

CONTINUITY OF THE SINE AND THE DISCRIMINANT

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1. CONTINUITY OF SINE

The following lemma will be used to prove Claim 1.

Lemma 1. $|\sin(x)| \leq |x| \forall x \in \mathbb{R}$.

Proof. Suppose instead that $|\sin(x)| > |x| \forall x$. Then $-|x| > \sin(x) > |x|$. Consider three cases as follows.

Case 1: $x = 0$. Then we have $0 > \sin(0) > 0$, or $0 > 0 > 0$. This is impossible.

Case 2: $x > 0$. We thus have $-x > \sin(x) > x$, which gives $-x > x$. This contradicts the assumption that $x > 0$.

Case 3: $x < 0$. Then we have $x > \sin(x) > -x$. It follows that $x > -x$, which fails to hold for $x < 0$ as assumed.

Each case gives an impossible mathematical statement. So, $|\sin(x)| > |x| \forall x$ is incorrect. Therefore, the original proposition is true. \square

Claim 1. *The function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) = \sin(x)$ is well defined and continuous everywhere on its domain.*

Proof. Let $x \in (-\infty, \infty)$. We know that for f to be well defined, we must have $\forall x$, there is exactly one $y \in \mathbb{R}$ such that $f(x) = y$. Let $y = \sin(x)$. Then $f(x) = y$. Let $x_1, x_2 \in (-\infty, \infty)$ such that $f(x_1) \neq f(x_2)$. Then $\sin(x_1) \neq \sin(x_2)$. Taking the inverse sine, we have $x_1 \neq x_2$. Therefore, $\forall x \in (-\infty, \infty)$, there exists a unique y such that $\sin(x) = y$. So, f is a well defined function.

For a function f to be continuous, we know that the limit

$$\lim_{x \rightarrow a} f(x) = f(a)$$

must hold for all a in the domain. Let $a \in (-\infty, \infty)$. We want to show that $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$\delta > |x - a| > 0 \implies |f(x) - f(a)| < \varepsilon.$$

Suppose ε is given. We want to find δ such that

$$|\sin(x) - \sin(a)| < \varepsilon.$$

We can see by trigonometric identity that

$$|\sin(x) - \sin(a)| = \left| 2 \cos\left(\frac{x+a}{2}\right) \sin\left(\frac{x-a}{2}\right) \right|.$$

Since $\cos(\alpha) \leq 1 \forall \alpha \in \mathbb{R}$, we can show that

$$\left| 2 \cos\left(\frac{x+a}{2}\right) \sin\left(\frac{x-a}{2}\right) \right| \leq 2 \left| \sin\left(\frac{x-a}{2}\right) \right|.$$

We want $\delta > |x-a| > 0$, but notice that

$$|x-a| \geq \left| \frac{x-a}{2} \right|,$$

so we can write

$$\delta > \left| \frac{x-a}{2} \right| > 0.$$

Due to lemma 1, we know

$$2 \left| \frac{x-a}{2} \right| \geq 2 \left| \sin\left(\frac{x-a}{2}\right) \right|$$

is true, so we will write

$$2\delta > 2 \left| \frac{x-a}{2} \right| \geq 2 \left| \sin\left(\frac{x-a}{2}\right) \right|.$$

If we take $\delta = \frac{\varepsilon}{2}$, then

$$2 \left| \sin\left(\frac{x-a}{2}\right) \right| < \varepsilon$$

holds true. As δ and ε do not depend on x , this is true over the entire domain \mathbb{R} . Therefore,

$$\lim_{x \rightarrow a} \sin(x) = \sin(a)$$

is true for all $a \in \mathbb{R}$, and we know that f is continuous. □

2. QUADRATIC DISCRIMINANT

A quadratic polynomial $f(x) = ax^2 + bx + c$ can be factored into $a(x - \alpha_1)(x - \alpha_2)$ for some complex numbers α_1 and α_2 . Define the discriminant of $f(x)$ to be

$$\Delta_f := \prod_{i \neq j} (\alpha_i - \alpha_j) = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_1).$$

Claim 2. Δ_f can be written in terms of a , b , and c by the expression

$$\Delta_f = \frac{4ac - b^2}{a^2}.$$

Proof. We know that the roots α_1 and α_2 are given by the general quadratic formula. So let

$$\alpha_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and let

$$\alpha_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Then we can write $(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_1)$ as

$$\left(\frac{(-b + \sqrt{b^2 - 4ac}) - (-b - \sqrt{b^2 - 4ac})}{2a} \right) \left(\frac{(-b - \sqrt{b^2 - 4ac}) - (-b + \sqrt{b^2 - 4ac})}{2a} \right).$$

Combining terms gives us

$$\left(\frac{2\sqrt{b^2 - 4ac}}{2a} \right) \left(\frac{-2\sqrt{b^2 - 4ac}}{2a} \right).$$

Multiplying, we have

$$-\frac{(b^2 - 4ac)}{a^2} = \frac{4ac - b^2}{a^2}.$$

□

3. CUBIC DISCRIMINANT

The following lemma will be used to prove Claim 3.

