HOMEWORK 3

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55. Take $G = \mathbb{Z}_6$, with subgroups $H = \{0, 2, 4\}$ and $K = \{0, 3\}$. Then $H \cup K$ is $\{0, 2, 3, 4\}$. This is not a subgroup since $2 + 3 = 5 \notin H \cup K$.

56. From Lagrange, we know that

$$\frac{|G|}{|H|} = [G : H], \qquad \frac{|G|}{|K|} = [G : K], \qquad \frac{|K|}{|H|} = [K : H].$$

Therefore

$$[G : K][K : H] = \frac{|G|}{|K|}\frac{|K|}{|H|} = \frac{|G|}{|H|} = [G : H].$$

57. We know $H \cap K$ is a subgroup of H, K, and G. From Lagrange's theorem, we know the order of $H \cap K$ divides $|H|$ and $|K|$. Since the order of H and K are coprime, $\gcd(|H|, |K|) = 1$ so $|H \cap K| = 1$.

59. Suppose that G is not cyclic. Then take $x \in G$, $x \neq 1$. Because G is not cyclic, $|x| \neq 4$. But from Corollary 2.85, $|x|$ divides $|G|$. The order of x is not 1, as it is not the identity, and not 4, so it must be 2. Exercise 44 was proven in homework 2, that if $x^2 = 1\ \forall x \in G$, G is abelian.

If G is cyclic, then it is necessarily abelian. Take g to be the generator. Then for a, $b \in G$, say $a = g^s$ and $b = g^t$, then

$$ab = g^s g^t = g^{s+t} = g^{t+s} = g^t g^s = ba.$$

63. Take $H = \{(1),\ (1\ 2)\}$, and $\alpha = (1\ 3)$. Then the left coset αH is

$$\{(1\ 3),\ (1\ 2\ 3)\}.$$

The right coset $H\alpha$ is found to be

$$\{(1\ 3),\ (1\ 3\ 2)\}.$$

The left and right coset are not equal. It follows that H is not normal in S_3.

68. Suppose G is abelian. Then for a, $b \in G$:

$$f(ab) = (ab)^{-1} = (ba)^{-1} = a^{-1}b^{-1} = f(a)f(b)$$

and f is a homomorphism.

Suppose f is a homomorphism. Then for a, $b \in G$:

$$ab = (b^{-1}a^{-1})^{-1} = (f(b)f(a))^{-1} = (f(ba))^{-1} = ba$$

and G is abelian.

69. Take $a \in G$ with order n and $f(a) \in H$ with order m. If n is finite, then

$$f(a)^n = f(a^n) = f(1_G) = 1_H.$$

Then we know $m \mid n$ and thus m is finite. Since f is bijective, we can try

$$f^{-1}(f(a))^m = f^{-1}(f(a)^m) = f^{-1}(1_H) = 1_G.$$

Thus $n \mid m$ and therefore $n = m$.

For the second part, let $a_1, \ldots, a_t$ be all the elements in G with order k. Then by the first part of this exercise, we know $f(a_1), \ldots, f(a_t)$ are elements in H of order k. Suppose there is some additional $b \in H \setminus \{f(a_1), \ldots, f(a_t)\}$ with order k. Then there must exist a $c \in G \setminus \{a_1, \ldots, a_t\}$ with order k such that $f(c) = b$, since f is a bijection. A contradiction. Therefore G and H have the same number of elements of order k.

71. Note that the dihedral group of order 4 contains, geometrically, a horizontal reflection, a vertical reflection, a rotation by 180 degrees, and the identity. Consider the mapping $f : V \to D_4$

$$f = \{((1,\ 1),\ 1),\ ((-1,\ 1),\ \text{h-flip}),\ ((1,\ -1),\ \text{v-flip}),\ ((-1,\ -1),\ \text{rotate})\}$$

where the second element of each pair corresponds to the geometric descriptions above. So f corresponds to transformations on the orientation of a 2-gon; the parity of each element in a pair from V corresponds to the horizontal and vertical orientation of the 2-gon.

For the second part, we know that the dihedral group of order 6 is the symmetry group of an equilateral triangle. The group S_3 is the set of permutations on three symbols; if vertices are "symbols", then S_3 permutes the vertices of an equilateral triangle, giving the dihedral group.

80. Let $H_1, H_2, \ldots$ be a family of normal subgroups, and denote their intersection by $\bigcap H$. Then take $x \in \bigcap H$. For some $g \in G$, we know that $gxg^{-1} \in H_i$ since each H_i is normal. Then $gxg^{-1} \in \bigcap H$. Therefore, $\bigcap H$ is normal.

82. Lemma: Suppose A is a finite set, and $f : A \to A$ is a function. If f is injective, it is surjective. *Proof.* Suppose f is injective. Then the image of f has at least $|A|$ elements, but the image of f is contained in A, so it must have exactly $|A|$ elements. Therefore f is surjective.

Suppose x, $y \in G$, $|G| = 2k - 1$, and $x^2 = y^2$. Then

$$x = x^{2k} = (x^2)^k = (y^2)^k = y^{2k} = y$$

so squaring is injective. Thus by the lemma, squaring is bijective, so every element has an inverse of squaring, or a square root.

If every element in G has a square root, we can take x_1, x_2 such that $x_1 \neq x_2$. Then suppose there is some y that is the square root of x_1 and x_2. Then $y^2 = x_1 = x_2$, a contradiction. Therefore every element has a unique square root.

87. We know that the dihedral group is generated by a reflection element of order 2, and a rotation element of order 4. Suppose these are a and b, respectively, such that $a^2 = 1$ and $b^4 = 1$. Then the dihedral group contains at least two elements of order two: b^2 and a. The only element of order two of the quaternion group is -1. Therefore, the groups cannot be isomorphic as they contain differing amounts of elements of order two.