Non-Markovian voter model in regular networks

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Abstract: Most voter models assume memoryless systems where the agents do not save information about their opinion evolution. Nevertheless, many real-life cases show strong correlation between the agent's behavior and his previous evolution. In this study we perform a qualitative evaluation of the effect of non-Markovian stochastic dynamics of the Voter Model in degree-regular networks.

I. INTRODUCTION

A person's opinion about some aspect of life (politics, economics, technology, etc.) is not fixed: usually it changes at some point. These changes use to be influenced by external points of view, specially those who have more impact in their closest environment.

The voter model was initially developed as a species' competition model in 1973 by Clifford and Sudbury [1], but in 1975 Holley and Liggett described it as a voter dynamics [2], coining the term "voter model". The first approaches of the voter model consider regular topologies (usually regular lattices) as a substrate. The first analytical study in nonregular topologies was in 2003 by Slanina and Lavicka [3].

Most voter models assume that the opinion spreading through the agents is a Markovian stochastic process, which implies that the next change in the system only depends of the actual state. Although these models have shown good results, it has been proven that the introduction of non-Markovian dynamics in models such as SIS implies significant changes in their dynamics [4].

In this essay we do an initial exploration of how the introduction of non-Markovian dynamics affect the behavior of the voter model in degree-regular networks.

II. MARKOVIAN VOTER MODEL

A. Description of the model

Consider a population of N agents, each of which can be in state A or B, according to the opinion they support. Also, these agents have social interactions, allowing to represent all the population as an indirect network, where each node represents an agent and the links between the nodes represent their social interactions. In the direct voter model [6], at each time step a random agent copy the opinion of a random neighbor, changing its state if the opinion is different or remaining equal if the opinion is the same. In this discrete-time model the system always reaches consensus. In this study we use as a Markovian voter model a continuous-time variation of the direct voter model: we consider every link as a two Poisson point processes, the activation of them implies the spread of the opinion in different direction. Due to the properties of Poisson point processes, these processes are independent, the interevent time (time between events) is given by a exponential distribution and it does not depend on previous interevent time. That last property gives us the Markovian behavior.

We consider two types of processes:

1. *Stay* processes, linking two agents with the same opinion. The exponential distribution of these processes has $\lambda = \lambda_s$. In the Markovian model the activation of a *stay* process does not change anything in the system.

2. *Change* processes, linking two agents with different opinion. The exponential distribution of these processes has $\lambda = \lambda_c$. The activation of a *change* process implies the spread of one agent opinion to a neighbor with the other opinion.

We expect that this continuous-time variation has similar properties to the discrete-time voter model, including the tendency to the consensus.

B. Gillespie algorithm

The Gillespie algorithm [7, 8] generates trajectories in systems with multiple stochastic processes running. At each step of the simulation the algorithm gives the time until the next event τ and the chosen process i. The distribution of τ is the distribution of the minimum between the intervention times for all processes, given Poisson point processes is

$$\psi(\tau) = n\hat{\lambda}e^{-n\hat{\lambda}\tau} \tag{1}$$

Where n is the number of processes, $\hat{\lambda} = \frac{1}{n}\sum_{k=1}^{n}\lambda_k$ and λ_k the rate of the process k. Applied to the Markovian voter model, we have $n = 2E$, with E the number of links of the network, and $\hat{\lambda} = \frac{E_s\lambda_s + E_c\lambda_c}{E}$, with E_s and E_c

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the number of *stay* and *change* links, respectively, and λ_s and λ_c the rates of the *stay* and *change* processes. The probability of a process i to be selected is

$$\prod(i) = \frac{\lambda_i}{n\hat{\lambda}}$$

Due to λ_i only depends on whether is a *stay* or *change* process, we can simplify

$$\begin{aligned}\prod(i \in stay) = \frac{2E_s\lambda_s}{n\hat{\lambda}},\\ \prod(i \in change) = \frac{2E_c\lambda_c}{n\hat{\lambda}}\end{aligned} \tag{2}$$

