Conjectures on Optimal Auctions in Multidimensional Settings

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Abstract

We explore the properties of optimal multi-dimensional auctions in a setting where a single object of multiple qualities is sold to several buyers. Using simulations, we test the hypothesis that the optimal mechanism is an *exclusive buyer mechanism*, where buyers compete to be the right to be the only buyer to choose between quality levels of a good. We find compelling evidence of the optimality of the exclusive buyer mechanism in multi-dimensional settings and explore a number of other conjectures. As part of this work, we provide the first open-source library for multidimensional auction simulations written entirely in Python.

Contents

1 Introduction

TODO structure: (1) multi-item, (2) single-item, (3) everything else

Since the seminal work of Roger Myerson (1981), revenue-maximizing auctions in the case of a single dimension (e.g., good) are well known. However, little is known about optimal auctions in settings with multiple dimensions of value. The problem is incredibly complex: despite sustained research efforts for decades, it is only in the past few years that it is known how to sell two goods to a single buyer (Daskalakis, Deckelbaum, and Tzamos 2017). In this chapter, I focus on the multidimensional setting of a single good with multiple quality levels and explore whether a specific mechanism—the *exclusive buyer mechanism* (EBM)—is approximately optimal. The EBM can be understood as an (either a second price or ascending bid) auction with a reserve price for the exclusive right to be the buyer of one of the quality levels of the good. Building off existing work at the intersection of economics and computer science, I adopt a novel approach involving extensive simulations to test the optimality of the EBM. Across a broad range of settings and simulations, I find evidence for the optimality of the EBM. From this, I conclude with a conjecture designed to motivate future theoretical work in this challenging domain: the exclusive buyer mechanism is optimal in the setting where one good with multiple quality levels is sold to an arbitrary number of buyers.

The setting of a single good with multiple quality levels can be understood as corresponding to the case of purchasing a single airplane ticket where the 'quality level' might be: economy class, business class, or first class. A buyer might prefer, say, business over economy class. Each quality level can be (and very often is) given a different price and might have different value to the buyer. Crucially, unlike the multidimensional setting where multiple goods are sold by a seller to an arbitrary number of buyers, here since only one good is sold no 'bundling'—selling subsets of all offered goods for a discount—cannot occur. The absence of bundling renders this a much simpler setting and an ideal point of departure for investigating the qualitative features of optimal multidimensional auctions.

The approach adopted in this chapter is to eschew analytic results in settings where few have been forthcoming in favor of exploring these complex, analytically intractable settings using simulations to approximate optimal mechanisms. The key idea is simple. Computer scientists and economists have made significant strides developing approximation algorithms that can arbitrarily well approximate optimal revenue using linear programming techniques (e.g., Cai, Daskalakis, and Weinberg 2012; Belloni, Lopomo, and Wang 2010). Taking these algorithmic developments as a point of departure, it is possible to explore a wide class of settings where a single good with multiple quality levels is sold to better understand important qualitative features of the optimal mechanism. Thus, once an approximation algorithm is run to uncover the optimal mechanism, a qualitative analysis of the key features of the optimal mechanism then proceeds.

Important questions explored in this thesis concern: Do these approximation algorithms yield deterministic optimal mechanisms or is randomization always required for revenue maximization? Is a positive measure of buyers always excluded from the allocation in equilibrium? Are there simple, intuitive mechanisms that might characterize the results from the approximation algorithms? These and related questions are explored in this chapter.

The exclusive buyer mechanism has previously been explored in multidimensional settings (e.g., Brusco, Lopomo, and Marx 2011; Belloni, Lopomo, and Wang 2010) but this thesis chapter represents the first sustained exploration of its optimality in a wide range of settings. This chapter first considers cases where significant prior research (of either analytic or computational character) exists. These settings are that of (Pavlov 2011), who conducted found the optimal mechanism for substitute goods in the case of a single buyer with symmetric, uniform valuations, and (Belloni, Lopomo, and Wang 2010), who developed the original approximation algorithm for multidimensional settings when a single good with multiple quality levels is sold to an arbitrary number of buyers. Both of these settings consider cases where buyer's valuations across quality grades are independent and uniformly distributed. We extend the range of cases to those involving other types of distributions as well as arbitrary correlations across quality grades. Where computationally feasible, we consider settings where the number of buyers is greater than two[1].

The structure of this chapter proceeds as follows. In section (2), we review the existing literature on multidimensional mechanism design, approximation algorithms, and specific works in the setting of a single good with multiple quality levels. The problem is formally introduced in section (3), and a description of the EBM can be found in section (3.2). Section (4) describes the approach of the chapter and explores several hypotheses concerning optimal multidimensional mechanisms. Finally, I explore and discuss these results in section (5) and conclude in section (6), highlighting the implications of these results for theoretical microeconomics. Appendices with a complete description of the approximation algorithm developed for this thesis and specific calculations are to be found in sections (8) and (9), respectively.

2 Literature Review

The specific case considered here of selling a single good with multiple quality levels to an arbitrary number of buyers is a special case of the more general multidimensional mechanism design problem of selling an arbitrary number of goods to an arbitrary number of buyers. Since the groundbreaking work of Roger Myerson (1981) who solved the optimal auction design problem in the case of single-dimensional types, economists have sought to characterize optimal auctions in the more general multidimensional setting, with limited success.

[1] TODO paragraph(s) on results after this

At this juncture, it is widely accepted that "[e]ssentially, nothing has been known about optimal auctions in this [multidimensional] setting" (Kolesnikov et al. 2022, p1). This literature review covers historical and recent developments by economists and computer scientists who have sought to uncover characteristics of optimal mechanisms in multidimensional settings, with a particular focus on the case of a single good with multiple quality levels.

In the case of single-dimensional types, early work on optimal mechanism design demonstrated the optimality of deterministic mechanisms (i.e., reserve or 'take-it-or-leave-it' prices) (Myerson 1981; Riley and Zeckhauser 1983). These approaches leveraged the approach of integration-by-parts (as used in Mussa and Rosen 1978) to solve the relaxed optimal mechanism design problem without directly considering incentive-compatibility constraints. These early results cannot be generalized to multi-dimensional settings because the integral solution to the optimization problem is path-dependent; any two points in the multi-dimensional typespace can be connected by a continuum of paths. Thus, a major breakthrough in multidimensional auction design came with the use of duality-based approaches to multidimensional screening developed by (Rochet and Choné 1998) which circumvents this problem.

The duality-based approach (Rochet and Choné 1998) builds on the single-dimensional nonlinear pricing framework of (Mussa and Rosen 1978), which was given its canonical formulation in multidimensional settings in (Wilson 1993) and (Armstrong 1996). This work takes as its point of departure the approach of (Mirrlees 1971) on optimal taxation and relies on results that establish the implementability of a decision rule in multidimensional settings (Rochet 1987). In this multidimensional screening problem, a few key findings emerge. The first is that 'bunching'—a situation where multiple types are treated identically in the optimal solution—is a "robust" feature of multidimensional screening (Rochet and Stole 2003; Rochet and Choné 1998). There are two types of bunching: in the first case, a set of types of positive measure are excluded from purchasing the goods in the optimal solution (this is commonly known as the 'exclusion region'); in the second case, a non-negligible set of types outside the exclusion region receive the same product although they have different tastes. In addition, the work of (Rochet and Choné 1998) illustrates that the optimal solution to multidimensional screening problems may involve "bundling" the goods, which involves selling multiple goods together.

In multi-item settings, authors have long sought to characterize when bundling multiple goods in a single contract is optimal for the seller. Bundling strategies available to a seller include 'pure' bundling, where only the bundle of all goods is offered to sellers, and 'mixed' bundling, where each different bundle of items is priced separately. Early results showed that offering mixed bundles strictly dominates offering pure bundles to the buyers (Adams and Yellen 1976; McAfee, McMillan, and Whinston 1989) and more recent results have demonstrated that randomized bundles may dominate mixed bundles (Thanassoulis 2004; Daskalakis, Deckelbaum, and Tzamos 2017). In these settings, the optimal menu of contracts may include infinitely many randomized bundles (Manelli and Vincent 2007; Hart and Nisan 2019). Additionally, recent work has demonstrated settings where simply offering only the grand bundle of all goods is optimal (Haghpanah and Hartline 2021).

