum harmonic oscillator part 2		1297 1298 1299
1277	$\omega = rac{p^2}{2m\hbar}$	(961)	[q,p] = qp - pq	(976)	1300
1278 1279	B.62 Quantum harmonic oscillator: La operator method 4	ıdder	$p = -i\frac{d}{dq}$	(977)	1301
1280	$aa^{\dagger} - a^{\dagger}a = \frac{i}{\hbar}(\hat{p}\hat{x} - \hat{x}\hat{p})$	(962)	$[q, p] = q(-i\frac{d}{dq}) - (-i\frac{d}{dq})q$	(978)	1302
1281	$[\hat{p},\hat{x}]=\hat{p}\hat{x}-\hat{x}\hat{p}$ $[\hat{p},\hat{x}]=-i\hbar$	(963) (964)	$[q,p] = -iq\frac{d}{dq} + i\frac{d}{dq}q$	(979)	1303
1283	$[p,x] = -in$ $-i \cdot i = 1$	(965)	$[q,p]f(q) = -iq\frac{d}{dq}f(q) + i\frac{d}{dq}(qf(q))$	(980)	1304
1284	$rac{i}{\hbar}[\hat{p},\hat{x}]=rac{i}{\hbar}(-i\hbar)$	(966)	[q,p]f(q) = if(q)	(981)	1305
		3	1		

 $\frac{i}{\hbar}[\hat{p},\hat{x}] = 1$ 

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**B.60** Particle in a box: Wavefunction angular

from Schrödinger's equation 7

velocity as a function of particle mass

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1309		$a = \frac{1}{\sqrt{2}}(\frac{d}{dq} + q)$	(982)	$[H,a] = -\hbar\omega(aa^{\dagger} - a^{\dagger}a)a$	(999)	1335
1310		$p = -i\frac{d}{dq}$	(983)	$\left[a,a^{\dagger}\right]=aa^{\dagger}-a^{\dagger}a$	(1000)	1336
1311		$ip = -\frac{d}{dq}$	(984)	$-\hbar\omega(aa^{\dagger}-a^{\dagger}a)a = -\hbar\omega\Big[a,a^{\dagger}\Big]a$	(1001)	1337
1312		$a = \frac{1}{\sqrt{2}}(-\frac{d}{dq} + q)$	(985)		(1001)	.007
1313		$a = \frac{1}{\sqrt{2}}(ip + q)$	(986)	$-\hbar\omega\Big[a,a^{\dagger}\Big]a = -\hbar\omega a\Big[a,a^{\dagger}\Big]$	(1002)	1338
1314	<b>B.66</b>	Creation and annihilation opera	ators:			
1315 1316		Ladder operators for the quant harmonic oscillator part 3.2	um	$[H,a] = -\hbar\omega a \Big[a,a^\dagger\Big]$	(1003)	1339
1317		$a^{\dagger} = \frac{1}{\sqrt{2}}(-\frac{d}{dq} + q)$	(987)	$[H,a] = -\hbar\omega a$	(1004)	1340
1318		$p = -i\frac{d}{dq}$	(988)	B.69 Creation and annihilation opera Ladder operators for the quant		1341 1342
		. 1		harmonic oscillator part 5.2		1343
1319		$a^{\dagger} = \frac{1}{\sqrt{2}}(-ip + q)$	(989)	$\left[H,a^{\dagger}\right]=\hbar\omega a^{\dagger}(aa^{\dagger}-a^{\dagger}a)$	(1005)	1344
1320	<b>B.67</b>	Creation and annihilation opera				
1321		Ladder operators for the quant	um	$\left[a,a^{\dagger} ight]=aa^{\dagger}-a^{\dagger}a$	(1006)	1345
1322		harmonic oscillator part 3.6	(000)		(1000)	
1323		$aa^{\dagger} - a^{\dagger}a = i(pq - qp)$	(990)	$\left[a,a^{\dagger} ight]=1$	(1007)	1346
1324		[p,q] = pq - qp	(991)	B.70 Heisenberg picture: time evolut	ion 4	1347
1325		[p,q]=-i	(992)	$\frac{d}{dt}\hat{x}(t) = \frac{i}{\hbar}(\hat{H}e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}} - e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}}$	$-irac{\hat{H}t}{\hbar}\hat{H})$	
1326		$-i \cdot i = 1$	(993)		(1008)	1348
				$\hat{x}(t) = e^{i\frac{\hat{H}t}{\hbar}} \hat{x}e^{-i\frac{\hat{H}t}{\hbar}}$	(1009)	1349
1327		i(pq - qp) = i[p, q]	(994)			
1328		$i[p,q] = -i^2$	(995)	$\hat{H}\hat{x}(t) - \hat{x}(t)\hat{H} = \hat{H}e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}} - e^{i\frac{\hat{H}t}{\hbar}}.$	$\hat{x}e^{-i\frac{\hat{H}t}{\hbar}}\hat{H}$ (1010)	1350
1329		$-i^2 = 1$	(996)		(1010)	1330
1330		$aa^{\dagger} - a^{\dagger}a = 1$	(997)	$\left[\hat{H}, \hat{x}(t)\right] = \hat{H}\hat{x}(t) - \hat{x}(t)\hat{H}$	(1011)	1351
ii				$\frac{d}{dt}\hat{x}(t) = \frac{i}{\hbar} \left[ \hat{H}, \hat{x}(t) \right]$		

**B.65** Creation and annihilation operators:

harmonic oscillator part 3.1

**Ladder operators for the quantum** 

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1308

**B.68** Creation and annihilation operators:

harmonic oscillator part 4.2

Ladder operators for the quantum

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1333

#### Heisenberg picture: momentum 1353 evolution 4 1354 $[\hat{p}(t_1), \hat{p}(t_2)] = m\omega \hat{p}_0 \hat{x}_0 \sin(\omega t_2 - \omega t_1) - m\omega \hat{x}_0 \hat{p}_0 \sin(\omega t_2 - \omega t_1)$ (1013) $\hat{p}(t) = A\cos(\omega t) + B\sin(\omega t)$ (1027) $\frac{d}{dt}\cos(\omega t) = -\omega\sin(\omega t)$ (1014) $[\hat{p}(t_1), \hat{p}(t_2)] = (m\omega \hat{p}_0 \hat{x}_0 - m\omega \hat{x}_0 \hat{p}_0) \sin(\omega t_2 - \omega t_1)$ 1374 $\frac{d}{dt}\sin(\omega t) = \omega\cos(\omega t)$ (1015) $[\hat{p}(t_1), \hat{p}(t_2)] = i\hbar m\omega \sin(\omega t_2 - \omega t_1)$ 1375 **B.74** Vacuum Rabi Oscillations: excited state 1376 $\frac{d\hat{p}(t)}{dt} = \frac{d}{dt}(A\cos(\omega t)) + \frac{d}{dt}(B\sin(\omega t))$ (1016) 1358 probability 1377 $|\Psi(t)\rangle = \cos\left(\frac{\Omega t}{2}\right)|e,0\rangle - i\sin\left(\frac{\Omega t}{2}\right)|g,1\rangle$ $\frac{d\hat{p}(t)}{dt} = A\frac{d}{dt}\cos(\omega t) + B\frac{d}{dt}\sin(\omega t)$ 1378 1359 $\langle e, 0 | \Psi(t) \rangle = \cos\left(\frac{\Omega t}{2}\right) \langle e, 0 | e, 0 \rangle - i \sin\left(\frac{\Omega t}{2}\right) \langle e, 0 | g, 1 \rangle$ (1031) $\frac{d\hat{p}(t)}{dt} = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$ 1379 1360 $\langle e, 0|e, 0\rangle = 1$ (1032)1380 **B.72** Heisenberg picture: position 1361 commutator 4 1362 $\langle e, 0|q, 1\rangle = 0$ $[\hat{x}(t_1), \hat{x}(t_2)] = \frac{1}{\omega m} (\hat{x}_0 \hat{p}_0 - \hat{p}_0 \hat{x}_0) \sin(\omega t_2 - \omega t_1)$ (1033)1381 1363 $\langle e, 0 | \Psi(t) \rangle = \cos\left(\frac{\Omega t}{2}\right)$ (1034)1382 $\hat{x}_0 \hat{p}_0 - \hat{p}_0 \hat{x}_0 = [\hat{x}_0, \hat{p}_0]$ (1020)1364 $P_e(t) = |\langle e, 0 | \Psi(t) \rangle|^2$ (1035)1383 $[\hat{x}(t_1), \hat{x}(t_2)] = \frac{1}{\omega_0} [\hat{x}_0, \hat{p}_0] \sin(\omega t_2 - \omega t_1)$ $P_e(t) = \cos^2(\frac{\Omega t}{2})$ (1036)1384 (1021)1365 **Vacuum Rabi Oscillations: ground state** 1385 $[\hat{x}_0, \hat{p}_0] = i\hbar$ (1022)1366 probability 2 1386 $P_g(t) = |\cos\left(\frac{\Omega t}{2}\right)\langle g, 1|e, 0\rangle - i\sin\left(\frac{\Omega t}{2}\right)\langle g, 1|g, 1\rangle|^2$ (1037) $[\hat{x}(t_1), \hat{x}(t_2)] = \frac{i\hbar}{\omega m} \sin(\omega t_2 - \omega t_1)$ 1387 (1023) $\langle q, 1|e, 0\rangle = 0$ (1038)1388 Heisenberg picture: momentum 1368 commutator 3 1369 $\langle g, 1|g, 1\rangle = 1$ (1039)1389 $[\hat{p}(t_1), \hat{p}(t_2)] = m\omega \hat{p}_0 \hat{x}_0 \sin(\omega t_2 - \omega t_1) + m\omega \hat{x}_0 \hat{p}_0 \sin(\omega t_1 - \omega t_2)$ 1370 $P_g(t) = |\cos\left(\frac{\Omega t}{2}\right) \cdot 0 - i\sin\left(\frac{\Omega t}{2}\right) \cdot 1|^2$ (1040) 1390 $\sin(\omega t_1 - \omega t_2) = -\sin(\omega t_2 - \omega t_1)$ (1025)1371

 $m\omega \hat{x}_0\hat{p}_0\sin(\omega t_1-\omega t_2)=-m\omega \hat{x}_0\hat{p}_0\sin(\omega t_2-\omega t_1)$ 

1372

 $P_g(t) = |-i\sin\left(\frac{\Omega t}{2}\right)|^2$ 

 $P_g(t) = \sin^2(\frac{\Omega t}{2})$ 

(1041)

(1042)