The algorithm works as follows:

1. The system is initialized setting random opinions to the agents, with equal probability to A and B.

2. Get the time interval τ using the probability distribution (1)

3. Get which kind of process is activated using the probability (2), and select at random one process of this type.

4. If a *change* process has been selected, update the system changing the pertinent agent opinion.

5. Repeat steps 2-4 until the system reaches consensus (all the agents with the same opinion)

III. NON-MARKOVIAN VOTER MODEL

A. Description of the model

In this model the system is represented as a network, equal to the previous model, and in addition each agent has two related variables, the opinion loads for A and B (ol_A and ol_B), that represent the exposition of the agent to these opinions. In some points we will map these variables to the stay opinion load ol_s and change opinion load ol_c, corresponding to the opinion load for the agent's opinion and the opposite opinion, respectively.

Instead of processes associated to the links, this model has two processes associated to each agent:

1. *Stay* process: The activation of this kind of process implies setting ol_c to zero and does not change the opinion of the agent.

2. *Change* process: The activation of this kind of process implies the change of the agent's opinion. Also ol_s takes the value of the ol_c and ol_c is set to zero.

Between events, the opinion loads for the node i grow as $dol_s = N_{s,i}(t)dt$ and $dol_c = N_{c,i}(t)dt$, where $N_{s,i}(t)$ and $N_{c,i}(t)$ are the number of neighbors in the agent i's opinion and in the opposite opinion, respectively, at a time t.

The interevent time distribution is given by

$$\psi_{s,i}(t) = N_{s,i}(t)\psi_s^*(ol_{s,i}(t)) \tag{3}$$

$$\psi_{c,i}(t) = N_{c,i}(t)\psi_c^*(ol_{c,i}(t)) \tag{4}$$

These distributions may vary from one agent to other due to the heterogeneity of the society, but in this study we consider an homogeneous population. For the activation processes we select the versatile Weibull distribution

$$\begin{aligned}\psi_s^*(ol_{s,i}) = \alpha_s\mu_s^{\alpha_s} ol_{s,i}^{\alpha_s - 1} e^{-(\mu_s ol_{s,i})^{\alpha_s}}\\ \psi_c^*(ol_{c,i}) = \alpha_c\mu_c^{\alpha_c} ol_{c,i}^{\alpha_c - 1} e^{-(\mu_c ol_{c,i})^{\alpha_c}}\end{aligned} \tag{5}$$

Here we introduce the instantaneous hazard rate of a process, obtained from its interevent time distribution, $\psi(t)$, and the corresponding survival probability, $\mathbf{\Psi}(t) = \int_t^\infty \psi(t')dt'$, as:

$$\omega(t) = \frac{\psi(t)}{\mathbf{\Psi}(t)} = N(t)\frac{\psi^*(ol(t))}{\mathbf{\Psi}^*(ol(t))} \tag{6}$$

The instantaneous hazard rate measures the probability per unit of time that the process is activated at a time t. With (5) and (6) we get:

$$\begin{aligned}\omega_{s,i}(t) = \alpha_s N_{s,i}(t)\mu_s^{\alpha_s} ol_{s,i}(t)^{\alpha_s - 1}\\ \omega_{c,i}(t) = \alpha_c N_{c,i}(t)\mu_c^{\alpha_c} ol_{c,i}(t)^{\alpha_c - 1}\end{aligned}$$

Depending on the values of α_s, α_c, μ_s and μ_c the system has a different behavior. In this work we will focus on the impact of α_s and α_c. The parameter α has the following effect on the Weibull distribution:

- $\alpha < 1$: The Weibull distribution has a power-law-like fat tail, with decreasing hazard rate.

- $\alpha = 1$: The Weibull distribution correspond to a Poisson distribution, that has a constant hazard rate.

- $\alpha > 1$: The Weibull distribution presents a peak and an increasing hazard rate.

