HOMEWORK 1

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22. Begin by factoring α; we obtain

$$(1\ 9)(2\ 8)(3\ 7)(4\ 6)(5).$$

We know $\operatorname{sgn}(\alpha) = (-1)^{n-t}$, and here we have $n = 9$ and $t = 5$. Thus, $\operatorname{sgn}(\alpha) = (-1)^4 = 1$. To find the inverse, swap the rows:

$$\alpha^{-1} = \begin{pmatrix} 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}.$$

Sorting on the first row, we observe

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = \alpha.$$

24. i) For every r-cycle in S_n, we can write the cycle r different ways, by starting on a different number. For example, $(1\ 2\ 3\ 4)$ can also be written as $(2\ 3\ 4\ 1)$. To create a cycle, we want to choose r numbers from $1, \dots, n$. The number of ways to do this is

$$\frac{n!}{(n-r)!}.$$

To compensate for overcounting, divide by the number of ways we write the same cycle: r. So, we have

$$\frac{n!}{r(n-r)!}$$

r-cycles in S_n.

ii) To create disjoint r-cycles, we want to choose kr elements from $1, \dots, n$. This is

$$\frac{n!}{(n-kr)!}.$$

We can permute the cycles in any order we like, giving $k!$ possibilities. In addition, each cycle can be written in r equivalent ways; the total for k r-cycles is r^k. So, to compensate for overcounting, divide.

$$\frac{1}{k!}\frac{1}{r^k}\frac{n!}{(n-kr)!}$$

25. i) Suppose we have distinct numbers $i_1, \dots, i_r$ and α sends i_k to i_{k+1} for $k < r$, and sends i_r to i_1. Taking $\alpha(i_k)$ gives i_{k+1}, $\alpha^2(i_k)$ gives i_{k+2}, and in general, $\alpha^r(i_k)$ gives i_{k+r}. But $k+r$ must be reduced modulo r since $k + r > r$. Then $k + r \equiv k \mod r$, and so $\alpha^r(i_k) = i_k\ \forall\ 1 \le k \le r$. Thus $\alpha^r = (1)$.

ii) We know from (i) that $\alpha^k(i_j) = i_{j+k}$. Suppose $k < r$. Then $\alpha^k(i_1) = i_{k+1} \neq i_1$, since each i is distinct. If $k = r$, then $\alpha^r(i_j) = i_{j+r} = i_j\ \forall j$. So there does not exist a $k < r$ satisfying $\alpha^k = (1)$ and r is the least positive integer that does satisfy the conditions.

26. Suppose we have $\alpha \in S_n$ such that $\alpha = (a_1\ a_2\ \dots\ a_r)$. We can write α as $(a_1\ a_2)(a_2\ a_3)\cdots(a_{r-1}\ a_r)$, which comes to $r-1$ transpositions. If $r-1$ is even, r is odd, and vice versa. We know from

Theorem 2.40 that the number of transpositions in a product give the parity of the product. So, α is even if r is odd, and α is odd if r is even.

27. Consider $j - i$. If $j - i = 1$, we have $(i\ i+1)$, clearly a product of one adjacency. Suppose the conclusion is true for $j - i = k$. Then consider $j - i = k + 1$. We have

$$(i\ i+k+1) = (i\ i+k)(i+k\ i+k+1)(i+k\ i).$$

We know from the induction hypothesis that $(i\ i+k)$ and $(i+k\ i)$ are the products of an odd number of adjacencies. $(i+k\ i+k+1)$ is one adjacency. The sum of two odd numbers is even, and the sum of an even number and one is odd, so $(i\ i+k+1)$ is a product of an odd number of adjacencies.

30. i) We write $(\alpha\beta)^k$ as

$$(\alpha\beta)(\alpha\beta)\cdots(\alpha\beta) \text{ k times.}$$

Composition of permutations is associative, so the parenthesis are irrelevant. Since the permutations commute, we can move the αs and βs to opposite sides of the product:

$$\alpha\alpha\cdots\alpha\beta\beta\cdots\beta = \alpha^k\beta^k.$$

ii) Take $\alpha = (1\ 2)$ and $\beta = (2\ 3)$. Then $(\alpha\beta)^2 = (1\ 3\ 2)$ while $\alpha^2\beta^2 = (1)$.

32. We can observe that S_n has cardinality $n!$, and so we want to show that the even permutations in S_n make up half of that. Then, the odd permutations must make up the other half, and the subsets containing the odd and even permutations have the same size. Let A_n contain the even permutations, and let O_n contain the odd ones.

Fix some $\beta \in O_n$. Define $f : A_n \rightarrowtail O_n$ such that

$$f(\alpha) = \beta\alpha.$$

We know that the product of an odd and an even permutation yields an odd permutation.

Suppose that $f(\alpha) = f(\alpha')$ for α, α' in A_n. Then we have $\beta\alpha = \beta\alpha'$, which gives $\alpha = \alpha'$. So f is injective.

Suppose we have a $\sigma \in O_n$. We know that $\beta^{-1}\sigma$ is an even permutation, so we compute

$$f(\beta^{-1}\sigma) = \beta\beta^{-1}\sigma = \sigma.$$

So f is surjective.

Therefore, f is a bijection from A_n to O_n, so the sets have the same size. Thus, each must have a size of one half of S_n, or $\frac{n!}{2}$.