Supplementary Model Derivations

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Abstract

This document contains derivations generated by GPT-4 following the few-shot prompting scheme discussed in the corresponding paper.

A $S = 0$ (no premises removed)

A.1 Gauss' law: equivalence between differential and integral forms

$$\oiint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0} \tag{1}$$

$$\oiint_S \mathbf{E} \cdot d\mathbf{A} = \iiint_V \nabla \cdot \mathbf{E} dV \tag{2}$$

$$\frac{Q}{\varepsilon_0} = \iiint_V \nabla \cdot \mathbf{E} dV \tag{3}$$

$$Q = \iiint_V \rho dV \tag{4}$$

$$\frac{\iiint_V \rho dV}{\varepsilon_0} = \iiint_V \nabla \cdot \mathbf{E} dV \tag{5}$$

$$\iiint_V \nabla \cdot \mathbf{E} dV = \iiint_V \frac{\rho}{\varepsilon_0} dV \tag{6}$$

A.2 Gauss' law: Equivalence of total and free charge statements

$$\rho_b = -\nabla \cdot \mathbf{P} \tag{7}$$

$$-\rho_b = \nabla \cdot \mathbf{P} \tag{8}$$

$$\rho_f = \nabla \cdot \mathbf{D} \tag{9}$$

$$\rho = \rho_b + \rho_f \tag{10}$$

$$\rho = -\nabla \cdot \mathbf{P} + \nabla \cdot \mathbf{D} \tag{11}$$

$$\rho = \nabla \cdot (\mathbf{D} - \mathbf{P}) \tag{12}$$

A.3 Uniqueness theorem for Poisson's equation

$$\nabla^2 \phi_1 = -\frac{\rho_f}{\varepsilon_0} \tag{13}$$

$$\nabla^2 \phi_2 = -\frac{\rho_f}{\varepsilon_0} \tag{14}$$

$$\nabla^2 (\phi_2 - \phi_1) = \nabla^2 \phi_2 - \nabla^2 \phi_1 \tag{15}$$

$$\nabla^2 (\phi_2 - \phi_1) = -\frac{\rho_f}{\varepsilon_0} - (-\frac{\rho_f}{\varepsilon_0}) \tag{16}$$

$$\nabla^2 (\phi_2 - \phi_1) = 0 \tag{17}$$

$$\phi = \phi_2 - \phi_1 \tag{18}$$

$$\nabla^2 \phi = 0 \tag{19}$$

A.4 Uniqueness theorem for Poisson's equation 2

$$\nabla \cdot (\phi \nabla \phi) = (\nabla \phi)^2 + \phi \nabla^2 \phi \tag{20}$$

$$\nabla^2 \phi = 0 \tag{21}$$

$$\nabla \cdot (\phi \nabla \phi) = (\nabla \phi)^2 \tag{22}$$

$$\int_V \nabla \cdot (\phi \nabla \phi) dV = \int_V (\nabla \phi)^2 dV \tag{23}$$

$$\int_V \nabla \cdot (\phi \nabla \phi) dV = \int_S \phi \nabla \phi \cdot d\mathbf{S} \tag{24}$$

$$\int_S \phi \nabla \phi \cdot d\mathbf{S} = \int_V (\nabla \phi)^2 dV \tag{25}$$

A.5 Uniqueness theorem for Poisson's equation 6

$$\frac{\partial \phi}{\partial r} = 0 \tag{26}$$

$$\int \frac{\partial \phi}{\partial r} dr = \int 0 dr \tag{27}$$

$$\int 0 dr = C_1 \tag{28}$$

$$\int \frac{\partial \phi}{\partial r} dr = \phi + C_2 \tag{29}$$

$$\phi + C_2 = C_1 \tag{30}$$

$$\phi = C_1 - C_2 \tag{31}$$

A.6 Uniqueness theorem for Poisson's equation 7

$$\phi = C_1 - C_2 \tag{32}$$

$$C = C_1 - C_2 \tag{33}$$

$$\phi = C \tag{34}$$

$$\phi = \phi_2 - \phi_1 \tag{35}$$

$$\phi_2 - \phi_1 = C \tag{36}$$

A.7 Poisson's equation: Newtonian gravity

$$\nabla \cdot \mathbf{g} = -4\pi G\rho \tag{37}$$

$$\mathbf{g} = -\nabla \phi \tag{38}$$

$$\nabla \cdot (-\nabla \phi) = -4\pi G\rho \tag{39}$$

$$-\nabla^2 \phi = -4\pi G\rho \tag{40}$$

$$\nabla^2 \phi = 4\pi G\rho \tag{41}$$

A.8 Poisson's equation: Gravitational potential from Poisson's equation 2

$$\int_V \nabla \cdot \nabla \phi dV = \int_V 4\pi G\rho dV \tag{42}$$

$$m = \int_V \rho dV \tag{43}$$

$$4\pi Gm = \int_V 4\pi G\rho dV \tag{44}$$

$$\int_V \nabla \cdot \nabla \phi dV = \int_S \nabla \phi \cdot d\mathbf{S} \tag{45}$$

$$\int_S \nabla \phi \cdot d\mathbf{S} = 4\pi Gm \tag{46}$$

A.9 Poisson's equation: Gravitational potential from Poisson's equation 6

$$\int_S \frac{\partial \phi}{\partial r} dS = \int_0^{2\pi} \int_0^{\pi} \frac{\partial \phi}{\partial r} r^2 \sin\theta d\theta d\varphi \tag{47}$$

$$\int_0^{\pi} \sin\theta d\theta = 2 \tag{48}$$

$$\int_0^{2\pi} d\varphi = 2\pi \tag{49}$$

$$\int_0^{2\pi} \int_0^{\pi} \frac{\partial \phi}{\partial r} r^2 \sin\theta d\theta d\varphi = \frac{\partial \phi}{\partial r} r^2 \int_0^{2\pi} \int_0^{\pi} \sin\theta d\theta d\varphi \tag{50}$$

$$\frac{\partial \phi}{\partial r} r^2 \int_0^{2\pi} \int_0^{\pi} \sin\theta d\theta d\varphi = \frac{\partial \phi}{\partial r} r^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \tag{51}$$

$$\frac{\partial \phi}{\partial r} r^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta = \frac{\partial \phi}{\partial r} r^2 (2\pi \cdot 2) \tag{52}$$

$$\frac{\partial \phi}{\partial r} r^2 (2\pi \cdot 2) = 4\pi \frac{\partial \phi}{\partial r} r^2 \tag{53}$$

$$\int_S \frac{\partial \phi}{\partial r} dS = 4\pi \frac{\partial \phi}{\partial r} r^2 \tag{54}$$

A.10 Poisson's equation: Gravitational potential from Poisson's equation 8

$$\frac{\partial \phi}{\partial r} = \frac{Gm}{r^2} \tag{55}$$

$$\int_{\infty}^{r} \frac{\partial \phi}{\partial r} dr = \phi(r) - \phi(\infty) \tag{56}$$

$$\int_{\infty}^{r} \frac{Gm}{r^2} dr = \frac{-Gm}{r} \tag{57}$$

$$\phi(\infty) = 0 \tag{58}$$

$$\phi(r) - \phi(\infty) = \frac{-Gm}{r} \tag{59}$$

$$\phi(r) = \frac{-Gm}{r} \tag{60}$$

A.11 Poisson's equation: Electrostatics

$$\nabla \cdot \mathbf{D} = \rho_f \tag{61}$$

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{62}$$

$$\nabla \cdot \varepsilon \mathbf{E} = \rho_f \tag{63}$$

$$\mathbf{E} = -\nabla \phi \tag{64}$$

$$\varepsilon(-\nabla \phi) = \rho_f \tag{65}$$

$$-\varepsilon \nabla \phi = \rho_f \tag{66}$$

$$-\nabla \cdot \varepsilon \nabla \phi = \rho_f \tag{67}$$

$$\nabla \cdot \nabla \phi = \nabla^2 \phi \tag{68}$$

$$-\varepsilon \nabla^2 \phi = \rho_f \tag{69}$$

$$\nabla^2 \phi = -\frac{\rho_f}{\varepsilon} \tag{70}$$

A.12 Poisson's equation: Electrostatic potential from Poisson's equation

$$\nabla^2 \phi = -\frac{\rho_f}{\varepsilon} \tag{71}$$

$$\nabla^2 \phi = \nabla \cdot \nabla \phi \tag{72}$$

$$\int_V \nabla^2 \phi dV = \int_V \nabla \cdot \nabla \phi dV \tag{73}$$

$$\int_V \nabla^2 \phi dV = -\frac{1}{\varepsilon} \int_V \rho_f dV \tag{74}$$

$$\int_V \nabla^2 \phi dV = -\frac{Q}{\varepsilon} \tag{75}$$

$$\int_V \nabla \cdot \nabla \phi dV = -\frac{Q}{\varepsilon} \tag{76}$$

A.13 Poisson's equation: Electrostatic potential from Poisson's equation 2

$$\int_V \nabla \cdot \nabla \phi dV = -\frac{Q}{\varepsilon} \tag{77}$$

$$\int_V \nabla \cdot \nabla \phi dV = \int_S \nabla \phi \cdot d\mathbf{S} \tag{78}$$

$$-\frac{Q}{\varepsilon} = \int_S \nabla \phi \cdot d\mathbf{S} \tag{79}$$

$$\nabla \phi \cdot d\mathbf{S} = \frac{\partial \phi}{\partial r} dS \tag{80}$$

$$\int_S \nabla \phi \cdot d\mathbf{S} = \int_S \frac{\partial \phi}{\partial r} dS \tag{81}$$

$$\int_S \frac{\partial \phi}{\partial r} dS = -\frac{Q}{\varepsilon} \tag{82}$$

A.14 Poisson's equation: Electrostatic potential from Poisson's equation 4

$$\int_{\infty}^{r} \frac{\partial \phi}{\partial r} dr = \int_{\infty}^{r} -\frac{Q}{4\pi\varepsilon r^2} dr \tag{83}$$

$$\int_{\infty}^{r} \frac{\partial \phi}{\partial r} dr = \phi(r) - \phi(\infty) \tag{84}$$

$$\int_{\infty}^{r} -\frac{Q}{4\pi\varepsilon r^2} dr = \frac{Q}{4\pi\varepsilon r} \tag{85}$$

$$\phi(\infty) = 0 \tag{86}$$

$$\phi(r) - \phi(\infty) = \frac{Q}{4\pi\varepsilon r} \tag{87}$$

$$\phi(r) - 0 = \frac{Q}{4\pi\varepsilon r} \tag{88}$$

$$\phi(r) = \frac{Q}{4\pi\varepsilon r} \tag{89}$$

A.15 Lorentz force: continuous charge distribution

$$\frac{d\mathbf{F}}{dV} = \frac{dq}{dV}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{90}$$

$$\mathbf{f} = \frac{d\mathbf{F}}{dV} \tag{91}$$

$$\rho = \frac{dq}{dV} \tag{92}$$

$$\mathbf{f} = \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{93}$$

$$\mathbf{f} = \rho\mathbf{E} + \rho\mathbf{v} \times \mathbf{B} \tag{94}$$

A.16 Lorentz force: continuous charge distribution 2

$$\mathbf{f} = \rho\mathbf{E} + \rho\mathbf{v} \times \mathbf{B} \tag{95}$$

$$\mathbf{J} = \rho\mathbf{v} \tag{96}$$

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{97}$$

$$\mathbf{f} = \frac{d\mathbf{F}}{dV} \tag{98}$$

$$\rho\mathbf{E} + \mathbf{J} \times \mathbf{B} = \frac{d\mathbf{F}}{dV} \tag{99}$$

$$\mathbf{F} = \iiint \frac{d\mathbf{F}}{dV} dV \tag{100}$$

$$\mathbf{F} = \iiint (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) dV \tag{101}$$

A.17 Lorentz force: Lorentz force in terms of potentials

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{102}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{103}$$

$$\mathbf{v} \times \mathbf{B} = \mathbf{v} \times (\nabla \times \mathbf{A}) \tag{104}$$

$$\mathbf{v} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A} \tag{105}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A} \tag{106}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{107}$$

$$\mathbf{F} = q(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A}) \tag{108}$$

A.18 Lorentz force: Potential energy derivation from scalar potential 3

$$U = q\int_{\infty}^{r} \nabla\phi \cdot d\mathbf{r} \tag{109}$$

$$\int_{\infty}^{r} \nabla\phi \cdot d\mathbf{r} = \int_{\infty}^{r} \frac{\partial \phi}{\partial r} dr \tag{110}$$

$$\int_{\infty}^{r} \frac{\partial \phi}{\partial r} dr = \phi(r) - \phi(\infty) \tag{111}$$

$$\phi(\infty) = 0 \tag{112}$$

$$\phi(r) - \phi(\infty) = \phi(r) \tag{113}$$

$$q\int_{\infty}^{r} \nabla\phi \cdot d\mathbf{r} = q\phi(r) \tag{114}$$

$$U = q\phi(r) \tag{115}$$

A.19 Laplace equation: Analytic functions (u)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{116}$$

$$\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial x^2} \tag{117}$$

$$\frac{\partial}{\partial x}(\frac{\partial v}{\partial y}) = \frac{\partial^2 v}{\partial x \partial y} \tag{118}$$

$$\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial v}{\partial y}) \tag{119}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{120}$$

A.20 Laplace equation: Analytic functions (u) 2

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{121}$$

$$\frac{\partial}{\partial y}(\frac{\partial v}{\partial x}) = \frac{\partial}{\partial y}(-\frac{\partial u}{\partial y}) \tag{122}$$

$$\frac{\partial}{\partial y}(-\frac{\partial u}{\partial y}) = -\frac{\partial}{\partial y}(\frac{\partial u}{\partial y}) \tag{123}$$

$$-\frac{\partial}{\partial y}(\frac{\partial u}{\partial y}) = -\frac{\partial^2 u}{\partial y^2} \tag{124}$$

$$\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial v}{\partial x}) \tag{125}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \tag{126}$$

A.21 Laplace equation: Analytic functions (u) 3

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad (127)$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad (128)$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} \quad (129)$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2} \quad (130)$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad (131)$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x^2} \quad (132)$$

$$\nabla^2 u = 0 \quad (133)$$

A.22 Laplace equation: Analytic functions (v)

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad (134)$$

$$\frac{\partial}{\partial x}(\frac{\partial v}{\partial x}) = \frac{\partial^2 v}{\partial x^2} \quad (135)$$

$$\frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) = \frac{\partial^2 u}{\partial x \partial y} \quad (136)$$

$$\frac{\partial}{\partial x}(-\frac{\partial u}{\partial y}) = -\frac{\partial^2 u}{\partial x \partial y} \quad (137)$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \quad (138)$$

A.23 Laplace equation: Analytic functions (v) 2

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (139)$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial v}{\partial y}) \quad (140)$$

$$\frac{\partial}{\partial y}(\frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial y \partial x} \quad (141)$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial u}{\partial x}) \quad (142)$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} \quad (143)$$

A.24 Laplace equation: Analytic functions (v) 3

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \quad (144)$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} \quad (145)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad (146)$$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \quad (147)$$

$$\nabla^2 v = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial x} \quad (148)$$

$$\nabla^2 v = 0 \quad (149)$$

A.25 Laplace equation: Electrostatics

$$\mathbf{E} = (u, v) \quad (150)$$

$$\nabla \cdot \mathbf{E} = \rho \quad (151)$$

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \quad (152)$$

$$\nabla \cdot \mathbf{E} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \cdot (u, v) \quad (153)$$

$$\nabla \cdot \mathbf{E} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad (154)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \rho \quad (155)$$

A.26 Laplace equation: Electrostatics 2

$$\frac{\partial \phi}{\partial x} = -u \quad (156)$$

$$\frac{\partial \phi}{\partial y} = -v \quad (157)$$

$$\frac{\partial u}{\partial x} = \rho - \frac{\partial v}{\partial y} \quad (158)$$

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{\partial u}{\partial x} \quad (159)$$

$$\frac{\partial^2 \phi}{\partial y^2} = -\frac{\partial v}{\partial y} \quad (160)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\rho \quad (161)$$

A.27 Laplace equation: Electrostatics 3

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \tag{162}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\rho \tag{163}$$

$$-\rho = 0 \tag{164}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{165}$$

$$\nabla^2\phi = 0 \tag{166}$$

A.28 Lorentz force: Potential energy derivation from vector potential

$$\mathbf{F} = q(\nabla(\mathbf{v}\cdot\mathbf{A}) - \frac{d\mathbf{A}}{dt}) \tag{167}$$

$$\frac{d\mathbf{A}}{dt} = 0 \tag{168}$$

$$\mathbf{F} = q\nabla(\mathbf{v}\cdot\mathbf{A}) \tag{169}$$

$$U = -\int_\infty^r \mathbf{F}\cdot d\mathbf{r} \tag{170}$$

$$U = -q\int_\infty^r \nabla(\mathbf{v}\cdot\mathbf{A})\cdot d\mathbf{r} \tag{171}$$

A.29 Lorentz force: Potential energy derivation from vector potential 3

$$\nabla(\mathbf{v}\cdot\mathbf{A})\cdot\hat{\mathbf{r}} = \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r} \tag{172}$$

$$d\mathbf{r} = \hat{\mathbf{r}}dr \tag{173}$$

$$\nabla(\mathbf{v}\cdot\mathbf{A})\cdot d\mathbf{r} = \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r}dr \tag{174}$$

$$U = -q\int_\infty^r \nabla(\mathbf{v}\cdot\mathbf{A})\cdot d\mathbf{r} \tag{175}$$

$$U = -q\int_\infty^r \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r}dr \tag{176}$$

A.30 Lorentz force: Potential energy derivation from vector potential 4

$$U = -q\int_\infty^r \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r}dr \tag{177}$$

$$\int_\infty^r \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r}dr = \mathbf{v}\cdot\mathbf{A}(r) - \mathbf{v}\cdot\mathbf{A}(\infty) \tag{178}$$

$$U = -q(\mathbf{v}\cdot\mathbf{A}(r) - \mathbf{v}\cdot\mathbf{A}(\infty)) \tag{179}$$

$$\mathbf{A}(\infty) = 0 \tag{180}$$

$$U = -q(\mathbf{v}\cdot\mathbf{A}(r) - 0) \tag{181}$$

$$U = -q\mathbf{v}\cdot\mathbf{A}(r) \tag{182}$$

A.31 Lorentz force: Derivation of classical Lagrangian of EM field

$$V = q\phi - q\dot{\mathbf{r}}\cdot\mathbf{A} \tag{183}$$

$$T = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} \tag{184}$$

$$L = T - V \tag{185}$$

$$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} - (q\phi - q\dot{\mathbf{r}}\cdot\mathbf{A}) \tag{186}$$

$$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} - q\phi + q\dot{\mathbf{r}}\cdot\mathbf{A} \tag{187}$$

$$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\dot{\mathbf{r}}\cdot\mathbf{A} - q\phi \tag{188}$$

A.32 Lorentz force: Derivation of classical Lagrangian of EM field 2

$$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\dot{\mathbf{r}}\cdot\mathbf{A} - q\phi \tag{189}$$

$$\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z}) \tag{190}$$

$$\mathbf{A} = (A_x, A_y, A_z) \tag{191}$$

$$L = \frac{m}{2}(\dot{x}, \dot{y}, \dot{z})\cdot(\dot{x}, \dot{y}, \dot{z}) + q(\dot{x}, \dot{y}, \dot{z})\cdot\mathbf{A} - q\phi \tag{192}$$

$$L = \frac{m}{2}(\dot{x}, \dot{y}, \dot{z})\cdot(\dot{x}, \dot{y}, \dot{z}) + q(\dot{x}, \dot{y}, \dot{z})\cdot(A_x, A_y, A_z) - q\phi \tag{193}$$

A.33 Lorentz force: Derivation of Lorentz force from classical Lagrangian (LHS) 4

$$dA_x = \frac{\partial A_x}{\partial t}dt + \frac{\partial A_x}{\partial x}dx + \frac{\partial A_x}{\partial y}dy + \frac{\partial A_x}{\partial z}dz \tag{194}$$

$$\frac{dx}{dt} = \dot{x} \tag{195}$$

$$\frac{dy}{dt} = \dot{y} \tag{196}$$

$$\frac{dz}{dt} = \dot{z} \tag{197}$$

$$dt = 1 \tag{198}$$

$$dx = \dot{x}dt \tag{199}$$

$$dy = \dot{y}dt \tag{200}$$

$$dz = \dot{z}dt \tag{201}$$

$$dA_x = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x}dt + \frac{\partial A_x}{\partial y}\dot{y}dt + \frac{\partial A_x}{\partial z}\dot{z}dt \tag{202}$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} \tag{203}$$

A.34 Lorentz force: Derivation of Lorentz force from classical Lagrangian (LHS) 5

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\frac{d}{dt}\dot{x} + q\frac{d}{dt}A_x \tag{204}$$

$$\frac{d}{dt}\dot{x} = \ddot{x} \tag{205}$$

$$m\frac{d}{dt}\dot{x} = m\ddot{x} \tag{206}$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} \tag{207}$$

$$q\frac{dA_x}{dt} = q(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}) \tag{208}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + q(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}) \tag{209}$$

A.35 Lorentz force: Derivation of Lorentz force from classical Lagrangian (RHS) 2

$$\frac{\partial L}{\partial x} = q\frac{\partial}{\partial x}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) - q\frac{\partial}{\partial x}\phi \tag{210}$$

$$I = \frac{\partial}{\partial x}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) \tag{211}$$

$$\frac{\partial}{\partial x}(\dot{x}A_x) + \frac{\partial}{\partial x}(\dot{y}A_y) + \frac{\partial}{\partial x}(\dot{z}A_z) = \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z} \tag{212}$$

$$q\frac{\partial}{\partial x}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) = q(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}) \tag{213}$$

$$\frac{\partial L}{\partial x} = q(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}) - q\frac{\partial}{\partial x}\phi \tag{214}$$

A.36 Lorentz force: Derivation of x component of electric field

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{215}$$

$$\nabla\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) \tag{216}$$

$$\mathbf{A} = (A_x, A_y, A_z) \tag{217}$$

$$\mathbf{E} = -((\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) - \frac{\partial}{\partial t}(A_x, A_y, A_z)) \tag{218}$$

$$\mathbf{E}\cdot(1,0,0) = -((\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) - \frac{\partial}{\partial t}(A_x, A_y, A_z))\cdot(1,0,0) \tag{219}$$

A.37 Lorentz force: Derivation of Lorentz force from classical Lagrangian 4

$$(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_x = \dot{y}(\nabla \times \mathbf{A})_z - \dot{z}(\nabla \times \mathbf{A})_y \tag{220}$$

$$(\nabla \times \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \tag{221}$$

$$(\nabla \times \mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \tag{222}$$

$$\dot{y}(\nabla \times \mathbf{A})_z - \dot{z}(\nabla \times \mathbf{A})_y = \dot{y}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) - \dot{z}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) \tag{223}$$

$$F_x = qE_x + q(\dot{y}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) + \dot{z}(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z})) \tag{224}$$

$$F_x = qE_x + q(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_x \tag{225}$$

