CIRCLES AND ELLIPSES

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Problem 1a. Find the perimeter of

$$x^2 + y^2 = r^2.$$

We know the arc length formula for polar coordinates is

$$C = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta,$$

and in this case we have a constant r. So the integral becomes

$$C = \int_0^{2\pi} \sqrt{r^2 + 0^2} d\theta = \int_0^{2\pi} r d\theta,$$

which evaluates to

$$C = r(2\pi - 0) = 2\pi r.$$

Problem 1b. Find the area of

$$x^2 + y^2 = r^2.$$

From above, we know that the perimeter is given by $C = 2\pi r$. We can use the shell method of integration to partition the circle into thin rings of radius s and width ds. Then the area of each ring is simply $2\pi s ds$, and we have

$$A = \int_0^r 2\pi s ds.$$

Integrating gives

$$A = 2\pi \frac{s^2}{2}\bigg|_0^r = \pi r^2 - 0 = \pi r^2.$$

Problem 2a. Find the perimeter of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

We know that the general formula for arc length of a parametric curve is given by

$$p = \int_c^d \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

The given equation can be parameterized such that $x = a\cos\theta$ and $y = b\sin\theta$. In addition, finding the arc length in the first quadrant and then multiplying by four will give the total circumference. Then we have the circumference of the curve as

$$p = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta d\theta}$$

Noting that $\sin^2\theta = 1 - cos^2\theta$, we can write

$$p = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2(1 - cos^2\theta) + b^2\cos^2\theta d\theta}$$

and simplify to get

$$p = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2 + (b^2 - a^2)cos^2\theta d\theta}.$$

We let the eccentricity ε be defined as

$$\varepsilon = \frac{\sqrt{a^2 - b^2}}{a}.$$

Now we can write the circumference as

$$p = 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - \varepsilon^2\cos^2\theta d\theta}.$$

We know that the binomial series, a special Maclaurin series, gives

$$(z+1)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \ldots$$

This series can be applied to the argument of the circumference integral with $\alpha = 1/2$ and $z = -\varepsilon^2\cos^2\theta$. Specifically, we have

$$\sqrt{1 - \varepsilon^2\cos^2\theta} = 1 - \frac{\varepsilon^2\cos^2\theta}{2} - \sum_{k=2}^{\infty} \binom{1/2}{k} (\varepsilon^2\cos^2\theta)^k$$

Note that since ε describes the eccentricity of the ellipse, its value must be strictly between 0 and 1. Also, $|\cos^2\theta| \leq 1$ for all θ. So, the series converges.

Returning to the circumference, we have

$$
\begin{aligned}
p &= 4a\int_0^{\frac{\pi}{2}}\left[1-\frac{\varepsilon^2\cos^2\theta}{2}-\sum_{k=2}^{\infty}\binom{1/2}{k}z^k\right]d\theta\\
&= 4a\int_0^{\frac{\pi}{2}}d\theta-4a\int_0^{\frac{\pi}{2}}\frac{\varepsilon^2\cos^2\theta}{2}d\theta-4a\int_0^{\frac{\pi}{2}}\sum_{k=2}^{\infty}\binom{1/2}{k}(\varepsilon^2\cos^2\theta)^k d\theta
\end{aligned}
$$

The first two terms are integrated easily. With regards to the integral of the summation, it is clear that the quantity must be finite, as we are calculating a perimeter. So we can apply Fubini's Theorem to exchange the integral and the summation to obtain

$$p = 4a\int_0^{\frac{\pi}{2}}d\theta-4a\int_0^{\frac{\pi}{2}}\frac{\varepsilon^2\cos^2\theta}{2}d\theta-4a\sum_{k=2}^{\infty}\int_0^{\frac{\pi}{2}}\binom{1/2}{k}(\varepsilon^2\cos^2\theta)^k d\theta.$$

Now we can rearrange the placement of terms, yielding a much nicer looking integral:

$$p = 4a\int_0^{\frac{\pi}{2}}d\theta-4a\int_0^{\frac{\pi}{2}}\frac{\varepsilon^2\cos^2\theta}{2}d\theta-4a\sum_{k=2}^{\infty}\varepsilon^{2k}\binom{1/2}{k}\int_0^{\frac{\pi}{2}}\cos^{2k}\theta d\theta.$$

Using Wolfram Alpha to assist with the last integral, we can evaluate the integrals to obtain

$$p = 2\pi a-4a\frac{\pi}{4}\frac{\varepsilon^2}{2}-4a\sum_{k=2}^{\infty}\varepsilon^{2k}\binom{1/2}{k}\frac{\pi}{2}\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}.$$

The last fraction is obtained from Wallis's cosine formula. We can make the observation that

$$\binom{1/2}{k}=\frac{(-1)^{k+1}1\cdot 3\cdot 5\cdots(2k-5)\cdot(2k-3)}{k!2^k}$$

So we can write

$$p = 2\pi a-4a\frac{\pi}{4}\frac{\varepsilon^2}{2}-4a\sum_{k=2}^{\infty}\varepsilon^{2k}\frac{\pi}{2}\frac{[1\cdot 3\cdot 5\cdots(2k-3)]^2(2k-1)}{(k!2^k)^2}.$$

Finally, we simplify to obtain

$$p = 2\pi a\left[1-\frac{\varepsilon^2}{4}-\sum_{k=2}^{\infty}\varepsilon^{2k}\frac{[1\cdot 3\cdot 5\cdots(2k-3)]^2(2k-1)}{(k!2^k)^2}\right].$$

In the case of a circle where $r=a=b$, eccentricity $\varepsilon=0$ and it is clear from the equation that circumference collapses to $p=2\pi r$.

Problem 2b. Find the area of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Note that we can obtain the total area by multiplying the area in the first quadrant by four. Rewriting the equation in terms of y, we have

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

and we would like to evaluate

$$A = 4\int_0^{\frac{\pi}{2}} ydx$$

which we can write as

$$A = 4b\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{x^2}{a^2}}dx.$$

Let $x = a\sin\theta$. Then we have

$$\frac{dx}{d\theta} = a\cos\theta$$

and we can substitute into the integral to obtain

$$A = 4ab\int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2\theta}\cos\theta d\theta.$$

We observe that since

$$1 - \sin^2\theta = cos^2\theta$$

we can write this as

$$A = 4ab\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\cos\theta d\theta = 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta = 4ab\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}d\theta$$

and finally integrate, yielding

$$A = 4ab\frac{1}{2}(\theta + \sin\theta\cos\theta)\Big|_0^{\frac{\pi}{2}} = 2ab(\frac{\pi}{2} - 0) = \pi ab.$$