HOMEWORK 5

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97iii. We want to show that if $f : (H \times K) \to H$ is an homomorphism, K^* is the kernel of f and therefore a normal subgroup of $(H \times K)$ by the first isomorphism theorem. In addition, we must show that the image of f is H; then it follows from the first isomorphism theorem that $(H \times K)/K^*$ is isomorphic to H. Namely, we take f that maps $(h, k) \mapsto h$ for $h \in H$ and $k \in K$.

First, f is a homomorphism since

$$f((h_1, k_1)(h_2, k_2)) = f(h_1h_2, k_1k_2) = h_1h_2 = f(h_1, k_1)f(h_2, k_2).$$

We know that we can write

$$\begin{aligned} \ker f &= \{(h, k) \in (H \times K) : f(h, k) = 1_H\} \\ &= \{(1_H, k) \in (H \times K)\} \\ &= K^* \end{aligned}$$

and therefore $K*$ is a normal subgroup of $(H \times K)$. Next, we want to show that the image of f is the entirety of H. In particular,

$$\begin{aligned} \text{im } f &= \{h \in H : \exists (h, k) \in (H \times K), f(h, k) = h\} \\ &= \{h \in H\} \\ &= H \end{aligned}$$

since the first element in (h, k) can be chosen from H arbitrarily. Thus, H is the image of f, and so is isomorphic to the quotient group $(H \times K)/K^*$.

98. Suppose $G/Z(G)$ is cyclic with generator g. Then the cosets of $Z(G)$ are of the form $g^iZ(G)$. Since these cosets partition G, every element of G belongs to one coset. Take $a = g^iz_1$ and $b = g^jz_2$ for some z_1, $z_2 \in Z(G)$. Then we can write

$$ab = g^iz_1g^jz_2 = g^ig^jz_1z_2 = g^{i+j}z_1z_2 = g^{j+i}z_2z_1 = g^jg^iz_2z_1 = g^jz_2g^iz_1 = ba$$

since z_1 and z_2 belong to the center and thus commute with every element in G. Thus G is abelian. The conclusion is exactly the contrapositive of this.

99. Since every element in G appears in one coset of H, it follows from Lagrange's theorem that

$$|G| = |G/H||H|.$$

Write $|G/H| = p^x$ and $|H| = p^y$. Then

$$|G| = |G/H||H| = p^xp^y = p^{x+y}$$

and so the order of G is a power of p.

105. Let H be any other subgroup of G with order $|K|$. Define $f : G \to G/K$. Then $f(H)$ is a quotient group of H and thus $|f(H)|$ divides $|H| = |K|$. By the first isomorphism theorem, $f(H)$ is a subgroup of G/K so $|f(H)|$ also divides $[G : K]$. Then it must be that $|f(H)| = 1$. Then $H \subseteq K$, but since they have the same cardinality, $H = K$.

106. First, suppose HK is a subgroup of G, with $h \in H$ and $g \in G$. Then $h = h1 \in HK$ and $k = 1k \in HK$, therefore $kh \in HK$. So $KH \subseteq HK$. We must also have $(hk)^{-1} \in HK$, so it must follow that $(hk)^{-1} = h'k'$ for $h' \in H$, $k' \in K$. Then $hk = (h'k')^{-1} = k'^{-1}h'^{-1} \in KH$ since $k'^{-1} \in K$, $h'^{-1} \in H$. Thus $HK \subseteq KH$, so if HK is a subgroup in G, $HK = KH$.

Conversely, suppose that $HK = KH$. Then let a, $b \in HK$. Then $a = h_a k_a$ and $b = h_b k_b$ for h_a, $h_b \in H$ and k_a, $k_b \in K$. Then $k_a h_b \in KH = HK$, so $k_a h_b = hk$ for some $h \in H$ and $k \in K$. Therefore

$$ab = h_a k_a h_b k_b = h_a h k k_b \in HK$$

as $h_a h \in H$ and $k k_b \in K$. Then

$$a^{-1} = (h_a k_a)^{-1} = k_a^{-1} h_a^{-1} \in KH = HK$$

so HK is closed under multiplication and inverses. Thus, HK is a subgroup of G.

113. i) Suppose x, $y \in G$. Then their commutator is $xyx^{-1}y^{-1}$. Take $g \in G$, then

$$g(xyx^{-1}y^{-1})g^{-1} = gxg^{-1}gyg^{-1}gx^{-1}g^{-1}gy^{-1}g^{-1} = (gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1}$$

and thus the conjugate of the commutator of x and y is the commutator of gxg^{-1} and gyg^{-1}. Then for x_i and y_i in G for $1 \le i \le n$ we have

$$g\bigg(\prod_1^n x_i y_i x_i^{-1} y_i^{-1}\bigg)g^{-1} = g(x_1 y_1 x_1^{-1} y_1^{-1})\bigg(\prod_2^n x_i y_i x_i^{-1} y_i^{-1}\bigg)g^{-1} = \cdots = \prod_1^n g x_i y_i x_i^{-1} y_i^{-1} g^{-1}$$

and since the conjugate of a commutator is a commutator, and the conjugate of a product of commutators is a product of commutators, and is therefore in G'. So G' is normal in G.

ii) Let aG' and $bG' \in G/G'$. Then $aG'bG' = (ab)G' = ba(a^{-1}b^{-1}ab)G' = (ba)G'$ since the commutator is in G'. Thus G/G' is abelian.

iii) We want to show that every commutator maps to the identity.

$$\varphi(xyx^{-1}y^{-1}) = \varphi(x)\varphi(y)\varphi(x)^{-1}\varphi(y)^{-1}$$

and since these elements commute and cancel to the identity, $G' \le \ker\varphi$.

iv) Since G/G' is abelian, every subgroup of it is normal. Subgroups of G/G' correspond to subgroups of G that contain G', and normal subgroups of G/G' correspond to normal subgroups of G containing G'. Thus, any subgroup containing G' is also normal.

30. Take $x \in R[x]$. If it had an inverse, say $f(x)$, then $xf(x) = 1$. But then if $f(x)$ is of degree n, $1 = xf(x)$ has degree $n+1$. But $\deg(xf(x)) = \deg(1) = 0$, a contradiction. So x does not have an inverse, and therefore $R[x]$ is not a field.

31.

32. i) If R is a domain, so is $R[x]$. Then the degree of a product of two polynomials is the sum of the degree of each. Thus, $f(x)$ can only be a unit if its degree is zero, and its inverse is thus of degree zero. But an invertible constant is a unit in R.

ii) $(2x+1)(2x+1) = 4x^2 + 4x + 1$. Modulo 4, the first two terms are zero, so this is always 1.

33. We know from Fermat's little theorem that for prime p and any x,

$$x^p \equiv x \mod p.$$

In other words,

$$x^p - x \equiv 0 \mod p$$

for all x, so letting $f(x) = x^p - x$, we have that f^b will always evaluate to zero.