A.38 Lorentz force: Derivation of Lorentz force from classical Lagrangian 5

$$F_x = qE_x + q(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_x \tag{226}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{227}$$

$$q(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_x = q(\dot{\mathbf{r}} \times \mathbf{B})_x \tag{228}$$

$$F_x = qE_x + q(\dot{\mathbf{r}} \times \mathbf{B})_x \tag{229}$$

$$F_x = \mathbf{F} \cdot \hat{\mathbf{x}} \tag{230}$$

$$qE_x = q\mathbf{E} \cdot \hat{\mathbf{x}} \tag{231}$$

$$q(\dot{\mathbf{r}} \times \mathbf{B})_x = q(\dot{\mathbf{r}} \times \mathbf{B}) \cdot \hat{\mathbf{x}} \tag{232}$$

$$\mathbf{F} \cdot \hat{\mathbf{x}} = q\mathbf{E} \cdot \hat{\mathbf{x}} + q(\dot{\mathbf{r}} \times \mathbf{B}) \cdot \hat{\mathbf{x}} \tag{233}$$

A.39 Electromagnetic wave equation: The origin of the electromagnetic wave equation in 2

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \tag{234}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{235}$$

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = \frac{\partial}{\partial t}(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}) \tag{236}$$

$$\frac{\partial}{\partial t}(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\frac{\partial \mathbf{E}}{\partial t}) \tag{237}$$

$$\frac{\partial}{\partial t}(\frac{\partial \mathbf{E}}{\partial t}) = \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{238}$$

$$\mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\frac{\partial \mathbf{E}}{\partial t}) = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{239}$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{240}$$

A.40 Electromagnetic wave equation: The origin of the electromagnetic wave equation in 3

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \tag{241}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{242}$$

$$\nabla(\nabla \cdot \mathbf{E}) = \nabla(0) \tag{243}$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{244}$$

$$-\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla(0) - \nabla^2 \mathbf{E} \tag{245}$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0 \tag{246}$$

A.41 Electromagnetic wave equation: The origin of the electromagnetic wave equation in 2

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) \tag{247}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{248}$$

$$\mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\frac{\partial \mathbf{B}}{\partial t}) \tag{249}$$

$$\frac{\partial}{\partial t}(\frac{\partial \mathbf{B}}{\partial t}) = \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{250}$$

$$-\mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\frac{\partial \mathbf{B}}{\partial t}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{251}$$

$$\nabla \times (\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{252}$$

A.42 Electromagnetic wave equation: The origin of the electromagnetic wave equation in 3

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \tag{253}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{254}$$

$$\nabla(\nabla \cdot \mathbf{B}) = \nabla(0) \tag{255}$$

$$\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{B} \tag{256}$$

$$\nabla \times (\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{257}$$

$$-\nabla^2 \mathbf{B} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{258}$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0 \tag{259}$$

A.43 Ampere's circuital law: Proof of equivalence 2

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{260}$$

$$\nabla \times \frac{1}{\mu_0} \mathbf{B} = \nabla \times \mathbf{H} + \mathbf{J}_M \tag{261}$$

$$\nabla \times \frac{1}{\mu_0} \mathbf{B} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}_M \tag{262}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{263}$$

$$\frac{\partial \mathbf{D}}{\partial t} = \frac{\partial}{\partial t} \varepsilon_0 \mathbf{E} + \frac{\partial}{\partial t} \mathbf{P} \tag{264}$$

$$\nabla \times \frac{1}{\mu_0} \mathbf{B} = \mathbf{J}_f + \frac{\partial}{\partial t} \varepsilon_0 \mathbf{E} + \frac{\partial}{\partial t} \mathbf{P} + \mathbf{J}_M \tag{265}$$

A.44 Ampere's circuital law: Proof of equivalence 4

$$\frac{1}{\mu_0} (\nabla \times \mathbf{B}) = \mathbf{J}_f + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_P + \mathbf{J}_M \tag{266}$$

$$\mathbf{J}_b = \mathbf{J}_P + \mathbf{J}_M \tag{267}$$

$$\mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + \mathbf{J}_P + \mathbf{J}_M \tag{268}$$

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b \tag{269}$$

$$\frac{1}{\mu_0} (\nabla \times \mathbf{B}) = \mathbf{J}_f + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_b \tag{270}$$

$$\frac{1}{\mu_0} (\nabla \times \mathbf{B}) = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{271}$$

A.45 Uncertainty principle: Kennard inequality proof part 1.1

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 \tag{272}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \cdot p(x) dx \tag{273}$$

$$p(x) = |\psi(x)|^2 \tag{274}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \cdot p(x) dx \tag{275}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \cdot |\psi(x)|^2 dx \tag{276}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \cdot |\psi(x)|^2 dx \tag{277}$$

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 \tag{278}$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} x^2 \cdot |\psi(x)|^2 dx - (\int_{-\infty}^{\infty} x \cdot |\psi(x)|^2 dx)^2 \tag{279}$$

A.46 Uncertainty principle: Kennard inequality proof part 1.4

$$f^*(x) \cdot f(x) = x^2 \cdot (\psi^*(x) \cdot \psi(x)) \tag{280}$$

$$\psi^*(x) \cdot \psi(x) = |\psi(x)|^2 \tag{281}$$

$$x^2 \cdot |\psi(x)|^2 = x^2 \cdot f^*(x) \cdot f(x) \tag{282}$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} x^2 \cdot |\psi(x)|^2 dx \tag{283}$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} x^2 \cdot f^*(x) \cdot f(x) dx \tag{284}$$

$$\langle f | f \rangle = \int_{-\infty}^{\infty} f^*(x) \cdot f(x) dx \tag{285}$$

$$\sigma_x^2 = \langle f | f \rangle \tag{286}$$

A.47 Uncertainty principle: Kennard inequality proof part 2.2

$$\frac{dv}{d\chi} = e^{\frac{-ip\chi}{\hbar}} \tag{287}$$

$$v = \int \frac{dv}{d\chi} d\chi \tag{288}$$

$$b = \frac{-ip\chi}{\hbar} \tag{289}$$

$$\frac{dv}{d\chi} = e^{b} \tag{290}$$

$$v = \int e^{b} d\chi \tag{291}$$

$$v = \frac{\hbar}{-ip} e^{b} + C \tag{292}$$

A.48 Uncertainty principle: Kennard inequality proof part 2.3

$$u = \psi(\chi) \tag{293}$$

$$v = \frac{\hbar}{-ip} e^{b} \tag{294}$$

$$b = \frac{-ip\chi}{\hbar} \tag{295}$$

$$v = \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \tag{296}$$

$$uv = \psi(\chi) \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \tag{297}$$

A.49 Uncertainty principle: Kennard inequality proof part 2.4

$$uv = \psi(\chi) \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \tag{298}$$

$$uv(\infty) = \psi(\infty) \frac{\hbar}{-ip} e^{\frac{-ip\infty}{\hbar}} \tag{299}$$

$$uv(-\infty) = \psi(-\infty) \frac{\hbar}{-ip} e^{\frac{-ip(-\infty)}{\hbar}} \tag{300}$$

$$\psi(\infty) = 0 \tag{301}$$

$$\psi(-\infty) = 0 \tag{302}$$

$$uv(\infty) = 0 \frac{\hbar}{-ip} e^{\frac{-ip\infty}{\hbar}} \tag{303}$$

$$uv(-\infty) = 0 \frac{\hbar}{-ip} e^{\frac{-ip(-\infty)}{\hbar}} \tag{304}$$

$$uv(\infty) = 0 \tag{305}$$

$$uv(-\infty) = 0 \tag{306}$$

$$(uv)\Big|_{-\infty}^{\infty} = 0 \tag{307}$$

A.50 Uncertainty principle: Kennard inequality proof part 2.5

$$I = (uv)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v \frac{du}{d\chi} d\chi \tag{308}$$

$$(uv)\Big|_{-\infty}^{\infty} = 0 \tag{309}$$

$$I = -\int_{-\infty}^{\infty} v \frac{du}{d\chi} d\chi \tag{310}$$

$$\frac{du}{d\chi} = \frac{d\psi(\chi)}{d\chi} \tag{311}$$

$$v = \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \tag{312}$$

$$I = -\int_{-\infty}^{\infty} \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \frac{d\psi(\chi)}{d\chi} d\chi \tag{313}$$

$$I = \frac{\hbar}{ip} \int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi} e^{\frac{-ip\chi}{\hbar}} d\chi \tag{314}$$

A.51 Uncertainty principle: Kennard inequality proof part 2.9

$$g(x) = \frac{\hbar}{2\pi i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\psi(\chi)}{d\chi}e^{i(x-\chi)b}d\chi db \tag{315}$$

$$\int_{-\infty}^{\infty}e^{i(x-\chi)b}db = 2\pi\delta(x-\chi) \tag{316}$$

$$\frac{\hbar}{2\pi i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\psi(\chi)}{d\chi}2\pi\delta(x-\chi)d\chi db = \frac{\hbar}{i}\int_{-\infty}^{\infty}\frac{d\psi(\chi)}{d\chi}\delta(x-\chi)d\chi \tag{317}$$

$$\int_{-\infty}^{\infty}\frac{d\psi(\chi)}{d\chi}2\pi\delta(x-\chi)d\chi = 2\pi\frac{d\psi(x)}{dx} \tag{318}$$

$$\frac{\hbar}{i}\int_{-\infty}^{\infty}\frac{d\psi(\chi)}{d\chi}\delta(x-\chi)d\chi = \frac{\hbar}{i}2\pi\frac{d\psi(x)}{dx} \tag{319}$$

$$g(x) = \frac{\hbar}{i}(\frac{d\psi(x)}{dx}) \tag{320}$$

A.52 Uncertainty principle: Kennard inequality proof part 3.2

$$\tilde{g}^*(p)\cdot\tilde{g}(p) = p^2\varphi^*(p)\cdot\varphi(p) \tag{321}$$

$$\varphi^*(p)\cdot\varphi(p) = |\varphi(p)|^2 \tag{322}$$

$$p^2\varphi^*(p)\cdot\varphi(p) = p^2|\varphi(p)|^2 \tag{323}$$

$$\tilde{g}^*(p)\cdot\tilde{g}(p) = |\tilde{g}(p)|^2 \tag{324}$$

$$|\tilde{g}(p)|^2 = p^2|\varphi(p)|^2 \tag{325}$$

A.53 Uncertainty principle: Kennard inequality proof part 3.3

$$\sigma_p^2 = \int_{-\infty}^{\infty}p^2|\varphi(p)|^2dp \tag{326}$$

$$|\tilde{g}(p)|^2 = p^2|\varphi(p)|^2 \tag{327}$$

$$\int_{-\infty}^{\infty}|\tilde{g}(p)|^2dp = \int_{-\infty}^{\infty}p^2|\varphi(p)|^2dp \tag{328}$$

$$\int_{-\infty}^{\infty}|\tilde{g}(p)|^2dp = \int_{-\infty}^{\infty}|g(x)|^2dx \tag{329}$$

$$\langle g|g\rangle = \int_{-\infty}^{\infty}|g(x)|^2dx \tag{330}$$

$$\sigma_p^2 = \langle g|g\rangle \tag{331}$$

A.54 Uncertainty principle: Kennard inequality proof part 4.1

$$\sigma_x^2 = \langle f|f\rangle \tag{332}$$

$$\sigma_p^2 = \langle g|g\rangle \tag{333}$$

$$\langle f|f\rangle\cdot\langle g|g\rangle = \sigma_x^2\sigma_p^2 \tag{334}$$

$$\sigma_x^2\sigma_p^2 \geq |\langle f|g\rangle|^2 \tag{335}$$

A.55 Uncertainty principle: Kennard inequality proof part 4.3

$$|z|^2 = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2 \tag{336}$$

$$(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2 \geq (\mathrm{Im}(z))^2 \tag{337}$$

$$(\mathrm{Im}(z))^2 = (\frac{\langle f|g\rangle - \langle g|f\rangle}{2i})^2 \tag{338}$$

$$|z|^2 \geq (\mathrm{Im}(z))^2 \tag{339}$$

$$|z|^2 \geq (\frac{\langle f|g\rangle - \langle g|f\rangle}{2i})^2 \tag{340}$$

A.56 Uncertainty principle: Kennard inequality proof part 4.4

$$\sigma_x^2\sigma_p^2 \geq |\langle f|g\rangle|^2 \tag{341}$$

$$|z|^2 \geq (\frac{\langle f|g\rangle - \langle g|f\rangle}{2i})^2 \tag{342}$$

$$z = \langle f|g\rangle \tag{343}$$

$$|\langle f|g\rangle|^2 = (\frac{\langle f|g\rangle - \langle g|f\rangle}{2i})^2 \tag{344}$$

$$\sigma_x^2\sigma_p^2 \geq (\frac{\langle f|g\rangle - \langle g|f\rangle}{2i})^2 \tag{345}$$

A.57 Uncertainty principle: Kennard inequality proof part 5.1

$$f(x) = x \cdot \psi(x) \tag{346}$$

$$g(x) = (-i\hbar\frac{d}{dx}) \cdot \psi(x) \tag{347}$$

$$\langle f|g\rangle = \int_{-\infty}^{\infty} f^*(x) \cdot g(x)dx \tag{348}$$

$$\langle f|g\rangle = \int_{-\infty}^{\infty} (x \cdot \psi^*(x)) \cdot ((-i\hbar\frac{d}{dx}) \cdot \psi(x))dx \tag{349}$$

$$\langle f|g\rangle = -i\hbar \int_{-\infty}^{\infty} x\psi^*(x)\frac{d\psi(x)}{dx}dx \tag{350}$$

A.58 Uncertainty principle: Kennard inequality proof part 5.2

$$f(x) = x \cdot \psi(x) \tag{351}$$

$$g^*(x) = \psi^*(x) \cdot (-i\hbar\frac{d}{dx}) \tag{352}$$

$$\langle g|f\rangle = \int_{-\infty}^{\infty} g^*(x) \cdot f(x)dx \tag{353}$$

$$\langle g|f\rangle = \int_{-\infty}^{\infty} \psi^*(x) \cdot (-i\hbar\frac{d}{dx}) \cdot (x \cdot \psi(x))dx \tag{354}$$

$$\langle g|f\rangle = -i\hbar \int_{-\infty}^{\infty} \psi^*(x)\frac{d}{dx}(x\psi(x))dx \tag{355}$$

A.59 Uncertainty principle: Kennard inequality proof part 5.6

$$\langle f|g\rangle - \langle g|f\rangle = i\hbar \int_{-\infty}^{\infty} |\psi(x)|^2 dx \tag{356}$$

$$p(x) = |\psi(x)|^2 \tag{357}$$

$$\int_{-\infty}^{\infty} p(x)dx = \int_{-\infty}^{\infty} |\psi(x)|^2 dx \tag{358}$$

$$i\hbar \int_{-\infty}^{\infty} |\psi(x)|^2 dx = i\hbar \int_{-\infty}^{\infty} p(x)dx \tag{359}$$

$$\int_{-\infty}^{\infty} p(x)dx = 1 \tag{360}$$

A.60 Particle in a box: Wavefunction angular velocity as a function of particle mass from Schrödinger's equation 7

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t) \tag{361}$$

$$V(x) = 0 \tag{362}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)+V(x)\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{363}$$

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = \hbar\omega\psi(x,t) \tag{364}$$

$$\hbar\omega\psi(x,t) = i\hbar\frac{\partial\psi(x,t)}{\partial t} \tag{365}$$

$$\hbar\omega\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{366}$$

$$\hbar\omega\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{367}$$

A.61 Particle in a box: Wavefunction angular velocity as a function of particle mass from Schrödinger's equation 8

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{\hbar^2 k^2}{2m}\psi(x,t) \tag{368}$$

$$\frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m\hbar} \tag{369}$$

$$\hbar\omega\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{370}$$

$$\hbar\omega = \frac{p^2}{2m} \tag{371}$$

$$\omega = \frac{p^2}{2m\hbar} \tag{372}$$

A.62 Quantum harmonic oscillator: Ladder operator method 4

$$aa^\dagger - a^\dagger a = \frac{i}{\hbar}(\hat{p}\hat{x} - \hat{x}\hat{p}) \tag{373}$$

$$[\hat{p}, \hat{x}] = \hat{p}\hat{x} - \hat{x}\hat{p} \tag{374}$$

$$[\hat{p}, \hat{x}] = -i\hbar \tag{375}$$

$$-i \cdot i = 1 \quad (376)$$

$$\left[a, a^{\dagger}\right] = aa^{\dagger} - a^{\dagger}a \quad (377)$$

$$\left[a, a^{\dagger}\right] = \frac{i}{\hbar}[\hat{p}, \hat{x}] \quad (378)$$

$$\left[a, a^{\dagger}\right] = \frac{i}{\hbar}(-i\hbar) \quad (379)$$

$$\left[a, a^{\dagger}\right] = 1 \quad (380)$$

A.63 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 1.6

$$\hbar\omega(\frac{1}{2}+\frac{1}{\sqrt{2}}(-\frac{d^2}{dq^2}+q^2)\frac{1}{\sqrt{2}}(\frac{d^2}{dq^2}+q^2))\psi(q) = E\psi(q) \quad (381)$$

$$a = \frac{1}{\sqrt{2}}(\frac{d^2}{dq^2} + q^2) \quad (382)$$

$$a^{\dagger} = \frac{1}{\sqrt{2}}(-\frac{d^2}{dq^2} + q^2) \quad (383)$$

$$\hbar\omega(\frac{1}{2} + a^{\dagger}a)\psi(q) = E\psi(q) \quad (384)$$

$$E = \hbar\omega(a^{\dagger}a + \frac{1}{2}) \quad (385)$$

A.64 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 2

$$[q, p] = qp - pq \quad (386)$$

$$p = -i\frac{d}{dq} \quad (387)$$

$$[q, p] = q(-i\frac{d}{dq}) - (-i\frac{d}{dq})q \quad (388)$$

$$[q, p] = -iq\frac{d}{dq} + i\frac{d}{dq}q \quad (389)$$

$$(\frac{d}{dq}q - q\frac{d}{dq}) = 1 \quad (390)$$

$$-iq\frac{d}{dq} + i\frac{d}{dq}q = i \quad (391)$$

$$[q, p] = i \quad (392)$$

$$[q, p]f(q) = if(q) \quad (393)$$

A.65 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 3.1

$$a = \frac{1}{\sqrt{2}}(\frac{d}{dq} + q) \quad (394)$$

$$p = -i\frac{d}{dq} \quad (395)$$

$$-i \cdot i = 1 \quad (396)$$

$$\frac{d}{dq} = -ip \quad (397)$$

$$a = \frac{1}{\sqrt{2}}(-ip + q) \quad (398)$$

$$a = \frac{1}{\sqrt{2}}(ip + q) \quad (399)$$

A.66 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 3.2

$$a^{\dagger} = \frac{1}{\sqrt{2}}(-\frac{d}{dq} + q) \quad (400)$$

$$p = -i\frac{d}{dq} \quad (401)$$

$$-i \cdot i = 1 \quad (402)$$

$$-\frac{d}{dq} = ip \quad (403)$$

$$a^{\dagger} = \frac{1}{\sqrt{2}}(-ip + q) \quad (404)$$

A.67 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 3.6

$$aa^{\dagger} - a^{\dagger}a = i(pq - qp) \quad (405)$$

$$[p, q] = pq - qp \quad (406)$$

$$aa^{\dagger} - a^{\dagger}a = i[p, q] \quad (407)$$

$$[p, q] = -i \quad (408)$$

$$aa^{\dagger} - a^{\dagger}a = -i \cdot i \quad (409)$$

$$-i \cdot i = 1 \quad (410)$$

$$aa^\dagger - a^\dagger a = 1 \tag{411}$$

$$\left[a, a^\dagger\right] = aa^\dagger - a^\dagger a \tag{412}$$

$$\left[a, a^\dagger\right] = 1 \tag{413}$$

A.68 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 4.2

$$[H, a] = -\hbar\omega(aa^\dagger - a^\dagger a)a \tag{414}$$

$$\left[a, a^\dagger\right] = aa^\dagger - a^\dagger a \tag{415}$$

$$\left[a, a^\dagger\right] = 1 \tag{416}$$

$$aa^\dagger - a^\dagger a = 1 \tag{417}$$

$$-\hbar\omega(aa^\dagger - a^\dagger a)a = -\hbar\omega(1)a \tag{418}$$

$$[H, a] = -\hbar\omega a \tag{419}$$

A.69 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 5.2

$$\left[H, a^\dagger\right] = \hbar\omega a^\dagger(aa^\dagger - a^\dagger a) \tag{420}$$

$$\left[a, a^\dagger\right] = aa^\dagger - a^\dagger a \tag{421}$$

$$\left[a, a^\dagger\right] = 1 \tag{422}$$

$$\left[a, a^\dagger\right] = 1 \tag{423}$$

A.70 Heisenberg picture: time evolution 4

$$\frac{d}{dt}\hat{x}(t) = \frac{i}{\hbar}(\hat{H}e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}} - e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}}\hat{H}) \tag{424}$$

$$\hat{x}(t) = e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}} \tag{425}$$

$$\hat{H}e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}} - e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}}\hat{H} = \hat{H}\hat{x}(t) - \hat{x}(t)\hat{H} \tag{426}$$

$$\left[\hat{H}, \hat{x}(t)\right] = \hat{H}\hat{x}(t) - \hat{x}(t)\hat{H} \tag{427}$$

$$\frac{i}{\hbar}(\hat{H}\hat{x}(t) - \hat{x}(t)\hat{H}) = \frac{i}{\hbar}\left[\hat{H}, \hat{x}(t)\right] \tag{428}$$

$$\frac{d}{dt}\hat{x}(t) = \frac{i}{\hbar}\left[\hat{H}, \hat{x}(t)\right] \tag{429}$$

A.71 Heisenberg picture: momentum evolution 4

$$\hat{p}(t) = A\cos(\omega t) + B\sin(\omega t) \tag{430}$$

$$\frac{d}{dt}\cos(\omega t) = -\omega\sin(\omega t) \tag{431}$$

$$\frac{d}{dt}\sin(\omega t) = \omega\cos(\omega t) \tag{432}$$

$$\frac{d}{dt}A\cos(\omega t) = -A\omega\sin(\omega t) \tag{433}$$

$$\frac{d}{dt}B\sin(\omega t) = B\omega\cos(\omega t) \tag{434}$$

$$\frac{d\hat{p}(t)}{dt} = \frac{d}{dt}A\cos(\omega t) + \frac{d}{dt}B\sin(\omega t) \tag{435}$$

$$\frac{d\hat{p}(t)}{dt} = -A\omega\sin(\omega t) + B\omega\cos(\omega t) \tag{436}$$

A.72 Heisenberg picture: position commutator 4

$$[\hat{x}(t_1), \hat{x}(t_2)] = \frac{1}{\omega m}(\hat{x}_0\hat{p}_0 - \hat{p}_0\hat{x}_0)\sin(\omega t_2 - \omega t_1) \tag{437}$$

$$\hat{x}_0\hat{p}_0 - \hat{p}_0\hat{x}_0 = [\hat{x}_0, \hat{p}_0] \tag{438}$$

$$[\hat{x}_0, \hat{p}_0] = i\hbar \tag{439}$$

$$\frac{1}{\omega m}[\hat{x}_0, \hat{p}_0]\sin(\omega t_2 - \omega t_1) = \frac{i\hbar}{\omega m}\sin(\omega t_2 - \omega t_1) \tag{440}$$

$$[\hat{x}(t_1), \hat{x}(t_2)] = \frac{i\hbar}{\omega m}\sin(\omega t_2 - \omega t_1) \tag{441}$$