For the configuration $\alpha_s = \alpha_c = 1$, all the hazard rates are constant, which implies that the processes do not take into account the opinion loads and the interevent time has a exponential distribution for each process. In this case, we have a Markovian system corresponding to the Markovian voter model described before.

B. Generalized Gillespie algorithm

Tos simulate the non-Markovian voter model, a generalized Gillespie Algorithm is needed, allowing the use of Weibull distributions instead of Poisson Point Processes. In this study we use the algorithm developed by Boguña et al. [5], that is shortly explained in this section.

Consider a set of n stochastic processes, with intervent time distributions $\psi_i(\tau), i = 1, ..., n$ and an elapsed time t_i since the last activation. Then the interevent time distribution conditioned by the elapsed time t_i can be expressed as

$$\psi_i(\tau|t_i) = \frac{\psi_i(t_i+\tau)}{\mathbf{\Psi}_i(t_i)}$$

Where $\mathbf{\Psi}(t_i)$ is the survival probability. Suppose that we know the elapsed time t_i for all processes, and ask the probability that the next event will be the activation of the process i at a time $\tau + d\tau$. Since $\psi_i(\tau|t_i)$ is the probability of the process i is activated at a time τ and $\mathbf{\Psi}_k(\tau|t_k)$ the probability that the process k is not activated before τ, we have

$$\varphi(\tau, i|t_k) = \psi_i(\tau|t_i) \prod_{k\neq i} \mathbf{\Psi}_k(\tau|t_k) = \frac{\psi_i(\tau|t_i)}{\mathbf{\Psi}_i(\tau+t_i)} \mathbf{\Phi}(\tau|t_k)$$

Where

$$\mathbf{\Phi}(\tau|t_k) = \prod_{k=1}^{n} \frac{\mathbf{\Psi}_k(\tau+t_i)}{\mathbf{\Psi}_k(t_i)} \tag{7}$$

is the survival probability of τ, the probability that no process is activated before $\tau + d\tau$. Given a value of τ, the probability that the next process to activate is i is

$$\prod(i|\tau, t_k) = \frac{\varphi(\tau, i|t_k)}{\sum_{j=1}^{n} \varphi(\tau, j|t_k)} = \frac{\omega_i(\tau+t_i)}{\sum_{k=1}^{n} \omega_k(\tau+t_k)} \tag{8}$$

Where $\omega_i(t)$ is the instantaneous hazard rate for the process i. As we work with a large number of processes, we can simplify equation (7) as

$$\mathbf{\Phi}(\tau|t_k) = exp\Big[-\sum_{k=1}^{n} ln\big(\frac{\mathbf{\Psi}_k(\tau+t_i)}{\mathbf{\Psi}_k(t_i)}\big)\Big] \approx e^{-\tau n\hat{\omega}(t_k)} \tag{9}$$

Where $\hat{\omega}(t_k) = n^{-1}\sum_{k=1}^{n}\omega_k(t_k)$.

The algorithm works as follows:

1. The system is initialized setting random opinions to the agents, with equal probability to opinions A and B. The opinion loads are initialized too assuming the system is in the steady state, which implies that the distribution of the initial opinion loads is defined by $\frac{\mathbf{\Psi}_i(ol)}{<ol_i>}$, where $\langle ol_i \rangle$ is the expected value of ol_i.
2. Get the time interval τ using the survival distribution (9)
3. Get the selected process from the probability distribution (8)
4. Update all the opinion loads and apply the selected process as described in section III A.
5. Repeat steps 2-4 until the system reaches consensus (all the agents with the same opinion).

IV. RESULTS

In this section we will show simulations of Markovian and non-Markovian voter model, all in a degree-regular network of 1000 nodes and degree 4.

As we can see in the Fig.(1), the Markovian voter model has a tendency to reaches consensus, as we expected in the section II A. Looking at the evolution of the proportion of agents with opinion A and the number of active links (that can be interpreted as the contact area between opinions) we can suppose that the only equilibrium point is the consensus.

