CONTINUITY OF THE SINE AND THE DISCRIMINANT

JASON MEDCOFF

1. Continuity of Sine

The following lemma will be used to prove Claim 1.

Lemma 1. $|\sin(x)| \leq |x| \; \forall x \in \mathbb{R}$.

Proof. Suppose instead that $|\sin(x)| > |x| \; \forall x$. Then $-|x| > \sin(x) > |x|$. Consider three cases as follows.

Case 1: $x = 0$. Then we have $0 > \sin(0) > 0$, or $0 > 0 > 0$. This is impossible.

Case 2: $x > 0$. We thus have $-x > \sin(x) > x$, which gives $-x > x$. This contradicts the assumption that $x > 0$.

Case 3: $x < 0$. Then we have $x > \sin(x) > -x$. It follows that $x > -x$, which fails to hold for $x < 0$ as assumed.

Each case gives an impossible mathematical statement. So, $|\sin(x)| > |x| \; \forall x$ is incorrect. Therefore, the original proposition is true. $\square$

Claim 1. *The function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \sin(x)$ is well defined and continuous everywhere on its domain.*

Proof. Let $x \in (-\infty, \infty)$. We know that for f to be well defined, we must have $\forall x$, there is exactly one $y \in \mathbb{R}$ such that $f(x) = y$. Let $y = \sin(x)$. Then $f(x) = y$. Let $x_1, x_2 \in (-\infty, \infty)$ such that $f(x_1) \neq f(x_2)$. Then $\sin(x_1) \neq \sin(x_2)$. Taking the inverse sine, we have $x_1 \neq x_2$. Therefore, $\forall x \in (-\infty, \infty)$, there exists a unique y such that $\sin(x) = y$. So, f is a well defined function.

For a function f to be continuous, we know that the limit

$$\lim_{x \to a} f(x) = f(a)$$

must hold for all a in the domain. Let $a \in (-\infty, \infty)$. We want to show that $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$\delta > |x - a| > 0 \implies |f(x) - f(a)| < \varepsilon.$$

Suppose ε is given. We want to find δ such that

$$|\sin(x) - \sin(a)| < \varepsilon.$$

We can see by trigonometric identity that

$$|\sin(x) - \sin(a)| = \left|2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)\right|.$$

Since $\cos(\alpha) \leq 1\ \forall \alpha \in \mathbb{R}$, we can show that

$$\left|2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)\right| \leq 2\left|\sin\left(\frac{x-a}{2}\right)\right|.$$

We want $\delta > |x-a| > 0$, but notice that

$$|x-a| \geq \left|\frac{x-a}{2}\right|,$$

so we can write

$$\delta > \left|\frac{x-a}{2}\right| > 0.$$

Due to lemma 1, we know

$$2\left|\frac{x-a}{2}\right| \geq 2\left|\sin\left(\frac{x-a}{2}\right)\right|$$

is true, so we will write

$$2\delta > 2\left|\frac{x-a}{2}\right| \geq 2\left|\sin\left(\frac{x-a}{2}\right)\right|.$$

If we take $\delta = \frac{\varepsilon}{2}$, then

$$2\left|\sin\left(\frac{x-a}{2}\right)\right| < \varepsilon$$

holds true. As δ and ε do not depend on x, this is true over the entire domain $\mathbb{R}$. Therefore,

$$\lim_{x \to a} \sin(x) = \sin(a)$$

is true for all $a \in \mathbb{R}$, and we know that f is continuous. □

2. Quadratic Discriminant

A quadratic polynomial $f(x) = ax^2+bx+c$ can be factored into $a(x-\alpha_1)(x-\alpha_2)$ for some complex numbers α_1 and α_2. Define the discriminant of $f(x)$ to be

$$\Delta_f := \prod_{i \neq j}(\alpha_i - \alpha_j) = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_1).$$

Claim 2. *Δ_f can be written in terms of a, b, and c by the expression*

$$\Delta_f = \frac{4ac - b^2}{a^2}.$$

Proof. We know that the roots α_1 and α_2 are given by the general quadratic formula. So let

$$\alpha_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and let

$$\alpha_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Then we can write $(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_1)$ as

$$\left(\frac{(-b + \sqrt{b^2 - 4ac}) - (-b - \sqrt{b^2 - 4ac})}{2a}\right)\left(\frac{(-b - \sqrt{b^2 - 4ac}) - (-b + \sqrt{b^2 - 4ac})}{2a}\right).$$

Combining terms gives us

$$\left(\frac{2\sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-2\sqrt{b^2 - 4ac}}{2a}\right).$$

Multiplying, we have

$$-\frac{(b^2 - 4ac)}{a^2} = \frac{4ac - b^2}{a^2}.$$

$\square$

3. Cubic Discriminant

The following lemmas will be used to prove Claim 3.