In the past few years, major breakthroughs in optimal multidimensional mechanism design have come from the use of the methods of optimal transport applied to the optimization problems of microeconomic theory (see Ekeland 2010). These results (Daskalakis, Deckelbaum, and Tzamos 2017; Kolesnikov et al. 2022) greatly aid the *certification* of optimality: the techniques of optimal transport facilitate the identification of the dual of the seller's optimization problem from which a given mechanism's optimality can be verified. Thus, previously existing results that characterize optimal mechanisms in specific settings (for example, where valuations for two goods are i.i.d on $U[0,1]^2$ (Pavlov 2011; Manelli and Vincent 2006)) can be shown to be optimal using a novel, more general approach. The success of the tools of optimal transport in mechanism design is due to the success of a 'guess-and-verify' approach where one guesses a solution to the primal problem and then the dual solution plays the role of a certificate of optimality for the initial guess.

These breakthroughs which facilitate the certification of optimality are particularly helpful when viewed in light of the growth in work at the intersection of economics and computer science[2]. One line of work (Chawla, Hartline, and Kleinberg 2007; Cai, Daskalakis, and Weinberg 2012; Cai, Devanur, and Weinberg 2016; Belloni, Lopomo, and Wang 2010; Alaei et al. 2019) provides an algorithmic approximation of optimal mechanisms in multidimensional settings. Here, the buyer's typespace is discretized, and linear programming techniques are used to approximate the optimal solution, often using simple mechanisms like posted prices. Work in this area aims to achieve a constant factor of the optimal revenue achievable by a Bayesian incentive-compatible mechanism through an approximation. Other work at the intersection of computer science and economics offers insights into the nature of the optimal mechanisms in multidimensional settings. These works show that in specific settings, optimal mechanisms contain only a few contract points (Wang and Tang 2014) or that menus with only a finite number of items cannot ensure any positive fraction of optimal revenue (Hart and Nisan 2019).

Returning to the specific context multidimensional mechanism design context of a single good with multiple quality levels, the work of (Belloni, Lopomo, and Wang 2010) provides insight into the character of the optimal mechanism in this particular setting. Applying their algorithm to concrete cases, they find a number of surprising results from their simulations. First, there is clear evidence that in the optimal solution, a measure-zero set of buyers is excluded from the allocation in equilibrium. This stands in marked contrast to results in the multi-item case which show that the optimal solution requires exclusion (Rochet and

[2] TODO vincent work after this paragraph.

Choné 1998; Armstrong 1996). Second, their results indicate the optimality an *exclusive-buyer mechanism*: it performs "quite well" relative to the numerical optimal solutions and that it "shares many of its defining features with its one-dimensional counterparts" (Belloni, Lopomo, and Wang 2010, p1085-6), including being implementable in dominant strategies. This mechanism involves an auction (with a reserve price) among buyers for who gets to be the sole recipient of the good. A premium can then be paid for whichever quality grade the winning buyer desires. Interestingly, this mechanism is entirely deterministic in the single-bidder case. This is particularly surprising because, in the neighboring multi-item case, randomized allocations are widely considered necessary for revenue maximization (Daskalakis 2015).

The theoretical study of exclusive-buyer mechanisms originates from the phenomenon of 'contingent re-auctions' where sellers will modify objects sold to benefit themselves or the general public (Brusco, Lopomo, and Marx 2011). For example, in the context of the US Spectrum License Auction 73 held in 2008[3], the US government adopted a contingent re-auction format where it offered restricted spectrum licenses first, and committed to re-auction the licenses without many of the restrictions in the case the reserve prices were not met. Brusco, Lopomo, and Marx (2011) show that an exclusive-buyer mechanism can always be parameterized such that the mechanism induces the efficient outcome in dominant strategies. However, outside of a restrictive context where all bidders' valuations for the restricted object are a fixed percentage of the unrestricted object, no general results concerning the optimality of the mechanism are provided.

Analytic results concerning optimal multidimensional auctions for a single good with multiple quality levels and a single bidder are scarce. Notably, (Pavlov 2011) investigates the case where the bidder's valuations for the object are uniformly distributed on the unit square $[c, c+1]^2$. Pavlov finds that the optimal mechanism varies considerably with c and sometimes requires randomization for revenue maximization. This approach was further generalized in the work of (Thirumulanathan, Sundaresan, and Narahari 2019a) who study an almost identical case where a bidder's valuations are distributed uniformly on the rectangle $[c, c+b_1] \times [c, c+b_2]$. Similarly to (Pavlov 2011), the solution to the optimal mechanism design problem entails both deterministic and stochastic contracts. Surprisingly, however, (Thirumulanathan, Sundaresan, and Narahari 2019a) find evidence of settings where optimal mechanisms do not exclude a position measure of buyers. Additionally, a working paper by (Haghpanah and Hartline 2014) gives sufficient conditions for the optimality of posting a single, uniform price for all quality levels of a good, albeit in a restricted class of settings.

Analytic results for optimally selling substitute goods have also been given in the Hotelling model (Hotelling 1929) where two horizontally differentiated goods are located at the endpoints of a segment. In this setting, (Balestrieri, Izmalkov, and Leao 2020) find that stochastic contracts are part of the optimal

[3]For more details see (Brusco, Lopomo, and Marx 2009).

mechanism. The economic intuition that arises from this body of research is clear: by offering a lottery over which good the bidder receives, a seller can offer a discount to entice marginal buyers who would otherwise choose the outside option. Similarly, (Loertscher and Muir 2023), find that in this setting randomization is required by the seller to maximize revenue. These results support earlier work (Thanassoulis 2004) which shows that in the standard auction design problem for two substitute goods, the seller can always increase revenue by including stochastic contracts alongside take-it-or-leave-it prices in the optimal mechanism. As noted above, these result supports the view that in multidimensional settings randomization is required to maximize revenue (see Daskalakis 2015).

The approaches of (Pavlov 2011; Thirumulanathan, Sundaresan, and Narahari 2019a) to solving the mechanism design problem for the case of a single good with multiple quality levels follows the work of (Guesnerie and Laffont 1984), where optimal control theory is used to address the fact the measure of participating types endogenously depends on the mechanism. The optimal control theory approach[4] has also been successfully applied to single-dimensional settings when the participation constraints are endogenously determined by the mechanism (Jullien 2000). Here, the bidder's reservation utility depends on their type. This approach generalizes to accommodate the fact that the measure of participating types in a given mechanism is endogenously determined (for example, in the multidimensional case of a single good with two quality levels and a single buyer considered by Pavlov 2011; Thirumulanathan, Sundaresan, and Narahari 2019a).

Finally, although it has long been believed that it is always profitable for the seller to exclude some measure of bidders (Rochet and Choné 1998; Armstrong 1996) in multidimensional settings, recent theoretical and computational work auction design in these settings has challenged these conclusions. The original result of (Armstrong 1996) demonstrated that in multi-product settings with a single bidder, the seller benefits from always excluding a positive measure of bidder types. By relaxing Armstrong's strong assumptions about the bidder's utility function and the convexity of the type space this result has been extended and it has been shown that "exclusion is generically optimal in a large class of models" (Barelli et al. 2014, p. 75). The intuition is as follows: in a multidimensional screening problem of dimension m, when the seller raises the price by $\epsilon > 0$ then they earn extra profits of order $O(\epsilon)$ from the remaining bidder types but the measure types excluded from the mechanism is of order $O(\epsilon^m)$. However, simulation results from (Belloni, Lopomo, and Wang 2010) suggest that in certain asymmetric settings, this intuition fails and it is optimal for the seller not to exclude any bidder types. This finding is corroborated by the theoretical work of (Thirumulanathan, Sundaresan, and Narahari 2019a) where the optimal mechanism for a single bidder

[4] See (Basov 2005, §7) for an extended discussion of the different approaches to multidimensional mechanism design and their respective strengths and weaknesses.

with valuations distributed uniformly on a rectangle will include settings without exclusion.