Lemma 2. *Given any cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$, let $x = y - \frac{b}{3a}$. $f(x)$ can be written as*

$$g(y) = y^3 + py + q$$

with

$$p = \frac{3ac - b^2}{3a^2}, \quad q = \frac{2b^3 - 9abc + 27a^2d}{27a^3}.$$

Proof. Begin by substituting $y - \frac{b}{3a}$ for x . Then we obtain

$$a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d.$$

Expanding and simplifying, we have

$$ay^3 + \left(c - \frac{b^2}{3a}\right)y + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right),$$

and we can divide by a and use common denominators to get the form

$$y^3 + \left(\frac{3ac - b^2}{3a^2}\right)y + \left(\frac{2b^3 - 9abc + 27a^2d}{27a^3}\right).$$

Substituting $p = \frac{3ac - b^2}{3a^2}$ and $q = \frac{2b^3 - 9abc + 27a^2d}{27a^3}$, we obtain

$$y^3 + py + q.$$

□

Claim 3. *Let $f(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial. Once we have obtained the depressed form*

$$g(y) = y^3 + py + q$$

as described in lemma 2, the discriminant is given by

$$\Delta_f = 4p^3 + 27q^2.$$

Proof. We know that for any cubic polynomial $f(x)$, we can write

$$\begin{aligned} f(x) &= ax^3 + bx^2 + cx + d \\ &= a(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \end{aligned}$$

if it has roots α_1 , α_2 , and α_3 . Expanding, we obtain

$$f(x) = ax^3 + ax^2(-\alpha_1 - \alpha_2 - \alpha_3) + ax(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + a(-\alpha_1\alpha_2\alpha_3)$$

which gives

$$\begin{aligned} b &= a(-\alpha_1 - \alpha_2 - \alpha_3) \\ c &= a(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) \\ d &= a(-\alpha_1\alpha_2\alpha_3). \end{aligned}$$

Considering the case for $y^3 + py + q$, we have $a = 1$ and $-\alpha_1 - \alpha_2 - \alpha_3 = 0$. Then let $\alpha_3 = -\alpha_1 - \alpha_2$. We now can expand as above to obtain

$$g(y) = \alpha_1^2\alpha_2 - \alpha_1^2y + \alpha_1\alpha_2^2 - \alpha_1\alpha_2y - \alpha_2^2y + y^3.$$

This gives

$$g(y) = y^3 + y(-\alpha_1^2 + \alpha_1\alpha_2 - \alpha_2^2) + (\alpha_1^2\alpha_2 + \alpha_1\alpha_2^2),$$

from which we may derive

$$\begin{aligned} p &= -\alpha_1^2 - \alpha_1\alpha_2 - \alpha_2^2 \\ q &= \alpha_1^2\alpha_2 + \alpha_1\alpha_2^2. \end{aligned}$$

For the cubic, we have the discriminant

$$\Delta_f := \prod_{i \neq j} (\alpha_i - \alpha_j) = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_1 - \alpha_3)(\alpha_3 - \alpha_1)$$

and since $\alpha_3 = -\alpha_1 - \alpha_2$, we can make the substitution

$$\Delta_f = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_1)((-\alpha_1 - \alpha_2) - \alpha_2)(\alpha_2 - (-\alpha_1 - \alpha_2))(\alpha_1 - (-\alpha_1 - \alpha_2))((-\alpha_1 - \alpha_2) - \alpha_1).$$

Expanding yields

$$\Delta_f = -4\alpha_1^6 - 12\alpha_1^5\alpha_2 + 3\alpha_1^4\alpha_2^2 + 26\alpha_1^3\alpha_2^3 + 3\alpha_1^2\alpha_2^4 - 12\alpha_1\alpha_2^5 - 4\alpha_2^6.$$

Consider the quantity

$$4p^3 + 27q^2.$$

Using the above expressions for p and q , we have

$$4p^3 + 27q^2 = 4(-\alpha_1^2 - \alpha_1\alpha_2 - \alpha_2^2)^3 + 27(\alpha_1^2\alpha_2 + \alpha_1\alpha_2^2)^2.$$

Expanding this expression gives

$$-4\alpha_1^6 - 12\alpha_1^5\alpha_2 + 3\alpha_1^4\alpha_2^2 + 26\alpha_1^3\alpha_2^3 + 3\alpha_1^2\alpha_2^4 - 12\alpha_1\alpha_2^5 - 4\alpha_2^6$$

which is exactly Δ_f as above.

So, given a cubic polynomial in the general form $f(x) = ax^3 + bx^2 + cx + d$, we can find the discriminant by evaluating

$$\Delta_f = 4p^3 + 27q^2$$

where as shown in lemma 2, p and q are given in terms of a , b , c , and d by

$$p = \frac{3ac - b^2}{3a^2} , \quad q = \frac{2b^3 - 9abc + 27a^2d}{27a^3}.$$

□