A.73 Heisenberg picture: momentum commutator 3

$$[\hat{p}(t_1), \hat{p}(t_2)] = m\omega\hat{p}_0\hat{x}_0\sin(\omega t_2 - \omega t_1) + m\omega\hat{x}_0\hat{p}_0\sin(\omega t_1 - \omega t_2) \tag{442}$$

$$\sin(\omega t_1 - \omega t_2) = -\sin(\omega t_2 - \omega t_1) \tag{443}$$

$$m\omega\hat{x}_0\hat{p}_0\sin(\omega t_1 - \omega t_2) = -m\omega\hat{x}_0\hat{p}_0\sin(\omega t_2 - \omega t_1) \tag{444}$$

$$[\hat{p}(t_1), \hat{p}(t_2)] = m\omega\hat{p}_0\hat{x}_0\sin(\omega t_2 - \omega t_1) - m\omega\hat{x}_0\hat{p}_0\sin(\omega t_2 - \omega t_1) \tag{445}$$

$$\hat{x}_0\hat{p}_0 - \hat{p}_0\hat{x}_0 = i\hbar \tag{446}$$

$$m\omega(\hat{x}_0\hat{p}_0 - \hat{p}_0\hat{x}_0)\sin(\omega t_2 - \omega t_1) = i\hbar m\omega\sin(\omega t_2 - \omega t_1) \tag{447}$$

$$[\hat{p}(t_1), \hat{p}(t_2)] = i\hbar m\omega\sin(\omega t_2 - \omega t_1) \tag{448}$$

A.74 Vacuum Rabi Oscillations: excited state probability

$$|\Psi(t)\rangle = \cos\left(\frac{\Omega t}{2}\right)|e, 0\rangle - i\sin\left(\frac{\Omega t}{2}\right)|g, 1\rangle \tag{449}$$

$$\langle e, 0|\Psi(t)\rangle = \cos\left(\frac{\Omega t}{2}\right)\langle e, 0|e, 0\rangle - i\sin\left(\frac{\Omega t}{2}\right)\langle e, 0|g, 1\rangle \tag{450}$$

$$\langle e, 0|e, 0\rangle = 1 \tag{451}$$

$$\langle e, 0|g, 1\rangle = 0 \tag{452}$$

$$\langle e, 0|\Psi(t)\rangle = \cos\left(\frac{\Omega t}{2}\right) \tag{453}$$

$$P_e(t) = |\langle e, 0|\Psi(t)\rangle|^2 \tag{454}$$

$$P_e(t) = \cos^2(\frac{\Omega t}{2}) \tag{455}$$

A.75 Vacuum Rabi Oscillations: ground state probability 2

$$P_g(t) = |\cos\left(\frac{\Omega t}{2}\right)\langle g, 1|e, 0\rangle - i\sin\left(\frac{\Omega t}{2}\right)\langle g, 1|g, 1\rangle|^2 \tag{456}$$

$$\langle g, 1|e, 0\rangle = 0 \tag{457}$$

$$P_g(t) = |\cos\left(\frac{\Omega t}{2}\right) * 0 - i\sin\left(\frac{\Omega t}{2}\right)\langle g, 1|g, 1\rangle|^2 \tag{458}$$

$$P_g(t) = |-i\sin\left(\frac{\Omega t}{2}\right)\langle g, 1|g, 1\rangle|^2 \tag{459}$$

$$\langle g, 1|g, 1\rangle = 1 \tag{460}$$

$$P_g(t) = |-i\sin\left(\frac{\Omega t}{2}\right) * 1|^2 \tag{461}$$

$$|-i|^2 = 1 \tag{462}$$

$$P_g(t) = |\sin\left(\frac{\Omega t}{2}\right)|^2 \tag{463}$$

$$P_g(t) = \sin^2(\frac{\Omega t}{2}) \tag{464}$$

A.76 Expectation value: integral expression

$$\left\langle \hat{X} \right\rangle_\Psi = \langle\Psi|\hat{X}|\Psi\rangle \tag{465}$$

$$\hat{X} = \mathbb{1}\hat{X}\mathbb{1} \tag{466}$$

$$\left\langle \hat{X} \right\rangle_\Psi = \langle\Psi|\mathbb{1}\hat{X}\mathbb{1}|\Psi\rangle \tag{467}$$

$$\mathbb{1} = \int |x\rangle\langle x|\,dx \tag{468}$$

$$\left\langle \hat{X} \right\rangle_\Psi = \langle\Psi| \left(\int |x\rangle \langle x| \, dx \right) \hat{X} \left(\int |x'\rangle \langle x'| \, dx' \right) |\Psi\rangle \tag{469}$$

$$\left\langle \hat{X} \right\rangle_\Psi = \int \int \langle \Psi | x \rangle \langle x | \, \hat{X} \, |x'\rangle \langle x' | \Psi \rangle \, dx dx' \tag{470}$$

A.77 Expectation value: integral expression 2

$$\left\langle \hat{X} \right\rangle_\Psi = \int \int \langle \Psi | x \rangle \langle x | \, \hat{X} \, |x'\rangle \langle x' | \Psi \rangle \, dx dx' \tag{471}$$

$$\hat{X} \, |x'\rangle = x' \, |x'\rangle \tag{472}$$

$$\langle \Psi | x \rangle = \langle x | \Psi \rangle^\dagger \tag{473}$$

$$\langle x | x' \rangle = \delta(x - x') \tag{474}$$

$$\int \int \langle \Psi | x \rangle \langle x | \, x' \, |x'\rangle \langle x' | \Psi \rangle \, dx dx' = \int \int \langle x | \Psi \rangle^\dagger \, x' \, |x'\rangle \langle x' | \Psi \rangle \, dx dx' \tag{475}$$

$$\int \int \langle x | \Psi \rangle^\dagger \, x' \delta(x - x') \langle x' | \Psi \rangle \, dx dx' = \left\langle \hat{X} \right\rangle_\Psi \tag{476}$$

A.78 Expectation value: integral expression 3

$$\left\langle \hat{X} \right\rangle_\Psi = \int \int \langle x | \Psi \rangle^\dagger \, x' \delta(x - x') \langle x' | \Psi \rangle \, dx dx' \tag{477}$$

$$\int \langle x | \Psi \rangle^\dagger \, x' \delta(x - x') \langle x' | \Psi \rangle \, dx' = \langle x | \Psi \rangle^\dagger \, x \, \langle x | \Psi \rangle \tag{478}$$

$$\langle x | \Psi \rangle = \Psi(x) \tag{479}$$

$$\langle x | \Psi \rangle^\dagger \, x \, \langle x | \Psi \rangle = \Psi^\dagger(x) x \Psi(x) \tag{480}$$

$$\Psi^\dagger(x)\Psi(x) = |\Psi(x)|^2 \tag{481}$$

$$\Psi^\dagger(x) x \Psi(x) = x |\Psi(x)|^2 \tag{482}$$

$$\left\langle \hat{X} \right\rangle_\Psi = \int x |\Psi(x)|^2 dx \tag{483}$$

A.79 Euler-lagrange equation: Full derivative of the perturbation Lagrangian with respect to ε 2

$$\frac{dg_\varepsilon}{d\varepsilon} = \frac{d}{d\varepsilon}(\varepsilon\eta(x)) \tag{484}$$

$$\frac{d}{d\varepsilon}(\varepsilon\eta(x)) = \eta(x) + \varepsilon \frac{d\eta(x)}{d\varepsilon} \tag{485}$$

$$\frac{d\eta(x)}{d\varepsilon} = 0 \tag{486}$$

$$\eta(x) + \varepsilon \frac{d\eta(x)}{d\varepsilon} = \eta(x) \tag{487}$$

$$\frac{dg_\varepsilon}{d\varepsilon} = \eta(x) \tag{488}$$

A.80 Euler-Lagrange equation: Derivation

$$J = \int_a^b L(x, f(x), f'(x)) \tag{489}$$

$$g_\varepsilon(x) = f(x) + \varepsilon\eta(x) \tag{490}$$

$$J_\varepsilon = \int_a^b L(x, g_\varepsilon(x), g'_\varepsilon(x)) dx \tag{491}$$

$$L_\varepsilon = L(x, g_\varepsilon(x), g'_\varepsilon(x)) \tag{492}$$

$$\frac{dJ_\varepsilon}{d\varepsilon} = \frac{d}{d\varepsilon} \int_a^b L_\varepsilon dx \tag{493}$$

$$\frac{dJ_\varepsilon}{d\varepsilon} = \int_a^b \frac{dL_\varepsilon}{d\varepsilon} dx \tag{494}$$

A.81 Euler-Lagrange equation: Derivation 4

$$\frac{dv}{dx} = \eta'(x) \tag{495}$$

$$v = \int \frac{dv}{dx} dx \tag{496}$$

$$\eta'(x) = \frac{d\eta(x)}{dx} \tag{497}$$

$$\frac{dv}{dx} = \frac{d\eta(x)}{dx} \tag{498}$$

$$v = \int \frac{d\eta(x)}{dx} dx \tag{499}$$

$$v = \eta(x) \tag{500}$$

A.82 Euler-Lagrange equation: Derivation 5

$$u = \frac{\partial L}{\partial f'} \tag{501}$$

$$v = \eta(x) \tag{502}$$

$$\eta(a) = 0 \tag{503}$$

$$\eta(b) = 0 \tag{504}$$

$$uv = \frac{\partial L}{\partial f'}\eta(x) \tag{505}$$

$$uv\Big|_a^b = \frac{\partial L}{\partial f'}\eta(x)\Big|_a^b \tag{506}$$

$$\frac{\partial L}{\partial f'}\eta(x)\Big|_a^b = \frac{\partial L}{\partial f'}\ \eta(x)\Big|_a^b \tag{507}$$

$$\frac{\partial L}{\partial f'}\ \eta(x)\Big|_a^b = \frac{\partial L}{\partial f'}(\eta(b) - \eta(a)) \tag{508}$$

$$\frac{\partial L}{\partial f'}(\eta(b) - \eta(a)) = \frac{\partial L}{\partial f'}(0 - 0) \tag{509}$$

$$\frac{\partial L}{\partial f'}(0 - 0) = 0 \tag{510}$$

$$(uv)\Big|_a^b = 0 \tag{511}$$

A.83 Euler-Lagrange equation: Derivation 6

$$I = \int_a^b \frac{\partial L}{\partial f'}\eta'(x)dx \tag{512}$$

$$I = (uv)\Big|_a^b - \int_a^b v\frac{du}{dx}dx \tag{513}$$

$$(uv)\Big|_a^b = 0 \tag{514}$$

$$\frac{du}{dx} = \frac{d}{dx}\frac{\partial L}{\partial f'} \tag{515}$$

$$I = -\int_a^b v\frac{du}{dx}dx \tag{516}$$

$$I = -\int_a^b \eta(x)\frac{d}{dx}\frac{\partial L}{\partial f'}dx \tag{517}$$

A.84 Euler-Lagrange equation: Straight line

$$S = \int_a^b ds \tag{518}$$

$$ds = \sqrt{dx^2 + dy^2} \tag{519}$$

$$dy = y'dx \tag{520}$$

$$ds = \sqrt{dx^2 + (y'dx)^2} \tag{521}$$

$$ds = dx\sqrt{1 + y'^2} \tag{522}$$

$$S = \int_a^b \sqrt{1 + y'^2}dx \tag{523}$$

A.85 Euler-Lagrange equation: Straight line 3

$$\frac{dL}{dy} - \frac{d}{dx}\frac{dL}{dy'} = 0 \tag{524}$$

$$\frac{dL}{dy} = 0 \tag{525}$$

$$\frac{dL}{dy'} = y'(1 + y'^2)^{-\frac{1}{2}} \tag{526}$$

$$\frac{dL}{dy} - \frac{d}{dx}y'(1 + y'^2)^{-\frac{1}{2}} = 0 \tag{527}$$

$$-\frac{d}{dx}y'(1 + y'^2)^{-\frac{1}{2}} = 0 \tag{528}$$

$$\int -\frac{d}{dx}(y'(1 + y'^2)^{-\frac{1}{2}})dx = \int 0dx \tag{529}$$

$$\int \frac{d}{dx}(y'(1 + y'^2)^{-\frac{1}{2}})dx = C \tag{530}$$

A.86 Euler-Lagrange equation: Straight line 6

$$\frac{dy}{dx} = C(1 - C^2)^{-1/2} \tag{531}$$

$$C(1 - C^2)^{-1/2} = A \tag{532}$$

$$\frac{dy}{dx} = A \tag{533}$$

$$\int \frac{dy}{dx}dx = \int Adx \tag{534}$$

$$\int Adx = Ax + C \tag{535}$$

$$y = Ax + C \tag{536}$$

A.87 Escape velocity

$$F = \frac{GMm}{r^2} \tag{537}$$

$$dW = Fdr \tag{538}$$

$$dW = \frac{GMm}{r^2}dr \tag{539}$$

$$W = \int_{r_0}^{\infty} dW \tag{540}$$

$$W = \int_{r_0}^{\infty} \frac{GMm}{r^2}dr \tag{541}$$

A.88 Escape velocity 2

$$W = \int_{r_0}^{\infty} \frac{GMm}{r^2}dr \tag{542}$$

$$F = \frac{GMm}{r^2} \tag{543}$$

$$F = mg \tag{544}$$

$$\frac{GMm}{r^2} = mg \tag{545}$$

$$GMm = mgr^2 \tag{546}$$

$$\int_{r_0}^{\infty} \frac{GMm}{r^2}dr = \int_{r_0}^{\infty} mgdr \tag{547}$$

$$W = mgr_0 \tag{548}$$

A.89 Escape velocity 3

$$W = mgr_0 \tag{549}$$

$$E = \frac{1}{2}mv_{esc}^2 \tag{550}$$

$$W = E \tag{551}$$

$$mgr_0 = \frac{1}{2}mv_{esc}^2 \tag{552}$$

$$2gr_0 = v_{esc}^2 \tag{553}$$

$$v_{esc} = \sqrt{2gr_0} \tag{554}$$

A.90 Snell's law: from Fermat's principle 2

$$\frac{dT}{dx} = \frac{x}{v_1(x^2+a^2)^{\frac{1}{2}}} + \frac{x-l}{v_2((x-l)^2+b^2)^{\frac{1}{2}}} \tag{555}$$

$$\frac{x}{(x^2+a^2)^{\frac{1}{2}}} = \sin\theta_1 \tag{556}$$

$$\frac{l-x}{((x-l)^2+b^2)^{\frac{1}{2}}} = \sin\theta_2 \tag{557}$$

$$\frac{dT}{dx} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} \tag{558}$$

$$\frac{dT}{dx} = 0 \tag{559}$$

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \tag{560}$$

A.91 Snell's law: from Fermat's principle 3

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \tag{561}$$

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \tag{562}$$

$$\frac{1}{v_1} = \frac{n_1}{c} \tag{563}$$

$$\frac{1}{v_2} = \frac{n_2}{c} \tag{564}$$

$$\frac{\sin\theta_1}{\frac{n_1}{c}} = \frac{\sin\theta_2}{\frac{n_2}{c}} \tag{565}$$

$$c\sin\theta_1 = n_1\sin\theta_1 = c\sin\theta_2 = n_2\sin\theta_2 \tag{566}$$

$$n_1\sin\theta_1 = n_2\sin\theta_2 \tag{567}$$

A.92 Wave equation: plane wave eigenmodes 2

$$\frac{\partial^2 u(x,t)}{\partial t^2} = (-i\omega)\frac{\partial}{\partial t}(e^{-i\omega t}f(x)) \tag{568}$$

$$\frac{\partial}{\partial t}(e^{-i\omega t}f(x)) = -i\omega e^{-i\omega t}f(x) \tag{569}$$

$$(-i\omega)(-i\omega e^{-i\omega t}f(x)) = \omega^2 e^{-i\omega t}f(x) \tag{570}$$

$$i \cdot i = -1 \tag{571}$$

$$-\omega^2 e^{-i\omega t} f(x) = -\omega^2 e^{-i\omega t} f(x) \tag{572}$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \tag{573}$$

$$-\omega^2 e^{-i\omega t} f(x) = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \tag{574}$$

A.93 Wave equation: plane wave eigenmodes 4

$$u(x,t) = Ae^{-i(kx-\omega t)} + Be^{i(kx-\omega t)} \tag{575}$$

$$u(x,t) = \int_{-\infty}^{\infty} s(\omega) u(x,t) d\omega \tag{576}$$

$$s_+(\omega) = As(\omega) \tag{577}$$

$$s_-(\omega) = Bs(\omega) \tag{578}$$

$$Ae^{-i(kx-\omega t)} = \int_{-\infty}^{\infty} s_+(\omega) e^{-i(kx-\omega t)} d\omega \tag{579}$$

$$Be^{i(kx-\omega t)} = \int_{-\infty}^{\infty} s_-(\omega) e^{i(kx-\omega t)} d\omega \tag{580}$$

$$u(x,t) = \int_{-\infty}^{\infty} s_+(\omega) e^{-i(kx-\omega t)} d\omega + \int_{-\infty}^{\infty} s_-(\omega) e^{i(kx-\omega t)} d\omega \tag{581}$$

A.94 Wave equation: plane wave eigenmodes 5

$$u(x,t) = \int_{-\infty}^{\infty} s_+(\omega) e^{-i(kx-\omega t)} d\omega + \int_{-\infty}^{\infty} s_-(\omega) e^{i(kx-\omega t)} d\omega \tag{582}$$

$$\omega = kc \tag{583}$$

$$\int_{-\infty}^{\infty} s_+(\omega) e^{-i(kx-\omega t)} d\omega = \int_{-\infty}^{\infty} s_+(\omega) e^{-ik(x-ct)} d\omega \tag{584}$$

$$\int_{-\infty}^{\infty} s_+(\omega) e^{-ik(x-ct)} d\omega = F(x-ct) \tag{585}$$

$$\int_{-\infty}^{\infty} s_-(\omega) e^{i(kx-\omega t)} d\omega = \int_{-\infty}^{\infty} s_-(\omega) e^{ik(x-ct)} d\omega \tag{586}$$

$$\int_{-\infty}^{\infty} s_-(\omega) e^{ik(x-ct)} d\omega = G(x+ct) \tag{587}$$

$$u(x,t) = F(x-ct) + G(x+ct) \tag{588}$$

A.95 Wave equation: Hooke's law

$$F_H = F_{x+2h} - F_x \tag{589}$$

$$F_N = ma(t) \tag{590}$$

$$F_N = F_H \tag{591}$$

$$a(t) = \frac{\partial^2}{\partial t^2} u(x+h,t) \tag{592}$$

$$ma(t) = F_H \tag{593}$$

$$m \frac{\partial^2}{\partial t^2} u(x+h,t) = F_H \tag{594}$$

$$m \frac{\partial^2}{\partial t^2} u(x+h,t) = F_{x+2h} - F_x \tag{595}$$

A.96 Wave equation: Hooke's law 2

$$m \frac{\partial^2}{\partial t^2} u(x+h,t) = F_{x+2h} - F_x \tag{596}$$

$$F_{x+2h} = ku(x+2h,t) - ku(x+h,t) \tag{597}$$

$$F_x = ku(x+h,t) - ku(x,t) \tag{598}$$

$$m \frac{\partial^2}{\partial t^2} u(x+h,t) = ku(x+2h,t) - ku(x+h,t) - ku(x+h,t) + ku(x \tag{599}$$

$$m \frac{\partial^2}{\partial t^2} u(x+h,t) = k\big(u(x+2h,t) - 2u(x+h,t) + u(x,t)\big) \tag{600}$$

$$\frac{\partial^2}{\partial t^2} u(x+h,t) = \frac{k}{m}\big(u(x+2h,t) - 2u(x+h,t) + u(x,t)\big) \tag{601}$$

A.97 Wave equation: Hooke's law 3

$$\frac{\partial^2}{\partial t^2}u(x+h,t) = \frac{k}{m}\big(u(x+2h,t)-2u(x+h,t)+u(x,t)\big) \tag{602}$$

$$N = \frac{L}{h} \tag{603}$$

$$m = \frac{M}{N} \tag{604}$$

$$k = KN \tag{605}$$

$$\frac{k}{m} = \frac{KN}{\frac{M}{N}} \tag{606}$$

$$\frac{k}{m} = \frac{KN^2}{M} \tag{607}$$

$$\frac{k}{m} = \frac{K\frac{L^2}{h^2}}{M} \tag{608}$$

$$\frac{\partial^2}{\partial t^2}u(x+h,t) = \frac{K\frac{L^2}{h^2}}{M}\big(u(x+2h,t)-2u(x+h,t)+u(x,t)\big) \tag{609}$$

$$\frac{\partial^2}{\partial t^2}u(x+h,t) = \frac{KL^2}{Mh^2}\big(u(x+2h,t)-2u(x+h,t)+u(x,t)\big) \tag{610}$$

A.98 Wave equation: stress pulse in a bar 2

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{KL^2}{M}\frac{\partial^2 u(x,t)}{\partial x^2} \tag{611}$$

$$K = \frac{EA}{L} \tag{612}$$

$$\rho = \frac{M}{V} \tag{613}$$

$$V = AL \tag{614}$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{EA}{L}\frac{L^2}{M}\frac{\partial^2 u(x,t)}{\partial x^2} \tag{615}$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{EAL}{M}\frac{\partial^2 u(x,t)}{\partial x^2} \tag{616}$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{E}{\frac{M}{AL}}\frac{\partial^2 u(x,t)}{\partial x^2} \tag{617}$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u(x,t)}{\partial x^2} \tag{618}$$

B $S = 1$ (one premise removed)

B.1 Gauss' law: equivalence between differential and integral forms

$$\oiint_S \mathbf{E}\cdot d\mathbf{A} = \frac{Q}{\varepsilon_0} \tag{619}$$

$$\oiint_S \mathbf{E}\cdot d\mathbf{A} = \iiint_V \nabla\cdot\mathbf{E}dV \tag{620}$$

$$\frac{Q}{\varepsilon_0} = \iiint_V \nabla\cdot\mathbf{E}dV \tag{621}$$

$$Q = \iiint_V \rho dV \tag{622}$$

$$\frac{\iiint_V \rho dV}{\varepsilon_0} = \iiint_V \nabla\cdot\mathbf{E}dV \tag{623}$$

$$\iiint_V \nabla\cdot\mathbf{E}dV = \iiint_V \frac{\rho}{\varepsilon_0}dV \tag{624}$$