3 Model & Setup

In this section, we introduce the optimal multidimensional auction design problem for a single with multiple quality levels in addition to the specific *exclusive buyer mechanism*. There is one seller wishing to sell one item with $j = 1, \ldots, K$ quality levels to $i = 1, \ldots, N$ bidders. Bidder i's valuation (their type) of quality level j is denoted $X_j^i = [\underline{x}_j^i, \overline{x}_j^i] \subset \mathbb{R}_+$. Each bidder's vector of valuations is given by $X^i = \prod_j X_j^i$ and I will denote by $X = \prod_i X^i$. I will denote by X^{-i} the types of all bidders except for i. Bidder i's type is private information and is known only to themselves.

Bidder i's valuation for quality level j is distributed according to the cumulative density function F_j^i. The joint density of all bidders' valuations of all quality grades is denoted F, and, again, denote by F^{-i} the distribution of types of all bidders except bidder i. The joint density is known to the seller. It is assumed that F is continuously differentiable. Furthermore, as is common in the setting of (Myerson 1981), it is assumed that the distributions of bidders' valuations are independent. However, a bidder's valuations across quality grades may be correlated.

A crucial step to solving the optimal auction design problem was the use of the *revelation principle* which simplifies the search space for optimal mechanisms (see Myerson 1981, Lemma 1). The revelation principle allows the auction designer to restrict their attention to a class of mechanisms called *direct mechanisms*. Direct mechanisms are those where the bidders simultaneously and confidentially reveal their types to the seller and the seller decides who gets the object and how much each bidder must pay, as a function of their types.

Thus, a direct mechanism is described by a pair of functions (q, p). The *allocation function* $q : X \to [0, 1]^{KN}$ specifies the probability $q_j^i(x)$ for some $x \in X$ that bidder i receives the good with quality level j. Note that in deterministic mechanisms $q_j^i(x) \in \{0, 1\}$. The *price function* $p : X \to \mathbb{R}^N$ specifies the amount each bidder pays (bidders might be required to pay even if they do not receive the good, as occurs in an 'all-pay' auction).

The utility functions of the seller and bidders are risk-neutral and additively separable. The bidders' utilities are given by

$$u^i(x) = \sum_j x_j^i q_j^i(x) - p^i(x) \tag{1}$$

for all $x \in X$. Denote bidder i's expected utility as

$$U^i(x^i) = \int_{X^{-i}} u^i(x^i, x^{-i}) dF^{-i}(x^{-i}) \tag{2}$$

for all $x^i \in X^i$. I assume for simplicity of presentation that costs are zero[5]. The seller's utility function is given by

$$u^0(x) = r\big(1 - \sum_i \sum_j q^i_j(x)\big) + \sum_i p^i(x) \tag{3}$$

where r is the seller's value estimate for the object, which is most commonly interpreted as the reserve price. Thus, the seller's expected utility is given by

$$\int_X u^0(x) dF(x) \tag{4}$$

However, not every pair of functions (q, p) represents a *feasible* auction mechanism. There are three types of constraints[6] that must be imposed on (q, p).

First, since there is only one object to be allocated, the allocation function must satisfy the following feasibility conditions (F):

$$\sum_i \sum_j q^i_j(x) \leq 1 \text{ and } q^i_j(x) \geq 0 \tag{F}$$

for all $i = 1, \ldots, N$, $j = 1, \ldots, K$ and $x \in X$. Note that, in contrast to the multidimensional setting of a single good with multiple quality levels, in the multi-item case where the seller has K goods to sell the probability conditions are given by:

$$\sum_i q^i_j(x) \leq 1 \text{ and } q^i_j(x) \geq 0 \tag{5}$$

for all $i = 1, \ldots, N$, $j = 1, \ldots, K$ and $x \in X$.

Second, the mechanism (p, q) must be *individually rational* (IR) in the sense that every bidder has non-negative expected utility from participating in the mechanism. More formally,

$$U^i(x^i) \geq 0 \tag{IR}$$

for all bidders $i = 1, \ldots, N$ and all $x^i \in X^i$.

Third, the revelation mechanism can only be implemented if no bidder can expect to gain from lying

[5] TODO add costs.

[6] Here, we outline (IR) and (IC) constraints when the solution concept is a Bayesian Nash equilibrium. TODO BIC-DIC (cite Alexey's paper + give definitions)

about their type. If bidder i misrepresents their true type x^i with the lie $\widehat{x}^i$ their expected utility would be

$$\int_{X^{-i}} \sum_j x^i_j q^i_j(\widehat{x}^i, x^{-i}) - p^i(\widehat{x}^i, x^{-i}) dF^{-i}(x^{-i}) \tag{6}$$

Thus, in a direct mechanism it is necessary to ensure

$$U^i(x^i) \geq \int_{X^{-i}} \sum_j x^i_j q^i_j(\widehat{x}^i, x^{-i}) - p^i(\widehat{x}^i, x^{-i}) dF^{-i}(x^{-i}) \tag{BIC}$$

for all $i = 1, \ldots, N$ and $x^i, \widehat{x}^i \in X^i$. This final condition is known as *Bayesian incentive compatibility* (BIC).

The revenue maximization problem faced by the seller is therefore

$$\begin{aligned} &\max_{p,q} \int_X \Big(r\big(1 - \sum_i \sum_j q^i_j(x)\big) + \sum_i p^i(x) \Big) dF(x) \\ &\qquad \text{subject to } (F), (IR), (BIC) \end{aligned} \tag{7}$$

Notice when the reserve price[7] r is 0 the problem simplifies to

$$\begin{aligned} &\max_{p,q} \int_X \sum_i p^i(x) dF(x) \\ &\qquad \text{subject to } (F), (IR), (BIC) \end{aligned} \tag{8}$$

which I will reference as the canonical formulation of the problem for convenience moving forward.

3.1 Convex Optimization Formulation

TODO (Daskalakis 2015) for $N > 1$; see also (Belloni, Lopomo, and Wang 2010) for notation. Use iterim formulation.

$$\begin{aligned} &\sup_u \int_X (\nabla u(x) \cdot x - u(x)) dF(x) \\ &\quad \text{s.t. } |u(x) - u(y)| \leq |x - y|_1, \forall x, y \in X \\ &\qquad u \text{ non-decreasing} \\ &\qquad u \text{ convex} \\ &\qquad u(x) \geq 0, \forall x \in X \end{aligned} \tag{9}$$

[7] TODO $r_1, r_2, \ldots$

3.2 Exclusive Buyer Mechanism

The exclusive buyer mechanism (EBM) was introduced in (Brusco, Lopomo, and Marx 2011) for the case of two quality levels. This can be understood as an auction where the buyers compete in a second price or ascending-bid auction (with reserve prices) for the right to be the only buyer and choose which quality grade to purchase.

More formally[8], in the case where there are two quality levels, for player i's bid x and any given *reserve price* $r = (r_1, r_2)$[9], let

$$\beta_1^i = x_1^i - r_1 \quad \text{and} \quad \beta_2^i = x_2^i - r_2 \tag{10}$$

Then, for bidder i denote by $h(x^i; r) = \arg\max_j \beta_j^i$ the largest difference between their bid x^i and the reserve price r. Then, let $H(x; r)$ be the largest of these differences and $M(x; r)$ be the set of winners (i.e., those with a bid $x_j^i = H(x)$):

$$H(x; r) = \max_i h(x^i; r) \tag{11}$$

$$M(x; r) = \{i | H(x; r) = \beta^i_{h(x^i)} \geq 0\} \tag{12}$$

and the allocation q is

$$q(x; r) = \begin{cases} \frac{1}{|M(x)|} & \beta_j^i = H(x) \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

Then for the winning bidder $i \in M(x; r)$, their payment is given by the function

$$p^i(x; r) = r_{h(x^i)} + \max\{H(x^{-i}), 0\} \tag{14}$$

Lastly, note the expected revenue for a given reserve price r is

$$R(r) = \int_X p(x; r) dF(x) \tag{15}$$

where p is defined above. Additionally, this can be numerical approximated as

$$R(r) \approx \sum_{x \in X_T} p(x; r) f(x) \tag{16}$$

This expression is used below for the calculations of the expected revenue from the EBM.