1391

#### $\frac{dg_{\varepsilon}}{d\varepsilon} = \eta(x)when\varepsilon = 0.$ $\hat{X} | x' \rangle = x' | x' \rangle$ (1049)(1065)1401 1421 **Euler-Lagrange equation: Derivation** 1422 $\langle x | \hat{X} | x' \rangle = x' \langle x | x' \rangle$ (1050)1402 $J = \int_{0}^{b} L(x, f(x), f'(x))$ (1066)1423 $\langle x | \hat{X} | x' \rangle = x' \delta(x - x')$ (1051) $g_{\varepsilon}(x) = f(x) + \varepsilon \eta(x)$ (1067)1424 $\langle \Psi | x \rangle = \langle x | \Psi \rangle^{\dagger}$ (1052)1404 $J_{\varepsilon} = \int_{-\infty}^{b} L(x, g_{\varepsilon}(x), g'_{\varepsilon}(x)) dx$ (1068)1425 $\langle \Psi | x \rangle \langle x | \hat{X} | x' \rangle = \langle x | \Psi \rangle^{\dagger} x' \delta(x - x') \quad (1053)$ 1405 $L_{\varepsilon}(x) = L(x, q_{\varepsilon}(x), q'_{\varepsilon}(x))$ (1069)1426 $\frac{dJ_{\varepsilon}}{d\varepsilon} = \frac{d}{d\varepsilon} \int_{-\infty}^{b} L_{\varepsilon}(x) dx$ $\left\langle \hat{X} \right\rangle_{\Psi} = \int \int \left\langle x | \Psi \right\rangle^{\dagger} x' \delta(x - x') \left\langle x' | \Psi \right\rangle dx dx'$ (1070)1427 (1054)1406 $\frac{dJ_{\varepsilon}}{d\varepsilon} = \int_{\varepsilon}^{b} \frac{dL_{\varepsilon}}{d\varepsilon} dx$ (1071)1428 **Expectation value: integral expression 3** 1407 $\left\langle \hat{X} \right\rangle_{\Psi} = \int \int \left\langle x | \Psi \right\rangle^{\dagger} x' \delta(x - x') \left\langle x' | \Psi \right\rangle dx dx'$ **Euler-Lagrange equation: Derivation 4** 1429 $\frac{dv}{dx} = \eta'(x)$ (1055)1408 (1072)1430 $v = \int \frac{dv}{dx} dx$ $\int \langle x|\Psi\rangle^{\dagger} x' \delta(x-x') \langle x'|\Psi\rangle dx' = \langle x|\Psi\rangle^{\dagger} x \langle x|\Psi\rangle$ (1073)(1056)1409 $v = \int \eta'(x) dx$ (1074)1432 $\left\langle \hat{X} \right\rangle_{\Psi} = \int \left\langle x | \Psi \right\rangle^{\dagger} x \left\langle x | \Psi \right\rangle dx$ (1057)1410 $v = \eta(x)$ (1075)1433

 $\langle x|\Psi\rangle = \Psi(x)$ 

 $\langle x|\Psi\rangle^{\dagger} = \Psi^*(x)$ 

 $\langle x|\Psi\rangle^{\dagger} x \langle x|\Psi\rangle = x|\Psi(x)|^2$ 

 $\left\langle \hat{X} \right\rangle_{\Psi} = \int x |\Psi(x)|^2 dx$ 

**Euler-lagrange equation: Full** 

derivative of the perturbation

Lagrangian with respect to  $\varepsilon$  2

 $\frac{d}{d\varepsilon}(\varepsilon\eta(x)) = \eta(x) + \varepsilon \frac{d\eta(x)}{d\varepsilon}$ 

 $\frac{dg_{\varepsilon}}{d\varepsilon} = \frac{d}{d\varepsilon}(\varepsilon \eta(x))$ 

 $\frac{dg_{\varepsilon}}{d\varepsilon} = \eta(x) + \varepsilon \frac{d\eta(x)}{d\varepsilon}$ 

(1058)

(1059)

(1060)

(1061)

(1062)

(1063)

(1064)

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**Expectation value: integral expression** 

 $\langle \Psi | \, \mathbb{I}\hat{X}\mathbb{I} \, | \Psi \rangle = \int \int \langle \Psi | x \rangle \, \langle x | \, \hat{X} \, \big| x' \big\rangle \, \big\langle x' \big| \Psi \big\rangle \, dx dx' \, \, \, \, \mathbf{B.79}$ 

(1043)

(1044)

(1045)

(1048)

 $\langle \hat{X} \rangle_{_{\mathcal{M}}} = \langle \Psi | \hat{X} | \Psi \rangle$ 

 $\hat{X} = \|\hat{X}\|$ 

 $\left\langle \hat{X} \right\rangle_{\Psi} = \int \int \left\langle \Psi | x \right\rangle \left\langle x | \hat{X} | x' \right\rangle \left\langle x' | \Psi \right\rangle dx dx'$ 

 $\left\langle \hat{X} \right\rangle_{\Psi} = \int \int \left\langle \Psi | x \right\rangle \left\langle x | \hat{X} | x' \right\rangle \left\langle x' | \Psi \right\rangle dx dx'$ 

**Expectation value: integral expression 2** 

 $\left\langle \Psi\right|\hat{X}\left|\Psi\right\rangle =\left\langle \Psi\right|\left|\hat{X}\right|\left|\Psi\right\rangle$ 

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#### $\frac{d}{dx}(y'(1+y'^2)^{-\frac{1}{2}}) = C$ (1099) $\left(\frac{\partial L}{\partial f'}\eta(x)\right)\Big|^b = 0$ 1462 (1081)1440 **Euler-Lagrange equation: Derivation 6 B.83** 1441 $\int \frac{d}{dx} (y'(1+y'^2)^{-\frac{1}{2}}) dx = \int C dx$ (1100)1463 $I = \int_{-a}^{b} \frac{\partial L}{\partial f'} \eta'(x) dx$ (1082)1442 $I = (uv) \Big|_{b}^{b} - \int_{a}^{b} v \frac{du}{dx} dx$ $\int \frac{d}{dx} (y'(1+y'^2)^{-\frac{1}{2}}) dx = C$ (1083)(1101)1464 1443 **B.86** Euler-Lagrange equation: Straight line $(uv)\Big|^b = 0$ 1465 (1084) $\frac{dy}{dx} = C(1 - C^2)^{-1/2}$ (1102)1467 $I = -\int_{-\infty}^{b} v \frac{du}{dx} dx$ 1445 (1085) $C(1 - C^2)^{-1/2} = A$ (1103)1468 $\frac{\partial L}{\partial f'}\eta'(x) = v\frac{du}{dx}$ (1086)1446 $\frac{dy}{dx} = A$ (1104)1469 $\eta(x) = v$ (1087)1447 $\int dy = \int Adx$ $\frac{d}{dx}\frac{\partial L}{\partial f'} = \frac{du}{dx}$ (1105)1470 (1088)1448 $I = -\int^{b} \eta(x) \frac{d}{dx} \frac{\partial L}{\partial f'} dx$ y = Ax + C(1106)1471 (1089)**B.87 Escape velocity** 1472 **Euler-Lagrange equation: Straight line** 1450 $F = \frac{GMm}{r^2}$ (1107)1473 $S = \int^b ds$ (1090)1451 dW = Fdr(1108)1474 $ds = \sqrt{dx^2 + dy^2}$ (1091)1452 $dW = \frac{GMm}{r^2}dr$ $ds = dx\sqrt{1 + v'^2}$ (1109)1475 (1092)1453 $S = \int^b dx \sqrt{1 + y'^2}$ $W = \int dW$ (1110)(1093)1476

**Euler-Lagrange equation: Straight line** 

 $\frac{dL}{du} - \frac{d}{dx}\frac{dL}{du'} = 0$ 

 $\frac{dL}{du} = 0$ 

 $-\frac{d}{dx}\frac{dL}{dv'} = 0$ 

 $\frac{dL}{du'} = C$ 

 $W = \int_{r_0}^{\infty} \frac{GMm}{r^2} dr$ 

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(1076)

(1077)

(1078)

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(1096)

(1097)

(1098)

**Euler-Lagrange equation: Derivation 5** 

 $u = \frac{\partial L}{\partial f'}$ 

 $v = \eta(x)$ 

n(a) = 0

 $uv = \frac{\partial L}{\partial f'} \eta(x)$ 

 $(uv)\Big|^b = \left(\frac{\partial L}{\partial f'}\eta(x)\right)\Big|^b$ 

 $S = \int^b \sqrt{(1 + y'^2)} dx$ 

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(1094)

### B.88 Escape velocity 2

$$W = \int_{r_0}^{\infty} \frac{GMm}{r^2} dr \tag{1112}$$

$$F = \frac{GMm}{r^2} \tag{1113}$$

$$W = \int_{r_0}^{\infty} F dr \tag{1114}$$

$$W = Fr_0 \tag{1115}$$

$$W = mgr_0 (1116)$$

### B.89 Escape velocity 3

$$W = mgr_0 (1117)$$

$$E = \frac{1}{2}mv_{esc}^2 \tag{1118}$$

$$2E = mv_{esc}^2 \tag{1119}$$

$$2mgr_0 = mv_{esc}^2 (1120)$$

$$v_{esc}^2 = 2gr_0 (1121)$$

$$v_{esc} = \sqrt{2gr_0} \tag{1122}$$

### B.90 Snell's law: from Fermat's principle 2

$$\frac{dT}{dx} = \frac{x}{v_1(x^2 + a^2)^{\frac{1}{2}}} + \frac{x - l}{v_2((x - l)^2 + b^2)^{\frac{1}{2}}}$$
(1123)

$$\frac{x}{(x^2 + a^2)^{\frac{1}{2}}} = \sin \theta_1 \tag{1124}$$

$$\frac{l-x}{((x-l)^2+b^2)^{\frac{1}{2}}} = \sin\theta_2 \tag{1125}$$

$$\frac{dT}{dx} = \frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} \tag{1126}$$

$$\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0 \tag{1127}$$

### B.91 Snell's law: from Fermat's principle 3

$$\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0 \tag{1128}$$

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2} \tag{1129}$$

$$\frac{1}{v_1} = \frac{n_1}{c} \tag{1130}$$

$$\frac{\sin \theta_1}{\frac{c}{n_1}} = \frac{\sin \theta_2}{v_2} \tag{1131}$$

$$n_1 \sin \theta_1 = v_2 \sin \theta_2 \tag{1132}$$

$$\frac{1}{v_2} = \frac{n_2}{c} \tag{1133}$$

$$n_1 \sin \theta_1 = \frac{c}{n_2} \sin \theta_2 \tag{1134}$$

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \tag{1135}$$

## B.92 Wave equation: plane wave eigenmodes 2

$$\frac{\partial^2 u(x,t)}{\partial t^2} = (-i\omega)\frac{\partial}{\partial t}(e^{-i\omega t}f(x))$$
 (1136)

$$i \cdot i = -1 \tag{1137}$$

$$-i\omega \cdot -i\omega = \omega^2 \tag{1138}$$

$$\omega^2 e^{-i\omega t} f(x) = \omega^2 e^{-i\omega t} f(x) \tag{1139}$$

$$-\omega^2 e^{-i\omega t} f(x) = -\omega^2 e^{-i\omega t} f(x) \qquad (1140)$$

$$-\omega^2 e^{-i\omega t} f(x) = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$
 (1141)

## B.93 Wave equation: plane wave eigenmodes 4 1514

$$u(x,t) = Ae^{-i(kx - \omega t)} + Be^{i(kx - \omega t)}$$
 (1142)

