HOMEWORK 2

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38. This permutation is not a product of disjoint cycles, so we begin by writing it in two line notation.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}$$

Now we can observe the disjoint cycles, and write the permutation as

$$\alpha = (1\ 5\ 4)(2\ 3).$$

Noting that $\mathrm{lcm}(3,\ 2) = 6$, the order of α is 6. The inverse is easily obtained from the two line notation, by swapping rows and sorting on the top row:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix}.$$

Finally, to compute the parity, we go back to α as a product of disjoint cycles. We have a permutation on 5 letters, with 2 cycles, which gives $\mathrm{sgn}(\alpha) = (-1)^{5-2} = -1$. So α has odd parity.

The permutation in 2.22 can be factored as

$$(1\ 9)(2\ 8)(3\ 7)(4\ 6)(5).$$

Then we observe that the least common multiple is 2. Thus, the order is 2. The permutation in 2.28 can be written as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 6 & 9 & 5 & 3 & 10 & 2 & 8 & 4 & 7 \end{pmatrix}$$

Factoring, we get

$$(1)(2\ 6\ 10\ 7)(3\ 9\ 4\ 5)(8).$$

The least common multiple is clearly 4, so the order is 4.

39. For S_5, we want all elements with order 2. It follows that since we can write any permutation as a product of disjoint cycles, and the order of such a product is the lcm of their orders, we need the elements whose disjoint cycles do not exceed length 2. In S_5, there are two ways to do this: either we have the form $(a_1\ a_2)$ or $(a_1\ a_2)(a_3\ a_4)$. The number of permutations of the first form plus the second form is

$$\binom{5}{2} + \frac{1}{2!}\binom{5}{2}\binom{3}{2} = 25.$$

We divide by 2! to account for the ordering of the two disjoint cycles, which does not change the permutation. Similarly for S_5, we can have the following forms: $(a_1\ a_2)$, $(a_1\ a_2)(a_3\ a_4)$, or $(a_1\ a_2)(a_3\ a_4)(a_5\ a_6)$. The respective number of possible permutations is

$$\binom{6}{2} + \frac{1}{2!}\binom{6}{2}\binom{4}{2} + \frac{1}{3!}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 75.$$

In general for S_n, we want to count the number of ways a product of 2-cycles can be written. If n is even, the most 2-cycles we can have is $n/2$, so we have

$$\binom{n}{2} + \frac{1}{2!}\binom{n}{2}\binom{n-2}{2} + \cdots + \frac{1}{(n/2)!}\binom{n}{2}\binom{n-2}{2}\cdots\binom{2}{2}$$

and if n is odd, the most 2-cycles we can have is $(n-1)/2$, giving

$$\binom{n}{2} + \frac{1}{2!}\binom{n}{2}\binom{n-2}{2} + \cdots + \frac{1}{((n-1)/2)!}\binom{n}{2}\binom{n-2}{2}\cdots\binom{3}{2}.$$

40. We know that $1 = y^m = y^{dt} = (y^t)^d$, so the order of y^t divides d. Suppose there is some $e > 0$ such that $(y^t)^e = 1$ and $e < d$. Then we can see that $et < m = dt$, and $(y^t)^e = y^{et} = 1$, but this contradicts the statement that y has order m. Therefore, d must be the smallest number such that $(y^t)^d = 1$. Thus, d is the order of y^t.

41. Let $x \in G$ such that $x^d = a$. Then $a^{dk} = (x^d)^{dk} = x^{d^2k} = 1$. Suppose there exists an integer e such that $e \leq d^2k$ and the order of x is e. Then $a^e = (x^d)^e = (x^e)^d = 1$, so $dk \mid e$. Then we can write $e = bdk$ for some integer b. So $1 = x^e = x^{bdk} = (x^d)^{bk} = a^{bk}$. This implies $dk \mid bk$, so $d \mid b$. Therefore, we can write e differently again as $e = cd^2k$ for some integer c. But we know that $e \leq d^2k$, so $c = 1$ and $e = d^2k$.

44. Suppose we have a and b in G. Since $x^2 = xx = 1$ implies $x = x^{-1}$ for all $x \in G$, every element is its own inverse. Therefore, $ab = (ab)^{-1} = (b^{-1})(a^{-1}) = ba$. Thus a and b commute and G is abelian.

46. Suppose we define a relation "$\cong$" on G such that for a and b in G, $a \cong b$ if $a = b$ or $a = b^{-1}$. This relation is reflexive since $a \cong a$. It is symmetric since $a \cong b$ means $a = b$ or $a = b^{-1}$, and $b \cong a$ means $b = a$ or $b = a^{-1}$, which are the same. It is transitive since $a \cong b$ means $a = b$ or $a = b^{-1}$, and $b \cong c$ means $b = c$ or $b = c^{-1}$, which implies $a = c$ or $a = c^{-1}$. Then $\cong$ is an equivalence relation and partitions G.

Next, we observe that every equivalence class contains an element and its inverse, nothing more. Order 2 elements are their own inverses, and as such are the only members of their equivalence classes. Suppose the order of G is $2n$. Let the number of classes with one element be s, and the number of classes with two elements be t. Then $2n = 2t + s$, so $s = 2n - 2t = 2(n-t)$. So s is even. We know that $\{1\}$ is the unique class with one element that does not have order 2; it has order 1. Then, the number of classes with order 2 is $s - 1$, an odd number.

48. Suppose we have two stochastic matrices

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

such that $a + b = c + d = e + g = f + h = 1$. Then their product is

$$\begin{bmatrix} ae + cg & af + ch \\ be + dg & bf + dh \end{bmatrix}.$$

We can show that

$$(ae + cg) + (be + dg) = e(a + b) + g(c + d) = e + g = 1$$

and similarly

$$(af + ch) + (bf + dh) = f(a + b) + h(c + d) = f + h = 1.$$

So, the stochastic matrices are closed under matrix multiplication.

Next, we can show that

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

Adding the elements of the left column gives

$$\frac{d-b}{ad-bc}.$$

Since $d = 1 - c$ we can write the denominator as

$$a(1-c) - bc = a - ac - bc = a - c(a+b) = a - c.$$

We can see that since

$$(a-c) - (d-b) = (a-c) + (b-d) = (a+b) - (c+d) = 1 - 1 = 0,$$

we know $a - c = d - b$ and therefore the left column of the inverse sums to 1. The right column sums to

$$\frac{a-c}{ad-bc}.$$

As above, we can write the denominator as $a - c$, and it follows that the right column also sums to 1. Thus, the inverse of a stochastic matrix is stochastic.