[8] TODO go through this notation with alexey...

[9] TODO what about $N > 2$?

We can alternatively define the interim allocations[10] as follows. When the distribution of bidder valuations is symmetric, we can omit the superscript and write the interim allocations as:

$$Q_1(x;r) = \mathbb{1}\{\beta_1 > \beta_2 \text{ and } \beta_1 \geq 0\}F^{N-1}(x_1, \min\{\overline{x}_2, r_2 + \beta_1\}) \tag{17}$$

$$Q_2(x;r) = \mathbb{1}\{\beta_2 > \beta_1 \text{ and } \beta_2 \geq 0\}F^{N-1}(\min\{\overline{x}_1, r_1 + \beta_2\}, x_2) \tag{18}$$

where, recall, $F^{N-1}(x_1, x_2) = F(x_1, x_2)^{N-1}$ since bidder's valuations are independent and identically distributed. Furthermore, the interim expected utility for each bidder can be calculated as:

$$U(x;r) = \max\left\{\int_{\underline{x}_1}^{x_1} Q_1(t, x_2; r)dt, \int_{\underline{x}_2}^{x_2} Q_2(x_1, t; r)dt\right\} \tag{19}$$

4 Conjectures and Simulations

In this section, I explore whether the exclusive buyer mechanism (EBM) approximates the optimal revenue in settings where analytic results are known or those where they can approximated arbitrarily well algorithmically. The goal is to uncover the qualitative features of the optimal mechanism in the multidimensional setting of a single good with multiple quality levels. The structure of this section is as follows. First, we introduce the strategy adopted to investigate the qualitative features of optimal mechanisms in section (4.1). Next, in section (4.2), we outline the specific conjectures investigated in this chapter. Finally, we investigate the optimal mechanisms that result from our approximation algorithms in a wide range of settings in section (4.3).

4.1 Methodology

The strategy we adopt to investigate the qualitative properties of optimal mechanisms in multidimensional settings is:

1. For a given setting, determine the optimal mechanism from the results of running the approximation algorithm (outlined in Appendix 8).

2. Once the optimal mechanism is found, examine the discretized interim allocation Q to construct a representation of the optimal allocation function q. The idea is to visually represent the discretized allocation for a quality level of the good (e.g., Figure 1)[11] and find a mathematical expression that corresponds to the approximately optimal mechanism.

[10] In the event of ties where $\beta_1 = \beta_2$ then both allocations Q_1, Q_2 are equal and are half the value of $F^{N-1}(x_1, x_2)$.

[11] TODO fix labels on this image.

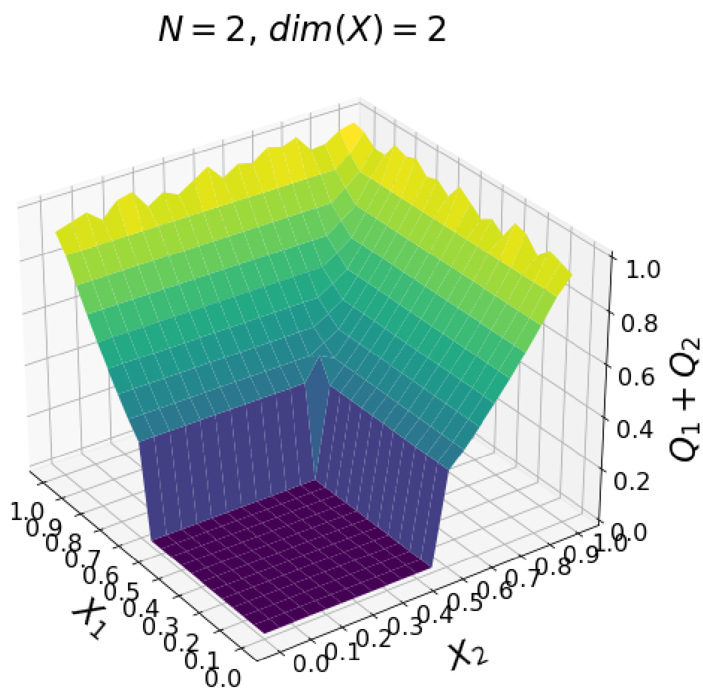

FIGURE 1: The interim allocation Q_1 for the first quality grade in the setting of symmetric, uniform setting of (Pavlov 2011). Notice, for example, the allocation is monotonic in values of X_1 and that the reserve price $p \in [.5, .6)$.

In Figure (1), the allocation can approximated by the interim allocation for the EBM (above):

$$Q_1(x; p) = \mathbb{1}\{\beta_1 > \beta_2 \text{ and } \beta_1 \geq 0\} F^{N-1}(x_1, \min\{\overline{x}_2, r_2 + \beta_1\}) \tag{20}$$

Examination of the interim allocations of mechanisms yielded by the approximation algorithms is the basis for hypotheses concerning the optimal mechanisms.

3. With the optimal allocation function q it is then possible to explore whether the conjectured mechanism (q, p) can recover the approximated revenue yielded by the algorithm. This can be used to confirm whether the conjectured mechanism is optimal.

By considering a wide enough range of cases and leveraging results from existing research in multidimensional mechanism design, we can develop some economic intuitions about the qualitative features of optimal mechanisms in a comparatively simpler multidimensional setting without worrying about the problem of bundling in the setting of multiple bidders *and* multiple goods. Since this approach is broadly susceptible to problems of discretization when approximating the optimal mechanism as well as precision issues with numerical computing, conclusions are, at best, a promising guide to developing further theoretical results. Thus, results are best interpreted as conjectures concerning the character of optimal mechanisms in multidimensional settings.

4.2 Conjectures

The principal conjecture investigated in this thesis chapter concerns the optimality of the EBM in the multidimensional setting of a single good with multiple quality levels:

Conjecture 1 (Revenue)**.** *The revenue of the exclusive buyer mechanism well-approximates the revenue of the optimal mechanism.*

By measuring the discrepancy between revenue from the EBM and that returned by the approximation algorithm it is possible to confirm or reject this conjecture. Additionally, it is important to explore the interim allocations yielded by the approximation algorithm and compare them to those of the EBM. There should be visual confirmation that the optimal mechanism yielded by the algorithm is qualitatively similar to the EBM:

Conjecture 2 (Allocations)**.** *The allocation of the exclusive buyer mechanism captures the same qualitative behavior as the allocation of the optimal mechanism yielded by the approximation algorithm.*

Additionally, a surprising feature of some optimal mechanisms in the setting of a single good with multiple quality levels noted by several economists is that the set of types excluded by the allocation—the *exclusion region*—in equilibrium sometimes has measure zero (e.g., Thirumulanathan, Sundaresan, and Narahari 2019b; Belloni, Lopomo, and Wang 2010). This surprising finding stands in opposition to the result of (Armstrong 1996), where it was shown that in the case of a multiproduct monopolist that it is always optimal to exclude a positive measure of buyers. Thus, we explore under what circumstances the exclusion region is measure zero. Specifically, we conjecture:

Conjecture 3 (Exclusion Region)**.** *The exclusion region of the optimal mechanism in the multidimensional setting of a single good with multiple quality levels remains the same for $N = 1, 2, 3, \ldots$ bidders.*

4.3 Simulations

In what follows we explore conjectures 1, 2, 3 in the following contexts:

1. The **symmetric, independent, and uniform setting**, where buyers' valuations are independent across quality grades, uniformly and symmetrically distributed. Analytic results when the single buyer case when $X \sim U[c, c+1]^2$ are known (Pavlov 2011) and serve as a benchmark.

2. The **symmetric, independent, and non-uniform setting**. Here, it is desirable to see if the conclusions reached in the first two settings extend to non-uniform distributions. In particular, we consider the case of the $Beta(\alpha, \beta)$ distribution, which was explored in (Daskalakis, Deckelbaum, and Tzamos 2017) in the multiple-good setting.