(1126) 
$$u(x,t) = \int_{-\infty}^{\infty} s(\omega) (Ae^{-i(kx-\omega t)} + Be^{i(kx-\omega t)}) d\omega$$
(1143)

$$s_{+}(\omega) = As(\omega) \tag{1144}$$

 $\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u(x,t)}{\partial x^2}$ 

 $\frac{\partial^2}{\partial t^2} u(x+h,t) = \frac{k}{m} \left( u(x+2h,t) - 2u(x+h,t) + u(x,t) \right)$ 

(1175)

(1176)

1555

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1538

equation 2 1580 C.1 Gauss' law: equivalence between 1558  $\nabla \cdot (\phi \nabla \phi) = (\nabla \phi)^2 + \phi \nabla^2 \phi$ differential and integral forms (1192)1581 1559  $\oint_C \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon_0}$ (1177)1560  $\int_{V} \nabla \cdot (\phi \nabla \phi) dV = \int_{V} ((\nabla \phi)^{2} + \phi \nabla^{2} \phi) dV$  $Q = \iiint_V \rho dV$ (1178)1561 1582  $\frac{Q}{\varepsilon_0} = \frac{\iiint_V \rho dV}{\varepsilon_0}$  $\int_{V} \nabla \cdot (\phi \nabla \phi) dV = \int_{S} \phi \nabla \phi \cdot d\mathbf{S}$ (1179)1583  $\iint_{S} \mathbf{E} \cdot d\mathbf{A} = \iiint_{V} \frac{\rho}{\varepsilon_{0}} dV$ (1180)1563  $\int_{\mathcal{S}} \phi \nabla \phi \cdot d\mathbf{S} = \int_{V} ((\nabla \phi)^2 + \phi \nabla^2 \phi) dV \quad (1195)$ 1584  $\iiint_{V} \nabla \cdot \mathbf{E} dV = \iiint_{V} \frac{\rho}{\epsilon_{0}} dV$ (1181)1564  $\int_{S} \phi \nabla \phi \cdot d\mathbf{S} = \int_{V} (\nabla \phi)^{2} dV$ (1196)1585 C.2 Gauss' law: Equivalence of total and free 1565 charge statements 1566 Uniqueness theorem for Poisson's 1586  $\rho_b = -\nabla \cdot \mathbf{P}$ 1567 (1182)equation 6 1587  $\frac{\partial \phi}{\partial x} = 0$ (1197)1588  $\rho = -\rho_b$ (1183)1568  $\int \frac{\partial \phi}{\partial r} dr = \int 0 dr$ (1198)1589  $\rho = \nabla \cdot \mathbf{P}$ (1184) $\phi = C_1 r + C_2$ (1199)1590  $\rho = \nabla \cdot (\mathbf{D} - \mathbf{P})$ (1185)**Uniqueness theorem for Poisson's**  $\phi = C_1 - C_2$ (1200)1591 equation  $\nabla^2 \phi_1 = -\frac{\rho_f}{\varepsilon_0}$ (1186)Uniqueness theorem for Poisson's 1573 1592 equation 7 1593  $\phi = C_1 - C_2$ (1201)1594  $\rho_f = 0$ (1187)1574  $-\frac{\rho_f}{\varepsilon_0} = 0$  $\phi_1 = C_1$ (1202)1595 1575 (1188) $\phi_2 = C_2$  $\nabla^2 \phi_1 = 0$ (1203)1596 (1189)1576  $\phi_2 - \phi_1 = C_2 - C_1$  $\nabla^2 \phi = \nabla^2 \phi_1$ (1190)(1204)1597 1577  $\nabla^2 \phi = 0$  $\phi_2 - \phi_1 = C$ (1191)(1205)1578 1598

**Uniqueness theorem for Poisson's** 

1579

S=2 (two premises removed)

#### C.7 Poisson's equation: Newtonian gravity

$$\nabla \cdot \mathbf{g} = -4\pi G \rho \tag{1206}$$

$$\mathbf{g} = -\nabla\phi \tag{1207}$$

$$\nabla \cdot (-\nabla \phi) = -4\pi G \rho \tag{1208}$$

$$-\nabla^2 \phi = -4\pi G\rho \tag{1209}$$

$$\nabla^2 \phi = 4\pi G \rho \tag{1210}$$

### C.8 Poisson's equation: Gravitational potential from Poisson's equation 2

$$\int_{V} \nabla \cdot \nabla \phi dV = \int_{V} 4\pi G \rho dV \qquad (1211)$$

$$\nabla \cdot \nabla \phi = 4\pi G \rho \tag{1212}$$

$$\int_{V} \nabla \cdot \nabla \phi dV = \int_{S} \nabla \phi \cdot d\mathbf{S}$$
 (1213)

$$4\pi G \int_{V} \rho dV = 4\pi Gm \tag{1214}$$

$$\int_{S} \nabla \phi \cdot d\mathbf{S} = 4\pi G m \tag{1215}$$

### C.9 Poisson's equation: Gravitational potential from Poisson's equation 6

$$\int_{S} \frac{\partial \phi}{\partial r} dS = \int_{0}^{2\pi} \int_{0}^{\pi} \frac{\partial \phi}{\partial r} r^{2} \sin \theta d\theta d\varphi \quad (1216)$$

$$\int_0^{\pi} \sin \theta d\theta = 2 \tag{1217}$$

$$\int_0^{2\pi} d\varphi = 2\pi \tag{1218}$$

$$\int_{0}^{2\pi} \int_{0}^{\pi} d\theta d\varphi = 4\pi \tag{1219}$$

$$\int_{S} \frac{\partial \phi}{\partial r} dS = 4\pi \frac{\partial \phi}{\partial r} r^2 \tag{1220}$$

### C.10 Poisson's equation: Gravitational potential from Poisson's equation 8

$$\frac{\partial \phi}{\partial r} = \frac{Gm}{r^2} \tag{1221}$$

$$\int \frac{\partial \phi}{\partial r} dr = \int \frac{Gm}{r^2} dr \qquad (1222)$$

$$\phi(r) - \phi(c) = -Gm \int \frac{1}{r} dr \qquad (1223)$$

$$\phi(r) - \phi(c) = -Gm \ln |r| + C \qquad (1224)$$

$$\phi(r) = -Gm \ln|r| + C + \phi(c)$$
 (1225)

$$\phi(r) = \frac{-Gm}{r} + C, where C is the constant of integration.$$
(1226) 1626

### C.11 Poisson's equation: Electrostatics

$$\nabla \cdot \mathbf{D} = \rho_f \tag{1227}$$

$$\mathbf{D} = \varepsilon \nabla \phi \tag{1228}$$

$$\nabla \cdot (\varepsilon \nabla \phi) = \rho_f \tag{1229}$$

$$\varepsilon \nabla^2 \phi = -\rho_f \tag{1230}$$

$$\nabla^2 \phi = -\frac{\rho_f}{\varepsilon} \tag{1231}$$

# C.12 Poisson's equation: Electrostatic potential from Poisson's equation

$$\nabla^2 \phi = -\frac{\rho_f}{\varepsilon} \tag{1232}$$

$$\int_{V} \nabla^{2} \phi dV = -\frac{1}{\varepsilon} \int_{V} \rho_{f} dV \qquad (1233)$$

$$\int_{V} \nabla \cdot \nabla \phi dV = -\frac{1}{\varepsilon} \int_{V} \rho_{f} dV \qquad (1234)$$

$$\int_{V} \nabla \cdot \nabla \phi dV = -\frac{Q}{\varepsilon} \tag{1235}$$

 $\int_{\mathcal{C}} \nabla \phi \cdot d\mathbf{S} = -\frac{Q}{\varepsilon}$ (1239)1644  $\mathbf{F} = \iiint (\rho \mathbf{E} + \mathbf{J} \times \mathbf{B}) dV$ (1255)1666  $\int_{S} \frac{\partial \phi}{\partial r} dS = -\frac{Q}{S}$ Lorentz force: Lorentz force in terms of 1667 (1240)1645 potentials 1668  $\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$ Poisson's equation: Electrostatic 1646 (1256)1669 potential from Poisson's equation 4  $\int_{-\infty}^{r} \frac{\partial \phi}{\partial r} dr = \int_{-\infty}^{r} -\frac{Q}{4\pi \varepsilon r^2} dr$  $q\mathbf{E} = q(-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t})$ 1648 (1241)(1257)1670  $\frac{\partial \phi}{\partial r} = -\frac{Q}{4\pi \varepsilon r^2}$  $q\mathbf{E} + q\nabla(\mathbf{v} \cdot \mathbf{A}) = q(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A}))$ (1242)1649 1671  $\phi(r) - \phi(\infty) = -\frac{Q}{4\pi\varepsilon} \int_{-\infty}^{r} \frac{1}{r^2} dr$  $q\mathbf{E} + q\nabla(\mathbf{v}\cdot\mathbf{A}) - q(\mathbf{v}\cdot\nabla)\mathbf{A} = q(-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} + \nabla(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\mathbf{A})$ (1259)
1672 (1243)1650  $\phi(r) - \phi(\infty) = \frac{Q}{4\pi\varepsilon} \left[\frac{1}{r}\right]^r$ (1244)1651  $\mathbf{F} = q(-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} + \nabla(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\mathbf{A}) \quad (1260)$ 1673  $\phi(r) - \phi(\infty) = \frac{Q}{4\pi\varepsilon r}$ 1652 (1245)C.18 **Lorentz force: Potential energy** 1674 derivation from scalar potential 3 1675  $U = q \int_{-r}^{r} \nabla \phi \cdot d\mathbf{r}$  $\phi(r) = \frac{Q}{4\pi\epsilon r}$ (1261)(1246)1676 1653 **Lorentz force: continuous charge**  $\nabla \phi = \frac{d\phi}{d\pi}$ 1654 (1262)1677 distribution 1655  $\frac{d\mathbf{F}}{d\mathbf{V}} = \frac{dq}{d\mathbf{V}}(\mathbf{E} + \mathbf{v} \times \mathbf{B})$  $U = q \int_{-r}^{r} \frac{d\phi}{dr} \cdot d\mathbf{r}$ (1247)1656 (1263)1678  $\frac{d\mathbf{F}}{d\mathbf{V}} = \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B})$  $U = q[\phi(r) - \phi(\infty)]$ (1264)(1248)1679 1657  $U = a\phi(r) - a\phi(\infty)$ (1265) $\mathbf{F} = \rho V(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ 

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**Lorentz force: continuous charge** 

 $\mathbf{f} = \rho \mathbf{E} + \rho \mathbf{v} \times \mathbf{B}$ 

 $J = \rho v$ 

 $\mathbf{f} = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}$ 

 $\mathbf{F} = \iiint \mathbf{f} dV$ 

distribution 2

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**Poisson's equation: Electrostatic** 

 $\int_{\mathbb{R}^2} \nabla \cdot \nabla \phi dV = -\frac{Q}{\epsilon}$ 

 $\nabla \cdot \nabla \phi = \nabla^2 \phi$ 

 $\int_{\mathcal{M}} \nabla^2 \phi dV = -\frac{Q}{\varepsilon}$ 

potential from Poisson's equation 2

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 $assuming \phi(\infty) = 0, then U = q\phi(r)$ 