Lemma 2. *Given any cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$, let $x = y - \frac{b}{3a}$. $f(x)$ can be written as*

$$y^3 + py + q$$

with

$$p = \frac{3ac - b^2}{3a^2}\ ,\ q = \frac{2b^3 - 9abc + 27a^2d}{27a^3}.$$

Proof. Begin by substituting $y - \frac{b}{3a}$ for x. Then we obtain

$$a\Big(y - \frac{b}{3a}\Big)^3 + b\Big(y - \frac{b}{3a}\Big)^2 + c\Big(y - \frac{b}{3a}\Big) + d.$$

Expanding and simplifying, we have

$$ay^3 + \Big(c - \frac{b^2}{3a}\Big)y + \Big(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\Big),$$

and we can divide by a and use common denominators to get the form

$$y^3 + \Big(\frac{3ac - b^2}{3a^2}\Big)y + \Big(\frac{2b^3 - 9abc + 27a^2d}{27a^3}\Big).$$

Substituting $p = \frac{3ac-b^2}{3a^2}$ and $q = \frac{2b^3-9abc+27a^2d}{27a^3}$, we obtain

$$y^3 + py + q.$$

$\square$

Lemma 3. *Let $f(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial. Once we have obtained the depressed form*

$$y^3 + py + q$$

as described in lemma 2, we can write the solutions to $f(x) = 0$ as

Proof. We need to solve $y^3 + py + q = 0$. Set $st = \frac{p}{3}$ and $s^3 - t^3 = -q$. Then we have

$$(s-t)^3 + 3st(s-t) - (s^3 - t^3) = 0.$$

which leads to

$$\begin{aligned}(s^3 - 3s^2t + 3st^2 - t^3) - (s^3 - 3s^2t + 3st^2 - t^3) &= 0\\ 0 &= 0.\end{aligned}$$

So we need to find s and t that satisfy $st = \frac{p}{3}$ and $s^3 - t^3 = -q$. The first equation can be written as $s = \frac{p}{3t}$, and substituting into the second, we obtain

$$\frac{p^3}{27t^3} - t^3 = -q.$$

Multiplying by $-t^3$, we have

$$t^6 - qt^3 - \frac{p^3}{27} = 0.$$

Let $u = t^3$. Then we have

$$u^2 - qu - \frac{p^3}{27} = 0,$$

a quadratic equation in u. From the quadratic formula, we have solutions

$$u = \frac{q}{2} + \sqrt{\frac{(q)^2}{4} + \frac{p^3}{27}}\ ,\ u = \frac{q}{2} - \sqrt{\frac{(q)^2}{4} + \frac{p^3}{27}}$$

which yields

$$t = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{(q)^2}{4} + \frac{p^3}{27}}}\ ,\ t = \sqrt[3]{\frac{q}{2} - \sqrt{\frac{(q)^2}{4} + \frac{p^3}{27}}}.$$

Recall the previously asserted equality $st = \frac{p}{3}$, which implies $s^3 = \frac{p^3}{27t^3}$. Begin by taking the positive root. We have

$$t^3 = \frac{q}{2} + \sqrt{\frac{(q)^2}{4} + \frac{p^3}{27}},$$

which gives

$$s^3 = \frac{p^3}{27\left(\frac{q}{2} + \sqrt{\frac{(q)^2}{4} + \frac{p^3}{27}}\right)}.$$

If we take the negative root, we similarly have

$$s^3 = \left(\frac{q}{2} - \sqrt{\frac{(q)^2}{4} + \frac{p^3}{27}}\right)\frac{p^3}{27}.$$

Let t_1 be a particular solution. Construct a root of unity $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Then the two other solutions are $t_2 = t_1 z$ and $t_3 = t_1 z^2$.

Let $s_1 = \frac{P}{3t_1}$. Since $st = \frac{p}{3}$, the solutions to the depressed form polynomial are $s_1 - t_1$, $s_1 z^2 - t_1 z$, and $s_1 z - t_1 z^2$.

$\square$