3. The **symmetric, correlated setting**. It is unknown how arbitrary correlations between a buyer's dimensions of value affect the revenue-maximization problem faced by the mechanism design in multi-dimensional settings. This setting aims to shed light on this problem.

4. The **asymmetric, independent, and uniform setting**. No analytic results are known in this similar setting; however, a very provisional analysis of the optimality of the EBM in this setting can be found in (Belloni, Lopomo, and Wang 2010).

4.3.1 Symmetric, independent, and uniform (Pavlov 2011)

Analytic results in the case of a single buyer exist in the symmetric, independent, and uniform setting considered here. (Pavlov 2011) studied the case of two substitute goods[12] independently and uniformly distributed on $U[c, c+1]^2$. In the specific case of a single buyer with valuations distributed according to $X \sim U[0,1]^2$ with zero costs, it is known that the optimal mechanism is deterministic and involves setting reserve price $p^* = \frac{1}{\sqrt{3}}$ for both goods (since the valuations are symmetric). Thus, the optimal allocation is given in Figure 2 and the auctioneer's revenue is simply $p^*(1 - p^{*2}) = 0.3849....$

[12]Note that when there is a single buyer an equivalent interpretation of the setting with one good and multiple quality levels is that there are multiple goods but the buyer has unit demand.

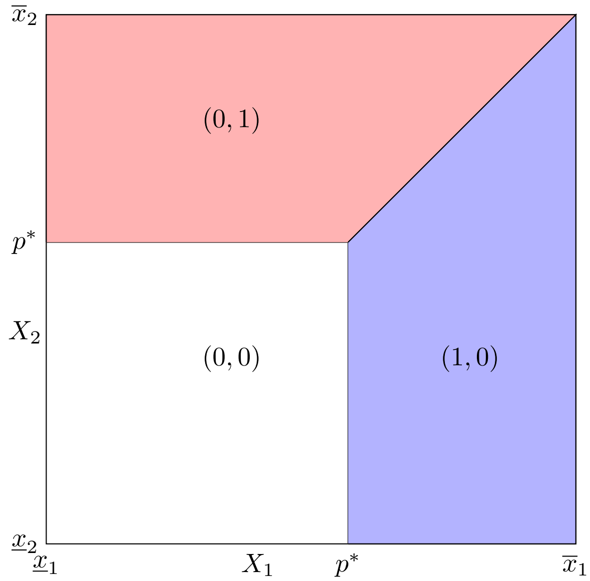

FIGURE 2: The optimal allocation of a single good with two quality symmetric levels to a single buyer with valuations $X \sim U[0,1]^2$ (Pavlov 2011). The area denoted $(0,0)$ is the 'exclusion region', where the good is not allocated.

In the case where there is more than one bidder we need to rely on the approximation algorithm to study the qualitative features of the optimal mechanism. First, we can confirm the approximation algorithm[13] yields similar revenue to that calculated by the appropriate EBM in this setting. These results are presented in Table 3.

Result Type	T	Revenue
approximation	5	0.681...
approximation	10	0.6383...
approximation	15	0.6217...
approximation	20	0.6129...
EBM	—	0.6028... (0.596)

TABLE 3: comparison of revenue generated by approximation algorithm with that of the EBM.

Note that the revenue generated by the EBM was computed using the *ex-post* description of the auction. Additionally, the EBM's revenue was computed by numerical integration on the same sized discretization grid as that used by the approximation algorithm. The trend for different T implies that approximation algorithm is converging to result provided by the EBM. This supports Conjecture 1.

[13] TODO fix bug in algorithm code

Using the *interim* description of the EBM in section (3.2), we can plot the allocations against those returned by the approximation algorithm. These are presented side-by-side in Figure 4.

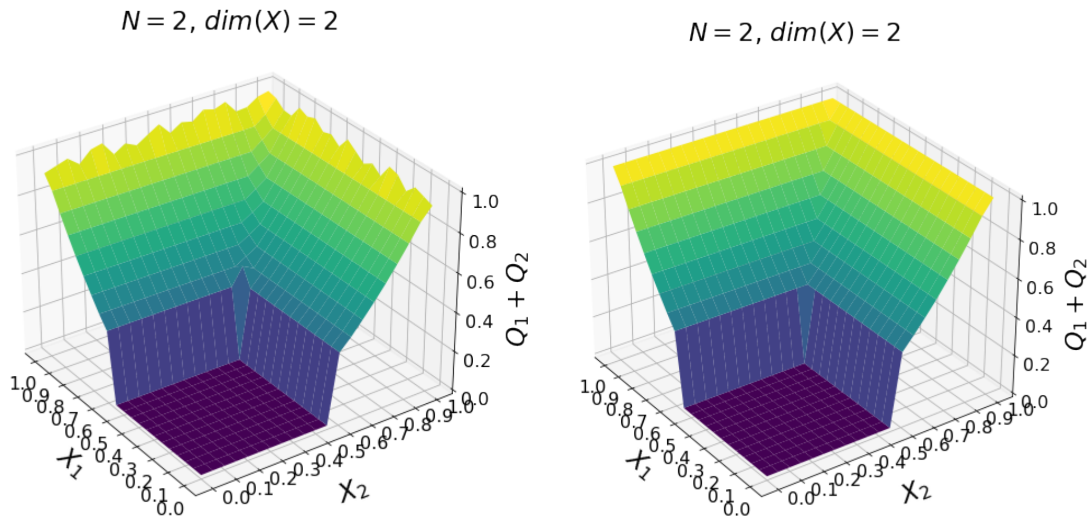

FIGURE 4: The allocations produced by the approximation algorithm (left) and EBM (right).

Conjecture 2 requires qualitatively evaluating the allocations returned from the approximation algorithm and those calculated from the EBM, without an obvious means to conclusively determine whether the conjecture is supported. However, the results in Figure 4 clearly imply that EBM captures the behavior of the (approximately) optimal mechanism[14]. Thus, these results offer support for Conjecture 2.

Notice that the exclusion region in Figure 4 is similar to that discovered in (Pavlov 2011). The price $p^* = \frac{1}{\sqrt{3}} = 0.577...$ that maximizes revenue in the case when $N = 1$ is consistent with the exclusion region defined by $p^* = 0.6$ returned by the approximation algorithm for the case of $N = 2$. (Note that when $[0, 1]$ is discretized into 20 intervals, the price is $p \in \{\dots, .5, .55, .6, \dots\}$ so the choice by the algorithm reflects its approximate optimality). We investigate Conjecture 3 for $N = 1, 2, 3$ in Figure 5 (below).

[14]TODO add marginals

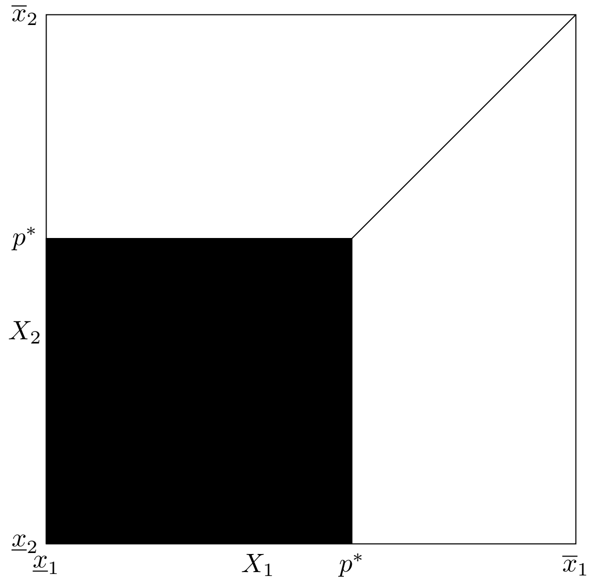

FIGURE 5: The exclusion region produced by the approximation algorithm when $T = 20$ for each of $N = 1, 2, 3$.

When $T = 20$, the approximation algorithm yields the same exclusion region for all of $N = 1, 2, 3$. This supports Conjecture 3.