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 $\mathbf{f} = \rho \mathbf{E} + \rho \mathbf{v} \times \mathbf{B}$ 

1682 1683	C.19	Laplace equation: Analytic func	etions	C.23	Laplace equation: Analytic function (v) 2	ctions	1707 1708
1684		$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$	(1267)		$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$	(1284)	1709
1685		$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right)$	(1268)		$\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}\frac{\partial v}{\partial y}$	(1285)	1710
1686		$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial y} \right)$	(1269)		$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2}$	(1286)	1711
1687		$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$	(1270)	C.24	Laplace equation: Analytic function (v) 3	etions	1712 1713
1688 1689	C.20	Laplace equation: Analytic function (u) 2	etions		$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$	(1287)	1714
1690		$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$	(1271)		$\frac{\partial^2 v}{\partial u^2} = \frac{\partial^2 u}{\partial x \partial y}$	(1288)	1715
1691		$\frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} \right) = \frac{\partial}{\partial y} \left( -\frac{\partial u}{\partial y} \right)$	(1272)		$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial u^2}$	(1289)	1716
1692		$\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$ $\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial y^2}$	(1273)		$\nabla^2 v = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y}$	(1290)	1717
1693		$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$	(1274)		$\nabla^2 v = 0$	(1291)	1718
1694 1695	C.21	Laplace equation: Analytic func (u) 3	etions	C.25	Laplace equation: Electrostatic		1719
1696		$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$	(1275)	0,20	$\mathbf{E} = (u, v)$	(1292)	1720
1697		$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x}$	(1276)		$\frac{\partial \mathbf{E}}{\partial x} = \frac{\partial u}{\partial x}$	(1293)	1721
1698		$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y}$	(1277)		$rac{\partial \mathbf{E}}{\partial y} = rac{\partial v}{\partial y}$	(1294)	1722
1699		$\frac{\partial^2 u}{\partial u^2} - \frac{\partial^2 u}{\partial x^2} = 0$	(1278)		$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \rho$	(1295)	1723
		- <i>9</i>		C.26	Laplace equation: Electrostatic	s 2	1724
1700		$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$	(1279)		$\frac{\partial \phi}{\partial x} = -u$	(1296)	1725
1701	C 22	$\nabla^2 u = 0$	(1280)		$\frac{\partial^2 \phi}{\partial x^2} = -\frac{\partial u}{\partial x}$	(1297)	1726
1702 1703	C.22	<b>Laplace equation: Analytic func</b> (v)	ctions				
1704		$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$	(1281)		$\frac{\partial^2 \phi}{\partial y^2} = 0$	(1298)	1727
1705		$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial}{\partial x} \frac{\partial u}{\partial y}$	(1282)		$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\frac{\partial u}{\partial x} + 0$	(1299)	1728
1706		$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y}$	(1283)		$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\rho$	(1300)	1729

1737	derivation from vector potential	$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\dot{\mathbf{r}}\cdot\mathbf{A} - q\phi \tag{1322}$	1760
1738	$\mathbf{F} = q(\nabla(\mathbf{v} \cdot \mathbf{A}) - \frac{d\mathbf{A}}{dt}) \tag{1306}$	2	
		C.32 Lorentz force: Derivation of classical Lagrangian of EM field 2	1761 1762
1739	$\mathbf{F} = -\nabla U \tag{1307}$		
	$I = \int_{-r}^{r} \mathbf{F} \cdot \mathbf{I} $ (1209)	$L = \frac{m}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + q\dot{\mathbf{r}} \cdot \mathbf{A} - q\phi \tag{1323}$	1763
1740	$U = -\int_{\infty}^{T} \mathbf{F} \cdot d\mathbf{r} \tag{1308}$	$\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z}) \tag{1324}$	1764
1741	$U = -q \int_{\infty}^{r} \nabla(\mathbf{v} \cdot \mathbf{A}) \cdot d\mathbf{r} $ (1309)	$\mathbf{I} = (x, y, z) \tag{1324}$	1704
1742	C.29 Lorentz force: Potential energy	$\mathbf{A} = (A_x, A_y, A_z) \tag{1325}$	1765
1743	derivation from vector potential 3		
1744	$\nabla(\mathbf{v}\cdot\mathbf{A})\cdot\hat{\mathbf{r}} = \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r} $ (1310)	$\frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} = \frac{m}{2}(\dot{x},\dot{y},\dot{z})\cdot(\dot{x},\dot{y},\dot{z}) $ (1326)	1766
	$-q \int_{-r}^{r} \nabla (\mathbf{v} \cdot \mathbf{A}) \cdot \hat{\mathbf{r}} dr = -q \int_{-r}^{r} \frac{\partial (\mathbf{v} \cdot \mathbf{A})}{\partial r} dr$		
1745	$\int_{\infty} \sqrt{(r-12)} \int_{\infty} dr \frac{\partial r}{\partial r} \frac{\partial r}{\partial r} $ (1311)	$q\dot{\mathbf{r}} \cdot \mathbf{A} = q(\dot{x}, \dot{y}, \dot{z}) \cdot (A_x, A_y, A_z) \tag{1327}$	1767
	$f^r \partial(\mathbf{y} \cdot \mathbf{A})$		
1746	$U = -q \int_{\infty}^{r} \frac{\partial (\mathbf{v} \cdot \mathbf{A})}{\partial r} dr \tag{1312}$		
1747	C.30 Lorentz force: Potential energy	$L = \frac{m}{2}(\dot{x}, \dot{y}, \dot{z}) \cdot (\dot{x}, \dot{y}, \dot{z}) + q(\dot{x}, \dot{y}, \dot{z}) \cdot (A_x, A_y, A_z) - q\phi$	
1748	derivation from vector potential 4	(1328)	1768
1749	$U = -q \int_{\infty}^{r} \frac{\partial (\mathbf{v} \cdot \mathbf{A})}{\partial r} dr \tag{1313}$	C.33 Lorentz force: Derivation of Lorentz force from classical Lagrangian (LHS) 4	1769 1770
1750	$\frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r} = \mathbf{v}\cdot\mathbf{A}'(r) \tag{1314}$	$dA_x = \frac{\partial A_x}{\partial t}dt + \frac{\partial A_x}{\partial x}dx + \frac{\partial A_x}{\partial y}dy + \frac{\partial A_x}{\partial z}dz$	
	$\int_{\mathbf{r}}^{r} \partial(\mathbf{v} \cdot \mathbf{A}) \int_{\mathbf{r}}^{r} \int_{\mathbf{A}}^{r} \int_{\mathbf{r}}^{r} \int_{\mathbf{A}}^{r} \int_{\mathbf{r}}^{r} \int_{\mathbf{A}}^{r} \int_{\mathbf{r}}^{r} \int$	(1329)	1771
	$-q \int_{\infty}^{r} \frac{\partial (\mathbf{v} \cdot \mathbf{A})}{\partial r} dr = -q \int_{\infty}^{r} \mathbf{v} \cdot \mathbf{A}'(r) dr$	$dA = \partial A = \partial A dx = \partial A dy = \partial A dz$	
1751	(1315)	$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_x}{\partial z}\frac{dz}{dt}$	
1752	$-q \int_{-\infty}^{r} \mathbf{v} \cdot \mathbf{A}'(r) dr = -q \mathbf{v} \cdot \mathbf{A}(r) \qquad (1316)$	(1330)	1772
1753	$U = -q\mathbf{v} \cdot \mathbf{A}(r) \tag{1317}$	$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} $ (1331)	1773
	4	12	

**Lorentz force: Derivation of classical** 

 $V = q\phi - q\dot{\mathbf{r}} \cdot \mathbf{A}$ 

 $q\phi = V + q\dot{\mathbf{r}} \cdot \mathbf{A}$ 

 $L = \frac{m}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - q\phi$ 

 $L = \frac{m}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} - (V + q\dot{\mathbf{r}} \cdot \mathbf{A})$ 

Lagrangian of EM field

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C.27 Laplace equation: Electrostatics 3

 $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$ 

 $\frac{\partial^2 \phi}{\partial x^2} = 0$ 

 $\frac{\partial^2 \phi}{\partial y^2} = 0$ 

 $\nabla^2 \phi = 0 + 0$ 

 $\nabla^2 \phi = 0$ 

C.28 Lorentz force: Potential energy

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1780 1781 1782	C.35 Lorentz force: Derivation of Lorentz force from classical Lagrangian (RHS)	$\mathbf{F} \cdot \hat{\mathbf{x}} = F_x \tag{1349}$	1802
1783	$\frac{\partial L}{\partial x} = q \frac{\partial}{\partial x} (\dot{x} A_x + \dot{y} A_y + \dot{z} A_z) - q \frac{\partial}{\partial x} \phi $ (1336)	$\mathbf{E} \cdot \hat{\mathbf{x}} = E_x \tag{1350}$	1803
1784	$\frac{\partial}{\partial x}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) = \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}$ (1337)	$\mathbf{B} = \nabla \times \mathbf{A} \tag{1351}$	1804
1785	$\frac{\partial L}{\partial x} = q(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}) - q\frac{\partial}{\partial x}\phi$ (1338)	$(\dot{\mathbf{r}} \times \mathbf{B}) \cdot \hat{\mathbf{x}} = (\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_x$ (1352)	1805
1786 1787	C.36 Lorentz force: Derivation of x component of electric field	$q\mathbf{E} \cdot \hat{\mathbf{x}} + q(\dot{\mathbf{r}} \times \mathbf{B}) \cdot \hat{\mathbf{x}} = qE_x + q(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_x$ (1353)	1806
1788	$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \tag{1339}$		
1789	$\nabla \phi = (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \tag{1340}$	$\mathbf{F} \cdot \hat{\mathbf{x}} = q\mathbf{E} \cdot \hat{\mathbf{x}} + q(\dot{\mathbf{r}} \times \mathbf{B}) \cdot \hat{\mathbf{x}} $ (1354)	1807
1790	$\frac{\partial \mathbf{A}}{\partial t} = \frac{\partial}{\partial t} (A_x, A_y, A_z) \tag{1341}$	C.39 Electromagnetic wave equation: The origin of the electromagnetic wave equation in 2	1808 1809 1810
	$oxed{\Delta A}$ , $\partial\phi$	$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) \qquad (1355)$	1811
1791	$-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = -(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) + \frac{\partial}{\partial t}(A_x, A_y, A_z)$ (1342)	$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1356}$	1812
1792	$\mathbf{E} = -\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) + \frac{\partial}{\partial t}(A_x, A_y, A_z)  (1343)$	$-\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} $ (1357)	1813

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**Lorentz force: Derivation of Lorentz** 

force from classical Lagrangian 4

 $(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_x = \dot{y}(\nabla \times \mathbf{A})_z - \dot{z}(\nabla \times \mathbf{A})_y$  (1345)

