DOUBLE ROOTS AND SYLVESTER MATRICES

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1. Presence of a Double Root

Let $f(x)$ be a degree 2 polynomial with real coefficients given by

$$f(x) = ax^2 + bx + c.$$

Without loss of generality, assume a is nonzero.

Claim 1. *The following are equivalent:*

(i) $f(x)$ *has a double root* α
(ii) $f'(\alpha) = 0$
(iii) $\Delta_f = 0$
(iv) $f(x)$ *has the same sign as the leading coefficient* $a \in \mathbb{R}$, *for all* $x \in \mathbb{R} \setminus \{\alpha\}$.

Proof. In (1) we will show that (i) implies (ii), and (2) shows the reverse. In (3) it is shown that (i) implies (iii) and in (4) the reverse is shown. In (5) and (6) it is demonstrated that (i) implies (iv) and (iv) implies (i), respectively.

(1) The quadratic polynomial $f(x)$ can be written as

$$f(x) = a(x - \alpha_1)(x - \alpha_2)$$

but in the case of a double root, this simplifies to

$$f(x) = a(x - \alpha)(x - \alpha) = a(x - \alpha)^2.$$

Consider the derivative $f'(x)$. Then we have

$$\begin{aligned} f'(x) &= \frac{d}{dx} a(x - \alpha)^2 \\ &= 2a(x - \alpha). \end{aligned}$$

It follows that if we evaluate $f'(\alpha)$, we have

$$f'(\alpha) = 2a(\alpha - \alpha) = 2a(0) = 0.$$

(2) Given that α is a root of $f(x)$, we want to show that $f'(\alpha) = 0$ implies that α is a double root. Suppose instead that $f(x)$ has two distinct roots, α_1 and α_2. Then we can write

$$f(x) = a(x - \alpha_1)(x - \alpha_2).$$

Taking the derivative, we have

$$\begin{aligned} f'(x) &= a[(x-\alpha_1)+(x-\alpha_2)] \\ &= a(2x-\alpha_1-\alpha_2). \end{aligned}$$

If we compute $f'(\alpha_1)$ and $f'(\alpha_2)$, we have

$$f'(\alpha_1) = a(\alpha_1-\alpha_2),\ \ f'(\alpha_2) = a(\alpha_2-\alpha_1)$$

and since we assumed that α_1 and α_2 are distinct and a is nonzero, we have $f'(\alpha_1) \neq 0$ and $f'(\alpha_2) \neq 0$. Then, by the contrapositive, we have that if α is a root of $f(x)$, $f'(\alpha) = 0$ implies α is a double root of $f(x)$.

(3) we write the discriminant as

$$\Delta_f = (\alpha_1-\alpha_2)(\alpha_2-\alpha_1).$$

If $f(x)$ has a double root, then $\alpha_1 = \alpha_2$, and we have

$$\Delta_f = (\alpha_1-\alpha_1)(\alpha_1-\alpha_1)$$

and it is shown that $\Delta_f = 0$.

(4) With the discriminant given as

$$\Delta_f = (\alpha_1-\alpha_2)(\alpha_2-\alpha_1),$$

it must be that if $\Delta_f = 0$, we have

$$0 = (\alpha_1-\alpha_2)(\alpha_2-\alpha_1)$$

and it follows that α_1 must equal α_2. Then we can write the polynomial $f(x)$ as

$$f(x) = a(x-\alpha_1)(x-\alpha_2)$$

and since $\alpha_1 = \alpha_2$ we have

$$f(x) = a(x-\alpha_1)^2.$$

Then the root α_1 has multiplicity two, which by definition, makes it a double root.

(5) As above, we can write the quadratic polynomial as

$$f(x) = a(x-\alpha)^2.$$

Consider the term $(x-\alpha)^2$. As this term is squared, it is always nonnegative. So, a is the only term in the expression that determines the sign of $f(x)$. It follows that for positive a, we have a positive term a multiplied by a positive term $(x-\alpha)^2$ which gives a positive product. For negative a, we have a negative term multiplied by a positive term, yielding a negative product.

The only case where this is not true is when $x = \alpha$, so we can say that $f(x)$ has the same sign as the leading coefficient a for all $x \in \mathbb{R} \setminus \{\alpha\}$.