4.3.2 Symmetric, independent, and non-uniform setting (Daskalakis, Deckelbaum, and Tzamos 2017)

I examine the three conjectures in the context of a symmetric, independent, and non-uniform setting. Following the multi-unit example in (Daskalakis, Deckelbaum, and Tzamos 2017), I consider the case where $X_1, X_2 \sim Beta(\alpha, \beta)$ where $\alpha = 1, \beta = 2$. Note, to the best of my knowledge, no prior work on analytic solutions to the optimal auction design problem exists in this setting. Therefore, we proceed by running the optimization algorithm and comparing the output of the algorithm to that provided by the EBM described above.

First, we compare the revenue generated by the approximation algorithm and the exclusive buyer mechanism. The results are presented in Table 6. Again, note the revenue from the EBM was calculated using *ex-post* from second price auction and the discretized grid was the same as that used by the approximation algorithm. The similarity of the revenues generated by the algorithm and the EBM provide support for Conjecture 1.

Result Type	T	Revenue
approximation	5	
approximation	10	
approximation	15	
approximation	20	0.3891...
EBM	—	0.3806...

TABLE 6: comparison of revenue generated by the approximation algorithm with that of the EBM when $X_1, X_2 \sim Beta(1, 2)$.

Next, we can compare the interim allocations from the approximation algorithm with those from the EBM. These are displayed graphically in Figure 7[15]. There is a clear similarity between both allocations, supporting Conjecture 2.

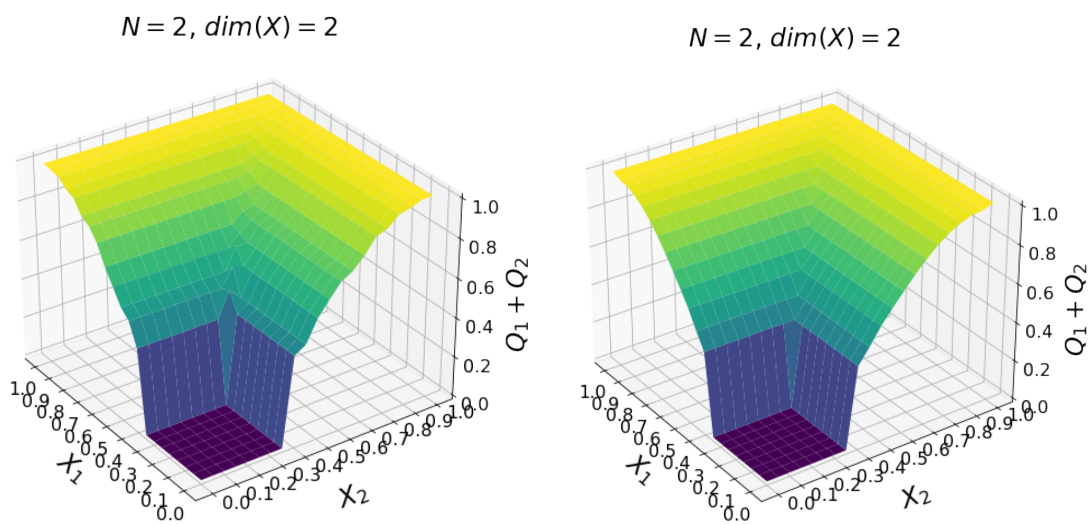

FIGURE 7: The allocations produced by the approximation algorithm (left) and EBM (right) when $X_1, X_2 \sim Beta(1, 2)$.

Finally, running the approximation algorithm for $N = 1, 2, 3$ confirms Conjecture 3. This is displayed in Figure 8, where the exclusion region for is the same for all N tested: the value of $p^* = 0.4$. (Recall, discretization of the grid into $T = 20$ intervals per quality level requires that $p^* \in \{\dots, 0.35, 0.4, 0.45, \dots\}$).

[15] TODO fix axis labels

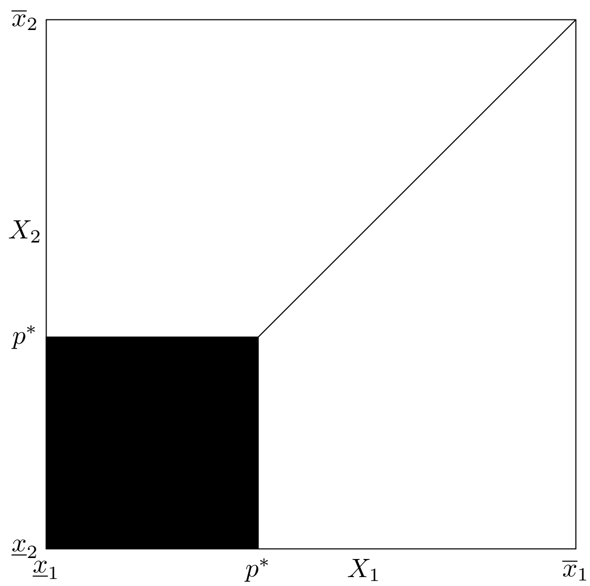

FIGURE 8: The exclusion region produced by the approximation algorithm when $T = 20$ for each of $N = 1, 2, 3$. Note, $p^* = 0.4$.

Thus, in conclusion, all three conjectures are supported in the setting of symmetric, independent and non-uniform distributions.

4.3.3 Symmetric, correlated setting

$X \sim F, f(x, y) = x + y$

4.3.4 Asymmetric, independent, and uniform setting (Belloni, Lopomo, and Wang 2010)

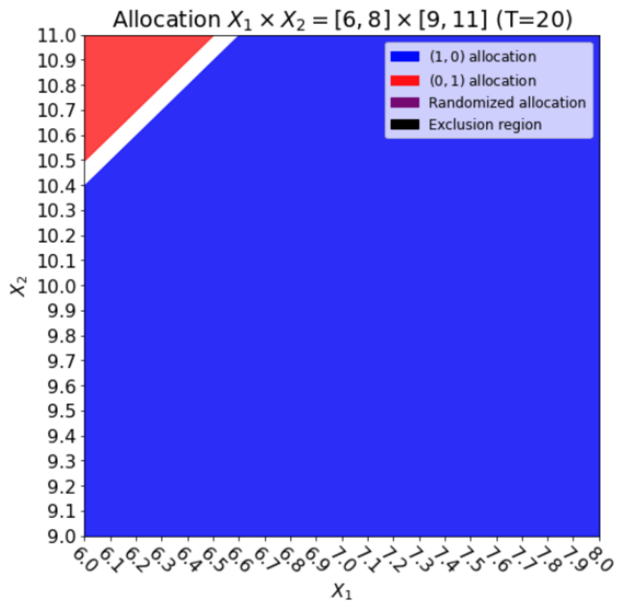

FIGURE 9: The optimal allocation in the setting of (Belloni, Lopomo, and Wang 2010) when $N = 1$. Note, the measure of the exclusion region is zero.

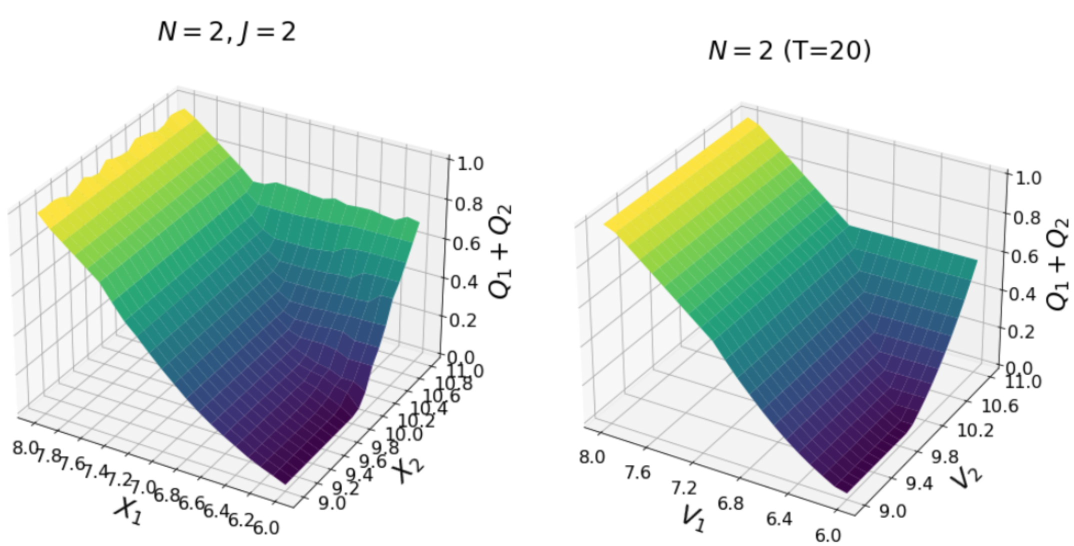

FIGURE 10: The allocations produced by the approximation algorithm (left) and the exclusive buyer mechanism (right).

