INDUCTION

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Problem 1. Prove by induction that $7^n - 1$ is divisible by 6 for all $n \geq 1$.

Proof. First, let $n = 1$. Then we have

$$7^1 - 1 = 6.$$

Clearly, 6 is divisible by 6, so the statement holds for $n = 1$.

Next, let $n = k$. Assuming the statement holds, we have

$$\begin{aligned} 7^k - 1 &= 6a \\ 7^k &= 6a + 1 \end{aligned}$$

for some integer a. Letting $n = k+1$, we can show by algebraic manipulation that

$$\begin{aligned} 7^{k+1} &= 7(6a+1) \\ 7^{k+1} &= 42a + 7 \\ 7^{k+1} - 1 &= 42a + 6. \end{aligned}$$

We can factor the right side of the equality to obtain

$$6(7a+1),$$

and then let $b = 7a + 1$. Now we have

$$7^{k+1} - 1 = 6b$$

for some integer b, and therefore $7^{k+1} - 1$ is divisible by 6. □

Problem 2. Show that, given $x_1 = 1$ and

$$x_{k+1} = \frac{x_k}{x_k + 2},$$

we can write

$$x_n = \frac{1}{2^n - 1}$$

for all $n \geq 1$.

Proof. Begin by calculating the first few values of x_n: namely, x_i for $i \in \{2,3,4,5\}$. We can see that

$$x_2 = \frac{1}{4-1} = \frac{1}{3},$$
$$x_3 = \frac{1}{8-1} = \frac{1}{7},$$
$$x_4 = \frac{1}{16-1} = \frac{1}{15},$$
$$x_5 = \frac{1}{32-1} = \frac{1}{31}.$$

Let $n = 1$. Then we have

$$x_1 = \frac{1}{2-1} = 1$$

which is as given.

Now let $n = k$. Then we can write

$$x_k = \frac{1}{2^k - 1} = \frac{x_{k-1}}{x_{k-1} + 2}.$$

Letting $n = k+1$, we have

$$x_{k+1} = \frac{x_k}{x_k + 2} = \frac{\frac{1}{2^k-1}}{\frac{1}{2^k-1} + 2}.$$

Writing the denominator as

$$\frac{1 + 2(2^k - 1)}{2^k - 1},$$

we obtain

$$\begin{aligned} x_{k+1} &= \left(\frac{1}{2^k - 1}\right)\left(\frac{2^k - 1}{1 + 2(2^k - 1)}\right) \\ &= \frac{1}{1 + 2(2^k - 1)} \\ &= \frac{1}{1 + 2^{k+1} - 2} \\ &= \frac{1}{2^{k+1} - 1} \end{aligned}$$

So, the statement holds for x_{k+1}. □

Problem 3. Find the equation of the set of all points twice as far from (0,1) as from (1,0).

Solution. Assume the Euclidean metric. Then the distance between two points (x, y) and (x_0, y_0) is defined as

$$d = \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

So, for each of the two points (0,1) and (1,0) we write the distance to a point (x, y) as

$$\begin{aligned} d_{(0,1)} &= \sqrt{x^2 + (y-1)^2}, \\ d_{(1,0)} &= \sqrt{(x-1)^2 + y^2}. \end{aligned}$$

We would like the set of points (x, y) twice as far from (0,1) as from (1,0). In other words,

$$d_{(1,0)} = 2d_{(0,1)}.$$

Using the above formulas, we can write

$$\begin{aligned} \sqrt{(x-1)^2 + y^2} &= 2\sqrt{x^2 + (y-1)^2} \\ (x-1)^2 + y^2 &= 4(x^2 + (y-1)^2) \\ x^2 - 2x + 1 + y^2 &= 4x^2 + 4y^2 - 8y + 4 \end{aligned}$$

Applying some algebraic simplification, we obtain

$$\begin{aligned} -2x &= 3x^2 + 3y^2 - 8y + 3 \\ 0 &= 3x^2 + 3y^2 + 2x - 8y + 3. \end{aligned}$$

So, those points (x, y) that are solutions to

$$0 = 3x^2 + 3y^2 + 2x - 8y + 3$$

are twice as far from (0,1) as from (1,0).