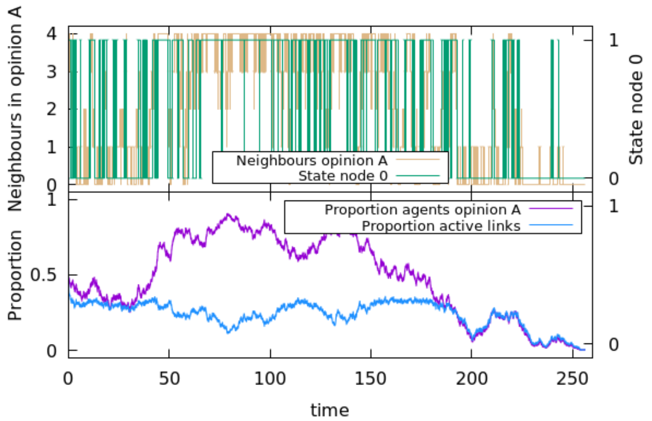
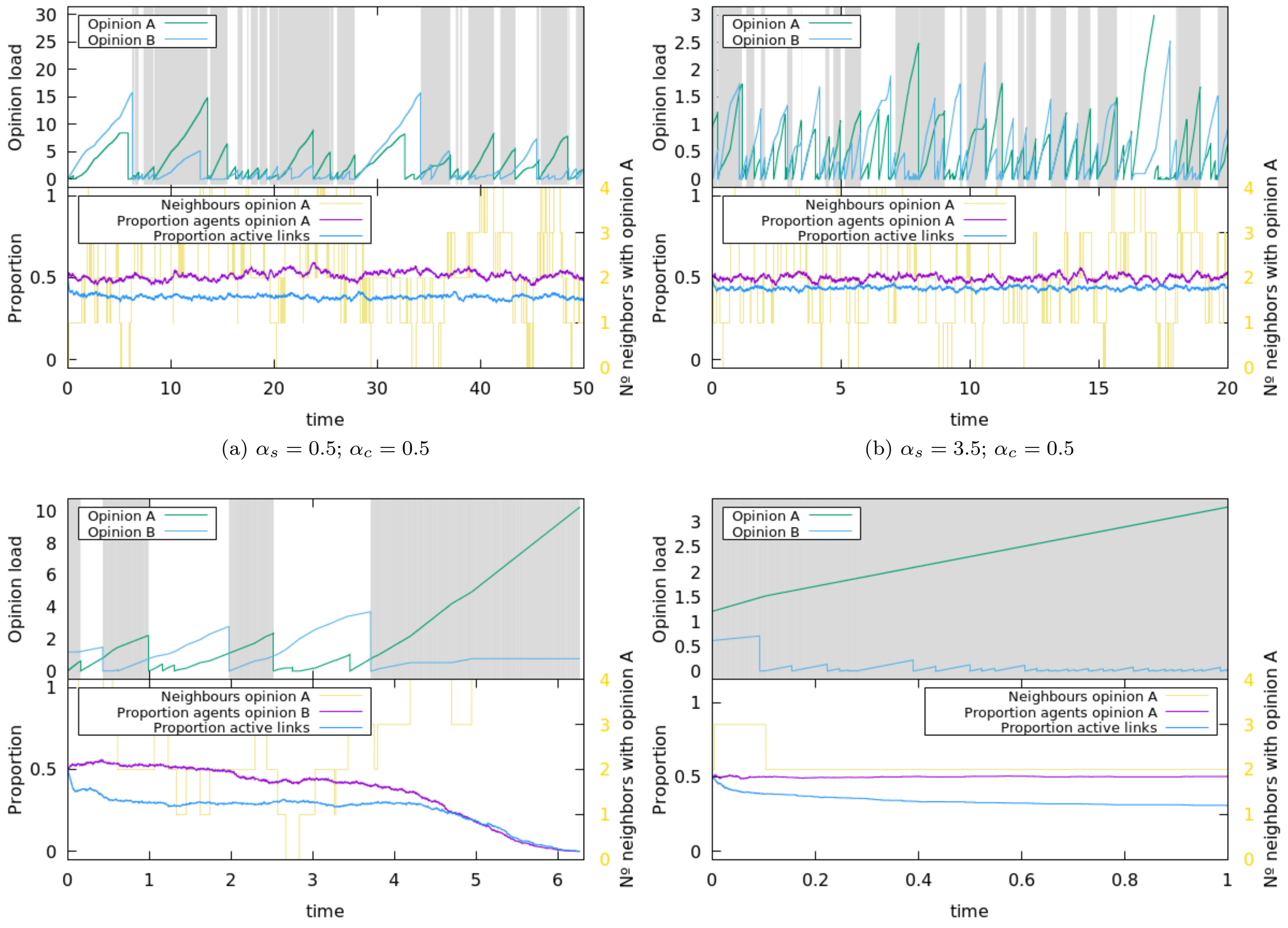