B.2 Gauss' law: Equivalence of total and free charge statements

$$\rho_b = -\nabla\cdot\mathbf{P} \tag{625}$$

$$-\rho_b = \nabla\cdot\mathbf{P} \tag{626}$$

$$\rho_f = \nabla\cdot\mathbf{D} \tag{627}$$

$$\rho = \rho_f - \rho_b \tag{628}$$

$$\rho = \nabla\cdot\mathbf{D} - \nabla\cdot\mathbf{P} \tag{629}$$

$$\rho = \nabla\cdot(\mathbf{D}-\mathbf{P}) \tag{630}$$

B.3 Uniqueness theorem for Poisson's equation

$$\nabla^2\phi_1 = -\frac{\rho_f}{\varepsilon_0} \tag{631}$$

$$\nabla^2\phi_2 = -\frac{\rho_f}{\varepsilon_0} \tag{632}$$

$$\nabla^2\phi_1 + \nabla^2\phi_2 = -\frac{\rho_f}{\varepsilon_0} + -\frac{\rho_f}{\varepsilon_0} \tag{633}$$

$$\nabla^2\phi_1 + \nabla^2\phi_2 = -2\frac{\rho_f}{\varepsilon_0} \tag{634}$$

$$\nabla^2(\phi_1+\phi_2) = -2\frac{\rho_f}{\varepsilon_0} \tag{635}$$

$$\nabla^2\phi = \nabla^2(\phi_1 + \phi_2) \tag{636}$$

$$\nabla^2\phi = -2\frac{\rho_f}{\varepsilon_0} \tag{637}$$

$$\nabla^2\phi = 0 \tag{638}$$

B.4 Uniqueness theorem for Poisson's equation 2

$$\nabla \cdot (\phi\nabla\phi) = (\nabla\phi)^2 + \phi\nabla^2\phi \tag{639}$$

$$\nabla^2\phi = 0 \tag{640}$$

$$\phi\nabla^2\phi = 0 \tag{641}$$

$$\nabla \cdot (\phi\nabla\phi) = (\nabla\phi)^2 \tag{642}$$

$$\int_V \nabla \cdot (\phi\nabla\phi)dV = \int_V (\nabla\phi)^2 dV \tag{643}$$

$$\int_V \nabla \cdot (\phi\nabla\phi)dV = \int_S \phi\nabla\phi \cdot d\mathbf{S} \tag{644}$$

$$\int_S \phi\nabla\phi \cdot d\mathbf{S} = \int_V (\nabla\phi)^2 dV \tag{645}$$

B.5 Uniqueness theorem for Poisson's equation 6

$$\frac{\partial\phi}{\partial r} = 0 \tag{646}$$

$$\int \frac{\partial\phi}{\partial r} dr = \int 0 dr \tag{647}$$

$$\phi = \int 0 dr + C_2 \tag{648}$$

$$\phi = C_1 + C_2 \tag{649}$$

$$\phi = C_1 - C_2 \tag{650}$$

B.6 Uniqueness theorem for Poisson's equation 7

$$\phi = C_1 - C_2 \tag{651}$$

$$C = C_1 - C_2 \tag{652}$$

$$\phi_2 = C_1 - C_2 \tag{653}$$

$$\phi_1 = C_1 - C_2 \tag{654}$$

$$\phi_2 - \phi_1 = C_1 - C_2 - (C_1 - C_2) \tag{655}$$

$$\phi_2 - \phi_1 = C \tag{656}$$

B.7 Poisson's equation: Newtonian gravity

$$\nabla \cdot \mathbf{g} = -4\pi G\rho \tag{657}$$

$$\mathbf{g} = -\nabla\phi \tag{658}$$

$$\nabla \cdot (-\nabla\phi) = -4\pi G\rho \tag{659}$$

$$-\nabla^2\phi = -4\pi G\rho \tag{660}$$

$$\nabla^2\phi = 4\pi G\rho \tag{661}$$

B.8 Poisson's equation: Gravitational potential from Poisson's equation 2

$$\int_V \nabla \cdot \nabla\phi dV = \int_V 4\pi G\rho dV \tag{662}$$

$$\int_V \nabla \cdot \nabla\phi dV = \int_S \nabla\phi \cdot d\mathbf{S} \tag{663}$$

$$m = \int_V \rho dV \tag{664}$$

$$4\pi Gm = 4\pi G \int_V \rho dV \tag{665}$$

$$4\pi Gm = \int_V 4\pi G\rho dV \tag{666}$$

$$\int_S \nabla\phi \cdot d\mathbf{S} = 4\pi Gm \tag{667}$$

B.9 Poisson's equation: Gravitational potential from Poisson's equation 6

$$\int_S \frac{\partial \phi}{\partial r} dS = \int_0^{2\pi} \int_0^{\pi} \frac{\partial \phi}{\partial r} r^2 \sin\theta d\theta d\varphi \quad (668)$$

$$\int_0^{\pi} \sin\theta d\theta = 2 \quad (669)$$

$$\int_0^{2\pi} d\varphi = 2\pi \quad (670)$$

$$\int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta d\theta d\varphi = 2\pi \cdot 2 \cdot r^2 = 4\pi r^2 \quad (671)$$

$$\int_S \frac{\partial \phi}{\partial r} dS = \int_0^{2\pi} \int_0^{\pi} \frac{\partial \phi}{\partial r} r^2 \sin\theta d\theta d\varphi = \frac{\partial \phi}{\partial r} \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta d\theta d\varphi = \frac{\partial \phi}{\partial r} \cdot 4\pi r^2 \quad (672)$$

$$\int_S \frac{\partial \phi}{\partial r} dS = 4\pi \frac{\partial \phi}{\partial r} r^2 \quad (673)$$

B.10 Poisson's equation: Gravitational potential from Poisson's equation 8

$$\frac{\partial \phi}{\partial r} = \frac{Gm}{r^2} \quad (674)$$

$$\int_{\infty}^{r} \frac{\partial \phi}{\partial r} dr = \phi(r) - \phi(\infty) \quad (675)$$

$$\int_{\infty}^{r} \frac{Gm}{r^2} dr = \frac{-Gm}{r} \quad (676)$$

$$\phi(r) - \phi(\infty) = \frac{-Gm}{r} \quad (677)$$

$$\phi(r) = \frac{-Gm}{r} + \phi(\infty) \quad (678)$$

$$\phi(\infty) = 0 \quad (679)$$

$$\phi(r) = \frac{-Gm}{r} \quad (680)$$

B.11 Poisson's equation: Electrostatics

$$\nabla \cdot \mathbf{D} = \rho_f \quad (681)$$

$$\mathbf{D} = \varepsilon \mathbf{E} \quad (682)$$

$$\nabla \cdot (\varepsilon \mathbf{E}) = \rho_f \quad (683)$$

$$\mathbf{E} = -\nabla \phi \quad (684)$$

$$\nabla \cdot (\varepsilon(-\nabla \phi)) = \rho_f \quad (685)$$

$$\varepsilon \nabla \cdot (-\nabla \phi) = \rho_f \quad (686)$$

$$-\varepsilon \nabla^2 \phi = \rho_f \quad (687)$$

$$\nabla^2 \phi = -\frac{\rho_f}{\varepsilon} \quad (688)$$

B.12 Poisson's equation: Electrostatic potential from Poisson's equation

$$\nabla^2 \phi = -\frac{\rho_f}{\varepsilon} \quad (689)$$

$$\nabla^2 \phi = \nabla \cdot \nabla \phi \quad (690)$$

$$\int_V \nabla^2 \phi dV = \int_V \nabla \cdot \nabla \phi dV \quad (691)$$

$$\int_V \nabla^2 \phi dV = -\frac{1}{\varepsilon} \int_V \rho_f dV \quad (692)$$

$$\int_V \nabla \cdot \nabla \phi dV = -\frac{1}{\varepsilon} \int_V \rho_f dV \quad (693)$$

$$\int_V \nabla \cdot \nabla \phi dV = -\frac{Q}{\varepsilon} \quad (694)$$

B.13 Poisson's equation: Electrostatic potential from Poisson's equation 2

$$\int_V \nabla \cdot \nabla \phi dV = -\frac{Q}{\varepsilon} \quad (695)$$

$$\int_V \nabla \cdot \nabla \phi dV = \int_S \nabla \phi \cdot d\mathbf{S} \quad (696)$$

$$-\frac{Q}{\varepsilon} = \int_S \nabla \phi \cdot d\mathbf{S} \quad (697)$$

$$\nabla \phi = \frac{\partial \phi}{\partial r} \quad (698)$$

$$\int_S \nabla \phi \cdot d\mathbf{S} = \int_S \frac{\partial \phi}{\partial r} dS \quad (699)$$

$$\int_S \frac{\partial \phi}{\partial r} dS = -\frac{Q}{\varepsilon} \quad (700)$$

B.14 Poisson's equation: Electrostatic potential from Poisson's equation 4

$$\int_{\infty}^{r} \frac{\partial \phi}{\partial r} dr = \int_{\infty}^{r} -\frac{Q}{4\pi\varepsilon r^2} dr \tag{701}$$

$$\int_{\infty}^{r} \frac{\partial \phi}{\partial r} dr = \phi(r) - \phi(\infty) \tag{702}$$

$$\int_{\infty}^{r} -\frac{Q}{4\pi\varepsilon r^2} dr = \frac{Q}{4\pi\varepsilon r} \tag{703}$$

$$\phi(r) - \phi(\infty) = \frac{Q}{4\pi\varepsilon r} \tag{704}$$

$$\phi(r) = \frac{Q}{4\pi\varepsilon r} + \phi(\infty) \tag{705}$$

$$assuming \phi(\infty) = 0, then \phi(r) = \frac{Q}{4\pi\varepsilon r} \tag{706}$$

B.15 Lorentz force: continuous charge distribution

$$\frac{d\mathbf{F}}{dV} = \frac{dq}{dV}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{707}$$

$$\mathbf{f} = \frac{d\mathbf{F}}{dV} \tag{708}$$

$$\mathbf{f} = \frac{dq}{dV}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{709}$$

$$\rho = \frac{dq}{dV} \tag{710}$$

$$\mathbf{f} = \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{711}$$

$$\mathbf{f} = \rho\mathbf{E} + \rho\mathbf{v} \times \mathbf{B} \tag{712}$$

B.16 Lorentz force: continuous charge distribution 2

$$\mathbf{f} = \rho\mathbf{E} + \rho\mathbf{v} \times \mathbf{B} \tag{713}$$

$$\mathbf{J} = \rho\mathbf{v} \tag{714}$$

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{715}$$

$$\frac{d\mathbf{F}}{dV} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{716}$$

$$\mathbf{F} = \iiint \frac{d\mathbf{F}}{dV} dV \tag{717}$$

$$\mathbf{F} = \iiint (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) dV \tag{718}$$

B.17 Lorentz force: Lorentz force in terms of potentials

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{719}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{720}$$

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{721}$$

$$\mathbf{F} = q(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times \nabla \times \mathbf{A}) \tag{722}$$

$$\mathbf{F} = q(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A}) \tag{723}$$

B.18 Lorentz force: Potential energy derivation from scalar potential 3

$$U = q \int_{\infty}^{r} \nabla\phi \cdot d\mathbf{r} \tag{724}$$

$$\int_{\infty}^{r} \nabla\phi \cdot d\mathbf{r} = \int_{\infty}^{r} \frac{\partial \phi}{\partial r} dr \tag{725}$$

$$\int_{\infty}^{r} \frac{\partial \phi}{\partial r} dr = \phi(r) - \phi(\infty) \tag{726}$$

$$U = q(\phi(r) - \phi(\infty)) \tag{727}$$

$$U = q\phi(r) - q\phi(\infty) \tag{728}$$

$$U = q\phi(r) if \phi(\infty) = 0 \tag{729}$$

B.19 Laplace equation: Analytic functions (u)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{730}$$

$$\frac{\partial}{\partial y}(\frac{\partial u}{\partial x}) = \frac{\partial}{\partial y}(\frac{\partial v}{\partial y}) \tag{731}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \tag{732}$$

$$\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \frac{\partial}{\partial x}(\frac{\partial^2 u}{\partial x \partial y}) \tag{733}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{734}$$

B.20 Laplace equation: Analytic functions (u) 2

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{735}$$

$$\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial}{\partial y}(\frac{\partial u}{\partial y}) \tag{736}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial u}{\partial y}) \tag{737}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \tag{738}$$

B.21 Laplace equation: Analytic functions (u) 3

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{739}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \tag{740}$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} \tag{741}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \tag{742}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{743}$$

$$\nabla^2 u = 0 \tag{744}$$

B.22 Laplace equation: Analytic functions (v)

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{745}$$

$$\frac{\partial}{\partial x}(\frac{\partial v}{\partial x}) = \frac{\partial}{\partial x}(-\frac{\partial u}{\partial y}) \tag{746}$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \tag{747}$$

B.23 Laplace equation: Analytic functions (v) 2

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{748}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial v}{\partial y}) \tag{749}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial u}{\partial x}) \tag{750}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} \tag{751}$$

B.24 Laplace equation: Analytic functions (v) 3

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \tag{752}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} \tag{753}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \tag{754}$$

$$-\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \tag{755}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{756}$$

$$\nabla^2 v = 0 \tag{757}$$

B.25 Laplace equation: Electrostatics

$$\mathbf{E} = (u, v) \tag{758}$$

$$\nabla \cdot \mathbf{E} = \rho \tag{759}$$

$$\nabla \cdot \mathbf{E} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{760}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \rho \tag{761}$$

B.26 Laplace equation: Electrostatics 2

$$\frac{\partial \phi}{\partial x} = -u \tag{762}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial(-u)}{\partial x} \tag{763}$$

$$\frac{\partial \phi}{\partial y} = -v \tag{764}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial(-v)}{\partial y} \tag{765}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial(-u)}{\partial x} + \frac{\partial(-v)}{\partial y} \tag{766}$$

$$let \rho = \frac{\partial(-u)}{\partial x} + \frac{\partial(-v)}{\partial y} \tag{767}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\rho \tag{768}$$

B.27 Laplace equation: Electrostatics 3

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \tag{769}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -\rho \tag{770}$$

$$-\rho = 0 \tag{771}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{772}$$

$$\nabla^2\phi = 0 \tag{773}$$

B.28 Lorentz force: Potential energy derivation from vector potential

$$\mathbf{F} = q(\nabla(\mathbf{v}\cdot\mathbf{A}) - \frac{d\mathbf{A}}{dt}) \tag{774}$$

$$U = -\int_{\infty}^{r} \mathbf{F}\cdot d\mathbf{r} \tag{775}$$

$$U = -q\int_{\infty}^{r} (\nabla(\mathbf{v}\cdot\mathbf{A}) - \frac{d\mathbf{A}}{dt})\cdot d\mathbf{r} \tag{776}$$

$$U = -q\int_{\infty}^{r} \nabla(\mathbf{v}\cdot\mathbf{A})\cdot d\mathbf{r} + q\int_{\infty}^{r} \frac{d\mathbf{A}}{dt}\cdot d\mathbf{r} \tag{777}$$

$$U = -q\int_{\infty}^{r} \nabla(\mathbf{v}\cdot\mathbf{A})\cdot d\mathbf{r} \tag{778}$$

B.29 Lorentz force: Potential energy derivation from vector potential 3

$$\nabla(\mathbf{v}\cdot\mathbf{A})\cdot\hat{\mathbf{r}} = \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r} \tag{779}$$

$$d\mathbf{r} = \hat{\mathbf{r}}dr \tag{780}$$

$$\int_{\infty}^{r} \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r}dr = \int_{\infty}^{r} \nabla(\mathbf{v}\cdot\mathbf{A})\cdot d\mathbf{r} \tag{781}$$

$$U = -q\int_{\infty}^{r} \nabla(\mathbf{v}\cdot\mathbf{A})\cdot d\mathbf{r} \tag{782}$$

$$U = -q\int_{\infty}^{r} \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r}dr \tag{783}$$

B.30 Lorentz force: Potential energy derivation from vector potential 4

$$U = -q\int_{\infty}^{r} \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r}dr \tag{784}$$

$$\int_{\infty}^{r} \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r}dr = \mathbf{v}\cdot\mathbf{A}(r) - \mathbf{v}\cdot\mathbf{A}(\infty) \tag{785}$$

$$U = -q(\mathbf{v}\cdot\mathbf{A}(r) - \mathbf{v}\cdot\mathbf{A}(\infty)) \tag{786}$$

$$U = -q\mathbf{v}\cdot\mathbf{A}(r) + q\mathbf{v}\cdot\mathbf{A}(\infty) \tag{787}$$

$$assuming\mathbf{v}\cdot\mathbf{A}(\infty) = 0, thenU = -q\mathbf{v}\cdot\mathbf{A}(r) \tag{788}$$

B.31 Lorentz force: Derivation of classical Lagrangian of EM field

$$V = q\phi - q\dot{\mathbf{r}}\cdot\mathbf{A} \tag{789}$$

$$q\phi = V + q\dot{\mathbf{r}}\cdot\mathbf{A} \tag{790}$$

$$T = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} \tag{791}$$

$$L = T - V \tag{792}$$

$$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} - (q\phi - q\dot{\mathbf{r}}\cdot\mathbf{A}) \tag{793}$$

$$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\dot{\mathbf{r}}\cdot\mathbf{A} - q\phi \tag{794}$$

B.32 Lorentz force: Derivation of classical Lagrangian of EM field 2

$$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\dot{\mathbf{r}}\cdot\mathbf{A} - q\phi \tag{795}$$

$$\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z}) \tag{796}$$

$$\frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} = \frac{m}{2}(\dot{x}, \dot{y}, \dot{z})\cdot(\dot{x}, \dot{y}, \dot{z}) \tag{797}$$

$$q\dot{\mathbf{r}}\cdot\mathbf{A} = q(\dot{x}, \dot{y}, \dot{z})\cdot(A_x, A_y, A_z) \tag{798}$$

$$L = \frac{m}{2}(\dot{x}, \dot{y}, \dot{z})\cdot(\dot{x}, \dot{y}, \dot{z}) + q(\dot{x}, \dot{y}, \dot{z})\cdot(A_x, A_y, A_z) - q\phi \tag{799}$$

B.33 Lorentz force: Derivation of Lorentz force from classical Lagrangian (LHS) 4

$$dA_x = \frac{\partial A_x}{\partial t}dt + \frac{\partial A_x}{\partial x}dx + \frac{\partial A_x}{\partial y}dy + \frac{\partial A_x}{\partial z}dz \tag{800}$$

$$\frac{dx}{dt} = \dot{x} \tag{801}$$

$$\frac{dy}{dt} = \dot{y} \tag{802}$$

$$dx = \dot{x}dt \tag{803}$$

$$dy = \dot{y}dt \tag{804}$$

$$dA_x = \frac{\partial A_x}{\partial t}dt + \frac{\partial A_x}{\partial x}\dot{x}dt + \frac{\partial A_x}{\partial y}\dot{y}dt + \frac{\partial A_x}{\partial z}\dot{z}dt \tag{805}$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} \tag{806}$$

B.34 Lorentz force: Derivation of Lorentz force from classical Lagrangian (LHS) 5

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\frac{d}{dt}\dot{x} + q\frac{d}{dt}A_x \tag{807}$$

$$\frac{d}{dt}\dot{x} = \ddot{x} \tag{808}$$

$$m\frac{d}{dt}\dot{x} = m\ddot{x} \tag{809}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + q\frac{d}{dt}A_x \tag{810}$$

$$\frac{d}{dt}A_x = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} \tag{811}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + q(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}) \tag{812}$$

B.35 Lorentz force: Derivation of Lorentz force from classical Lagrangian (RHS) 2

$$\frac{\partial L}{\partial x} = q\frac{\partial}{\partial x}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) - q\frac{\partial}{\partial x}\phi \tag{813}$$

$$I = \frac{\partial}{\partial x}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) \tag{814}$$

$$\frac{\partial}{\partial x}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) = \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z} \tag{815}$$

$$q\frac{\partial}{\partial x}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) = q(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}) \tag{816}$$

$$\frac{\partial L}{\partial x} = q(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}) - q\frac{\partial}{\partial x}\phi \tag{817}$$

B.36 Lorentz force: Derivation of x component of electric field

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{818}$$

$$\nabla\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) \tag{819}$$

$$\mathbf{E} = -((\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) - \frac{\partial}{\partial t}(A_x, A_y, A_z)) \tag{820}$$

$$\mathbf{E}\cdot(1,0,0) = -((\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) - \frac{\partial}{\partial t}(A_x, A_y, A_z))\cdot(1,0,0) \tag{821}$$

B.37 Lorentz force: Derivation of Lorentz force from classical Lagrangian 4

$$(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_x = \dot{y}(\nabla \times \mathbf{A})_z - \dot{z}(\nabla \times \mathbf{A})_y \tag{822}$$

$$(\nabla \times \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \tag{823}$$

$$(\nabla \times \mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \tag{824}$$

$$\dot{y}(\nabla \times \mathbf{A})_z - \dot{z}(\nabla \times \mathbf{A})_y = \dot{y}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) - \dot{z}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) \tag{825}$$

$$F_x = qE_x + q(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_x \tag{826}$$

$$F_x = qE_x + q(\dot{y}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) - \dot{z}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})) \tag{827}$$

B.38 Lorentz force: Derivation of Lorentz force from classical Lagrangian 5

$$F_x = qE_x + q(\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}))_x \tag{828}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{829}$$

$$F_x = qE_x + q(\dot{\mathbf{r}} \times \mathbf{B})_x \tag{830}$$

$$F_x = \mathbf{F} \cdot \hat{\mathbf{x}} \tag{831}$$

$$E_x = \mathbf{E} \cdot \hat{\mathbf{x}} \tag{832}$$

$$qE_x = q\mathbf{E} \cdot \hat{\mathbf{x}} \tag{833}$$

$$q(\dot{\mathbf{r}} \times \mathbf{B})_x = q(\dot{\mathbf{r}} \times \mathbf{B}) \cdot \hat{\mathbf{x}} \tag{834}$$

$$\mathbf{F} \cdot \hat{\mathbf{x}} = q\mathbf{E} \cdot \hat{\mathbf{x}} + q(\dot{\mathbf{r}} \times \mathbf{B}) \cdot \hat{\mathbf{x}} \tag{835}$$

B.39 Electromagnetic wave equation: The origin of the electromagnetic wave equation in 2

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \tag{836}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{837}$$

$$-\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\frac{\partial}{\partial t}(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}) \tag{838}$$

$$-\frac{\partial}{\partial t}(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{839}$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{840}$$

B.40 Electromagnetic wave equation: The origin of the electromagnetic wave equation in 3

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \tag{841}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{842}$$

$$\nabla(\nabla \cdot \mathbf{E}) = \nabla(0) \tag{843}$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2 \mathbf{E} \tag{844}$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla \times (\nabla \times \mathbf{E}) \tag{845}$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\nabla^2 \mathbf{E} \tag{846}$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0 \tag{847}$$

B.41 Electromagnetic wave equation: The origin of the electromagnetic wave equation in 2

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) \tag{848}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{849}$$

$$\mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t} \frac{\partial \mathbf{B}}{\partial t} \tag{850}$$

$$\mu_0 \varepsilon_0 \frac{\partial}{\partial t} \frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{851}$$

$$\nabla \times (\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{852}$$

B.42 Electromagnetic wave equation: The origin of the electromagnetic wave equation in 3

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \tag{853}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{854}$$

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(0) - \nabla^2 \mathbf{B} \tag{855}$$

$$\nabla \times (\nabla \times \mathbf{B}) = -\nabla^2 \mathbf{B} \tag{856}$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = \nabla \times (\nabla \times \mathbf{B}) \tag{857}$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0 \tag{858}$$

B.43 Ampere's circuital law: Proof of equivalence 2

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{859}$$

$$\frac{\partial \mathbf{D}}{\partial t} = \frac{\partial}{\partial t}(\varepsilon_0 \mathbf{E} + \mathbf{P}) \tag{860}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial}{\partial t}(\varepsilon_0 \mathbf{E} + \mathbf{P}) \tag{861}$$

$$\nabla \times \frac{1}{\mu_0}\mathbf{B} = \nabla \times \mathbf{H} + \mathbf{J}_M \tag{862}$$

$$\nabla \times \frac{1}{\mu_0}\mathbf{B} = \mathbf{J}_f + \frac{\partial}{\partial t}(\varepsilon_0 \mathbf{E} + \mathbf{P}) + \mathbf{J}_M \tag{863}$$

$$\nabla \times \frac{1}{\mu_0}\mathbf{B} = \mathbf{J}_f + \frac{\partial}{\partial t}\varepsilon_0 \mathbf{E} + \frac{\partial}{\partial t}\mathbf{P} + \mathbf{J}_M \tag{864}$$