5 Discussion

6 Conclusion

7 References

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8 Appendix: Approximation Algorithm

TODO fix references+links+notation

We adopt and improve the original finite-dimensional approximation algorithm of (Belloni, Lopomo, and Wang 2010) by focusing on local and downward-sloping incentive-compatibility constraint (ICC) violations. Although these local constraints are often violated in this approximate setting, we drastically reduce the number of times *all* incentive-compatible constraints need to be checked.

In order to approximate an optimal solution to **??**, we discretize the type space V. Let T denote a positive integer that controls the granularity of the discretization. For each $j \in J$, let $V_T(j)$ denote the discretization of the interval $[\underline{v}_j, \overline{v}_j]$ given by $V_T(j) = \{\underline{v}_j, \underline{v}_j + \epsilon, \underline{v}_j + 2\epsilon, \dots, \overline{v}_j\}$ where $\epsilon = \min_{j \in J}\{(\overline{v}_j - \underline{v}_j)/T\}$. Our discretized version of the type space V is given by $V_T := \prod_{j \in J} V_T(j)$. Furthermore, we define a probability density function on V_T by setting $\hat{f}(v) = f(v)/(\sum_{t \in V_T} f(t))$. We thus obtain a linear program which is a finite-dimensional approximation of **??** for each $T > 0$ by replacing V with V_T.

Belloni et al. (2010) use a plane-putting algorithm which works with a randomly chosen subset of incentive-compatibility (ICC) and Border (B) constraints at each iteration. They provide an efficient reduction in the growth in T of the Border constraints (B) from $O(2^{T^J})$ to $O(T^J \log(T^J))$ (Belloni, Lopomo, and Wang 2010, Lemma 10). We adopt their solution to checking (B) constraints; however, our approach to checking (ICC) constraints involves iteratively growing the 'local' region of the type space around each point

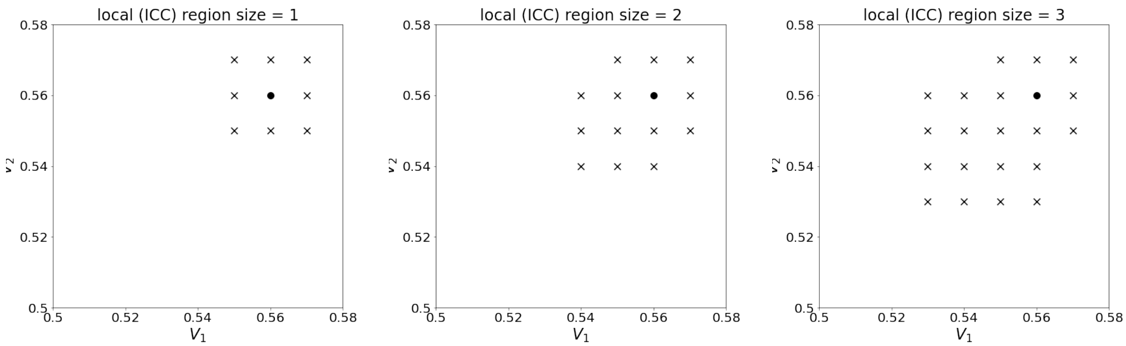

FIGURE 11: We iteratively grow the local region of the discretized type space checked for downwards-sloping constraint violations. Notice that immediately adjacent (ICC) constraints are always checked ($\times$) when the local region increases in size around a point ($\bullet$).

v in the discretized set of types V_T. We do two things. First, all the immediately adjacent points in the discretized type space are always checked for incentive compatibility. Secondly, downwards-sloping points in the discretized type space are also checked. Furthermore, the downwards-sloping region of the type space grows until all (ICC) constraints are ultimately satisfied. This procedure is illustrated visually in Figure 11. Thus, for a fixed-size local region around each point in the discretized type space, we first satisfy *local* (ICC) and (B) constraints as in the iterative plane-cutting algorithm of (Belloni, Lopomo, and Wang 2010). Then we run the separation oracle with *all* (ICC) and (B) constraints. We then restart the solver with any previously violated constraints, this time increasing the size of the local region around each point in the discretized type space. This procedure iterates until no constraints are violated. This modified version of (Belloni, Lopomo, and Wang 2010)'s algorithm is described in Algorithm 1.

Our algorithm[16] is written in Python 3.10 and uses Google's open source linear programming solver 'GLOP' available in their `or-tools` package (Perron and Furnon 2023).

[16]For more details see: `https://github.com/jmemich/optimal-auction-multidim`

$L = 1, S = \emptyset, A = \emptyset, \overline{OPT} = \infty$;
`violated_any_icc` $\leftarrow$ TRUE;
while *violated_any_icc* **do**
    `violated_local_icc` $\leftarrow$ TRUE;
    while *violated_local_icc* **do**
        $k = 1, A^k = A, S^k = S$;
        Solve the linear program associated with S^k. Let OPT^k denote the optimal value.;
        Solve the separation oracle using only local (ICC) constraints in region L. Let A^k donate all violated local (ICC) and (B) constraints.;
        if $A^k = \emptyset$ **then**
            `violated_local_icc` $\leftarrow$ FALSE;
            Break;
        end
        Select a subset $I^k \subset S^k$ of inactive (ICC) and (B) constraints;
        if $OPT^k < \overline{OPT}$ **then**
            $S^{k+1} \leftarrow (S^k \setminus I^k) \cup A^k \cup A$;
            $\overline{OPT} \leftarrow OPT^k$;
        else
            $S^{k+1} \leftarrow S^k \cup A^k \cup A$;
        end
        $k \leftarrow k + 1$;
    end
    Solve the separation oracle using all (ICC) constraints. Let A^* donate all violated (ICC) constraints.;
    if $A^* = \emptyset$ **then**
        `violated_any_icc` $\leftarrow$ FALSE;
        Break;
    end
    $A \leftarrow A \cup A^*$;
    $S \leftarrow S^k$;
    $L \leftarrow L + 1$;
end

Algorithm 1: Iterative plane-cutting algorithm with local and downwards-sloping (ICC) constraints

9 Appendix: Calculations

9.1 $N = 1, X = U[0,1]^2, c = [0,0]$ (Pavlov 2011)

From (Pavlov 2011, Example 1), we know that the optimal reserve price is $p^* = (\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})$. The revenue according to the exclusive buyer mechanism is given by:

$$\begin{aligned} Rev(p) &= \int_{x \in A} Q_1(x;p)(p^* - c_1)dF(x) + \int_{x \in B} Q_2(x;p)(p^* - c_2)dF(x) && (21) \\ &= \int_{p^*}^1 \int_0^x (p^* - c_1) f(x_1, x_2) dx_2 dx_1 + \int_{p^*}^1 \int_0^y (p^* - c_2) f(x_1, x_2) dx_1 dx_2 && (22) \\ &= p^* \left(\int_{p^*}^1 \int_0^x dx_2 dx_1 + \int_{p^*}^1 \int_0^y dx_1 dx_2 \right) && (23) \\ &= p^*(1 - p^{*2}) && (24) \end{aligned}$$

9.2 $N = 1, X = U[6,8] \times U[9,11], c = [.9, 5]$ (Belloni, Lopomo, and Wang 2010)

Note, for all calculations that follow we compute the revenue from a mechanism defined by reserve prices $p^* = (6, 9.9)$.

