INDUCTION

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Problem 1. Prove by induction that $7^n - 1$ is divisible by 6 for all $n \geq 1$.

Proof. First, let $n = 1$. Then we have

$$7^1 - 1 = 6.$$

Clearly, 6 is divisible by 6, so the statement holds for $n = 1$.

Next, let $n = k$. Assuming the statement holds, we have

$$\begin{aligned} 7^k - 1 &= 6a \\ 7^k &= 6a + 1 \end{aligned}$$

for some integer a. Letting $n = k + 1$, we can show by algebraic manipulation that

$$\begin{aligned} 7^{k+1} &= 7(6a + 1) \\ 7^{k+1} &= 42a + 7 \\ 7^{k+1} - 1 &= 42a + 6. \end{aligned}$$

We can factor the right side of the equality to obtain

$$6(7a + 1),$$

and then let $b = 7a + 1$. Now we have

$$7^{k+1} - 1 = 6b$$

for some integer b, and therefore $7^{k+1} - 1$ is divisible by 6. $\square$

Problem 2. Show that, given $x_1 = 1$ and

$$x_{k+1} = \frac{x_k}{x_k + 2},$$

we can write

$$x_n = \frac{1}{2^n - 1}$$

for all $n \geq 1$.

Proof. Begin by calculating the first few values of x_n: namely, x_i for $i \in \{2, 3, 4, 5\}$. We can see that

$$\begin{aligned} x_2 &= \frac{1}{4-1} = \frac{1}{3}, \\ x_3 &= \frac{1}{8-1} = \frac{1}{7}, \\ x_4 &= \frac{1}{16-1} = \frac{1}{15}, \\ x_5 &= \frac{1}{32-1} = \frac{1}{31}. \end{aligned}$$

Let $n = 1$. Then we have

$$x_1 = \frac{1}{2-1} = 1$$

which is as given.

Now let $n = k$. Then we can write

$$x_k = \frac{1}{2^k - 1} = \frac{x_{k-1}}{x_{k-1} + 2}.$$

Letting $n = k + 1$, we have

$$x_{k+1} = \frac{x_k}{x_k + 2} = \frac{\frac{1}{2^k-1}}{\frac{1}{2^k-1} + 2}.$$

Writing the denominator as

$$\frac{1 + 2(2^k - 1)}{2^k - 1},$$

we obtain

$$\begin{aligned} x_{k+1} &= \left(\frac{1}{2^k - 1}\right)\left(\frac{2^k - 1}{1 + 2(2^k - 1)}\right) \\ &= \frac{1}{1 + 2(2^k - 1)} \\ &= \frac{1}{1 + 2^{k+1} - 2} \\ &= \frac{1}{2^{k+1} - 1} \end{aligned}$$

So, the statement holds for x_{k+1}.

$\square$