B.44 Ampere's circuital law: Proof of equivalence 4

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J}_f + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_P + \mathbf{J}_M \tag{865}$$

$$\mathbf{J}_b = \mathbf{J}_P + \mathbf{J}_M \tag{866}$$

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J}_f + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_b \tag{867}$$

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b \tag{868}$$

$$\frac{1}{\mu_0}(\nabla \times \mathbf{B}) = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{869}$$

B.45 Uncertainty principle: Kennard inequality proof part 1.1

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 \tag{870}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \cdot p(x) dx \tag{871}$$

$$p(x) = |\psi(x)|^2 \tag{872}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \cdot |\psi(x)|^2 dx \tag{873}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \cdot p(x) dx \tag{874}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \cdot |\psi(x)|^2 dx \tag{875}$$

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 \tag{876}$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} x^2 \cdot |\psi(x)|^2 dx - (\int_{-\infty}^{\infty} x \cdot |\psi(x)|^2 dx)^2 \tag{877}$$

B.46 Uncertainty principle: Kennard inequality proof part 1.4

$$f^*(x) \cdot f(x) = x^2 \cdot (\psi^*(x) \cdot \psi(x)) \tag{878}$$

$$x^2 \cdot (\psi^*(x) \cdot \psi(x)) = x^2 \cdot |\psi(x)|^2 \tag{879}$$

$$\int_{-\infty}^{\infty} f^*(x) \cdot f(x) dx = \int_{-\infty}^{\infty} x^2 \cdot |\psi(x)|^2 dx \tag{880}$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} x^2 \cdot |\psi(x)|^2 dx \tag{881}$$

$$\sigma_x^2 = \langle f | f \rangle \tag{882}$$

B.47 Uncertainty principle: Kennard inequality proof part 2.2

$$\frac{dv}{d\chi} = e^{\frac{-ip\chi}{\hbar}} \tag{883}$$

$$v = \int e^{\frac{-ip\chi}{\hbar}} d\chi \tag{884}$$

$$v = \frac{\hbar}{-ip} \int e^{\frac{-ip\chi}{\hbar}} d\chi \tag{885}$$

$$v = \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} + C \tag{886}$$

$$v = \frac{\hbar}{-ip} e^{b} + C \tag{887}$$

B.48 Uncertainty principle: Kennard inequality proof part 2.3

$$u = \psi(\chi) \tag{888}$$

$$v = \frac{\hbar}{-ip} e^{b} \tag{889}$$

$$uv = \psi(\chi) \frac{\hbar}{-ip} e^{b} \tag{890}$$

$$uv = \psi(\chi) \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \tag{891}$$

B.49 Uncertainty principle: Kennard inequality proof part 2.4

$$uv = \psi(\chi)\frac{\hbar}{-ip}e^{\frac{-ip\chi}{\hbar}} \tag{892}$$

$$\psi(\infty) = 0 \tag{893}$$

$$uv(\infty) = \psi(\infty)\frac{\hbar}{-ip}e^{\frac{-ip\infty}{\hbar}} \tag{894}$$

$$uv(\infty) = 0\frac{\hbar}{-ip}e^{\frac{-ip\infty}{\hbar}} \tag{895}$$

$$uv(\infty) = 0 \tag{896}$$

$$(uv)\Big|_{-\infty}^{\infty} = uv(\infty) - uv(-\infty) \tag{897}$$

$$(uv)\Big|_{-\infty}^{\infty} = 0 - uv(-\infty) \tag{898}$$

$$(uv)\Big|_{-\infty}^{\infty} = 0 \tag{899}$$

B.50 Uncertainty principle: Kennard inequality proof part 2.5

$$I = (uv)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v\frac{du}{d\chi}d\chi \tag{900}$$

$$(uv)\Big|_{-\infty}^{\infty} = 0 \tag{901}$$

$$I = -\int_{-\infty}^{\infty} v\frac{du}{d\chi}d\chi \tag{902}$$

$$\frac{du}{d\chi} = \frac{d\psi(\chi)}{d\chi} \tag{903}$$

$$I = -\int_{-\infty}^{\infty} v\frac{d\psi(\chi)}{d\chi}d\chi \tag{904}$$

$$I = \frac{\hbar}{ip}\int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi}e^{\frac{-ip\chi}{\hbar}}d\chi \tag{905}$$

B.51 Uncertainty principle: Kennard inequality proof part 2.9

$$g(x) = \frac{\hbar}{2\pi i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi}e^{i(x-\chi)b}d\chi db \tag{906}$$

$$\int_{-\infty}^{\infty} e^{i(x-\chi)b}db = 2\pi\delta(x-\chi) \tag{907}$$

$$g(x) = \frac{\hbar}{2\pi i}\int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi}\int_{-\infty}^{\infty} e^{i(x-\chi)b}db d\chi \tag{908}$$

$$g(x) = \frac{\hbar}{2\pi i}\int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi}2\pi\delta(x-\chi)d\chi \tag{909}$$

$$g(x) = \frac{\hbar}{i}\int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi}\delta(x-\chi)d\chi \tag{910}$$

$$g(x) = \frac{\hbar}{i}(\frac{d\psi(x)}{dx}) \tag{911}$$

B.52 Uncertainty principle: Kennard inequality proof part 3.2

$$\tilde{g}^*(p)\cdot\tilde{g}(p) = p^2\varphi^*(p)\cdot\varphi(p) \tag{912}$$

$$\varphi^*(p)\cdot\varphi(p) = |\varphi(p)|^2 \tag{913}$$

$$p^2\varphi^*(p)\cdot\varphi(p) = p^2|\varphi(p)|^2 \tag{914}$$

$$\tilde{g}^*(p)\cdot\tilde{g}(p) = p^2|\varphi(p)|^2 \tag{915}$$

$$|\tilde{g}(p)|^2 = p^2|\varphi(p)|^2 \tag{916}$$

B.53 Uncertainty principle: Kennard inequality proof part 3.3

$$\sigma_p^2 = \int_{-\infty}^{\infty} p^2|\varphi(p)|^2 dp \tag{917}$$

$$|\tilde{g}(p)|^2 = p^2|\varphi(p)|^2 \tag{918}$$

$$\int_{-\infty}^{\infty} |\tilde{g}(p)|^2 dp = \int_{-\infty}^{\infty} |g(x)|^2 dx \tag{919}$$

$$\sigma_p^2 = \int_{-\infty}^{\infty} |\tilde{g}(p)|^2 dp \tag{920}$$

$$\sigma_p^2 = \int_{-\infty}^{\infty} |g(x)|^2 dx \tag{921}$$

$$\sigma_p^2 = \langle g|g\rangle \tag{922}$$

B.54 Uncertainty principle: Kennard inequality proof part 4.1

$$\sigma_x^2 = \langle f|f\rangle \tag{923}$$

$$\sigma_p^2 = \langle g|g\rangle \tag{924}$$

$$\sigma_x^2\sigma_p^2 = \langle f|f\rangle \langle g|g\rangle \tag{925}$$

$$\sigma_x^2\sigma_p^2 \geq |\langle f|g\rangle|^2 \tag{926}$$

B.55 Uncertainty principle: Kennard inequality proof part 4.3

$$|z|^2 = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2 \tag{927}$$

$$(\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2 \geq (\mathrm{Im}(z))^2 \tag{928}$$

$$|z|^2 \geq (\mathrm{Im}(z))^2 \tag{929}$$

$$let z = \frac{\langle f|g\rangle - \langle g|f\rangle}{2i} \tag{930}$$

$$|z|^2 = \left(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i}\right)^2 \tag{931}$$

$$|z|^2 \geq \left(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i}\right)^2 \tag{932}$$

B.56 Uncertainty principle: Kennard inequality proof part 4.4

$$\sigma_x^2\sigma_p^2 \geq |\langle f|g\rangle|^2 \tag{933}$$

$$|z|^2 \geq (\frac{\langle f|g\rangle - \langle g|f\rangle}{2i})^2 \tag{934}$$

$$\sigma_x^2\sigma_p^2 \geq |z|^2 \tag{935}$$

$$\sigma_x^2\sigma_p^2 \geq (\frac{\langle f|g\rangle - \langle g|f\rangle}{2i})^2 \tag{936}$$

B.57 Uncertainty principle: Kennard inequality proof part 5.1

$$f(x) = x \cdot \psi(x) \tag{937}$$

$$g(x) = (-i\hbar\frac{d}{dx}) \cdot \psi(x) \tag{938}$$

$$\langle f|g\rangle = \int_{-\infty}^{\infty} f^*(x)g(x)dx \tag{939}$$

$$\langle f|g\rangle = \int_{-\infty}^{\infty} (x \cdot \psi^*(x))((-i\hbar\frac{d}{dx}) \cdot \psi(x))dx \tag{940}$$

$$\langle f|g\rangle = -i\hbar\int_{-\infty}^{\infty} x\psi^*(x)\frac{d\psi(x)}{dx}dx \tag{941}$$

B.58 Uncertainty principle: Kennard inequality proof part 5.2

$$f(x) = x \cdot \psi(x) \tag{942}$$

$$g^*(x) = \psi^*(x) \cdot (-i\hbar\frac{d}{dx}) \tag{943}$$

$$\langle g|f\rangle = \int_{-\infty}^{\infty} g^*(x)f(x)dx \tag{944}$$

$$\langle g|f\rangle = \int_{-\infty}^{\infty} \psi^*(x)(-i\hbar\frac{d}{dx})(x\psi(x))dx \tag{945}$$

$$\langle g|f\rangle = -i\hbar\int_{-\infty}^{\infty} \psi^*(x)\frac{d}{dx}(x\psi(x))dx \tag{946}$$

B.59 Uncertainty principle: Kennard inequality proof part 5.6

$$\langle f|g\rangle - \langle g|f\rangle = i\hbar\int_{-\infty}^{\infty} |\psi(x)|^2dx \tag{947}$$

$$p(x) = |\psi(x)|^2 \tag{948}$$

$$i\hbar\int_{-\infty}^{\infty} p(x)dx = i\hbar\int_{-\infty}^{\infty} |\psi(x)|^2dx \tag{949}$$

$$\int_{-\infty}^{\infty} p(x)dx = \frac{1}{i\hbar}(\langle f|g\rangle - \langle g|f\rangle) \tag{950}$$

$$\int_{-\infty}^{\infty} p(x)dx = 1 \tag{951}$$

B.60 Particle in a box: Wavefunction angular velocity as a function of particle mass from Schrödinger's equation 7

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t) \tag{952}$$

$$V(x) = 0 \tag{953}$$

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{954}$$

$$\frac{\partial}{\partial t}\psi(x,t) = -\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{955}$$

$$\hbar\omega\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{956}$$

B.61 Particle in a box: Wavefunction angular velocity as a function of particle mass from Schrödinger's equation 8

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{\hbar^2 k^2}{2m}\psi(x,t) \tag{957}$$

$$\frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m\hbar} \tag{958}$$

$$\hbar\omega\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) \tag{959}$$

$$\hbar\omega = \frac{p^2}{2m} \tag{960}$$

$$\omega = \frac{p^2}{2m\hbar} \tag{961}$$

B.62 Quantum harmonic oscillator: Ladder operator method 4

$$aa^\dagger - a^\dagger a = \frac{i}{\hbar}(\hat{p}\hat{x} - \hat{x}\hat{p}) \tag{962}$$

$$[\hat{p},\hat{x}] = \hat{p}\hat{x} - \hat{x}\hat{p} \tag{963}$$

$$[\hat{p},\hat{x}] = -i\hbar \tag{964}$$

$$-i\cdot i = 1 \tag{965}$$

$$\frac{i}{\hbar}[\hat{p},\hat{x}] = \frac{i}{\hbar}(-i\hbar) \tag{966}$$

$$\frac{i}{\hbar}[\hat{p},\hat{x}] = 1 \tag{967}$$

$$aa^\dagger - a^\dagger a = \frac{i}{\hbar}[\hat{p},\hat{x}] \tag{968}$$

$$aa^\dagger - a^\dagger a = 1 \tag{969}$$

$$\left[a, a^\dagger\right] = 1 \tag{970}$$

B.63 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 1.6

$$\hbar\omega(\frac{1}{2}+\frac{1}{\sqrt{2}}(-\frac{d^2}{dq^2}+q^2)\frac{1}{\sqrt{2}}(\frac{d^2}{dq^2}+q^2))\psi(q) = E\psi(q) \tag{971}$$

$$a = \frac{1}{\sqrt{2}}(\frac{d^2}{dq^2} + q^2) \tag{972}$$

$$a^\dagger = \frac{1}{\sqrt{2}}(-\frac{d^2}{dq^2} + q^2) \tag{973}$$

$$\hbar\omega(\frac{1}{2} + a^\dagger a)\psi(q) = E\psi(q) \tag{974}$$

$$E = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{975}$$

B.64 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 2

$$[q,p] = qp - pq \tag{976}$$

$$p = -i\frac{d}{dq} \tag{977}$$

$$[q,p] = q(-i\frac{d}{dq}) - (-i\frac{d}{dq})q \tag{978}$$

$$[q,p] = -iq\frac{d}{dq} + i\frac{d}{dq}q \tag{979}$$

$$[q,p]f(q) = -iq\frac{d}{dq}f(q) + i\frac{d}{dq}(qf(q)) \tag{980}$$

$$[q,p]f(q) = if(q) \tag{981}$$

B.65 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 3.1

$$a = \frac{1}{\sqrt{2}}(\frac{d}{dq} + q) \tag{982}$$

$$p = -i\frac{d}{dq} \tag{983}$$

$$ip = -\frac{d}{dq} \tag{984}$$

$$a = \frac{1}{\sqrt{2}}(-\frac{d}{dq} + q) \tag{985}$$

$$a = \frac{1}{\sqrt{2}}(ip + q) \tag{986}$$

B.66 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 3.2

$$a^\dagger = \frac{1}{\sqrt{2}}(-\frac{d}{dq} + q) \tag{987}$$

$$p = -i\frac{d}{dq} \tag{988}$$

$$a^\dagger = \frac{1}{\sqrt{2}}(-ip + q) \tag{989}$$

B.67 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 3.6

$$aa^\dagger - a^\dagger a = i(pq - qp) \tag{990}$$

$$[p, q] = pq - qp \tag{991}$$

$$[p, q] = -i \tag{992}$$

$$-i \cdot i = 1 \tag{993}$$

$$i(pq - qp) = i[p, q] \tag{994}$$

$$i[p, q] = -i^2 \tag{995}$$

$$-i^2 = 1 \tag{996}$$

$$aa^\dagger - a^\dagger a = 1 \tag{997}$$

$$\left[a, a^\dagger\right] = 1 \tag{998}$$

B.68 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 4.2

$$[H, a] = -\hbar\omega(aa^\dagger - a^\dagger a)a \tag{999}$$

$$\left[a, a^\dagger\right] = aa^\dagger - a^\dagger a \tag{1000}$$

$$-\hbar\omega(aa^\dagger - a^\dagger a)a = -\hbar\omega\left[a, a^\dagger\right]a \tag{1001}$$

$$-\hbar\omega\left[a, a^\dagger\right]a = -\hbar\omega a\left[a, a^\dagger\right] \tag{1002}$$

$$[H, a] = -\hbar\omega a\left[a, a^\dagger\right] \tag{1003}$$

$$[H, a] = -\hbar\omega a \tag{1004}$$

B.69 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 5.2

$$\left[H, a^\dagger\right] = \hbar\omega a^\dagger(aa^\dagger - a^\dagger a) \tag{1005}$$

$$\left[a, a^\dagger\right] = aa^\dagger - a^\dagger a \tag{1006}$$

$$\left[a, a^\dagger\right] = 1 \tag{1007}$$

B.70 Heisenberg picture: time evolution 4

$$\frac{d}{dt}\hat{x}(t) = \frac{i}{\hbar}(\hat{H}e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}} - e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}}\hat{H}) \tag{1008}$$

$$\hat{x}(t) = e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}} \tag{1009}$$

$$\hat{H}\hat{x}(t) - \hat{x}(t)\hat{H} = \hat{H}e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}} - e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}}\hat{H} \tag{1010}$$

$$\left[\hat{H}, \hat{x}(t)\right] = \hat{H}\hat{x}(t) - \hat{x}(t)\hat{H} \tag{1011}$$

$$\frac{d}{dt}\hat{x}(t) = \frac{i}{\hbar}\left[\hat{H}, \hat{x}(t)\right] \tag{1012}$$

B.71 Heisenberg picture: momentum evolution 4

$$\hat{p}(t) = A\cos(\omega t) + B\sin(\omega t) \tag{1013}$$

$$\frac{d}{dt}\cos(\omega t) = -\omega\sin(\omega t) \tag{1014}$$

$$\frac{d}{dt}\sin(\omega t) = \omega\cos(\omega t) \tag{1015}$$

$$\frac{d\hat{p}(t)}{dt} = \frac{d}{dt}(A\cos(\omega t)) + \frac{d}{dt}(B\sin(\omega t)) \tag{1016}$$

$$\frac{d\hat{p}(t)}{dt} = A\frac{d}{dt}\cos(\omega t) + B\frac{d}{dt}\sin(\omega t) \tag{1017}$$

$$\frac{d\hat{p}(t)}{dt} = -A\omega\sin(\omega t) + B\omega\cos(\omega t) \tag{1018}$$

B.72 Heisenberg picture: position commutator 4

$$[\hat{x}(t_1), \hat{x}(t_2)] = \frac{1}{\omega m}(\hat{x}_0\hat{p}_0 - \hat{p}_0\hat{x}_0)\sin(\omega t_2 - \omega t_1) \tag{1019}$$

$$\hat{x}_0\hat{p}_0 - \hat{p}_0\hat{x}_0 = [\hat{x}_0, \hat{p}_0] \tag{1020}$$

$$[\hat{x}(t_1), \hat{x}(t_2)] = \frac{1}{\omega m}[\hat{x}_0, \hat{p}_0]\sin(\omega t_2 - \omega t_1) \tag{1021}$$

$$[\hat{x}_0, \hat{p}_0] = i\hbar \tag{1022}$$

$$[\hat{x}(t_1), \hat{x}(t_2)] = \frac{i\hbar}{\omega m}\sin(\omega t_2 - \omega t_1) \tag{1023}$$

B.73 Heisenberg picture: momentum commutator 3

$$[\hat{p}(t_1), \hat{p}(t_2)] = m\omega\hat{p}_0\hat{x}_0\sin(\omega t_2 - \omega t_1) + m\omega\hat{x}_0\hat{p}_0\sin(\omega t_1 - \omega t_2) \tag{1024}$$

$$\sin(\omega t_1 - \omega t_2) = -\sin(\omega t_2 - \omega t_1) \tag{1025}$$

$$m\omega\hat{x}_0\hat{p}_0\sin(\omega t_1 - \omega t_2) = -m\omega\hat{x}_0\hat{p}_0\sin(\omega t_2 - \omega t_1) \tag{1026}$$

$$[\hat{p}(t_1), \hat{p}(t_2)] = m\omega\hat{p}_0\hat{x}_0\sin(\omega t_2 - \omega t_1) - m\omega\hat{x}_0\hat{p}_0\sin(\omega t_2 - \omega t_1 \tag{1027}$$

$$[\hat{p}(t_1), \hat{p}(t_2)] = (m\omega\hat{p}_0\hat{x}_0 - m\omega\hat{x}_0\hat{p}_0)\sin(\omega t_2 - \omega t_1) \tag{1028}$$

$$[\hat{p}(t_1), \hat{p}(t_2)] = i\hbar m\omega\sin(\omega t_2 - \omega t_1) \tag{1029}$$

B.74 Vacuum Rabi Oscillations: excited state probability

$$|\Psi(t)\rangle = \cos\left(\frac{\Omega t}{2}\right)|e,0\rangle - i\sin\left(\frac{\Omega t}{2}\right)|g,1\rangle \tag{1030}$$

$$\langle e,0|\Psi(t)\rangle = \cos\left(\frac{\Omega t}{2}\right)\langle e,0|e,0\rangle - i\sin\left(\frac{\Omega t}{2}\right)\langle e,0|g,1\rangle \tag{1031}$$

$$\langle e,0|e,0\rangle = 1 \tag{1032}$$

$$\langle e,0|g,1\rangle = 0 \tag{1033}$$

$$\langle e,0|\Psi(t)\rangle = \cos\left(\frac{\Omega t}{2}\right) \tag{1034}$$

$$P_e(t) = |\langle e,0|\Psi(t)\rangle|^2 \tag{1035}$$

$$P_e(t) = \cos^2(\frac{\Omega t}{2}) \tag{1036}$$

B.75 Vacuum Rabi Oscillations: ground state probability 2

$$P_g(t) = |\cos\left(\frac{\Omega t}{2}\right)\langle g,1|e,0\rangle - i\sin\left(\frac{\Omega t}{2}\right)\langle g,1|g,1\rangle|^2 \tag{1037}$$

$$\langle g,1|e,0\rangle = 0 \tag{1038}$$

$$\langle g,1|g,1\rangle = 1 \tag{1039}$$

$$P_g(t) = |\cos\left(\frac{\Omega t}{2}\right)\cdot 0 - i\sin\left(\frac{\Omega t}{2}\right)\cdot 1|^2 \tag{1040}$$

$$P_g(t) = |-i\sin\left(\frac{\Omega t}{2}\right)|^2 \tag{1041}$$

$$P_g(t) = \sin^2(\frac{\Omega t}{2}) \tag{1042}$$

B.76 Expectation value: integral expression

$$\left\langle \hat{X} \right\rangle_{\Psi} = \langle \Psi | \hat{X} | \Psi \rangle \tag{1043}$$

$$\hat{X} = \mathbb{1} \hat{X} \mathbb{1} \tag{1044}$$

$$\langle \Psi | \hat{X} | \Psi \rangle = \langle \Psi | \mathbb{1} \hat{X} \mathbb{1} | \Psi \rangle \tag{1045}$$

$$\langle \Psi | \mathbb{1} \hat{X} \mathbb{1} | \Psi \rangle = \int \int \langle \Psi | x \rangle \langle x | \hat{X} \left| x' \right\rangle \left\langle x' \middle| \Psi \right\rangle dx dx' \tag{1046}$$

$$\left\langle \hat{X} \right\rangle_{\Psi} = \int \int \langle \Psi | x \rangle \langle x | \hat{X} \left| x' \right\rangle \left\langle x' \middle| \Psi \right\rangle dx dx' \tag{1047}$$

B.77 Expectation value: integral expression 2

$$\left\langle \hat{X} \right\rangle_{\Psi} = \int \int \langle \Psi | x \rangle \langle x | \hat{X} \left| x' \right\rangle \left\langle x' \middle| \Psi \right\rangle dx dx' \tag{1048}$$

$$\hat{X} \left| x' \right\rangle = x' \left| x' \right\rangle \tag{1049}$$

$$\langle x | \hat{X} \left| x' \right\rangle = x' \left\langle x \middle| x' \right\rangle \tag{1050}$$

$$\langle x | \hat{X} \left| x' \right\rangle = x' \delta(x - x') \tag{1051}$$

$$\langle \Psi | x \rangle = \langle x | \Psi \rangle^{\dagger} \tag{1052}$$

$$\langle \Psi | x \rangle \langle x | \hat{X} \left| x' \right\rangle = \langle x | \Psi \rangle^{\dagger} x' \delta(x - x') \tag{1053}$$

$$\left\langle \hat{X} \right\rangle_{\Psi} = \int \int \langle x | \Psi \rangle^{\dagger} x' \delta(x - x') \left\langle x' \middle| \Psi \right\rangle dx dx' \tag{1054}$$