FIG. 1: Simulation of Markovian voter model. The top section shows the evolution of one node, meaning the state 1 is in opinion A and the state 0 is in opinion B, and the number of neighbors with opinion A. The bottom section shows the evolution of the proportion of active links and the proportion of agents with opinion A

The non-Markovian voter model shows a radically different behavior depending on the values of α_s and α_s, as we can see in Fig.(2).

- For $\alpha_s = 0.5, \alpha_c = 0.5$ the hazard rates $\omega_s(ol_s)$ and $\omega_c(ol_c)$ decrease with the related opinion load, which implies that when an opinion load is set to zero (due to the activation of a process) the hazard rate reach its maximum value, increasing the probability of the activation of the related process soon after. Due to this behavior, when a change process



(a) $\alpha_s = 0.5$; $\alpha_c = 0.5$

(b) $\alpha_s = 3.5$; $\alpha_c = 0.5$

(c) $\alpha_s = 0.5$; $\alpha_c = 3.5$

(d) $\alpha_s = 3.5$; $\alpha_c = 3.5$

FIG. 2: Simulations of non-Markovian voter model. The upper half of the figures shows the evolution of the opinion loads one agent and its opinion (gray regions denote when the agent has opinion A and white regions in opinion B). The lower part of the figures shows the number of neighbors of the previous node with opinion A and the proportion of agents with opinion A or B and the proportion of active links

is activated (implying that ol_c is set to zero, as is explained in section III A) is highly probable that the agent changes his opinion again. This implies that the evolution of the agent has periods with very fast changes between opinions, as we can see around time 7 in Fig.(2).a. Also the activation of a stay process sets ol_c to zero, increasing the change hazard rate. When both ol_s and ol_c are large, the hazard rates of both processes are small, causing long times of stability, for example around time 30 in Fig.(2).a. This microscopic behavior causes a macroscopic dynamic stability for the global opinion and the contact area (proportion active links) impeding that the system reaches consensus.

- For $\alpha_s = 3.5, \alpha_c = 0.5$ the hazard rates $\omega_s(ol_s)$ and $\omega_c(ol_c)$ increases and decreases, respectively, with the related opinion load. The change hazard rate behavior causes that the evolution of the agent has periods with very fast changes between opinions as in the configuration $\alpha_s = 0.5, \alpha_c = 0.5$. We can see one of these periods around time 4 in Fig.(2).b. When ol_c grows enough the hazard rate is small, causing that the activation of the change process is small. At the same time ol_s grows, increasing its hazard rate until the stay process is activated, which implies that ol_c is set to zero, increasing radically its probability of activation. This causes long periods of inactivity that end with the activation of a stay processes and a change process soon after, as we can see in time 1 in Fig.(2).b. The global behavior is a dynamic stability for the global opinion and the contact area similar to the configuration $\alpha_s = 0.5, \alpha_c = 0.5$, which also impedes that the system reaches consensus.

