DEMORGAN'S LAWS AND HEXAGONAL SYMMETRY

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1. DeMorgan's Laws

Claim 1. *For any two sets A and B, $(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}$.*

Proof. Let $x \in (A \cup B)^{\complement}$. Then $x \notin A \cup B$. It must be the case that $x \notin A$ and $x \notin B$. Thus, $x \in A^{\complement}$ and $x \in B^{\complement}$, so $x \in A^{\complement} \cap B^{\complement}$. This means that $\forall x \in (A \cup B)^{\complement}$, $x \in A^{\complement} \cap B^{\complement}$. By the definition of set inclusion, $(A \cup B)^{\complement} \subset A^{\complement} \cap B^{\complement}$.

Let $y \in A^{\complement} \cap B^{\complement}$. Then $y \in A^{\complement}$ and $y \in B^{\complement}$. So $y \notin A$ and $y \notin B$. Therefore, $y \notin A \cup B$, so it must be that $y \in (A \cup B)^{\complement}$. Similar to above, it follows that $A^{\complement} \cap B^{\complement} \subset (A \cup B)^{\complement}$.

By definition of set equality, $(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}$. $\square$

Claim 2. *For any two sets A and B, $(A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}$.*

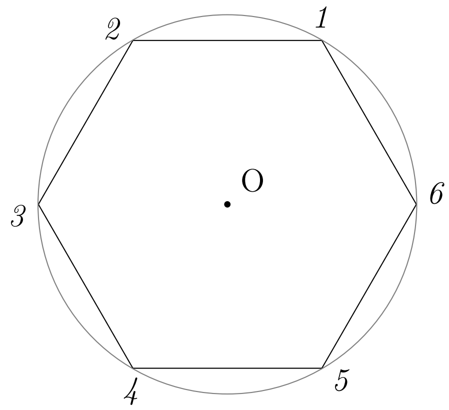
Proof. Let $x \in (A \cap B)^{\complement}$. Then $x \notin A \cap B$. So $x \notin A$ or $x \notin B$. Therefore $x \in A^{\complement}$ or $x \in B^{\complement}$. Thus, $x \in A^{\complement} \cup B^{\complement}$. By the definition of set inclusion, $(A \cap B)^{\complement} \subset A^{\complement} \cup B^{\complement}$.

Let $y \in A^{\complement} \cup B^{\complement}$. It follows that $y \in A^{\complement}$ or $y \in B^{\complement}$. Then $y \notin A$ or $y \notin B$. So $y \notin A \cap B$, therefore $y \in (A \cap B)^{\complement}$. Similar to above, we find that $A^{\complement} \cup B^{\complement} \subset (A \cap B)^{\complement}$.

By definition of set equality, $(A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}$. $\square$

2. The Regular Hexagon

A regular hexagon is a six sided convex polygon such that all interior angles are equal in measure and all sides have the same length. Consider a regular hexagon inscribed in a circle with center O and radius r. Label the vertices of the hexagon *1*, *2*, *3*, *4*, *5*, and *6*.



Claim 3. *The hexagon has side length r.*

Proof. Consider the triangle $O12$. We know this is an isosceles triangle because $\overline{O1}$ and $\overline{O2}$ are both radii of the circle. Since each interior angle of the regular hexagon is 120 degrees, we know $\angle O12 = \angle O21 = 60°$. Therefore, because the sum of the interior angles of a triangle is 180 degrees, $\angle 1O2 = 180 - 2(60) = 60$. Thus, $O12$ is an equilateral triangle, and $\overline{O1} = \overline{O2} = \overline{12}$. But $\overline{12}$ is the side of the hexagon, and $\overline{O1}$ is a radius r. So, the hexagon has side length r. □

The regular hexagon has twelve symmetries. Six are rotational, and the remaining six are reflectional. Let the pre-image be defined as the polygon before it is transformed, and the image is the polygon after it is transformed. With each transformation, a vertex will be mapped to the position of another. Using this notation we can write the reflectional symmetries.

pre-image	image
1	2
2	3
3	4
4	5
5	6
6	1

pre-image	image
1	3
2	4
3	5
4	6
5	1
6	2

pre-image	image
1	4
2	5
3	6
4	1
5	2
6	3

pre-image	image
1	5
2	6
3	1
4	2
5	3
6	4

pre-image	image
1	6
2	1
3	2
4	3
5	4
6	5

pre-image	image
1	1
2	2
3	3
4	4
5	5
6	6

An observation here is that every rotation shown is a multiple of 60 degrees, with the 360 degree rotation being the identity of the transformation.

In a similar way, the reflectional symmetries can be shown.

pre-image	image
1	*6*
2	*5*
3	*4*
4	*3*
5	*2*
6	*1*

pre-image	image
1	*5*
2	*4*
3	*3*
4	*2*
5	*1*
6	*6*

pre-image	image
1	*4*
2	*3*
3	*2*
4	*1*
5	*6*
6	*5*

pre-image	image
1	*3*
2	*2*
3	*1*
4	*6*
5	*5*
6	*4*

pre-image	image
1	*2*
2	*1*
3	*6*
4	*5*
5	*4*
6	*3*

pre-image	image
1	*1*
2	*6*
3	*5*
4	*4*
5	*3*
6	*2*

These symmetries are obtained by first reflecting the hexagon, then rotating it as above.