CIRCLES AND ELLIPSES

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We know that the general formula for arc length of a parametric curve is given by

$$p = \int_c^d \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

The given equation can be parameterized such that $x = a\cos\theta$ and $y = b\sin\theta$. In addition, finding the arc length in the first quadrant and then multiplying by four will give the total circumference. Then we have the circumference of the curve as

$$p = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2 \sin^2\theta + b^2\cos^2\theta} d\theta$$

Noting that $\sin^2\theta = 1 - cos^2\theta$, we can write

$$p = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2(1 - cos^2\theta) + b^2\cos^2\theta} d\theta$$

and simplify to get

$$p = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2 + (b^2 - a^2)cos^2\theta} d\theta.$$

We let the eccentricity ε be defined as

$$\varepsilon = \frac{\sqrt{a^2 - b^2}}{a}.$$

Now we can write the circumference as

$$p = 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - \varepsilon^2\cos^2\theta} d\theta.$$

We know that the binomial series, a special Maclaurin series, gives

$$(z+1)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \dots$$

This series can be applied to the argument of the circumference integral with $\alpha = 1/2$ and $z = -\varepsilon^2\cos^2\theta$. Specifically, we have

$$\sqrt{1 - \varepsilon^2\cos^2\theta} = 1 - \frac{\varepsilon^2\cos^2\theta}{2} - \sum_{k=2}^{\infty} \binom{1/2}{k} (\varepsilon^2\cos^2\theta)^k$$

Note that since ε describes the eccentricity of the ellipse, its value must be strictly between 0 and 1. Also, $|\cos^2\theta| \leq 1$ for all θ. So, the series converges.

Returning to the circumference, we have

$$\begin{aligned} p &= 4a \int_0^{\frac{\pi}{2}} \left[1 - \frac{\varepsilon^2 \cos^2 \theta}{2} - \sum_{k=2}^{\infty} \binom{1/2}{k} z^k \right] d\theta \\ &= 4a \int_0^{\frac{\pi}{2}} d\theta - 4a \int_0^{\frac{\pi}{2}} \frac{\varepsilon^2 \cos^2 \theta}{2} d\theta - 4a \int_0^{\frac{\pi}{2}} \sum_{k=2}^{\infty} \binom{1/2}{k} (\varepsilon^2 \cos^2 \theta)^k \end{aligned}$$