9.2.1 $\int_X Q(x;p) \cdot (p - c) dF(x)$

For ease of calculation, we divide up the type space according to which quality level of the good is preferred:

$$A := \{x \in X | x_1 - p_1 > x_2 - p_2 \text{ and } x_1 - p_1 \geq 0\} \quad (25)$$

$$B := \{x \in X | x_2 - p_2 > x_1 - p_1 \text{ and } x_2 - p_2 \geq 0\} \quad (26)$$

Thus, the objective function becomes

$$\begin{aligned} Rev_1(p) &= \int_{x \in A} Q_1(x;p)(x_1 - c_1)dF(x) + \int_{x \in B} Q_2(x;p)(x_2 - c_2)dF(x) && (27) \\ &= \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\min\{p_2^* + x_1 - p_1^*, \overline{x}_2\}} (p_1 - c_1) f(x_1, x_2) dx_2 dx_2 + \int_{\underline{x}_1}^{\underline{x}_1 + \overline{x}_2 - p_2^*} \int_{p_2^* + x_1 - p_1^*}^{\overline{x}_2} (p_2 - c_2) f(x_1, x_2) dx_2 dx_2 && (28) \\ &= \frac{1}{4} \left(\int_6^8 \int_9^{\min\{9.9 + x_1 - 6, 11\}} (6 - .9) dx_2 dx_2 + \int_6^{7.1} \int_{9.9 + x_1 - 6}^{11} (9.9 - 5) dx_2 dx_2 \right) && (29) \\ &= \frac{1}{4} \left(3.395(6 - .9) + 0.605(9.9 - 5) \right) && (30) \\ &= 5.06975 && (31) \end{aligned}$$

9.2.2 $\int_X Q(x;p)\cdot(x-c) - U(x;p)dF(x)$

Alternatively, using the definition of A, B, note that for all $x \in A$, $U(x;p) = \int_{\underline{x}_1}^{x_1} \mathbb{1}\{t \geq p_1^*\}dt$ and similarly, for $x \in B$, $U(x;p) = \int_{\underline{x}_2}^{x_2} \mathbb{1}\{t \geq p_2^*\}dt$. Therefore,

$$Rev_2(p) = \int_{x\in A} Q_1(x;p)(x_1 - c_1) - \left(\int_{\underline{x}_1}^{x_1} \mathbb{1}\{t \geq p_1^*\}dt\right)dF(x) \tag{32}$$

$$+ \int_{x\in B} Q_2(x;p)(x_2 - c_2) - \left(\int_{\underline{x}_2}^{x_2} \mathbb{1}\{t \geq p_2^*\}dt\right)dF(x) \tag{33}$$

$$= \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\min\{p_2^*+x_1-p_1^*,\overline{x}_2\}} (x_1 - c_1) - \mathbb{1}\{x_1 \geq p_1^*\}(x_1 - p_1^*)f(x_1,x_2)dx_2dx_1 + \tag{34}$$

$$+ \int_{\underline{x}_1}^{\underline{x}_1+\overline{x}_2-p_2^*} \int_{p_2^*+x_1-p_1^*}^{\overline{x}_2} (x_2 - c_2) - \mathbb{1}\{x_2 \geq p_2^*\}(x_2 - p_2^*)f(x_1,x_2)dx_2dx_2 \tag{35}$$

$$= \frac{1}{4}\left(\int_6^8 \int_9^{\min\{9.9+x_1-6,11\}} (6-.9)dx_2dx_1 + \int_6^{7.1}\int_{9.9+x_1-6}^{11} (9.9-5)dx_2dx_2\right) \tag{36}$$

$$= 5.06975 \tag{37}$$

9.3 $N = 2, X = U[6,8]\times U[9,11], c = [.9,5]$ (Belloni, Lopomo, and Wang 2010)

Again, we use the reserve price $p* = (6, 9.9)$.

9.3.1 $\int_X Q(x;p)\cdot(p-c)dF(x)$

$$
\begin{aligned}
Rev_1(p) &= N\int_{x\in A} Q_1(x;p)(p_1-c_1)dF(x) + \int_{x\in B} Q_2(x;p)(p_2-c_2)dF(x) && (38)\\
&= N\bigg(\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\min\{p_2^*+x_1-p_1^*,\overline{x}_2\}} F(x_1,\min\{p_2^*+x_1-p_1,\overline{x}_2\})(p_1-c_1)f(x_1,x_2)dx_2dx_2 && (39)\\
&\quad+\int_{\underline{x}_1}^{\underline{x}_1+\overline{x}_2-p_2^*}\int_{p_2^*+x_1-p_1^*}^{\overline{x}_2} F(\min\{\overline{x}_1,p_1^*+x_2-p_2^*\},x_2)(p_2-c_2)f(x_1,x_2)dx_2dx_2\bigg) && (40)\\
&= \frac{N}{4}\bigg(\int_6^8\int_9^{\min\{9.9+x_1-6,11\}}\left(\frac{x_1-6}{2}\right)\left(\frac{\min\{9.9+x_1-6,11\}-9}{2}\right)(6-.9)dx_2dx_2 && (41)\\
&\quad+\int_6^{7.1}\int_{9.9+x_1-6}^{11}\left(\frac{x_2-9.9}{2}\right)\left(\frac{x_2-9}{2}\right)(9.9-5)dx_2dx_2\bigg) && (42)\\
&= \frac{N}{4}\bigg(\int_6^{7.1}\int_9^{9.9+x_1-6}\left(\frac{x_1-6}{2}\right)\left(\frac{9.9+x_1-6-9}{2}\right)(6-.9)dx_2dx_2 && (43)\\
&\quad+\int_{7.1}^8\int_9^{11}\left(\frac{x_1-6}{2}\right)(6-.9)dx_2dx_2 && (44)\\
&\quad+\int_6^{7.1}\int_{9.9+x_1-6}^{11}\left(\frac{x_2-9.9}{2}\right)\left(\frac{x_2-9}{2}\right)(9.9-5)dx_2dx_2\bigg) && (45)\\
&= \frac{2}{4}\bigg(2.10971+7.1145+.937523\bigg) && (46)\\
&= 5.0808665 && (47)
\end{aligned}
$$

9.3.2 $\int_X Q(x;p)\cdot(x-c)-U(x;p)dF(x)$

Now, consider the revenue in B:

$$
\begin{aligned}
&N\int_{x\in B} Q(x;p)\cdot(x-c)-U(x;p)dF(x) && (48)\\
&= \int_{x\in B} Q_2(x;p)(x_2-c_2)-\left(\int_{\underline{x}_2}^{x_2} Q_2(x_1,t)dt\right)dF(x) && (49)\\
&= N\int_{x\in B} F(\min\{\overline{x}_1,p_1^*+x_2-p_2^*\},x_2)(x_2-c_2)-\left(\int_{\underline{x}_2}^{x_2}\mathbb{1}\{t\ge p_2^*\}F(\min\{\overline{x}_1,p_1^*+x_2-p_2^*\},x_2)dt\right)dF(x) && (50)\\
&= N\int_6^{7.1}\int_{9.9+x_1-6}^{11}\left(\frac{x_2-9.9}{2}\right)\left(\frac{x_2-9}{2}\right)(x_2-5)-\left(\int_{9.9}^{x_2}\left(\frac{t-9.9}{2}\right)\left(\frac{t-9}{2}\right)dt\right)f(x_1,x_2)dx_2dx_1 && (51)\\
&= \frac{N}{4}\int_6^{7.1}\int_{9.9+x_1-6}^{11}\left(\frac{x_2-9.9}{2}\right)\left(\frac{x_2-9}{2}\right)(x_2-5)-\left(\frac{1}{24}(x_2-9.9)^2(2x_2+9.9-27)\right)dx_2dx_1 && (52)\\
&= 0.516192 && (53)
\end{aligned}
$$

This is not equivalent[17] to the revenue from $N \int_{x \in B} Q(x;p) \cdot (p-c) dF(x)$.

[17] See here for computation.