B.78 Expectation value: integral expression 3

$$\left\langle \hat{X} \right\rangle_{\Psi} = \int \int \langle x | \Psi \rangle^{\dagger} x' \delta(x - x') \left\langle x' \middle| \Psi \right\rangle dx dx' \tag{1055}$$

$$\int \langle x | \Psi \rangle^{\dagger} x' \delta(x - x') \left\langle x' \middle| \Psi \right\rangle dx' = \langle x | \Psi \rangle^{\dagger} x \langle x | \Psi \rangle \tag{1056}$$

$$\left\langle \hat{X} \right\rangle_{\Psi} = \int \langle x | \Psi \rangle^{\dagger} x \langle x | \Psi \rangle \, dx \tag{1057}$$

$$\langle x | \Psi \rangle = \Psi(x) \tag{1058}$$

$$\langle x | \Psi \rangle^{\dagger} = \Psi^{*}(x) \tag{1059}$$

$$\langle x | \Psi \rangle^{\dagger} x \langle x | \Psi \rangle = x |\Psi(x)|^2 \tag{1060}$$

$$\left\langle \hat{X} \right\rangle_{\Psi} = \int x |\Psi(x)|^2 dx \tag{1061}$$

B.79 Euler-lagrange equation: Full derivative of the perturbation Lagrangian with respect to ε 2

$$\frac{dg_{\varepsilon}}{d\varepsilon} = \frac{d}{d\varepsilon}(\varepsilon \eta(x)) \tag{1062}$$

$$\frac{d}{d\varepsilon}(\varepsilon \eta(x)) = \eta(x) + \varepsilon \frac{d\eta(x)}{d\varepsilon} \tag{1063}$$

$$\frac{dg_{\varepsilon}}{d\varepsilon} = \eta(x) + \varepsilon \frac{d\eta(x)}{d\varepsilon} \tag{1064}$$

$$\frac{dg_{\varepsilon}}{d\varepsilon} = \eta(x) when \varepsilon = 0. \tag{1065}$$

B.80 Euler-Lagrange equation: Derivation

$$J = \int_a^b L(x, f(x), f'(x)) \tag{1066}$$

$$g_{\varepsilon}(x) = f(x) + \varepsilon \eta(x) \tag{1067}$$

$$J_{\varepsilon} = \int_a^b L(x, g_{\varepsilon}(x), g'_{\varepsilon}(x)) dx \tag{1068}$$

$$L_{\varepsilon}(x) = L(x, g_{\varepsilon}(x), g'_{\varepsilon}(x)) \tag{1069}$$

$$\frac{dJ_{\varepsilon}}{d\varepsilon} = \frac{d}{d\varepsilon} \int_a^b L_{\varepsilon}(x) dx \tag{1070}$$

$$\frac{dJ_{\varepsilon}}{d\varepsilon} = \int_a^b \frac{dL_{\varepsilon}}{d\varepsilon} dx \tag{1071}$$

B.81 Euler-Lagrange equation: Derivation 4

$$\frac{dv}{dx} = \eta'(x) \tag{1072}$$

$$v = \int \frac{dv}{dx} dx \tag{1073}$$

$$v = \int \eta'(x) dx \tag{1074}$$

$$v = \eta(x) \tag{1075}$$

B.82 Euler-Lagrange equation: Derivation 5

$$u = \frac{\partial L}{\partial f'} \tag{1076}$$

$$v = \eta(x) \tag{1077}$$

$$\eta(a) = 0 \tag{1078}$$

$$uv = \frac{\partial L}{\partial f'}\eta(x) \tag{1079}$$

$$(uv)\Big|_a^b = (\frac{\partial L}{\partial f'}\eta(x))\Big|_a^b \tag{1080}$$

$$(\frac{\partial L}{\partial f'}\eta(x))\Big|_a^b = 0 \tag{1081}$$

B.83 Euler-Lagrange equation: Derivation 6

$$I = \int_a^b \frac{\partial L}{\partial f'}\eta'(x)dx \tag{1082}$$

$$I = (uv)\Big|_a^b - \int_a^b v\frac{du}{dx}dx \tag{1083}$$

$$(uv)\Big|_a^b = 0 \tag{1084}$$

$$I = -\int_a^b v\frac{du}{dx}dx \tag{1085}$$

$$\frac{\partial L}{\partial f'}\eta'(x) = v\frac{du}{dx} \tag{1086}$$

$$\eta(x) = v \tag{1087}$$

$$\frac{d}{dx}\frac{\partial L}{\partial f'} = \frac{du}{dx} \tag{1088}$$

$$I = -\int_a^b \eta(x)\frac{d}{dx}\frac{\partial L}{\partial f'}dx \tag{1089}$$

B.84 Euler-Lagrange equation: Straight line

$$S = \int_a^b ds \tag{1090}$$

$$ds = \sqrt{dx^2 + dy^2} \tag{1091}$$

$$ds = dx\sqrt{1 + y'^2} \tag{1092}$$

$$S = \int_a^b dx\sqrt{1 + y'^2} \tag{1093}$$

$$S = \int_a^b \sqrt{(1 + y'^2)}dx \tag{1094}$$

B.85 Euler-Lagrange equation: Straight line 3

$$\frac{dL}{dy} - \frac{d}{dx}\frac{dL}{dy'} = 0 \tag{1095}$$

$$\frac{dL}{dy} = 0 \tag{1096}$$

$$-\frac{d}{dx}\frac{dL}{dy'} = 0 \tag{1097}$$

$$\frac{dL}{dy'} = C \tag{1098}$$

$$\frac{d}{dx}(y'(1 + y'^2)^{-\frac{1}{2}}) = C \tag{1099}$$

$$\int \frac{d}{dx}(y'(1 + y'^2)^{-\frac{1}{2}})dx = \int Cdx \tag{1100}$$

$$\int \frac{d}{dx}(y'(1 + y'^2)^{-\frac{1}{2}})dx = C \tag{1101}$$

B.86 Euler-Lagrange equation: Straight line 6

$$\frac{dy}{dx} = C(1 - C^2)^{-1/2} \tag{1102}$$

$$C(1 - C^2)^{-1/2} = A \tag{1103}$$

$$\frac{dy}{dx} = A \tag{1104}$$

$$\int dy = \int Adx \tag{1105}$$

$$y = Ax + C \tag{1106}$$

B.87 Escape velocity

$$F = \frac{GMm}{r^2} \tag{1107}$$

$$dW = Fdr \tag{1108}$$

$$dW = \frac{GMm}{r^2}dr \tag{1109}$$

$$W = \int dW \tag{1110}$$

$$W = \int_{r_0}^{\infty} \frac{GMm}{r^2}dr \tag{1111}$$

B.88 Escape velocity 2

$$W = \int_{r_0}^{\infty} \frac{GMm}{r^2} dr \tag{1112}$$

$$F = \frac{GMm}{r^2} \tag{1113}$$

$$W = \int_{r_0}^{\infty} F dr \tag{1114}$$

$$W = F r_0 \tag{1115}$$

$$W = m g r_0 \tag{1116}$$

B.89 Escape velocity 3

$$W = m g r_0 \tag{1117}$$

$$E = \frac{1}{2} m v_{esc}^2 \tag{1118}$$

$$2E = m v_{esc}^2 \tag{1119}$$

$$2 m g r_0 = m v_{esc}^2 \tag{1120}$$

$$v_{esc}^2 = 2 g r_0 \tag{1121}$$

$$v_{esc} = \sqrt{2 g r_0} \tag{1122}$$

B.90 Snell's law: from Fermat's principle 2

$$\frac{dT}{dx} = \frac{x}{v_1 (x^2 + a^2)^{\frac{1}{2}}} + \frac{x - l}{v_2 ((x-l)^2 + b^2)^{\frac{1}{2}}} \tag{1123}$$

$$\frac{x}{(x^2 + a^2)^{\frac{1}{2}}} = \sin\theta_1 \tag{1124}$$

$$\frac{l - x}{((x-l)^2 + b^2)^{\frac{1}{2}}} = \sin\theta_2 \tag{1125}$$

$$\frac{dT}{dx} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} \tag{1126}$$

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \tag{1127}$$

B.91 Snell's law: from Fermat's principle 3

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \tag{1128}$$

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \tag{1129}$$

$$\frac{1}{v_1} = \frac{n_1}{c} \tag{1130}$$

$$\frac{\sin\theta_1}{\frac{c}{n_1}} = \frac{\sin\theta_2}{v_2} \tag{1131}$$

$$n_1 \sin\theta_1 = v_2 \sin\theta_2 \tag{1132}$$

$$\frac{1}{v_2} = \frac{n_2}{c} \tag{1133}$$

$$n_1 \sin\theta_1 = \frac{c}{n_2} \sin\theta_2 \tag{1134}$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{1135}$$

B.92 Wave equation: plane wave eigenmodes 2

$$\frac{\partial^2 u(x,t)}{\partial t^2} = (-i\omega) \frac{\partial}{\partial t} (e^{-i\omega t} f(x)) \tag{1136}$$

$$i \cdot i = -1 \tag{1137}$$

$$-i\omega \cdot -i\omega = \omega^2 \tag{1138}$$

$$\omega^2 e^{-i\omega t} f(x) = \omega^2 e^{-i\omega t} f(x) \tag{1139}$$

$$-\omega^2 e^{-i\omega t} f(x) = -\omega^2 e^{-i\omega t} f(x) \tag{1140}$$

$$-\omega^2 e^{-i\omega t} f(x) = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1141}$$

B.93 Wave equation: plane wave eigenmodes 4

$$u(x,t) = A e^{-i(kx - \omega t)} + B e^{i(kx - \omega t)} \tag{1142}$$

$$u(x,t) = \int_{-\infty}^{\infty} s(\omega) (A e^{-i(kx-\omega t)} + B e^{i(kx-\omega t)}) d\omega \tag{1143}$$

$$s_+(\omega) = A s(\omega) \tag{1144}$$

$$s_-(\omega) = Bs(\omega) \tag{1145}$$

$$u(x,t) = \int_{-\infty}^{\infty} s_+(\omega) e^{-i(kx-\omega t)} d\omega + \int_{-\infty}^{\infty} s_-(\omega) e^{i(kx-\omega t)} d\omega \tag{1146}$$

B.94 Wave equation: plane wave eigenmodes 5

$$u(x,t) = \int_{-\infty}^{\infty} s_+(\omega) e^{-i(kx-\omega t)} d\omega + \int_{-\infty}^{\infty} s_-(\omega) e^{i(kx-\omega t)} d\omega \tag{1147}$$

$$\omega = kc \tag{1148}$$

$$\int_{-\infty}^{\infty} s_+(\omega) e^{-ik(x-ct)} d\omega = F(x-ct) \tag{1149}$$

$$\int_{-\infty}^{\infty} s_-(\omega) e^{ik(x+ct)} d\omega = G(x+ct) \tag{1150}$$

$$u(x,t) = F(x-ct) + G(x+ct) \tag{1151}$$

B.95 Wave equation: Hooke's law

$$F_H = F_{x+2h} - F_x \tag{1152}$$

$$F_N = ma(t) \tag{1153}$$

$$F_N = F_H \tag{1154}$$

$$ma(t) = F_{x+2h} - F_x \tag{1155}$$

$$m\frac{\partial^2}{\partial t^2} u(x+h,t) = F_{x+2h} - F_x \tag{1156}$$

B.96 Wave equation: Hooke's law 2

$$m\frac{\partial^2}{\partial t^2} u(x+h,t) = F_{x+2h} - F_x \tag{1157}$$

$$F_{x+2h} = ku(x+2h,t) - ku(x+h,t) \tag{1158}$$

$$F_x = ku(x+h,t) - ku(x,t) \tag{1159}$$

$$m\frac{\partial^2}{\partial t^2} u(x+h,t) = ku(x+2h,t) - 2ku(x+h,t) + ku(x,t) \tag{1160}$$

$$\frac{\partial^2}{\partial t^2} u(x+h,t) = \frac{k}{m}\big(u(x+2h,t) - 2u(x+h,t) + u(x,t)\big) \tag{1161}$$

B.97 Wave equation: Hooke's law 3

$$\frac{\partial^2}{\partial t^2} u(x+h,t) = \frac{k}{m}\big(u(x+2h,t) - 2u(x+h,t) + u(x,t)\big) \tag{1162}$$

$$N = \frac{L}{h} \tag{1163}$$

$$m = \frac{M}{N} \tag{1164}$$

$$\frac{k}{m} = \frac{kN}{M} \tag{1165}$$

$$\frac{kN}{M} = \frac{kL}{Mh} \tag{1166}$$

$$\frac{\partial^2}{\partial t^2} u(x+h,t) = \frac{kL}{Mh}\big(u(x+2h,t) - 2u(x+h,t) + u(x,t)\big) \tag{1167}$$

$$\frac{kL}{Mh} = \frac{KL}{Mh^2} \tag{1168}$$

$$\frac{\partial^2}{\partial t^2} u(x+h,t) = \frac{KL^2}{Mh^2}\big(u(x+2h,t) - 2u(x+h,t) + u(x,t)\big) \tag{1169}$$

B.98 Wave equation: stress pulse in a bar 2

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{KL^2}{M}\frac{\partial^2 u(x,t)}{\partial x^2} \tag{1170}$$

$$K = \frac{EA}{L} \tag{1171}$$

$$\rho = \frac{M}{V} \tag{1172}$$

$$\frac{KL^2}{M} = \frac{EA}{L}\frac{L^2}{M} \tag{1173}$$

$$\frac{EA}{L}\frac{L^2}{M} = \frac{EAL}{M} \tag{1174}$$

$$\frac{EAL}{M} = \frac{E}{\rho} \tag{1175}$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u(x,t)}{\partial x^2} \tag{1176}$$

C $S = 2$ (two premises removed)

C.1 Gauss' law: equivalence between differential and integral forms

$$\oiint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\varepsilon_0} \tag{1177}$$

$$Q = \iiint_V \rho dV \tag{1178}$$

$$\frac{Q}{\varepsilon_0} = \frac{\iiint_V \rho dV}{\varepsilon_0} \tag{1179}$$

$$\oiint_S \mathbf{E} \cdot d\mathbf{A} = \iiint_V \frac{\rho}{\varepsilon_0} dV \tag{1180}$$

$$\iiint_V \nabla \cdot \mathbf{E} dV = \iiint_V \frac{\rho}{\varepsilon_0} dV \tag{1181}$$

C.2 Gauss' law: Equivalence of total and free charge statements

$$\rho_b = -\nabla \cdot \mathbf{P} \tag{1182}$$

$$\rho = -\rho_b \tag{1183}$$

$$\rho = \nabla \cdot \mathbf{P} \tag{1184}$$

$$\rho = \nabla \cdot (\mathbf{D} - \mathbf{P}) \tag{1185}$$

C.3 Uniqueness theorem for Poisson's equation

$$\nabla^2 \phi_1 = -\frac{\rho_f}{\varepsilon_0} \tag{1186}$$

$$\rho_f = 0 \tag{1187}$$

$$-\frac{\rho_f}{\varepsilon_0} = 0 \tag{1188}$$

$$\nabla^2 \phi_1 = 0 \tag{1189}$$

$$\nabla^2 \phi = \nabla^2 \phi_1 \tag{1190}$$

$$\nabla^2 \phi = 0 \tag{1191}$$

C.4 Uniqueness theorem for Poisson's equation 2

$$\nabla \cdot (\phi \nabla \phi) = (\nabla \phi)^2 + \phi \nabla^2 \phi \tag{1192}$$

$$\int_V \nabla \cdot (\phi \nabla \phi) dV = \int_V ((\nabla \phi)^2 + \phi \nabla^2 \phi) dV \tag{1193}$$

$$\int_V \nabla \cdot (\phi \nabla \phi) dV = \int_S \phi \nabla \phi \cdot d\mathbf{S} \tag{1194}$$

$$\int_S \phi \nabla \phi \cdot d\mathbf{S} = \int_V ((\nabla \phi)^2 + \phi \nabla^2 \phi) dV \tag{1195}$$

$$\int_S \phi \nabla \phi \cdot d\mathbf{S} = \int_V (\nabla \phi)^2 dV \tag{1196}$$

C.5 Uniqueness theorem for Poisson's equation 6

$$\frac{\partial \phi}{\partial r} = 0 \tag{1197}$$

$$\int \frac{\partial \phi}{\partial r} dr = \int 0 dr \tag{1198}$$

$$\phi = C_1 r + C_2 \tag{1199}$$

$$\phi = C_1 - C_2 \tag{1200}$$

C.6 Uniqueness theorem for Poisson's equation 7

$$\phi = C_1 - C_2 \tag{1201}$$

$$\phi_1 = C_1 \tag{1202}$$

$$\phi_2 = C_2 \tag{1203}$$

$$\phi_2 - \phi_1 = C_2 - C_1 \tag{1204}$$

$$\phi_2 - \phi_1 = C \tag{1205}$$

C.7 Poisson's equation: Newtonian gravity

$$\nabla \cdot \mathbf{g} = -4\pi G\rho \tag{1206}$$

$$\mathbf{g} = -\nabla\phi \tag{1207}$$

$$\nabla \cdot (-\nabla\phi) = -4\pi G\rho \tag{1208}$$

$$-\nabla^2\phi = -4\pi G\rho \tag{1209}$$

$$\nabla^2\phi = 4\pi G\rho \tag{1210}$$

C.8 Poisson's equation: Gravitational potential from Poisson's equation 2

$$\int_V \nabla \cdot \nabla\phi dV = \int_V 4\pi G\rho dV \tag{1211}$$

$$\nabla \cdot \nabla\phi = 4\pi G\rho \tag{1212}$$

$$\int_V \nabla \cdot \nabla\phi dV = \int_S \nabla\phi \cdot d\mathbf{S} \tag{1213}$$

$$4\pi G\int_V \rho dV = 4\pi Gm \tag{1214}$$

$$\int_S \nabla\phi \cdot d\mathbf{S} = 4\pi Gm \tag{1215}$$

C.9 Poisson's equation: Gravitational potential from Poisson's equation 6

$$\int_S \frac{\partial\phi}{\partial r} dS = \int_0^{2\pi}\int_0^{\pi} \frac{\partial\phi}{\partial r} r^2 \sin\theta d\theta d\varphi \tag{1216}$$

$$\int_0^{\pi} \sin\theta d\theta = 2 \tag{1217}$$

$$\int_0^{2\pi} d\varphi = 2\pi \tag{1218}$$

$$\int_0^{2\pi}\int_0^{\pi} d\theta d\varphi = 4\pi \tag{1219}$$

$$\int_S \frac{\partial\phi}{\partial r} dS = 4\pi \frac{\partial\phi}{\partial r} r^2 \tag{1220}$$

C.10 Poisson's equation: Gravitational potential from Poisson's equation 8

$$\frac{\partial\phi}{\partial r} = \frac{Gm}{r^2} \tag{1221}$$

$$\int \frac{\partial\phi}{\partial r} dr = \int \frac{Gm}{r^2} dr \tag{1222}$$

$$\phi(r) - \phi(c) = -Gm\int \frac{1}{r} dr \tag{1223}$$

$$\phi(r) - \phi(c) = -Gm\ln|r| + C \tag{1224}$$

$$\phi(r) = -Gm\ln|r| + C + \phi(c) \tag{1225}$$

$$\phi(r) = \frac{-Gm}{r} + C, where C is the constant of integration. \tag{1226}$$

C.11 Poisson's equation: Electrostatics

$$\nabla \cdot \mathbf{D} = \rho_f \tag{1227}$$

$$\mathbf{D} = \varepsilon\nabla\phi \tag{1228}$$

$$\nabla \cdot (\varepsilon\nabla\phi) = \rho_f \tag{1229}$$

$$\varepsilon\nabla^2\phi = -\rho_f \tag{1230}$$

$$\nabla^2\phi = -\frac{\rho_f}{\varepsilon} \tag{1231}$$

C.12 Poisson's equation: Electrostatic potential from Poisson's equation

$$\nabla^2\phi = -\frac{\rho_f}{\varepsilon} \tag{1232}$$

$$\int_V \nabla^2\phi dV = -\frac{1}{\varepsilon}\int_V \rho_f dV \tag{1233}$$

$$\int_V \nabla \cdot \nabla\phi dV = -\frac{1}{\varepsilon}\int_V \rho_f dV \tag{1234}$$

$$\int_V \nabla \cdot \nabla\phi dV = -\frac{Q}{\varepsilon} \tag{1235}$$

C.13 Poisson's equation: Electrostatic potential from Poisson's equation 2

$$\int_V \nabla \cdot \nabla \phi dV = -\frac{Q}{\varepsilon} \tag{1236}$$

$$\nabla \cdot \nabla \phi = \nabla^2 \phi \tag{1237}$$

$$\int_V \nabla^2 \phi dV = -\frac{Q}{\varepsilon} \tag{1238}$$

$$\int_S \nabla \phi \cdot d\mathbf{S} = -\frac{Q}{\varepsilon} \tag{1239}$$

$$\int_S \frac{\partial \phi}{\partial r} dS = -\frac{Q}{\varepsilon} \tag{1240}$$

C.14 Poisson's equation: Electrostatic potential from Poisson's equation 4

$$\int_\infty^r \frac{\partial \phi}{\partial r} dr = \int_\infty^r -\frac{Q}{4\pi\varepsilon r^2} dr \tag{1241}$$

$$\frac{\partial \phi}{\partial r} = -\frac{Q}{4\pi\varepsilon r^2} \tag{1242}$$

$$\phi(r) - \phi(\infty) = -\frac{Q}{4\pi\varepsilon} \int_\infty^r \frac{1}{r^2} dr \tag{1243}$$

$$\phi(r) - \phi(\infty) = \frac{Q}{4\pi\varepsilon} \left[\frac{1}{r}\right]_\infty^r \tag{1244}$$

$$\phi(r) - \phi(\infty) = \frac{Q}{4\pi\varepsilon r} \tag{1245}$$

$$\phi(r) = \frac{Q}{4\pi\varepsilon r} \tag{1246}$$

C.15 Lorentz force: continuous charge distribution

$$\frac{d\mathbf{F}}{dV} = \frac{dq}{dV}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1247}$$

$$\frac{d\mathbf{F}}{dV} = \rho(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1248}$$

$$\mathbf{F} = \rho V(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1249}$$

$$\mathbf{f} = \rho\mathbf{E} + \rho\mathbf{v} \times \mathbf{B} \tag{1250}$$

C.16 Lorentz force: continuous charge distribution 2

$$\mathbf{f} = \rho\mathbf{E} + \rho\mathbf{v} \times \mathbf{B} \tag{1251}$$

$$\mathbf{J} = \rho\mathbf{v} \tag{1252}$$

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{1253}$$

$$\mathbf{F} = \iiint \mathbf{f} dV \tag{1254}$$

$$\mathbf{F} = \iiint (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B}) dV \tag{1255}$$