 $F_x = qE_x + q(\dot{y}(\nabla \times \mathbf{A})_z - \dot{z}(\nabla \times \mathbf{A})_y)$  (1347)

force from classical Lagrangian 5

**C.38** Lorentz force: Derivation of Lorentz

 $F_r = qE_r + q(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_r$ 

 $F_r = qE_r + q(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_r$ 

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**C.34** Lorentz force: Derivation of Lorentz

 $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\frac{d}{dt}\dot{x} + q\frac{d}{dt}A_x$ 

 $\frac{d}{dt}\dot{x} = \ddot{x}$ 

 $\frac{d}{dt}A_x = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} \quad (1334)$ 

 $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + q(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z})$ 

force from classical Lagrangian (LHS) 5

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 $\mathbf{E} \cdot (1,0,0) = -((\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) - \frac{\partial}{\partial t} (A_x, A_y, A_z)) \cdot (1,0,0)$   $(1344) \qquad \nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$ 

	2217		$\mu_0$ . The second of $t$		
1822	$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0$	(1363)			
	Ot		$\mathbf{J}_M = \nabla \times \mathbf{M}$	(1378)	1845
1823	C.41 Electromagnetic wave equation:				
1824	origin of the electromagnetic wa	ve			
1825	equation in $\ 2$		$ abla  imes rac{1}{\mu_0} \mathbf{B} = \mathbf{J}_f + rac{\partial}{\partial t} \varepsilon_0 \mathbf{E} + rac{\partial}{\partial t} \mathbf{P} + \mathbf{J}_M$	(1379)	1846
1826	$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E})$	(1364)	C.44 Ampere's circuital law: Proof of		1847
			equivalence 4		1848
1007	$ abla imes \mathbf{E} = -rac{\partial \mathbf{B}}{\partial t}$	(1365)	-		1040
1827	$\mathbf{V} \wedge \mathbf{E} = -\frac{1}{\partial t}$	(1303)	$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J}_f + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_P + \mathbf{J}_M$	(1380)	1849
1828	$\mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$	(1366)	$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_P + \mathbf{J}_M$	(1381)	1850
	$\partial t$ ( $\partial t^2$	(====)	$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	(1382)	1851
	$\nabla \times (\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$	(12(7)	C.45 Uncertainty principle: Kennard		1852
1829	$\mathbf{V} \times (\mathbf{V} \times \mathbf{B}) = -\mu_0 \varepsilon_0 \frac{\partial t^2}{\partial t^2}$	(1367)	inequality proof part 1.1		1853
1830	C.42 Electromagnetic wave equation:		$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$	(1383)	1854
1831	origin of the electromagnetic wa	ve			
1832	equation in 3		$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \cdot  \psi(x) ^2 dx$	(1384)	1855
1833	$\nabla \times (\nabla \times \mathbf{B}) = \nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$	(1368)	$\langle x \rangle = \int_{-\infty}^{\infty} x^{-\epsilon}  \psi(x)  dx$	(1304)	1000
			$\langle x \rangle = \int_{-\infty}^{\infty} x \cdot  \psi(x) ^2 dx$	(1385)	1856
1834	$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = \nabla \times (\nabla \times \mathbf{B})$	(1369)	$\int_{-\infty}^{\infty} u^{- \psi(u) } du$	(1303)	1000
	027		$\langle x \rangle^2 = \left( \int_{-\infty}^{\infty} x \cdot  \psi(x) ^2 dx \right)^2$	(1386)	1857
1835	$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = \nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$	(1370)	<del></del>		
	$\partial^2 {f B}$ = ${f F}^2 {f F}$	(1051)	$\sigma_x^2 = \int_{-\infty}^{\infty} x^2 \cdot  \psi(x) ^2 dx - \left(\int_{-\infty}^{\infty} x \cdot  \psi(x) ^2 dx\right)$	$ x ^2 dx$	

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Ampere's circuital law: Proof of

 $\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$ 

 $\mathbf{H} = \frac{1}{\mu_0} \mathbf{B}$ 

 $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ 

 $\nabla \times \frac{1}{\mu_0} \mathbf{B} = \mathbf{J}_f + \frac{\partial}{\partial t} \frac{1}{\mu_0} \mathbf{D}$ 

 $\frac{\partial}{\partial t} \frac{1}{\mu_0} \mathbf{D} = \frac{\partial}{\partial t} \varepsilon_0 \mathbf{E} + \frac{\partial}{\partial t} \mathbf{P}$ 

 $\nabla \times \frac{1}{\mu_0} \mathbf{B} = \mathbf{J}_f + \frac{\partial}{\partial t} \varepsilon_0 \mathbf{E} + \frac{\partial}{\partial t} \mathbf{P}$ 

equivalence 2

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**Electromagnetic wave equation: The** 

origin of the electromagnetic wave

 $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ 

 $\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla \times (\nabla \times \mathbf{E})$ 

 $\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ 

 $\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} - \nabla^2 \mathbf{E}$ 

 $\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0$ 

equation in 3

(1371)

## C.46 Uncertainty principle: Kennard inequality proof part 1.4

$$f^*(x) \cdot f(x) = x^2 \cdot (\psi^*(x) \cdot \psi(x)) \qquad (1388)$$

$$\sigma_x^2 = \langle f|f\rangle \tag{1389}$$

$$\sigma_x^2 = \langle x^2 | x^2 \rangle \tag{1390}$$

$$\sigma_x^2 = \langle f^*(x) \cdot f(x) | f^*(x) \cdot f(x) \rangle \qquad (1391)$$

$$\sigma_x^2 = \left\langle x^2 \cdot (\psi^*(x) \cdot \psi(x)) \middle| x^2 \cdot (\psi^*(x) \cdot \psi(x)) \right\rangle$$
(1392)

$$\sigma_x^2 = \langle f | f \rangle \tag{1393}$$

## C.47 Uncertainty principle: Kennard inequality proof part 2.2

$$\frac{dv}{d\chi} = e^{\frac{-ip\chi}{\hbar}} \tag{1394}$$

$$\int dv = \int e^{\frac{-ip\chi}{\hbar}} d\chi \tag{1395}$$

$$v = \frac{\hbar}{-ip} \int e^{\frac{-ip\chi}{\hbar}} d\chi \tag{1396}$$

$$v = \frac{\hbar}{-ip}e^b + C \tag{1397}$$

## C.48 Uncertainty principle: Kennard inequality proof part 2.3

$$u = \psi(\chi) \tag{1398}$$

$$v = \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \tag{1399}$$

$$uv = \psi(\chi) \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \tag{1400}$$

### C.49 Uncertainty principle: Kennard inequality proof part 2.4

$$uv = \psi(\chi) \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}}$$
 (1401)

$$(uv)\Big|_{-ip}^{\infty} = \psi(\chi) \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \Big|_{-ip}^{\infty}$$
 (1402)

$$(uv) \Big|_{-\infty}^{\infty} = 0$$
 (1403)

### C.50 Uncertainty principle: Kennard inequality proof part 2.5

$$I = (uv)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v \frac{du}{d\chi} d\chi \qquad (1404)$$

$$letu = \psi(\chi) \tag{1405}$$

$$v = e^{\frac{-ip\chi}{\hbar}} \tag{1406}$$

$$du = \frac{d\psi(\chi)}{d\chi}d\chi \tag{1407}$$

$$dv = \frac{-ip}{\hbar}e^{\frac{-ip\chi}{\hbar}}d\chi \tag{1408}$$

$$\int_{-\infty}^{\infty} v \frac{du}{d\chi} d\chi = \int_{-\infty}^{\infty} e^{\frac{-ip\chi}{\hbar}} \frac{d\psi(\chi)}{d\chi} d\chi \quad (1409)$$

$$(uv)\Big|_{-\infty}^{\infty} = \frac{\hbar}{ip} (uv)\Big|_{-\infty}^{\infty}$$
 (1410)

$$I = \frac{\hbar}{ip} \int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi} e^{\frac{-ip\chi}{\hbar}} d\chi$$
 (1411)

## C.51 Uncertainty principle: Kennard inequality proof part 2.9

$$g(x) = \frac{\hbar}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi} e^{i(x-\chi)b} d\chi db$$
(1412)

$$\frac{d\psi(\chi)}{d\chi} = \frac{d\psi(x)}{dx} \tag{1413}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\psi(x)}{dx} e^{i(x-\chi)b} d\chi db = \frac{d\psi(x)}{dx} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x-\chi)b} d\chi db$$
1897

$$\frac{\hbar}{2\pi i} \frac{d\psi(x)}{dx} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x-\chi)b} d\chi db = \frac{\hbar}{2\pi i} \frac{d\psi(x)}{dx}$$
(1415)

$$\frac{\hbar}{2\pi i} \frac{d\psi(x)}{dx} = \frac{\hbar}{i} \left(\frac{d\psi(x)}{dx}\right) \tag{1416}$$

$$g(x) = \frac{\hbar}{i} \left( \frac{d\psi(x)}{dx} \right) \tag{1417}$$

1901	inequality proof part 3.2		inequality proof part 4.4		1925
1903	$\tilde{g}^*(p) \cdot \tilde{g}(p) = p^2 \varphi^*(p) \cdot \varphi(p)$	(1418)	$\sigma_x^2 \sigma_p^2 \ge  raket{f g} ^2$	(1433)	1926
1904	$ \tilde{g}(p) ^2 = \tilde{g}^*(p) \cdot \tilde{g}(p)$	(1419)	$ \langle f g\rangle  = rac{\langle f g\rangle - \langle g f angle}{2i}$	(1434)	1927
1905	$ \tilde{g}(p) ^2 = p^2 \varphi^*(p) \cdot \varphi(p)$	(1420)	$\left(\frac{\langle f g\rangle - \langle g f\rangle}{2i}\right)^2 =  \langle f g\rangle ^2$	(1435)	1928
1906	$ \tilde{g}(p) ^2 = p^2  \varphi(p) ^2$	(1421)	$\sigma_x^2\sigma_p^2\geq (rac{\langle f g angle-\langle g f angle}{2i})^2$ C.57 Uncertainty principle: Kennard	(1436)	1929 1930
1907	C.53 Uncertainty principle: Kennard		inequality proof part 5.1		1931
1908	inequality proof part 3.3		$\sigma_x^2 \sigma_p^2 \ge (\frac{\langle f g \rangle - \langle g f \rangle}{2i})^2$	(1437)	1932
1909	$\sigma_p^2 = \int_{-\infty}^{\infty} p^2  \varphi(p) ^2 dp$	(1422)	$\langle f g angle = \langle g f angle$	(1438)	1933
1910	$\langle g  = \int_{-\infty}^{\infty} p^2  \varphi(p) ^2 dp$	(1423)	$\sigma_x^2 \sigma_p^2 \ge (\frac{\langle f g\rangle - \langle f g\rangle}{2i})^2$	(1439)	1934
1911	$ g\rangle = 1$	(1424)	$\sigma_x^2 \sigma_p^2 \ge 0$	(1440)	1935
1912	$\sigma_p^2 = \langle g   g  angle$	(1425)	$\langle f g\rangle = \langle g f\rangle$	(1441)	1936
1913 1914	C.54 Uncertainty principle: Kennard inequality proof part 4.1		$\langle f g\rangle = -i\hbar \int_{-\infty}^{\infty} x\psi^*(x) \frac{d\psi(x)}{dx} dx$	(1442)	1937
1915	$\sigma_x^2 = \langle f   f  angle$	(1426)	C.58 Uncertainty principle: Kennard inequality proof part 5.2		1938 1939
			$f(x) = x \cdot \psi(x)$	(1443)	1940
1916	$ \left\langle f g\right\rangle  ^{2}=\sigma_{x}^{2}\sigma_{p}^{2}$	(1427)	$\langle g f\rangle = \langle g x\cdot\psi(x)\rangle$	(1444)	1941
1917	$\sigma_x^2\sigma_p^2 \geq  raket{f g} ^2$	(1428)	$\langle g f\rangle = -i\hbar \int_{-\infty}^{\infty} \psi^*(x) \frac{d}{dx} (x\psi(x)) dx$	(1445)	1942
	C.55 Uncertainty principle: Kennard		$\langle g J/=-in\int_{-\infty}\psi(x)\frac{dx}{dx}(x\psi(x))dx$	(1443)	1342
1918			C 50 Uncontainty principle, Vannand		
1918 1919	inequality proof part 4.3		C.59 Uncertainty principle: Kennard		1943
	inequality proof part 4.3 $ z ^2 = (\text{Re}(z))^2 + (\text{Im}(z))^2$	(1429)	inequality proof part 5.6		1943 1944
1919		(1429)	~ ~ ~ ~	(1446)	
1919		(1429) (1430)	inequality proof part 5.6	(1446) (1447)	1944
1919 1920	$ z ^2 = (\text{Re}(z))^2 + (\text{Im}(z))^2$		inequality proof part 5.6 $\langle f g\rangle - \langle g f\rangle = i\hbar \int_{-\infty}^{\infty}  \psi(x) ^2 dx$		1944 1945