- For $\alpha_s = 0.5, \alpha_c = 3.5$ the hazard rates $\omega_s(ol_s)$ and $\omega_c(ol_c)$ decreases and increases, respectively, with the related opinion load. When the two opinion loads grow (heterogeneous neighborhood) the increase of ol_s causes the decreases of the activation probability for the stay process and the increase of ol_c causes the increase of the activation probability of the change process. This causes that the dynamic has periods where the agent accumulates opinion loads with some stay activations in the beginning until the change process is activated. One of these periods its shown between time 1 and 2 in Fig.(2).c. the global system has a clear tendency to the consensus.

- For $\alpha_s = 3.5, \alpha_c = 3.5$ the hazard rates $\omega_s(ol_s)$ and $\omega_c(ol_c)$ increase with the related opinion load. This causes that when a stay process is activated (setting ol_c to zero) the change hazard rate has to grow from zero again. Meanwhile the stay hazard rate is always growing, decreasing the time between activations. Asymptotically the agent reach a situation where the change hazard rate is always zero. The consequences of this local behavior at a global level are the freezing of the system after some time where the system is rearranged, as we can see in Fig.(2).d.

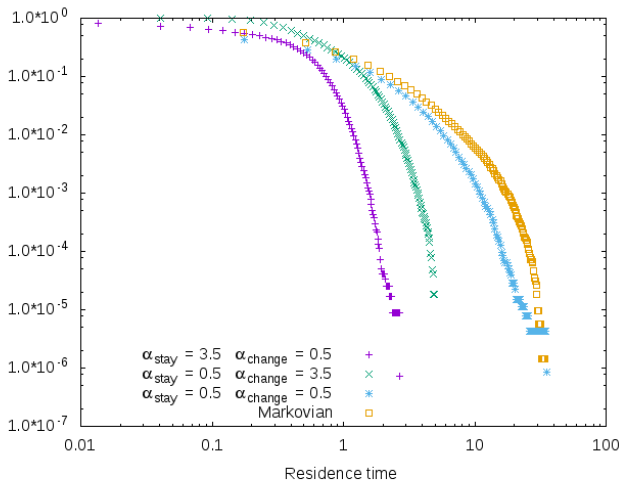


FIG. 3: Survival distribution of the residence time for different configurations

We define the residence time as the time between opinion changes in an agent. In Fig.(3) we can see the survival distribution of the residence time for the studied configurations excluding $\alpha_s = \alpha_c = 3.5$ due to the freezing of this systems, which impedes the definition of a residence time. All the non-Markovian configurations have smaller expected residence time values than the Markovian configuration.

It is important to consider that the residence time is not homogeneous during the simulation for configurations that reaches consensus: in the final convergence the agents will have longer residence time. This behavior is probably the cause of the sharp decay at larger times in the survival residence time distribution in the Markovian configuration and $\alpha_s = 0.5, \alpha_c = 3.5$ (the configurations that reaches consensus) in Fig.(3). Also it is important to consider the difference between timescale between the scenarios that reach the consensus: the consensus time for the Markovian configuration is two orders of magnitude bigger than the configuration $\alpha_s = 0.5, \alpha_c = 3.5$.

V. CONCLUSIONS

In this work we have introduced a new version of the voter model which endows the agent with memory. Preliminary results shows radically different behavior beside the Markovian voter model, in microscopic properties and macroscopic evolution, including equilibrium points outside the consensus and frozen systems, situations that are impossible in the Markovian voter model.

Nonetheless, additional analysis of the non-Markovian voter model dynamics is required. For example, the combination between Markovian and non-Markovian dynamics in the stay and change processes and the mathematical analysis of the effective potential that affect the evolution of the global state. Also another network topologies and heterogeneous populations might further improve our understanding of opinion spreading.

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