C.17 Lorentz force: Lorentz force in terms of potentials

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{1256}$$

$$q\mathbf{E} = q(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}) \tag{1257}$$

$$q\mathbf{E} + q\nabla(\mathbf{v} \cdot \mathbf{A}) = q(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A})) \tag{1258}$$

$$q\mathbf{E} + q\nabla(\mathbf{v} \cdot \mathbf{A}) - q(\mathbf{v} \cdot \nabla)\mathbf{A} = q(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A}) \tag{1259}$$

$$\mathbf{F} = q(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A}) \tag{1260}$$

C.18 Lorentz force: Potential energy derivation from scalar potential 3

$$U = q \int_\infty^r \nabla\phi \cdot d\mathbf{r} \tag{1261}$$

$$\nabla\phi = \frac{d\phi}{dr} \tag{1262}$$

$$U = q \int_\infty^r \frac{d\phi}{dr} \cdot d\mathbf{r} \tag{1263}$$

$$U = q[\phi(r) - \phi(\infty)] \tag{1264}$$

$$U = q\phi(r) - q\phi(\infty) \tag{1265}$$

$$assuming \phi(\infty) = 0, then U = q\phi(r) \tag{1266}$$

C.19 Laplace equation: Analytic functions (u)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1267}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \tag{1268}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) \tag{1269}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{1270}$$

C.20 Laplace equation: Analytic functions (u) 2

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{1271}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) \tag{1272}$$

$$\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2} \tag{1273}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \tag{1274}$$

C.21 Laplace equation: Analytic functions (u) 3

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \tag{1275}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} \tag{1276}$$

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} \tag{1277}$$

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{1278}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{1279}$$

$$\nabla^2 u = 0 \tag{1280}$$

C.22 Laplace equation: Analytic functions (v)

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{1281}$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y} \tag{1282}$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \tag{1283}$$

C.23 Laplace equation: Analytic functions (v) 2

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1284}$$

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}\frac{\partial v}{\partial y} \tag{1285}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \tag{1286}$$

C.24 Laplace equation: Analytic functions (v) 3

$$\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} \tag{1287}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} \tag{1288}$$

$$\nabla^2 v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \tag{1289}$$

$$\nabla^2 v = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} \tag{1290}$$

$$\nabla^2 v = 0 \tag{1291}$$

C.25 Laplace equation: Electrostatics

$$\mathbf{E} = (u, v) \tag{1292}$$

$$\frac{\partial \mathbf{E}}{\partial x} = \frac{\partial u}{\partial x} \tag{1293}$$

$$\frac{\partial \mathbf{E}}{\partial y} = \frac{\partial v}{\partial y} \tag{1294}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \rho \tag{1295}$$

C.26 Laplace equation: Electrostatics 2

$$\frac{\partial \phi}{\partial x} = -u \tag{1296}$$

$$\frac{\partial^2 \phi}{\partial x^2} = -\frac{\partial u}{\partial x} \tag{1297}$$

$$\frac{\partial^2 \phi}{\partial y^2} = 0 \tag{1298}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\frac{\partial u}{\partial x} + 0 \tag{1299}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\rho \tag{1300}$$

C.27 Laplace equation: Electrostatics 3

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \quad (1301)$$

$$\frac{\partial^2\phi}{\partial x^2} = 0 \quad (1302)$$

$$\frac{\partial^2\phi}{\partial y^2} = 0 \quad (1303)$$

$$\nabla^2\phi = 0 + 0 \quad (1304)$$

$$\nabla^2\phi = 0 \quad (1305)$$

C.28 Lorentz force: Potential energy derivation from vector potential

$$\mathbf{F} = q(\nabla(\mathbf{v}\cdot\mathbf{A}) - \frac{d\mathbf{A}}{dt}) \quad (1306)$$

$$\mathbf{F} = -\nabla U \quad (1307)$$

$$U = -\int_{\infty}^{r} \mathbf{F}\cdot d\mathbf{r} \quad (1308)$$

$$U = -q\int_{\infty}^{r} \nabla(\mathbf{v}\cdot\mathbf{A})\cdot d\mathbf{r} \quad (1309)$$

C.29 Lorentz force: Potential energy derivation from vector potential 3

$$\nabla(\mathbf{v}\cdot\mathbf{A})\cdot\hat{\mathbf{r}} = \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r} \quad (1310)$$

$$-q\int_{\infty}^{r} \nabla(\mathbf{v}\cdot\mathbf{A})\cdot\hat{\mathbf{r}}dr = -q\int_{\infty}^{r} \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r}dr \quad (1311)$$

$$U = -q\int_{\infty}^{r} \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r}dr \quad (1312)$$

C.30 Lorentz force: Potential energy derivation from vector potential 4

$$U = -q\int_{\infty}^{r} \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r}dr \quad (1313)$$

$$\frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r} = \mathbf{v}\cdot\mathbf{A}'(r) \quad (1314)$$

$$-q\int_{\infty}^{r} \frac{\partial(\mathbf{v}\cdot\mathbf{A})}{\partial r}dr = -q\int_{\infty}^{r} \mathbf{v}\cdot\mathbf{A}'(r)dr \quad (1315)$$

$$-q\int_{\infty}^{r} \mathbf{v}\cdot\mathbf{A}'(r)dr = -q\mathbf{v}\cdot\mathbf{A}(r) \quad (1316)$$

$$U = -q\mathbf{v}\cdot\mathbf{A}(r) \quad (1317)$$

C.31 Lorentz force: Derivation of classical Lagrangian of EM field

$$V = q\phi - q\dot{\mathbf{r}}\cdot\mathbf{A} \quad (1318)$$

$$q\phi = V + q\dot{\mathbf{r}}\cdot\mathbf{A} \quad (1319)$$

$$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} - q\phi \quad (1320)$$

$$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} - (V + q\dot{\mathbf{r}}\cdot\mathbf{A}) \quad (1321)$$

$$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\dot{\mathbf{r}}\cdot\mathbf{A} - q\phi \quad (1322)$$

C.32 Lorentz force: Derivation of classical Lagrangian of EM field 2

$$L = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\dot{\mathbf{r}}\cdot\mathbf{A} - q\phi \quad (1323)$$

$$\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z}) \quad (1324)$$

$$\mathbf{A} = (A_x, A_y, A_z) \quad (1325)$$

$$\frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} = \frac{m}{2}(\dot{x}, \dot{y}, \dot{z})\cdot(\dot{x}, \dot{y}, \dot{z}) \quad (1326)$$

$$q\dot{\mathbf{r}}\cdot\mathbf{A} = q(\dot{x}, \dot{y}, \dot{z})\cdot(A_x, A_y, A_z) \quad (1327)$$

$$L = \frac{m}{2}(\dot{x}, \dot{y}, \dot{z})\cdot(\dot{x}, \dot{y}, \dot{z}) + q(\dot{x}, \dot{y}, \dot{z})\cdot(A_x, A_y, A_z) - q\phi \quad (1328)$$

C.33 Lorentz force: Derivation of Lorentz force from classical Lagrangian (LHS) 4

$$dA_x = \frac{\partial A_x}{\partial t}dt + \frac{\partial A_x}{\partial x}dx + \frac{\partial A_x}{\partial y}dy + \frac{\partial A_x}{\partial z}dz \quad (1329)$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_x}{\partial y}\frac{dy}{dt} + \frac{\partial A_x}{\partial z}\frac{dz}{dt} \quad (1330)$$

$$\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} \quad (1331)$$

C.34 Lorentz force: Derivation of Lorentz force from classical Lagrangian (LHS) 5

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\frac{d}{dt}\dot{x} + q\frac{d}{dt}A_x \quad (1332)$$

$$\frac{d}{dt}\dot{x} = \ddot{x} \quad (1333)$$

$$\frac{d}{dt}A_x = \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z} \quad (1334)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + q(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}) \quad (1335)$$

C.35 Lorentz force: Derivation of Lorentz force from classical Lagrangian (RHS) 2

$$\frac{\partial L}{\partial x} = q\frac{\partial}{\partial x}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) - q\frac{\partial}{\partial x}\phi \quad (1336)$$

$$\frac{\partial}{\partial x}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) = \frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z} \quad (1337)$$

$$\frac{\partial L}{\partial x} = q(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_y}{\partial x}\dot{y} + \frac{\partial A_z}{\partial x}\dot{z}) - q\frac{\partial}{\partial x}\phi \quad (1338)$$

C.36 Lorentz force: Derivation of x component of electric field

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \quad (1339)$$

$$\nabla\phi = (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) \quad (1340)$$

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\partial}{\partial t}(A_x, A_y, A_z) \quad (1341)$$

$$-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} = -(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) + \frac{\partial}{\partial t}(A_x, A_y, A_z) \quad (1342)$$

$$\mathbf{E} = -(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) + \frac{\partial}{\partial t}(A_x, A_y, A_z) \quad (1343)$$

$$\mathbf{E}\cdot(1,0,0) = -((\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}) - \frac{\partial}{\partial t}(A_x, A_y, A_z))\cdot(1,0,0) \quad (1344)$$

C.37 Lorentz force: Derivation of Lorentz force from classical Lagrangian 4

$$(\dot{\mathbf{r}}\times(\nabla\times\mathbf{A}))_x = \dot{y}(\nabla\times\mathbf{A})_z - \dot{z}(\nabla\times\mathbf{A})_y \quad (1345)$$

$$F_x = qE_x + q(\dot{\mathbf{r}}\times(\nabla\times\mathbf{A}))_x \quad (1346)$$

$$F_x = qE_x + q(\dot{y}(\nabla\times\mathbf{A})_z - \dot{z}(\nabla\times\mathbf{A})_y) \quad (1347)$$

C.38 Lorentz force: Derivation of Lorentz force from classical Lagrangian 5

$$F_x = qE_x + q(\dot{\mathbf{r}}\times(\nabla\times\mathbf{A}))_x \quad (1348)$$

$$\mathbf{F}\cdot\hat{\mathbf{x}} = F_x \quad (1349)$$

$$\mathbf{E}\cdot\hat{\mathbf{x}} = E_x \quad (1350)$$

$$\mathbf{B} = \nabla\times\mathbf{A} \quad (1351)$$

$$(\dot{\mathbf{r}}\times\mathbf{B})\cdot\hat{\mathbf{x}} = (\dot{\mathbf{r}}\times(\nabla\times\mathbf{A}))_x \quad (1352)$$

$$q\mathbf{E}\cdot\hat{\mathbf{x}} + q(\dot{\mathbf{r}}\times\mathbf{B})\cdot\hat{\mathbf{x}} = qE_x + q(\dot{\mathbf{r}}\times(\nabla\times\mathbf{A}))_x \quad (1353)$$

$$\mathbf{F}\cdot\hat{\mathbf{x}} = q\mathbf{E}\cdot\hat{\mathbf{x}} + q(\dot{\mathbf{r}}\times\mathbf{B})\cdot\hat{\mathbf{x}} \quad (1354)$$

C.39 Electromagnetic wave equation: The origin of the electromagnetic wave equation in 2

$$\nabla\times(\nabla\times\mathbf{E}) = -\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) \quad (1355)$$

$$\nabla\times\mathbf{B} = \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} \quad (1356)$$

$$-\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) = -\mu_0\varepsilon_0\frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (1357)$$

$$\nabla\times(\nabla\times\mathbf{E}) = -\mu_0\varepsilon_0\frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (1358)$$

C.40 Electromagnetic wave equation: The origin of the electromagnetic wave equation in 3

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \quad (1359)$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla \times (\nabla \times \mathbf{E}) \quad (1360)$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \quad (1361)$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} - \nabla^2 \mathbf{E} \quad (1362)$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0 \quad (1363)$$

C.41 Electromagnetic wave equation: The origin of the electromagnetic wave equation in 2

$$\nabla \times (\nabla \times \mathbf{B}) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E}) \quad (1364)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1365)$$

$$\mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \times \mathbf{E}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \quad (1366)$$

$$\nabla \times (\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \quad (1367)$$

C.42 Electromagnetic wave equation: The origin of the electromagnetic wave equation in 3

$$\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \quad (1368)$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = \nabla \times (\nabla \times \mathbf{B}) \quad (1369)$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \quad (1370)$$

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0 \quad (1371)$$

C.43 Ampere's circuital law: Proof of equivalence 2

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \quad (1372)$$

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} \quad (1373)$$

$$\nabla \times \frac{1}{\mu_0} \mathbf{B} = \mathbf{J}_f + \frac{\partial}{\partial t} \frac{1}{\mu_0} \mathbf{D} \quad (1374)$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \quad (1375)$$

$$\frac{\partial}{\partial t} \frac{1}{\mu_0} \mathbf{D} = \frac{\partial}{\partial t} \varepsilon_0 \mathbf{E} + \frac{\partial}{\partial t} \mathbf{P} \quad (1376)$$

$$\nabla \times \frac{1}{\mu_0} \mathbf{B} = \mathbf{J}_f + \frac{\partial}{\partial t} \varepsilon_0 \mathbf{E} + \frac{\partial}{\partial t} \mathbf{P} \quad (1377)$$

$$\mathbf{J}_M = \nabla \times \mathbf{M} \quad (1378)$$

$$\nabla \times \frac{1}{\mu_0} \mathbf{B} = \mathbf{J}_f + \frac{\partial}{\partial t} \varepsilon_0 \mathbf{E} + \frac{\partial}{\partial t} \mathbf{P} + \mathbf{J}_M \quad (1379)$$

C.44 Ampere's circuital law: Proof of equivalence 4

$$\frac{1}{\mu_0} (\nabla \times \mathbf{B}) = \mathbf{J}_f + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_P + \mathbf{J}_M \quad (1380)$$

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_P + \mathbf{J}_M \quad (1381)$$

$$\frac{1}{\mu_0} (\nabla \times \mathbf{B}) = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (1382)$$

C.45 Uncertainty principle: Kennard inequality proof part 1.1

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 \quad (1383)$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \cdot |\psi(x)|^2 dx \quad (1384)$$

$$\langle x \rangle = \int_{-\infty}^{\infty} x \cdot |\psi(x)|^2 dx \quad (1385)$$

$$\langle x \rangle^2 = (\int_{-\infty}^{\infty} x \cdot |\psi(x)|^2 dx)^2 \quad (1386)$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} x^2 \cdot |\psi(x)|^2 dx - (\int_{-\infty}^{\infty} x \cdot |\psi(x)|^2 dx)^2 \quad (1387)$$

C.46 Uncertainty principle: Kennard inequality proof part 1.4

$$f^*(x) \cdot f(x) = x^2 \cdot (\psi^*(x) \cdot \psi(x)) \tag{1388}$$

$$\sigma_x^2 = \langle f|f\rangle \tag{1389}$$

$$\sigma_x^2 = \left\langle x^2 \middle| x^2 \right\rangle \tag{1390}$$

$$\sigma_x^2 = \langle f^*(x) \cdot f(x)|f^*(x) \cdot f(x)\rangle \tag{1391}$$

$$\sigma_x^2 = \left\langle x^2 \cdot (\psi^*(x) \cdot \psi(x)) \middle| x^2 \cdot (\psi^*(x) \cdot \psi(x)) \right\rangle \tag{1392}$$

$$\sigma_x^2 = \langle f|f\rangle \tag{1393}$$

C.47 Uncertainty principle: Kennard inequality proof part 2.2

$$\frac{dv}{d\chi} = e^{\frac{-ip\chi}{\hbar}} \tag{1394}$$

$$\int dv = \int e^{\frac{-ip\chi}{\hbar}} d\chi \tag{1395}$$

$$v = \frac{\hbar}{-ip} \int e^{\frac{-ip\chi}{\hbar}} d\chi \tag{1396}$$

$$v = \frac{\hbar}{-ip} e^b + C \tag{1397}$$

C.48 Uncertainty principle: Kennard inequality proof part 2.3

$$u = \psi(\chi) \tag{1398}$$

$$v = \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \tag{1399}$$

$$uv = \psi(\chi) \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \tag{1400}$$

C.49 Uncertainty principle: Kennard inequality proof part 2.4

$$uv = \psi(\chi) \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \tag{1401}$$

$$(uv)\Big|_{-\infty}^{\infty} = \psi(\chi) \frac{\hbar}{-ip} e^{\frac{-ip\chi}{\hbar}} \Big|_{-\infty}^{\infty} \tag{1402}$$

$$(uv)\Big|_{-\infty}^{\infty} = 0 \tag{1403}$$

C.50 Uncertainty principle: Kennard inequality proof part 2.5

$$I = (uv)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v \frac{du}{d\chi} d\chi \tag{1404}$$

$$letu = \psi(\chi) \tag{1405}$$

$$v = e^{\frac{-ip\chi}{\hbar}} \tag{1406}$$

$$du = \frac{d\psi(\chi)}{d\chi} d\chi \tag{1407}$$

$$dv = \frac{-ip}{\hbar} e^{\frac{-ip\chi}{\hbar}} d\chi \tag{1408}$$

$$\int_{-\infty}^{\infty} v \frac{du}{d\chi} d\chi = \int_{-\infty}^{\infty} e^{\frac{-ip\chi}{\hbar}} \frac{d\psi(\chi)}{d\chi} d\chi \tag{1409}$$

$$(uv)\Big|_{-\infty}^{\infty} = \frac{\hbar}{ip} \, (uv)\Big|_{-\infty}^{\infty} \tag{1410}$$

$$I = \frac{\hbar}{ip} \int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi} e^{\frac{-ip\chi}{\hbar}} d\chi \tag{1411}$$

C.51 Uncertainty principle: Kennard inequality proof part 2.9

$$g(x) = \frac{\hbar}{2\pi i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\psi(\chi)}{d\chi} e^{i(x-\chi)b} d\chi db \tag{1412}$$

$$\frac{d\psi(\chi)}{d\chi} = \frac{d\psi(x)}{dx} \tag{1413}$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\psi(x)}{dx} e^{i(x-\chi)b} d\chi db = \frac{d\psi(x)}{dx} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x-\chi)b} d\chi db \tag{1414}$$

$$\frac{\hbar}{2\pi i} \frac{d\psi(x)}{dx} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(x-\chi)b} d\chi db = \frac{\hbar}{2\pi i} \frac{d\psi(x)}{dx} \tag{1415}$$

$$\frac{\hbar}{2\pi i} \frac{d\psi(x)}{dx} = \frac{\hbar}{i} (\frac{d\psi(x)}{dx}) \tag{1416}$$

$$g(x) = \frac{\hbar}{i} (\frac{d\psi(x)}{dx}) \tag{1417}$$

C.52 Uncertainty principle: Kennard inequality proof part 3.2

$$\tilde{g}^*(p) \cdot \tilde{g}(p) = p^2\varphi^*(p) \cdot \varphi(p) \tag{1418}$$

$$|\tilde{g}(p)|^2 = \tilde{g}^*(p) \cdot \tilde{g}(p) \tag{1419}$$

$$|\tilde{g}(p)|^2 = p^2\varphi^*(p) \cdot \varphi(p) \tag{1420}$$

$$|\tilde{g}(p)|^2 = p^2|\varphi(p)|^2 \tag{1421}$$

C.53 Uncertainty principle: Kennard inequality proof part 3.3

$$\sigma_p^2 = \int_{-\infty}^{\infty} p^2|\varphi(p)|^2 dp \tag{1422}$$

$$\langle g| = \int_{-\infty}^{\infty} p^2|\varphi(p)|^2 dp \tag{1423}$$

$$|g\rangle = 1 \tag{1424}$$

$$\sigma_p^2 = \langle g|g\rangle \tag{1425}$$

C.54 Uncertainty principle: Kennard inequality proof part 4.1

$$\sigma_x^2 = \langle f|f\rangle \tag{1426}$$

$$|\langle f|g\rangle|^2 = \sigma_x^2\sigma_p^2 \tag{1427}$$

$$\sigma_x^2\sigma_p^2 \geq |\langle f|g\rangle|^2 \tag{1428}$$

C.55 Uncertainty principle: Kennard inequality proof part 4.3

$$|z|^2 = (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2 \tag{1429}$$

$$|z|^2 \geq 0 \tag{1430}$$

$$(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i})^2 \geq 0 \tag{1431}$$

$$|z|^2 \geq (\frac{\langle f|g\rangle - \langle g|f\rangle}{2i})^2 \tag{1432}$$

C.56 Uncertainty principle: Kennard inequality proof part 4.4

$$\sigma_x^2\sigma_p^2 \geq |\langle f|g\rangle|^2 \tag{1433}$$

$$|\langle f|g\rangle| = \frac{\langle f|g\rangle - \langle g|f\rangle}{2i} \tag{1434}$$

$$(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i})^2 = |\langle f|g\rangle|^2 \tag{1435}$$

$$\sigma_x^2\sigma_p^2 \geq (\frac{\langle f|g\rangle - \langle g|f\rangle}{2i})^2 \tag{1436}$$

C.57 Uncertainty principle: Kennard inequality proof part 5.1

$$\sigma_x^2\sigma_p^2 \geq (\frac{\langle f|g\rangle - \langle g|f\rangle}{2i})^2 \tag{1437}$$

$$\langle f|g\rangle = \langle g|f\rangle \tag{1438}$$

$$\sigma_x^2\sigma_p^2 \geq (\frac{\langle f|g\rangle - \langle f|g\rangle}{2i})^2 \tag{1439}$$

$$\sigma_x^2\sigma_p^2 \geq 0 \tag{1440}$$

$$\langle f|g\rangle = \langle g|f\rangle \tag{1441}$$

$$\langle f|g\rangle = -i\hbar \int_{-\infty}^{\infty} x\psi^*(x)\frac{d\psi(x)}{dx}dx \tag{1442}$$

C.58 Uncertainty principle: Kennard inequality proof part 5.2

$$f(x) = x \cdot \psi(x) \tag{1443}$$

$$\langle g|f\rangle = \langle g|x \cdot \psi(x)\rangle \tag{1444}$$

$$\langle g|f\rangle = -i\hbar \int_{-\infty}^{\infty} \psi^*(x)\frac{d}{dx}(x\psi(x))dx \tag{1445}$$

C.59 Uncertainty principle: Kennard inequality proof part 5.6

$$\langle f|g\rangle - \langle g|f\rangle = i\hbar \int_{-\infty}^{\infty} |\psi(x)|^2 dx \tag{1446}$$

$$p(x) = |\psi(x)|^2 \tag{1447}$$

$$\int_{-\infty}^{\infty} p(x)dx = \frac{\langle f|g\rangle - \langle g|f\rangle}{i\hbar} \tag{1448}$$

$$\int_{-\infty}^{\infty} p(x)dx = 1 \tag{1449}$$

C.60 Particle in a box: Wavefunction angular velocity as a function of particle mass from Schrödinger's equation 7

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t) \tag{1450}$$

$$\frac{i\hbar}{\hbar}\frac{\partial}{\partial t}\psi(x,t) = \frac{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t)}{\hbar} \tag{1451}$$

$$i\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + \frac{V(x)\psi(x,t)}{\hbar} \tag{1452}$$

$$\hbar\omega\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + \frac{V(x)\psi(x,t)}{\hbar} \tag{1453}$$

$$\hbar\omega\psi(x,t) = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) if V(x) = \hbar\omega. \tag{1454}$$

C.61 Particle in a box: Wavefunction angular velocity as a function of particle mass from Schrödinger's equation 8