C.56 Uncertainty principle: Kennard

C.52 Uncertainty principle: Kennard

1950	velocity as a function of particle mass	Ladder operators for the quantum	1973
1951	from Schrödinger's equation 7	harmonic oscillator part 1.6	1974
	$\hbar^2$ $\partial^2$	$1  1  d^2  2  1  d^2  2  \dots  \qquad \qquad$	
	$i\hbar \frac{\partial}{\partial t} \psi(x,t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x,t) + V(x)\psi(x,t)$	$\hbar\omega(\frac{1}{2} + \frac{1}{\sqrt{2}}(-\frac{d^2}{dg^2} + q^2)\frac{1}{\sqrt{2}}(\frac{d^2}{dg^2} + q^2))\psi(q) = E\psi(q)$	
1952	(1450)	(1465)	1975
	$i\hbar \partial = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t)$	$1  1  d^2  \qquad \dots$	
	$\frac{i\hbar}{\hbar}\frac{\partial}{\partial t}\psi(x,t) = \frac{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t)}{\hbar}$	$\hbar\omega(\frac{1}{2} + \frac{1}{2}(-\frac{d^2}{da^2} + q^2))\psi(q) = E\psi(q)  (1466)$	1976
1953	(1451)	2 2 44	
		1 1 .	
	$\hbar \partial_{x} V(x,t) = \hbar \partial^{2} V(x)\psi(x,t)$	$\hbar\omega(\frac{1}{2} + \frac{1}{2}(a^{\dagger}a))\psi(q) = E\psi(q) \qquad (1467)$	1977
	$i\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + \frac{V(x)\psi(x,t)}{\hbar}$	2 2	
1954	(1452)	. 1	
		$E\psi(q) = \hbar\omega(a^{\dagger}a + \frac{1}{2})\psi(q) \tag{1468}$	1978
	$\hbar\omega\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + \frac{V(x)\psi(x,t)}{\hbar}$ (1453)	2	
1055	$1 m \varphi(x,t) = 2m \partial x^2 \varphi(x,t) + \hbar $	$E = \hbar\omega(a^{\dagger}a + \frac{1}{2}) \tag{1469}$	1070
1955	(1433)	$E = \hbar\omega(a^{\dagger}a + \frac{1}{2}) \tag{1469}$	1979
		C.64 Creation and annihilation operators:	1980
	$\hbar\omega\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)ifV(x) = \hbar\omega.$	Ladder operators for the quantum	1981
1956	$2m \partial x^2 $ (1454)	harmonic oscillator part 2	1982
1930	(1434)	•	
1957	C.61 Particle in a box: Wavefunction angular	$[q, p] = qp - pq \tag{1470}$	1983
1958	velocity as a function of particle mass		
1959	from Schrödinger's equation 8	[q, p]f(q) = (qp - pq)f(q) (1471)	1984
	$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{\hbar^2 k^2}{2m}\psi(x,t) $ (1455)		
1960	$-\frac{n}{2m}\frac{\partial \psi(x,t)}{\partial x^2} = \frac{n}{2m}\psi(x,t) \tag{1455}$	$\begin{bmatrix} a & a \end{bmatrix} f(a) & a & f(a) & a & f(a) & (1472) \end{bmatrix}$	1005
	22 / ( )	[q, p]f(q) = qpf(q) - pqf(q)  (1472)	1985
1961	$\frac{\partial^2 \psi(x,t)}{\partial x^2} = k^2 \psi(x,t) \tag{1456}$		
	$\partial x^2$	$[q, p]f(q) = if(q) \tag{1473}$	1986
	$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{p^2}{\hbar^2} \psi(x,t) \tag{1457}$	C.65 Creation and annihilation operators:	1007
1962	$\frac{\partial \varphi(x,t)}{\partial x^2} = \frac{P}{\hbar^2} \psi(x,t) \tag{1457}$	C.65 Creation and annihilation operators:  Ladder operators for the quantum	1987 1988
	$2^{2} + (1) = 2$	harmonic oscillator part 3.1	1989
1963	$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{p^2}{\hbar^2} \psi(x,t) \tag{1458}$	. <del>-</del>	1303
	$\partial x^2$ $h^2$	$a = \frac{1}{\sqrt{2}}(\frac{d}{dq} + q) \tag{1474}$	1990
	$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{p^2}{\hbar^2} \psi(x,t) \tag{1459}$	$\sqrt{2} \ aq$	
1964	$\frac{\partial \varphi(x,t)}{\partial x^2} = \frac{P}{\hbar^2} \psi(x,t) \tag{1459}$	$ip = \frac{d}{da} \tag{1475}$	1991
	2	dq = dq	1331
1965	$\omega = \frac{p^2}{2m\hbar} \tag{1460}$	1	
	$2m\hbar$	$a = \frac{1}{\sqrt{2}}(ip+q)$ (1476)	1992
1966	C.62 Quantum harmonic oscillator: Ladder	V Z	
1967	operator method 4	C.66 Creation and annihilation operators:	1993
1968	$Givenaa^{\dagger} - a^{\dagger}a = \frac{i}{\hbar}(\hat{p}\hat{x} - \hat{x}\hat{p}) \qquad (1461)$	Ladder operators for the quantum	1994
1300	$Givenua - u \ u = \frac{1}{\hbar}(px - xp) \tag{1401}$	harmonic oscillator part 3.2	1995
	[ +] + + (1462)	$a^{\dagger} = \frac{1}{\sqrt{2}} \left( -\frac{d}{dq} + q \right) \tag{1477}$	1996
1969	$\left[a, a^{\dagger}\right] = aa^{\dagger} - a^{\dagger}a \tag{1462}$	$\sqrt{2}$ $dq$	
	4	d	
1970	$\frac{i}{\hbar}(\hat{p}\hat{x} - \hat{x}\hat{p}) = 1 \tag{1463}$	$-\frac{d}{dq} = -ip \tag{1478}$	1997
	n	1	
1971	$\left[a, a^{\dagger}\right] = 1 \tag{1464}$	$a^{\dagger} = \frac{1}{\sqrt{2}}(-ip+q)$ (1479)	1998
		$\sqrt{2}$	

C.63 Creation and annihilation operators:

1972

C.60 Particle in a box: Wavefunction angular

1999	<b>C.67</b>	Creation and annihilation open	rators:			
2000		Ladder operators for the quan	tum	$i_{(\hat{x},\hat{x},\hat{y})}$ $i_{(\hat{x},\hat{y},\hat{y})}$ $i_{(\hat{x},\hat{y},\hat{y})}$	(1.405)	
2001		harmonic oscillator part 3.6		$\frac{\imath}{\hbar}(\hat{H}\hat{x}(t) - \hat{x}(t)\hat{H}) = \frac{\imath}{\hbar} \Big[\hat{H}, \hat{x}(t)\Big]$	(1497)	2026
2002		$aa^{\dagger} - a^{\dagger}a = i(pq - qp)$	(1480)			
				$\frac{d}{dt}\hat{x}(t) = \frac{i}{\hbar} \left[ \hat{H}, \hat{x}(t) \right]$	(1498)	2027
2003		pq - qp = 1	(1481)	$dt^{w(t)} = \hbar \left[ \frac{11}{2}, w(t) \right]$	(11)0)	
		r 1 1r -	(-10-)	C.71 Heisenberg picture: momentum		2028
0004		:( :	(1402)	evolution 4		2029
2004		i(pq - qp) = i	(1482)	$\hat{p}(t) = A\cos(\omega t) + B\sin(\omega t)$	(1499)	2030
2005		$aa^{\dagger} - a^{\dagger}a = i$	(1483)	d		
		r 7		$\frac{d}{dt}A\cos(\omega t) = -A\omega\sin(\omega t)$	(1500)	2031
2006		$\left a,a^{\dagger}\right =1$	(1484)	$a\iota$		
000=	C (9	Cusation and annihilation and		$d_{R} = (\cdot, \cdot, \cdot)$	(1501)	0000
2007	C.68	Creation and annihilation oper Ladder operators for the quan		$\frac{d}{dt}B\sin(\omega t) = B\omega\cos(\omega t)$	(1501)	2032
2009		harmonic oscillator part 4.2	tuiii			
2010		$[H,a] = -\hbar\omega(aa^{\dagger} - a^{\dagger}a)a$	(1485)	$\frac{d\hat{p}(t)}{dt} = \frac{d}{dt}(A\cos(\omega t) + B\sin(\omega t))$	(1502)	0000
2010		$[\Pi, u] = -i\omega(uu^* - u^*u)u$	(1403)	$\frac{dt}{dt} = \frac{1}{dt} (A\cos(\omega t) + B\sin(\omega t))$	(1302)	2033
		$aa^{\dagger} - a^{\dagger}a = 1$	(1.406)			
2011		aa' - a'a = 1	(1486)	$d\hat{p}(t)$	(4.500)	
				$\frac{d\hat{p}(t)}{dt} = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$	(1503)	2034
2012		$-\hbar\omega(1)a = -\hbar\omega a$	(1487)	C.72 Heisenberg picture: position		2035
				commutator 4		2036
2013		$[H,a] = -\hbar\omega a$	(1488)	$[\hat{x}(t_1), \hat{x}(t_2)] = \frac{1}{\omega m} (\hat{x}_0 \hat{p}_0 - \hat{p}_0 \hat{x}_0) \sin(\omega t_2)$	$a = (at_a)$	
2014	C.69	Creation and annihilation oper	rators	$[x(t_1), x(t_2)] = \frac{1}{\omega m} (x_0 p_0 - p_0 x_0) \sin(\omega t_2)$	(1504)	2027
2015	C.07	Ladder operators for the quan			(1304)	2037
2016		harmonic oscillator part 5.2		$\hat{x}_0\hat{p}_0-\hat{p}_0\hat{x}_0=i\hbar$	(1505)	2038
2017		$\left[H, a^{\dagger}\right] = \hbar \omega a^{\dagger} (a a^{\dagger} - a^{\dagger} a)$	(1489)			
2017		$[n,a^*] = n\omega a^*(aa^* - a^*a)$	(1409)			
		[ +] + +	(1.400)	$\frac{1}{\omega m}(i\hbar)\sin(\omega t_2 - \omega t_1) = \frac{i\hbar}{\omega m}\sin(\omega t_2 - \omega t_2)$	$-\omega t_1)$	
2018		$\left[a,a^{\dagger} ight]=aa^{\dagger}-a^{\dagger}a$	(1490)	$\omega m$	(1506)	2039
2019		$aa^{\dagger} - a^{\dagger}a = 1$	(1491)	$i\hbar$	(4.505)	
				$[\hat{x}(t_1), \hat{x}(t_2)] = \frac{i\hbar}{\omega m} \sin(\omega t_2 - \omega t_1)$	(1507)	2040
2020		$\left a,a^{\dagger}\right =1$	(1492)	C.73 Heisenberg picture: momentum		2041
	C 70	III	4: 4	commutator 3		2042
2021	C.70	Heisenberg picture: time evolu		$[\hat{p}(t_1), \hat{p}(t_2)] = m\omega \hat{p}_0 \hat{x}_0 \sin(\omega t_2 - \omega t_1) +$	$-m\omega\hat{x}_0\hat{p}_0\sin(\omega)$	
	$\frac{a}{dt}\hat{x}$	$r(t) = \frac{i}{\hbar} (\hat{H}e^{i\frac{\hat{H}t}{\hbar}} \hat{x}e^{-i\frac{\hat{H}t}{\hbar}} - e^{i\frac{\hat{H}t}{\hbar}} \hat{x}e^{-i\frac{\hat{H}t}{\hbar}})$	$^{-irac{Ht}{\hbar}}\hat{H})$	$[p(v_1),p(v_2)]$ $maxpoworm(av_2 av_1)$	(1508)	2043
2022	$a\iota$	n	(1493)		( )	
		$\hat{A} \cdot \hat{H}t + \hat{H}t + \hat{A}$			^ ^ • (	,
2023		$\hat{H}e^{i\frac{Ht}{\hbar}}\hat{x}e^{-i\frac{Ht}{\hbar}} = \hat{H}\hat{x}(t)$	(1494)	$[\hat{p}(t_1), \hat{p}(t_2)] = m\omega \hat{p}_0 \hat{x}_0 \sin(\omega t_2 - \omega t_1) -$		
		û, û.			(1509)	2044
2024		$e^{i\frac{Ht}{\hbar}}\hat{x}e^{-i\frac{Ht}{\hbar}}\hat{H} = \hat{x}(t)\hat{H}$	(1495)	$[\hat{p}(t_1),\hat{p}(t_2)]=0$	(1510)	2045
				μ ( 1//1 ( 2/) -	` '	-
		, , , , , , ,				
2025		$\hat{H}\hat{x}(t) - \hat{x}(t)\hat{H} = \left \hat{H}, \hat{x}(t)\right $	(1496)	$[\hat{p}(t_1), \hat{p}(t_2)] = i\hbar m\omega \sin(\omega t_2 - \omega t_1)$	(1511)	2046

$\langle \Psi(t)  = \cos\left(\frac{\Omega t}{2}\right) \langle e, 0  + i \sin\left(\frac{\Omega t}{2}\right) \langle g, 1  \qquad \langle x \hat{X} x'\rangle = x'\delta(x - x') $ (15)	529) 20	:070
$P_e(t) =  \langle e, 0   \Psi(t) \rangle ^2 \qquad (1514) \qquad \left\langle \hat{X} \right\rangle_{\Psi} = \int \int \langle x   \Psi \rangle^{\dagger}  x' \delta(x - x')  \left\langle x'   \Psi \right\rangle dx dx$	edx'	
$P_e(t) =  \cos\left(\frac{\Omega t}{2}\right) ^2 \qquad (1515)$ C.78 Expectation value: integral expression		2071
$P_{e}(t) = \cos^{2}(\frac{\Omega t}{2}) \qquad (1516) \qquad \left\langle \hat{X} \right\rangle_{\Psi} = \int \int \left\langle x   \Psi \right\rangle^{\dagger} x' \delta(x - x') \left\langle x'   \Psi \right\rangle dx dx$		
2054 C.75 Vacuum Rabi Oscillations: ground state	, ,	2073
$P_g(t) =  \cos\left(\frac{\Omega t}{2}\right)\langle g, 1 e, 0\rangle - i\sin\left(\frac{\Omega t}{2}\right)\langle g, 1 g, 1\rangle ^2$		075
$P_g(t) =  \cos\left(\frac{\Omega t}{2}\right) ^2 \langle g, 1 e, 0\rangle^2 +  \sin\left(\frac{\Omega t}{2}\right) ^2 \langle g, 1 g, 1\rangle^2 \langle g, 1 g, 1\rangle^2 \langle g, 1 \Psi\rangle^{\dagger} x' \delta(x-x') \langle x' \Psi\rangle dx dx' = \int_{(1518)}^{(1518)} x' \delta(x-x') \langle x' \Psi\rangle dx dx' = \int_{(1518)}^{(1518)} x' \delta(x-x') \langle x' \Psi\rangle dx dx' = \int_{(1518)}^{(1518)} x' \delta(x-x') \langle x' \Psi\rangle dx dx' = \int_{(1518)}^{(1518)} x' \delta(x-x') \langle x' \Psi\rangle dx dx' = \int_{(1518)}^{(1518)} x' \delta(x-x') \langle x' \Psi\rangle dx dx' = \int_{(1518)}^{(1518)} x' \delta(x-x') \langle x' \Psi\rangle dx dx' = \int_{(1518)}^{(1518)} x' \delta(x-x') \langle x' \Psi\rangle dx' dx' = \int_{(1518)}^{(1518)} x' \delta(x-x') \langle x' \Psi\rangle dx' dx' = \int_{(1518)}^{(1518)} x' \delta(x-x') \langle x' \Psi\rangle dx' dx' = \int_{(1518)}^{(1518)} x' \delta(x-x') \langle x' \Psi\rangle dx' dx' = \int_{(1518)}^{(1518)} x' \delta(x-x') \langle x' \Psi\rangle dx' dx' = \int_{(1518)}^{(1518)} x' \delta(x-x') \langle x' \Psi\rangle dx' dx' = \int_{(1518)}^{(1518)} x' \delta(x-x') \langle x' \Psi\rangle dx' dx' dx' = \int_{(1518)}^{(1518)} x' \delta(x-x') \langle x' \Psi\rangle dx' dx' dx' dx' dx' dx' dx' dx' dx' dx'$	$ \Psi(x) ^2 dx$ 534) 20	2076
$P_g(t) = \cos^2(\frac{3e}{2}) \langle g, 1 e, 0\rangle^2 + \sin^2(\frac{3e}{2}) \langle g, 1 g, 1\rangle^2 \qquad \qquad \downarrow $	,	2077
2058 $(1519)  \textbf{C.79}  \textbf{Euler-lagrange equation: Full} \\ \textbf{derivative of the perturbation} \\ \textbf{2059} \qquad \langle g,1 e,0\rangle=0 \qquad (1520) \qquad \textbf{Lagrangian with respect to } \varepsilon \textbf{ 2}$	20	2078 2079 2080
$\langle g, 1 g, 1\rangle = 1 \qquad (1521) \qquad \frac{dg_{\varepsilon}}{d\varepsilon} = \frac{d}{d\varepsilon}(\varepsilon \eta(x)) \qquad (1521)$	536) 20	.081
$P_g(t) = \cos^2(\frac{\Omega t}{2}) * 0 + \sin^2(\frac{\Omega t}{2}) * 1  (1522)$ $\frac{d}{d\varepsilon}(\varepsilon \eta(x)) = \eta(x)  (1522)$	537) 20	082
$\frac{dg_{\varepsilon}}{d\varepsilon} = \eta(x) \tag{15}$		.083
2 C.00 Euler-Lagrange equation. Derivation	n 20	084
2063 C.76 Expectation value: integral expression $J = \int_{a}^{b} L(x, f(x), f'(x)) $ (15)	(20)	085

**Expectation value: integral expression 2** 

 $\left\langle \hat{X} \right\rangle_{\Psi} = \int \int \left\langle \Psi | x \right\rangle \left\langle x | \hat{X} | x' \right\rangle \left\langle x' | \Psi \right\rangle dx dx'$ 

 $\langle \Psi | x \rangle = \langle x | \Psi \rangle^{\dagger}$ 

 $J_{\varepsilon} = \int_{a}^{b} L_{\varepsilon}(x, f(x), f'(x))$ 

 $\frac{dJ_{\varepsilon}}{d\varepsilon} = \frac{d}{d\varepsilon} \int_{a}^{b} L_{\varepsilon}(x, f(x), f'(x))$ 

 $\frac{dJ_{\varepsilon}}{d\varepsilon} = \int_{\varepsilon}^{b} \frac{dL_{\varepsilon}}{d\varepsilon} dx$ 

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(1528)

(1540)

(1541)

(1542)

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Vacuum Rabi Oscillations: excited state

 $|\Psi(t)\rangle = \cos\left(\frac{\Omega t}{2}\right)|e,0\rangle - i\sin\left(\frac{\Omega t}{2}\right)|g,1\rangle$ 

 $\left\langle \hat{X} \right\rangle_{\Psi} = \left\langle \Psi \right| \hat{X} \left| \Psi \right\rangle$ 

 $\langle \Psi | \hat{X} | \Psi \rangle = \int \int \langle \Psi | x \rangle \langle x | \hat{X} | x' \rangle \langle x' | \Psi \rangle dx dx'$ 

 $\left\langle \hat{X} \right\rangle_{\Psi} = \int \int \left\langle \Psi | x \right\rangle \left\langle x | \hat{X} | x' \right\rangle \left\langle x' | \Psi \right\rangle dx dx'$ 

probability

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(1525)

(1526)

