HOMEWORK 6

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44. i) The inverse of a bijection is a bijection, so we need only show ψ is an isomorphism. Since $\varphi(a)\varphi(b) = \varphi(ab)$ and $\varphi(a^{-1}) = \varphi(a)^{-1}$,

$$\psi(\varphi(a)\varphi(b)) = \psi(\varphi(ab)) = ab = \psi(\varphi(a))\psi(\varphi(b)).$$

ii) The compositions of bijections is a bijection. Let $f : R \to S$ and $g : S \to T$, with $a, b \in R$. Then

$$(g \circ f)(ab) = g(f(ab)) = g(f(a)f(b)) = g(f(a))g(f(b)) = (g \circ f)(a)(g \circ f)(b).$$

iii) Say A, R, B belong to some set of commutative rings S. Then

- $A \cong R \implies R \cong A$ by (i) above,
- $A \cong A$ by the identity map $f : A \to A$, $a \mapsto a$ being an isomorphism,
- $A \cong R$, $R \cong B \implies A \cong B$ by (ii) above.

So isomorphism gives an equivalence relation on S.

46. $\eta(1) = 1$; that is, the constant polynomial 1 is taken to 1 in the ring R. In addition, for $f = \sum_i a_i x^i$, $g = \sum_j b_j x^j \in R[x]$,

$$\eta(f + g) = \eta(\sum_k (a_k + b_k)x^k) = a_0 + b_0 = \eta(f) + \eta(g)$$

and

$$\eta(fg) = \eta(\sum_i \Big(\sum_{j=0}^{i} a_j b_{i-j}\Big)x^i) = a_0 b_0 = \eta(f)\eta(g).$$

The kernel

$$\ker \eta = \{f \in R[x] : \eta(f) = 0\} = \{f \in R[x] : a_0 = 0\}$$

is the ideal of polynomials over R with a zero constant term. Thus the kernel can be described as those polynomials which have 0 as a root.

47. We can write

$$\psi^{-1}(J) = \{r \in R : \psi(r) = j \in J\}.$$

We know this contains zero since $\{0\} \subseteq J$ and

$$\psi^{-1}(\{0\}) = \ker \psi \subseteq \psi^{-1}(J).$$

Then let $a, b \in \psi^{-1}(J)$ with $a = \psi(x)$ and $b = \psi(y)$. Then

$$a - b = \psi(x) + \psi(y) = \psi(x - y)$$

and $x - y \in J$ so $\psi(x - y) \in \psi^{-1}(J)$. Similarly, for $r \in R$,

$$\psi(ra) = \psi(r)\psi(a)$$

and since $\psi(a) \in J$ and $J \in S$, $\psi(r)\psi(a) \in J$. So $\psi^{-1}(J)$ is an ideal in R.

49. Let $f : R \to \mathrm{Frac}(R)$, with $r \mapsto \frac{r}{1}$. This map is a homomorphism since for $r = 1$ we have $\frac{1}{1}$, and

$$f(r+s) = \frac{r+s}{1} = \frac{r}{1} + \frac{s}{1} = f(r) + f(s)$$

and furthermore

$$f(rs) = \frac{rs}{1} = \left(\frac{r}{1}\right)\left(\frac{s}{1}\right) = f(r)f(s).$$

We can demonstrate that f is bijective by observing $g : \mathrm{Frac}(R) \to R$, $\frac{a}{b} \mapsto ab^{-1}$. Then

$$(g \circ f)(r+s) = g(f(r+s)) = g(\frac{r}{1} + \frac{s}{1}) = g(\frac{r}{1}) + g(\frac{s}{1}) = r + s$$

and

$$(g \circ f)(rs) = g(f(rs)) = g(\frac{r}{1}\frac{s}{1}) = g(\frac{r}{1})g(\frac{s}{1}) = rs.$$

So g is the inverse of f and f is an isomorphism.

55.

- Associativity of matrix addition: inherited.
- Commutativity of matrix addition: inherited.
- Additive identity: zero matrix.
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} 0+a & 0+b \\ 0-b & 0+a \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$
- Additive inverse:
$$\begin{bmatrix} -a & -b \\ b & -a \end{bmatrix} + \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
- Associativity of matrix multiplication: inherited.
- Commutativity of matrix multiplication:

$$\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} ca-db & cb+da \\ -da-cb & -db+ca \end{bmatrix} = \begin{bmatrix} ac-bd & ad+bc \\ -bc-ad & -bd+ac \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$

- Multiplicative identity: 2×2 identity matrix.
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in F$$
since $0 = -0$.
- Distributive law: inherited.
- Nonzero elements have inverses: Consider $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. We know A^{-1} exists since $\det A = a^2 + b^2 \neq 0$ as long as $a \neq 0$ and $b \neq 0$. Then
$$A^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} \frac{a}{a^2+b^2} & -\frac{b}{a^2+b^2} \\ \frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix} \in F$$
since each entry in A^{-1} is in $\mathbb{R}$.

To show F is isomorphic to $\mathbb{C}$, we construct the obvious map $f : F \to \mathbb{C}$, $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mapsto a + bi$. We can see that the identity matrix maps to $1 + 0i = 1$, and furthermore

$$f(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}) + f(\begin{bmatrix} c & d \\ -d & c \end{bmatrix}) = a+bi+c+di = (a+c)+(b+d)i = f(\begin{bmatrix} a+c & b+d \\ -b-d & a+c \end{bmatrix}) = f(\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} c & d \\ -d & c \end{bmatrix})$$

and
$f(\begin{bmatrix} c & d \\ -d & c \end{bmatrix})f(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}) = (c+di)(a+bi) = ca-db+(da+cb)i = f(\begin{bmatrix} ca-db & cb+da \\ -da-cb & -db+ca \end{bmatrix}) = f(\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\begin{bmatrix} a & b \\ -b & a \end{bmatrix})$
so f is a homomorphism. To show f is injective, choose $a+bi = c+di$. Then $a+bi = f(\begin{bmatrix} a & b \\ -b & a \end{bmatrix})$ and $c+di = f(\begin{bmatrix} c & d \\ -d & c \end{bmatrix})$ but since $a = c$ and $b = d$, the matrices are equal. To show f is surjective, take $z = a+bi$. Then there does exist $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ such that $f(A) = z$. Thus f is bijective and a homomorphism, so F and $\mathbb{C}$ are isomorphic.

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