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{\hbar^2 k^2}{2m}\psi(x,t) \tag{1455}$$

$$\frac{\partial^2\psi(x,t)}{\partial x^2} = k^2\psi(x,t) \tag{1456}$$

$$\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{p^2}{\hbar^2}\psi(x,t) \tag{1457}$$

$$\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{p^2}{\hbar^2}\psi(x,t) \tag{1458}$$

$$\frac{\partial^2\psi(x,t)}{\partial x^2} = \frac{p^2}{\hbar^2}\psi(x,t) \tag{1459}$$

$$\omega = \frac{p^2}{2m\hbar} \tag{1460}$$

C.62 Quantum harmonic oscillator: Ladder operator method 4

$$Given aa^\dagger - a^\dagger a = \frac{i}{\hbar}(\hat{p}\hat{x} - \hat{x}\hat{p}) \tag{1461}$$

$$\left[a, a^\dagger\right] = aa^\dagger - a^\dagger a \tag{1462}$$

$$\frac{i}{\hbar}(\hat{p}\hat{x} - \hat{x}\hat{p}) = 1 \tag{1463}$$

$$\left[a, a^\dagger\right] = 1 \tag{1464}$$

C.63 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 1.6

$$\hbar\omega(\frac{1}{2}+\frac{1}{\sqrt{2}}(-\frac{d^2}{dq^2}+q^2)\frac{1}{\sqrt{2}}(\frac{d^2}{dq^2}+q^2))\psi(q) = E\psi(q) \tag{1465}$$

$$\hbar\omega(\frac{1}{2}+\frac{1}{2}(-\frac{d^2}{dq^2}+q^2))\psi(q) = E\psi(q) \tag{1466}$$

$$\hbar\omega(\frac{1}{2}+\frac{1}{2}(a^\dagger a))\psi(q) = E\psi(q) \tag{1467}$$

$$E\psi(q) = \hbar\omega(a^\dagger a + \frac{1}{2})\psi(q) \tag{1468}$$

$$E = \hbar\omega(a^\dagger a + \frac{1}{2}) \tag{1469}$$

C.64 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 2

$$[q,p] = qp - pq \tag{1470}$$

$$[q,p]f(q) = (qp - pq)f(q) \tag{1471}$$

$$[q,p]f(q) = qpf(q) - pqf(q) \tag{1472}$$

$$[q,p]f(q) = if(q) \tag{1473}$$

C.65 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 3.1

$$a = \frac{1}{\sqrt{2}}(\frac{d}{dq}+q) \tag{1474}$$

$$ip = \frac{d}{dq} \tag{1475}$$

$$a = \frac{1}{\sqrt{2}}(ip+q) \tag{1476}$$

C.66 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 3.2

$$a^\dagger = \frac{1}{\sqrt{2}}(-\frac{d}{dq}+q) \tag{1477}$$

$$-\frac{d}{dq} = -ip \tag{1478}$$

$$a^\dagger = \frac{1}{\sqrt{2}}(-ip+q) \tag{1479}$$

C.67 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 3.6

$$aa^\dagger - a^\dagger a = i(pq - qp) \tag{1480}$$

$$pq - qp = 1 \tag{1481}$$

$$i(pq - qp) = i \tag{1482}$$

$$aa^\dagger - a^\dagger a = i \tag{1483}$$

$$\left[a, a^\dagger\right] = 1 \tag{1484}$$

C.68 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 4.2

$$[H, a] = -\hbar\omega(aa^\dagger - a^\dagger a)a \tag{1485}$$

$$aa^\dagger - a^\dagger a = 1 \tag{1486}$$

$$-\hbar\omega(1)a = -\hbar\omega a \tag{1487}$$

$$[H, a] = -\hbar\omega a \tag{1488}$$

C.69 Creation and annihilation operators: Ladder operators for the quantum harmonic oscillator part 5.2

$$\left[H, a^\dagger\right] = \hbar\omega a^\dagger(aa^\dagger - a^\dagger a) \tag{1489}$$

$$\left[a, a^\dagger\right] = aa^\dagger - a^\dagger a \tag{1490}$$

$$aa^\dagger - a^\dagger a = 1 \tag{1491}$$

$$\left[a, a^\dagger\right] = 1 \tag{1492}$$

C.70 Heisenberg picture: time evolution 4

$$\frac{d}{dt}\hat{x}(t) = \frac{i}{\hbar}(\hat{H}e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}} - e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}}\hat{H}) \tag{1493}$$

$$\hat{H}e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}} = \hat{H}\hat{x}(t) \tag{1494}$$

$$e^{i\frac{\hat{H}t}{\hbar}}\hat{x}e^{-i\frac{\hat{H}t}{\hbar}}\hat{H} = \hat{x}(t)\hat{H} \tag{1495}$$

$$\hat{H}\hat{x}(t) - \hat{x}(t)\hat{H} = \left[\hat{H}, \hat{x}(t)\right] \tag{1496}$$

$$\frac{i}{\hbar}(\hat{H}\hat{x}(t) - \hat{x}(t)\hat{H}) = \frac{i}{\hbar}\left[\hat{H}, \hat{x}(t)\right] \tag{1497}$$

$$\frac{d}{dt}\hat{x}(t) = \frac{i}{\hbar}\left[\hat{H}, \hat{x}(t)\right] \tag{1498}$$

C.71 Heisenberg picture: momentum evolution 4

$$\hat{p}(t) = A\cos(\omega t) + B\sin(\omega t) \tag{1499}$$

$$\frac{d}{dt}A\cos(\omega t) = -A\omega\sin(\omega t) \tag{1500}$$

$$\frac{d}{dt}B\sin(\omega t) = B\omega\cos(\omega t) \tag{1501}$$

$$\frac{d\hat{p}(t)}{dt} = \frac{d}{dt}(A\cos(\omega t) + B\sin(\omega t)) \tag{1502}$$

$$\frac{d\hat{p}(t)}{dt} = -A\omega\sin(\omega t) + B\omega\cos(\omega t) \tag{1503}$$

C.72 Heisenberg picture: position commutator 4

$$[\hat{x}(t_1), \hat{x}(t_2)] = \frac{1}{\omega m}(\hat{x}_0\hat{p}_0 - \hat{p}_0\hat{x}_0)\sin(\omega t_2 - \omega t_1) \tag{1504}$$

$$\hat{x}_0\hat{p}_0 - \hat{p}_0\hat{x}_0 = i\hbar \tag{1505}$$

$$\frac{1}{\omega m}(i\hbar)\sin(\omega t_2 - \omega t_1) = \frac{i\hbar}{\omega m}\sin(\omega t_2 - \omega t_1) \tag{1506}$$

$$[\hat{x}(t_1), \hat{x}(t_2)] = \frac{i\hbar}{\omega m}\sin(\omega t_2 - \omega t_1) \tag{1507}$$

C.73 Heisenberg picture: momentum commutator 3

$$[\hat{p}(t_1), \hat{p}(t_2)] = m\omega\hat{p}_0\hat{x}_0\sin(\omega t_2 - \omega t_1) + m\omega\hat{x}_0\hat{p}_0\sin(\omega t_1 - \omega t_2 \tag{1508}$$

$$[\hat{p}(t_1), \hat{p}(t_2)] = m\omega\hat{p}_0\hat{x}_0\sin(\omega t_2 - \omega t_1) - m\omega\hat{p}_0\hat{x}_0\sin(\omega t_2 - \omega t_1 \tag{1509}$$

$$[\hat{p}(t_1), \hat{p}(t_2)] = 0 \tag{1510}$$

$$[\hat{p}(t_1), \hat{p}(t_2)] = i\hbar m\omega\sin(\omega t_2 - \omega t_1) \tag{1511}$$

C.74 Vacuum Rabi Oscillations: excited state probability

$$|\Psi(t)\rangle = \cos\left(\frac{\Omega t}{2}\right)|e,0\rangle - i\sin\left(\frac{\Omega t}{2}\right)|g,1\rangle \tag{1512}$$

$$\langle\Psi(t)| = \cos\left(\frac{\Omega t}{2}\right)\langle e,0| + i\sin\left(\frac{\Omega t}{2}\right)\langle g,1| \tag{1513}$$

$$P_e(t) = |\langle e,0|\Psi(t)\rangle|^2 \tag{1514}$$

$$P_e(t) = |\cos\left(\frac{\Omega t}{2}\right)|^2 \tag{1515}$$

$$P_e(t) = \cos^2(\frac{\Omega t}{2}) \tag{1516}$$

C.75 Vacuum Rabi Oscillations: ground state probability 2

$$P_g(t) = |\cos\left(\frac{\Omega t}{2}\right)\langle g,1|e,0\rangle - i\sin\left(\frac{\Omega t}{2}\right)\langle g,1|g,1\rangle|^2 \tag{1517}$$

$$P_g(t) = |\cos\left(\frac{\Omega t}{2}\right)|^2\langle g,1|e,0\rangle^2 + |\sin\left(\frac{\Omega t}{2}\right)|^2\langle g,1|g,1\rangle^2 \tag{1518}$$

$$P_g(t) = \cos^2(\frac{\Omega t}{2})\langle g,1|e,0\rangle^2 + \sin^2(\frac{\Omega t}{2})\langle g,1|g,1\rangle^2 \tag{1519}$$

$$\langle g,1|e,0\rangle = 0 \tag{1520}$$

$$\langle g,1|g,1\rangle = 1 \tag{1521}$$

$$P_g(t) = \cos^2(\frac{\Omega t}{2}) * 0 + \sin^2(\frac{\Omega t}{2}) * 1 \tag{1522}$$

$$P_g(t) = \sin^2(\frac{\Omega t}{2}) \tag{1523}$$

C.76 Expectation value: integral expression

$$\left\langle \hat{X}\right\rangle_\Psi = \langle\Psi|\hat{X}|\Psi\rangle \tag{1524}$$

$$\langle\Psi|\hat{X}|\Psi\rangle = \int\int \langle\Psi|x\rangle\langle x|\hat{X}|x'\rangle\langle x'|\Psi\rangle\,dxdx' \tag{1525}$$

$$\left\langle \hat{X}\right\rangle_\Psi = \int\int \langle\Psi|x\rangle\langle x|\hat{X}|x'\rangle\langle x'|\Psi\rangle\,dxdx' \tag{1526}$$

C.77 Expectation value: integral expression 2

$$\left\langle \hat{X}\right\rangle_\Psi = \int\int \langle\Psi|x\rangle\langle x|\hat{X}|x'\rangle\langle x'|\Psi\rangle\,dxdx' \tag{1527}$$

$$\langle\Psi|x\rangle = \langle x|\Psi\rangle^\dagger \tag{1528}$$

$$\langle x|\hat{X}|x'\rangle = x'\delta(x-x') \tag{1529}$$

$$\left\langle \hat{X}\right\rangle_\Psi = \int\int \langle x|\Psi\rangle^\dagger x'\delta(x-x')\langle x'|\Psi\rangle\,dxdx' \tag{1530}$$

C.78 Expectation value: integral expression 3

$$\left\langle \hat{X}\right\rangle_\Psi = \int\int \langle x|\Psi\rangle^\dagger x'\delta(x-x')\langle x'|\Psi\rangle\,dxdx' \tag{1531}$$

$$\delta(x-x') = 1 \text{ when } x = x' \tag{1532}$$

$$0 \text{ otherwise} \tag{1533}$$

$$\int\int \langle x|\Psi\rangle^\dagger x'\delta(x-x')\langle x'|\Psi\rangle\,dxdx' = \int x|\Psi(x)|^2dx \tag{1534}$$

$$\left\langle \hat{X}\right\rangle_\Psi = \int x|\Psi(x)|^2dx \tag{1535}$$

C.79 Euler-lagrange equation: Full derivative of the perturbation Lagrangian with respect to ε 2

$$\frac{dg_\varepsilon}{d\varepsilon} = \frac{d}{d\varepsilon}(\varepsilon\eta(x)) \tag{1536}$$

$$\frac{d}{d\varepsilon}(\varepsilon\eta(x)) = \eta(x) \tag{1537}$$

$$\frac{dg_\varepsilon}{d\varepsilon} = \eta(x) \tag{1538}$$

C.80 Euler-Lagrange equation: Derivation

$$J = \int_a^b L(x, f(x), f'(x)) \tag{1539}$$

$$J_\varepsilon = \int_a^b L_\varepsilon(x, f(x), f'(x)) \tag{1540}$$

$$\frac{dJ_\varepsilon}{d\varepsilon} = \frac{d}{d\varepsilon}\int_a^b L_\varepsilon(x, f(x), f'(x)) \tag{1541}$$

$$\frac{dJ_\varepsilon}{d\varepsilon} = \int_a^b \frac{dL_\varepsilon}{d\varepsilon}dx \tag{1542}$$

C.81 Euler-Lagrange equation: Derivation 4

$$\frac{dv}{dx} = \eta'(x) \tag{1543}$$

$$\int \frac{dv}{dx} dx = \int \eta'(x) dx \tag{1544}$$

$$v = \eta(x) \tag{1545}$$

C.82 Euler-Lagrange equation: Derivation 5

$$u = \frac{\partial L}{\partial f'} \tag{1546}$$

$$uv = \frac{\partial L}{\partial f'} \cdot v \tag{1547}$$

$$\int_a^b uv dx = \int_a^b \frac{\partial L}{\partial f'} \cdot v dx \tag{1548}$$

$$(uv)\Big|_a^b = (\frac{\partial L}{\partial f'} \cdot v)\Big|_a^b \tag{1549}$$

$$(uv)\Big|_a^b = 0 \tag{1550}$$

C.83 Euler-Lagrange equation: Derivation 6

$$I = \int_a^b \frac{\partial L}{\partial f'} \eta'(x) dx \tag{1551}$$

$$\frac{d}{dx}\frac{\partial L}{\partial f'} = \frac{\partial^2 L}{\partial f'^2}\frac{df'}{dx} \tag{1552}$$

$$\frac{\partial L}{\partial f'}\eta'(x) = -\eta(x)\frac{d}{dx}\frac{\partial L}{\partial f'} \tag{1553}$$

$$I = -\int_a^b \eta(x)\frac{d}{dx}\frac{\partial L}{\partial f'} dx \tag{1554}$$

C.84 Euler-Lagrange equation: Straight line

$$S = \int_a^b ds \tag{1555}$$

$$ds = \sqrt{(1+y'^2)} dx \tag{1556}$$

$$S = \int_a^b \sqrt{(1+y'^2)} dx \tag{1557}$$

C.85 Euler-Lagrange equation: Straight line 3

$$\frac{dL}{dy} - \frac{d}{dx}\frac{dL}{dy'} = 0 \tag{1558}$$

$$\frac{dL}{dy} = \frac{d}{dx}\frac{dL}{dy'} \tag{1559}$$

$$\frac{d}{dx}(y'(1+y'^2)^{-\frac{1}{2}}) = \frac{dL}{dy} \tag{1560}$$

$$\int \frac{d}{dx}(y'(1+y'^2)^{-\frac{1}{2}}) dx = \int \frac{dL}{dy} dx \tag{1561}$$

$$\int \frac{d}{dx}(y'(1+y'^2)^{-\frac{1}{2}}) dx = C \tag{1562}$$

C.86 Euler-Lagrange equation: Straight line 6

$$\frac{dy}{dx} = C(1-C^2)^{-1/2} \tag{1563}$$

$$\int dy = \int C(1-C^2)^{-1/2} dx \tag{1564}$$

$$y = Cx + \int (1-C^2)^{-1/2} dx \tag{1565}$$

$$y = Cx + \sqrt{1-C^2} + B \tag{1566}$$

$$y = Ax + C where A = C \tag{1567}$$

$$B = \sqrt{1-C^2} \tag{1568}$$

C.87 Escape velocity

$$F = \frac{GMm}{r^2} \tag{1569}$$

$$W = \int F dr \tag{1570}$$

$$W = \int \frac{GMm}{r^2} dr \tag{1571}$$

$$W = \int_{r_0}^{\infty} \frac{GMm}{r^2} dr \tag{1572}$$

C.88 Escape velocity 2

$$W = \int_{r_0}^{\infty} \frac{GMm}{r^2} dr \tag{1573}$$

$$let u = r \tag{1574}$$

$$dv = \frac{GMm}{r^2} dr \tag{1575}$$

$$du = dr \tag{1576}$$

$$v = -\frac{GMm}{r} \tag{1577}$$

$$by integration by parts, \int u dv = uv - \int v du \tag{1578}$$

$$W = \left[-\frac{GMm}{r} \right]_{r_0}^{\infty} - \int_{r_0}^{\infty} -\frac{GMm}{r} dr \tag{1579}$$

$$W = \left[-\frac{GMm}{r} \right]_{r_0}^{\infty} + GMm \int_{r_0}^{\infty} \frac{1}{r} dr \tag{1580}$$

$$W = \left[-\frac{GMm}{r} \right]_{r_0}^{\infty} + GMm[\ln r]_{r_0}^{\infty} \tag{1581}$$

$$W = \left[-\frac{GMm}{r} \right]_{r_0}^{\infty} + GMm(\ln \infty - \ln r_0) \tag{1582}$$

$$W = \left[-\frac{GMm}{r} \right]_{r_0}^{\infty} + GMm \ln \left(\frac{\infty}{r_0} \right) \tag{1583}$$

$$W = \left[-\frac{GMm}{r} \right]_{r_0}^{\infty} \tag{1584}$$

$$W = -\frac{GMm}{\infty} + \frac{GMm}{r_0} \tag{1585}$$

$$W = 0 + \frac{GMm}{r_0} \tag{1586}$$

$$W = \frac{GMm}{r_0} \tag{1587}$$

$$since GM = g, W = mgr_0. \tag{1588}$$

C.89 Escape velocity 3

$$W = mgr_0 \tag{1589}$$

$$v_{esc}^2 = 2W/m \tag{1590}$$

$$v_{esc}^2 = 2mgr_0/m \tag{1591}$$

$$v_{esc}^2 = 2gr_0 \tag{1592}$$

$$v_{esc} = \sqrt{2gr_0} \tag{1593}$$

C.90 Snell's law: from Fermat's principle 2

$$\frac{dT}{dx} = \frac{x}{v_1(x^2+a^2)^{\frac{1}{2}}} + \frac{x-l}{v_2((x-l)^2+b^2)^{\frac{1}{2}}} \tag{1594}$$

$$\frac{dT}{dx} = \frac{\sin\theta_1}{v_1} + \frac{\sin\theta_2}{v_2} \tag{1595}$$

$$\frac{\sin\theta_1}{v_1} = \frac{x}{v_1(x^2+a^2)^{\frac{1}{2}}} \tag{1596}$$

$$\frac{\sin\theta_2}{v_2} = \frac{x-l}{v_2((x-l)^2+b^2)^{\frac{1}{2}}} \tag{1597}$$

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = \frac{x}{v_1(x^2+a^2)^{\frac{1}{2}}} - \frac{x-l}{v_2((x-l)^2+b^2)^{\frac{1}{2}}} \tag{1598}$$

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \tag{1599}$$

C.91 Snell's law: from Fermat's principle 3

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \tag{1600}$$

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2} \tag{1601}$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{1602}$$

C.92 Wave equation: plane wave eigenmodes 2

$$\frac{\partial^2 u(x,t)}{\partial t^2} = (-i\omega)\frac{\partial}{\partial t}(e^{-i\omega t} f(x)) \tag{1603}$$

$$\frac{\partial}{\partial t}(e^{-i\omega t} f(x)) = -i\omega e^{-i\omega t} f(x) \tag{1604}$$

$$(-i\omega)\frac{\partial}{\partial t}(e^{-i\omega t} f(x)) = -\omega^2 e^{-i\omega t} f(x) \tag{1605}$$

$$-\omega^2 e^{-i\omega t} f(x) = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1606}$$

C.93 Wave equation: plane wave eigenmodes 4

$$u(x,t) = Ae^{-i(kx-\omega t)} + Be^{i(kx-\omega t)} \tag{1607}$$

$$s_+(\omega) = A \tag{1608}$$

$$s_-(\omega) = B \tag{1609}$$

$$\int_{-\infty}^{\infty} s_+(\omega)e^{-i(kx-\omega t)}d\omega = Ae^{-i(kx-\omega t)} \tag{1610}$$

$$\int_{-\infty}^{\infty} s_-(\omega)e^{i(kx-\omega t)}d\omega = Be^{i(kx-\omega t)} \tag{1611}$$

$$u(x,t) = \int_{-\infty}^{\infty} s_+(\omega)e^{-i(kx-\omega t)}d\omega + \int_{-\infty}^{\infty} s_-(\omega)e^{i(kx-\omega t)}d\omega \tag{1612}$$

C.94 Wave equation: plane wave eigenmodes 5

$$u(x,t) = \int_{-\infty}^{\infty} s_+(\omega)e^{-i(kx-\omega t)}d\omega + \int_{-\infty}^{\infty} s_-(\omega)e^{i(kx-\omega t)}d\omega \tag{1613}$$

$$letF(x-ct) = \int_{-\infty}^{\infty} s_+(\omega)e^{-i(kx-\omega t)}d\omega \tag{1614}$$

$$letG(x+ct) = \int_{-\infty}^{\infty} s_-(\omega)e^{i(kx-\omega t)}d\omega \tag{1615}$$

$$u(x,t) = F(x-ct) + G(x+ct) \tag{1616}$$

C.95 Wave equation: Hooke's law

$$F_H = F_{x+2h} - F_x \tag{1617}$$

$$m\frac{\partial^2}{\partial t^2}u(x+h,t) = F_H \tag{1618}$$

$$m\frac{\partial^2}{\partial t^2}u(x+h,t) = F_{x+2h} - F_x \tag{1619}$$

C.96 Wave equation: Hooke's law 2

$$m\frac{\partial^2}{\partial t^2}u(x+h,t) = F_{x+2h} - F_x \tag{1620}$$

$$F_{x+2h} - F_x = k\big(u(x+2h,t) - u(x+h,t)\big) \tag{1621}$$

$$m\frac{\partial^2}{\partial t^2}u(x+h,t) = k\big(u(x+2h,t) - u(x+h,t)\big) \tag{1622}$$

$$\frac{\partial^2}{\partial t^2}u(x+h,t) = \frac{k}{m}\big(u(x+2h,t) - u(x+h,t)\big) \tag{1623}$$

$$F_x = k\big(u(x+h,t) - u(x,t)\big) \tag{1624}$$

$$m\frac{\partial^2}{\partial t^2}u(x{+}h,t) = k\big(u(x{+}2h,t){-}2u(x{+}h,t){+}u(x,t)\big) \tag{1625}$$

$$\frac{\partial^2}{\partial t^2}u(x{+}h,t) = \frac{k}{m}\big(u(x{+}2h,t){-}2u(x{+}h,t){+}u(x,t)\big) \tag{1626}$$

C.97 Wave equation: Hooke's law 3

$$\frac{\partial^2}{\partial t^2}u(x{+}h,t) = \frac{k}{m}\big(u(x{+}2h,t){-}2u(x{+}h,t){+}u(x,t)\big) \tag{1627}$$

$$\frac{KL^2}{Mh^2} = \frac{k}{m} \tag{1628}$$

$$\frac{\partial^2}{\partial t^2}u(x{+}h,t) = \frac{KL^2}{Mh^2}\big(u(x{+}2h,t){-}2u(x{+}h,t){+}u(x,t)\big) \tag{1629}$$

C.98 Wave equation: stress pulse in a bar 2

$$Let KL^2 = E \tag{1630}$$

$$M = \rho, then \frac{KL^2}{M} = \frac{E}{\rho} \tag{1631}$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{KL^2}{M}\frac{\partial^2 u(x,t)}{\partial x^2} \tag{1632}$$

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u(x,t)}{\partial x^2} \tag{1633}$$