(6) If $f(x)$ has the same sign as its leading coefficient a for all $x \in \mathbb{R} \setminus \{\alpha\}$, we want to show that $f(x)$ has a double root. Suppose instead that $f(x)$ has two unique roots. Assume without loss of generality that $\alpha_1 < \alpha_2$. Then if we write

$$f(x) = a(x - \alpha_1)(x - \alpha_2)$$

we can see that there would exist some values of x such that

$$\alpha_1 < x < \alpha_2$$

and we would have

$$x - \alpha_1 > 0, \quad x - \alpha_2 < 0$$

and so the product $(x - \alpha_1)(x - \alpha_2)$ would be negative. This would lead to $f(x)$ not having the same sign as a for $\alpha_1 < x < \alpha_2$.

This contradicts our assumption that $f(x)$ has the same sign as its leading coefficient a for all $x \in \mathbb{R} \setminus \{\alpha\}$, so it must be the case that $\alpha_1 = \alpha_2$ and $f(x)$ has a double root.

Each statement has been proven true if and only if (i) is true. Therefore, all four statements are equivalent.

□

2. Sylvester Matrices

2.1. **Problem 2.** Let

$$\begin{aligned} f(x) &= x^5 - 3x^4 - 2x^3 + 3x^2 + 7x + 6 \\ g(x) &= x^4 + x^2 + 1. \end{aligned}$$

Then the Sylvester matrix of f and g is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & -3 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 3 & -2 & -3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 7 & 3 & -2 & -3 & 1 & 0 & 1 & 0 & 1 \\ 6 & 7 & 3 & -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 6 & 7 & 3 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 6 & 7 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$Res(f, g, x)$ is the determinant of this matrix. Computing the determinant, we have

$$Res(f, g, x) = 0.$$

2.2. **Problem 3.** Consider

$$f(x) = 6x^4 - 23x^3 - 19x + 4.$$

$f(x)$ has multiple roots if the discriminant $Res(f, f', x)$ is shown to be zero. We can compute $f'(x)$ to be

$$24x^3 - 69x^2 - 19.$$

The Sylvester matrix of f and f' is

$$\begin{bmatrix} 6 & 0 & 0 & 24 & 0 & 0 & 0 \\ -23 & 6 & 0 & -69 & 24 & 0 & 0 \\ 0 & -23 & 6 & 0 & -69 & 24 & 0 \\ -19 & 0 & -23 & -19 & 0 & -69 & 24 \\ 4 & -19 & 0 & 0 & -19 & 0 & -69 \\ 0 & 4 & -19 & 0 & 0 & -19 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & -19 \end{bmatrix}$$

and the determinant of this matrix is

$$Res(f, f', x) = -3921998592 \neq 0.$$

So, $f(x)$ does not have any multiple roots.

2.3. **Problem 4.** Consider

$$f(x) = x^4 - bx + 1.$$

We want b such that $f(x)$ has a double root. For f to have a double root we need

$$\frac{-1^{\frac{4(3)}{2}}}{1} Res(f, f', x) = (1)Res(f, f', x) = 0.$$

The Sylvester matrix of f and f' is

$$\begin{bmatrix} 1 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 4 & 0 \\ -b & 0 & 0 & -b & 0 & 0 & 4 \\ 1 & -b & 0 & 0 & -b & 0 & 0 \\ 0 & 1 & -b & 0 & 0 & -b & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -b \end{bmatrix}$$

and the determinant gives

$$Res(f, f', x) = -27b^4 + 256.$$

Using this in the above equation, we have

$$(1)(-27b^4 + 256) = 0$$

and it follows that
$$27b^4 = 256.$$
So, for all b that satisfy
$$b^4 = \frac{256}{27},$$
$f(x)$ will have a double root.

2.4. **Problem 5.** Consider
$$f(x) = x^3 - px + 1.$$
We want p such that $f(x)$ has a double root. So we need
$$\frac{-1^{\frac{3(2)}{2}}}{1} Res(f, f', x) = (-1)Res(f, f', x) = 0.$$
The Sylvester matrix of f and f' is
$$\begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ -p & 0 & -p & 0 & 3 \\ 1 & -p & 0 & -p & 0 \\ 0 & 1 & 0 & 0 & -p \end{bmatrix}$$
and computing the determinant yields
$$Res(f, f', x) = -4p^3 + 27.$$
Setting this equal to zero, we have
$$0 = -4p^3 + 27$$
and it follows that
$$4p^3 = 27.$$
So for all p such that
$$p^3 = \frac{27}{4}$$
is satisfied, $f(x)$ will have a double root.