2100 
$$I = \int_{a}^{b} \frac{\partial L}{\partial f'} \eta'(x) dx \qquad (1551) \qquad y = Cx + \int (1 - C^{2})^{-1/2} dx \qquad (1565) \qquad 2119$$
2101 
$$\frac{d}{dx} \frac{\partial L}{\partial f'} = \frac{\partial^{2} L}{\partial f'^{2}} \frac{df'}{dx} \qquad (1552) \qquad y = Cx + \sqrt{1 - C^{2}} + B \qquad (1566) \qquad 2120$$
2102 
$$\frac{\partial L}{\partial f'} \eta'(x) = -\eta(x) \frac{d}{dx} \frac{\partial L}{\partial f'} \qquad (1553)$$

$$y = Ax + CwhereA = C \qquad (1567) \qquad 2121$$
2103 
$$I = -\int_{a}^{b} \eta(x) \frac{d}{dx} \frac{\partial L}{\partial f'} dx \qquad (1554) \qquad \mathbf{C.87} \quad \mathbf{Escape velocity}$$
2104 
$$\mathbf{C.84} \quad \mathbf{Euler-Lagrange equation: Straight line} \qquad F = \frac{GMm}{r^{2}} \qquad (1569) \qquad 2124$$

(1555)

(1556)

(1557)

**Euler-Lagrange equation: Derivation 4** 

 $\frac{dv}{dx} = \eta'(x)$ 

 $\int \frac{dv}{dx} dx = \int \eta'(x) dx$ 

 $v = \eta(x)$ 

 $u = \frac{\partial L}{\partial f'}$ 

 $uv = \frac{\partial L}{\partial f'} \cdot v$ 

 $(uv)\Big|^b = \left(\frac{\partial L}{\partial f'} \cdot v\right)\Big|^b$ 

 $(uv)\Big|^b = 0$ 

 $S = \int^b ds$ 

 $ds = \sqrt{(1 + y'^2)} dx$ 

 $S = \int^b \sqrt{(1 + y'^2)} dx$ 

**Euler-Lagrange equation: Derivation 6** 

 $\int_{a}^{b} uvdx = \int_{a}^{b} \frac{\partial L}{\partial f'} \cdot vdx$ 

**Euler-Lagrange equation: Derivation 5** 

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**Euler-Lagrange equation: Straight line** 

 $\frac{dL}{du} - \frac{d}{dx}\frac{dL}{dy'} = 0$ 

 $\frac{dL}{du} = \frac{d}{dx} \frac{dL}{du'}$ 

 $\frac{d}{dx}(y'(1+y'^2)^{-\frac{1}{2}}) = \frac{dL}{dx}$ 

 $\int \frac{d}{dx} (y'(1+y'^2)^{-\frac{1}{2}}) dx = \int \frac{dL}{du} dx$ 

 $\int \frac{d}{dx} (y'(1+y'^2)^{-\frac{1}{2}}) dx = C$ 

 $\int dy = \int C(1 - C^2)^{-1/2} dx$ 

 $y = Cx + \int (1 - C^2)^{-1/2} dx$ 

 $W = \int F dr$ 

 $W = \int \frac{GMm}{r^2} dr$ 

 $W = \int_{r_0}^{\infty} \frac{GMm}{r^2} dr$ 

C.86 Euler-Lagrange equation: Straight line

 $\frac{dy}{dx} = C(1 - C^2)^{-1/2}$ 

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(1543)

(1544)

(1545)

(1546)

(1547)

(1548)

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(1550)

2108

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2126

2127

(1558)

(1559)

(1560)

(1562)

(1563)

(1564)

(1565)

(1570)

(1571)

(1572)

#### C.88 Escape velocity 2

$$W = \int_{r_0}^{\infty} \frac{GMm}{r^2} dr \tag{1573}$$

$$let u = r (1574)$$

$$dv = \frac{GMm}{r^2}dr \tag{1575}$$

$$du = dr (1576)$$

$$v = -\frac{GMm}{r} \tag{1577}$$

$$by integration by parts, \int u dv = uv - \int v du$$
(1578)

$$W = \left[ -\frac{GMm}{r} \right]_{r_0}^{\infty} - \int_{r_0}^{\infty} -\frac{GMm}{r} dr \quad (1579)$$

$$W = \left[ -\frac{GMm}{r} \right]_{r_0}^{\infty} + GMm \int_{r_0}^{\infty} \frac{1}{r} dr \quad (1580)$$

$$W = \left[ -\frac{GMm}{r} \right]_{r_0}^{\infty} + GMm[\ln r]_{r_0}^{\infty} \quad (1581)$$

$$W = \left[ -\frac{GMm}{r} \right]_{r_0}^{\infty} + GMm(\ln \infty - \ln r_0)$$
(1582)

$$W = \left[ -\frac{GMm}{r} \right]_{r_0}^{\infty} + GMm \ln \left( \frac{\infty}{r_0} \right) \quad (1583)$$

$$W = \left[ -\frac{GMm}{r} \right]_{r_0}^{\infty} \tag{1584}$$

$$W = -\frac{GMm}{\infty} + \frac{GMm}{r_0} \tag{1585}$$

$$W = 0 + \frac{GMm}{r_0}$$
 (1586)

$$W = \frac{GMm}{r_0} \tag{1587}$$

$$sinceGM = g, W = mgr_0. (1588)$$

#### C.89 Escape velocity 3

$$W = mgr_0 (1589)$$

$$v_{esc}^2 = 2W/m (1590)$$

$$v_{esc}^2 = 2mgr_0/m$$
 (1591)

$$v_{esc}^2 = 2gr_0 (1592)$$

$$v_{esc} = \sqrt{2gr_0}$$
 (1593) 2150

#### C.90 Snell's law: from Fermat's principle 2

$$\frac{dT}{dx} = \frac{x}{v_1(x^2 + a^2)^{\frac{1}{2}}} + \frac{x - l}{v_2((x - l)^2 + b^2)^{\frac{1}{2}}}$$
(1594)

$$\frac{dT}{dx} = \frac{\sin \theta_1}{v_1} + \frac{\sin \theta_2}{v_2} \tag{1595}$$

$$\frac{\sin \theta_1}{v_1} = \frac{x}{v_1(x^2 + a^2)^{\frac{1}{2}}} \tag{1596}$$

$$\frac{\sin \theta_2}{v_2} = \frac{x - l}{v_2((x - l)^2 + b^2)^{\frac{1}{2}}}$$
 (1597)

$$\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = \frac{x}{v_1(x^2 + a^2)^{\frac{1}{2}}} - \frac{x - l}{v_2((x - l)^2 + b^2)^{\frac{1}{2}}}$$
(1598)

$$\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0 \tag{1599}$$

#### C.91 Snell's law: from Fermat's principle 3

$$\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0 \tag{1600}$$

$$\frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2} \tag{1601}$$

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$
 (1602)

### C.92 Wave equation: plane wave eigenmodes

$$\frac{\partial^2 u(x,t)}{\partial t^2} = (-i\omega)\frac{\partial}{\partial t}(e^{-i\omega t}f(x)) \qquad (1603)$$

$$\frac{\partial}{\partial t}(e^{-i\omega t}f(x)) = -i\omega e^{-i\omega t}f(x) \qquad (1604)$$

$$(-i\omega)\frac{\partial}{\partial t}(e^{-i\omega t}f(x)) = -\omega^2 e^{-i\omega t}f(x) \quad (1605)$$

$$-\omega^2 e^{-i\omega t} f(x) = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$
 (1606)

#### C.93 Wave equation: plane wave eigenmodes

$$u(x,t) = Ae^{-i(kx-\omega t)} + Be^{i(kx-\omega t)}$$
 (1607)

$$s_{+}(\omega) = A \tag{1608}$$

$$s_{-}(\omega) = B \tag{1609}$$

$$\int_{-\infty}^{\infty} s_{+}(\omega)e^{-i(kx-\omega t)}d\omega = Ae^{-i(kx-\omega t)}$$
(1610)

$$\int_{-\infty}^{\infty} s_{-}(\omega)e^{i(kx-\omega t)}d\omega = Be^{i(kx-\omega t)} \quad (1611)$$

$$u(x,t) = \int_{-\infty}^{\infty} s_{+}(\omega)e^{-i(kx-\omega t)}d\omega + \int_{-\infty}^{\infty} s_{-}(\omega)e^{i(kx-\omega t)}d\omega$$
(1612)

### C.94 Wave equation: plane wave eigenmodes

$$u(x,t) = \int_{-\infty}^{\infty} s_{+}(\omega)e^{-i(kx-\omega t)}d\omega + \int_{-\infty}^{\infty} s_{-}(\omega)e^{i(kx-\omega t)}d\omega$$
(1613)

$$letF(x-ct) = \int_{-\infty}^{\infty} s_{+}(\omega)e^{-i(kx-\omega t)}d\omega$$
 (1614)

$$letG(x+ct) = \int_{-\infty}^{\infty} s_{-}(\omega)e^{i(kx-\omega t)}d\omega \quad (1615)$$

$$u(x,t) = F(x-ct) + G(x+ct)$$
 (1616)

#### C.95 Wave equation: Hooke's law

$$F_H = F_{x+2h} - F_x {(1617)}$$

$$m\frac{\partial^2}{\partial t^2}u(x+h,t) = F_H \tag{1618}$$

2185 
$$m\frac{\partial^2}{\partial t^2}u(x+h,t) = F_{x+2h} - F_x \qquad (1619)$$

#### C.96 Wave equation: Hooke's law 2

$$m\frac{\partial^2}{\partial t^2}u(x+h,t) = F_{x+2h} - F_x \qquad (1620)$$

$$F_{x+2h} - F_x = k(u(x+2h,t) - u(x+h,t))$$
(1621)

$$m\frac{\partial^2}{\partial t^2}u(x+h,t) = k\left(u(x+2h,t) - u(x+h,t)\right)$$
(1622)

$$\frac{\partial^2}{\partial t^2}u(x+h,t) = \frac{k}{m} (u(x+2h,t) - u(x+h,t))$$
(1623)

$$F_x = k(u(x+h,t) - u(x,t))$$
 (1624)

$$m\frac{\partial^2}{\partial t^2}u(x+h,t) = k\left(u(x+2h,t) - 2u(x+h,t) + u(x,t)\right)$$
(1625)

$$\frac{\partial^2}{\partial t^2}u(x+h,t) = \frac{k}{m} \left( u(x+2h,t) - 2u(x+h,t) + u(x,t) \right)$$
(1626)

#### C.97 Wave equation: Hooke's law 3

$$\frac{\partial^2}{\partial t^2}u(x+h,t) = \frac{k}{m} \left( u(x+2h,t) - 2u(x+h,t) + u(x,t) \right)$$
(1627)

$$\frac{KL^2}{Mh^2} = \frac{k}{m} \tag{1628}$$

$$\frac{\partial^2}{\partial t^2}u(x+h,t) = \frac{KL^2}{Mh^2} \left( u(x+2h,t) - 2u(x+h,t) + u(x,t) \right) \tag{1629}$$

#### C.98 Wave equation: stress pulse in a bar 2

$$Let KL^2 = E (1630)$$

$$M = \rho, then \frac{KL^2}{M} = \frac{E}{\rho}$$
 (1631)

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{KL^2}{M} \frac{\partial^2 u(x,t)}{\partial x^2}$$
 (1632)

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{E}{\rho} \frac{\partial^2 u(x,t)}{\partial x^2}$$
 (1633)