1 Exclusive Buyer Mechanism

Suppose there are $i = 1, \ldots, N$ bidders in an auction for a good with $j = 1, 2$ quality levels. Suppose each bidder's valuation for the good is given by $x^i = (x_1^i, x_2^i) \in [\underline{x}_1^i, \overline{x}_1^i] \times [\underline{x}_2^i, \overline{x}_2^i]$. We assume bidders' valuations are identical and independently distributed but we allow an individual bidder's valuations for quality grades to be arbitrarily correlated.

We now construct an *exclusive-buyer mechanism*. This mechanism is a little different[1] than that of (Belloni, Lopomo, and Wang 2010). Since bidder valuations are symmetric, we omit the superscript i (i.e., $X = X^i$). For any given *reserve price* $p = (p_1, p_2)$, let

$$\beta_1 = x_1 - p_1 \quad \text{and} \quad \beta_2 = x_2 - p_2 \tag{1}$$

and define the interim allocations[2] as:

$$Q_1(x;p) = \mathbb{1}\{\beta_1 > \beta_2 \wedge \beta_1 \geq 0\}\left(\int_{\underline{x}}^{(x_1, \min\{\overline{x}_2, p_2+\beta_1\})} \mathbb{1}\{t_1 - p_1 > t_2 - p_2 \wedge t_1 - p_1 \geq 0\} dF(t)\right)^{N-1} \tag{2}$$

$$Q_2(x;p) = \mathbb{1}\{\beta_2 > \beta_1 \wedge \beta_2 \geq 0\}\left(\int_{\underline{x}}^{(\min\{\overline{x}_1, p_1+\beta_2\}, x_2)} \mathbb{1}\{t_2 - p_2 > t_1 - p_1 \wedge t_2 - p_2 \geq 0\} dF(t)\right)^{N-1} \tag{3}$$

Then the expected revenue from price p is given by:

$$R_{EBM}(p) = N \int_X Q(x;p) \cdot (p - c) dF(x) \tag{4}$$

where costs are given by $c = (c_1, c_2)$.

2 Calculations

For ease of calculation, we divide up the type space according to which quality level of the good is preferred:

$$A := \{x \in X | x_1 - p_1 > x_2 - p_2 \text{ and } x_1 - p_1 \geq 0\} \tag{5}$$

$$B := \{x \in X | x_2 - p_2 > x_1 - p_1 \text{ and } x_2 - p_2 \geq 0\} \tag{6}$$

[1] It does not rely on a "add-on price" in addition to a "reserve price". Instead, we assume a price is given for each quality grade.

[2] In the event of ties where $\beta_1 = \beta_2$ then both allocations Q_1, Q_2 are equal.

2.1 $N = 1$

Notice that for $N = 1$,

$$R(p) = \int_{x \in A} (p_1 - c_1) dF(x) + \int_{x \in B} (p_2 - c_2) dF(x) \tag{7}$$

2.1.1 $X = U[0,1]^2$ (Pavlov 2011)

$p* = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ DONE

2.1.2 $X = U[6,8] \times U[9,11], c = [.9, 5]$ (Belloni, Lopomo, and Wang 2010)

$p^* = (9, 10.5)$ DONE

$revenue^* \approx 5.12$ DONE

2.2 $N = 2$

2.2.1 $X = U[6,8] \times U[9,11], c = [.9, 5]$ (Belloni, Lopomo, and Wang 2010)

$p^* = (6, 10.1)$ TODO

$revenue^* \approx 5.89$ TODO

$$R_{EBM}(p) = N\Big(\int_{x\in A} Q_1(x;p)(p_1 - c_1)dF(x) + \int_{x\in B} Q_2(x;p)(p_2 - c_2)dF(x)\Big) \tag{8}$$

$$= N\int_{x\in A}\Big(\int_{\underline{x}}^{(x_1, \min\{\overline{x}_2, p_2 + x_1 - p_1\})} \mathbb{1}\{t_1 - p_1 > t_2 - p_2 \wedge t_1 - p_1 \geq 0\}dF(t)\Big)^{N-1}(p_1 - c_1)dF(x) \tag{9}$$

$$+ N\int_{x\in B}\Big(\int_{\underline{x}}^{(\min\{\overline{x}_1, p_1 + x_2 - p_2\}, x_2)} \mathbb{1}\{t_2 - p_2 > t_1 - p_1 \wedge t_2 - p_2 \geq 0\}dF(t)\Big)^{N-1}(p_2 - c_2)dF(x) \tag{10}$$

$$= 2\int_6^8\int_9^{\min\{11, 10.1 + x_1 - 6\}}\Big(\int_6^{x_1}\int_9^{\min\{11, 10.1 + x_1 - 6\}} \mathbb{1}\{t_1 - 6 > t_2 - 10.1\}f(t_1, t_2)dt_2dt_1\Big)(6 - .9)f(x_1, x_2)dx_2dx_1 \tag{11}$$

$$+ 2\int_6^{6.9}\int_{10.1 + x_1 - 6}^{11}\Big(\int_6^{6 + x_2 - 10.1}\int_{10.1}^{x_2} \mathbb{1}\{t_2 - 10.1 > t_1 - 6\}f(t_1, t_2)dt_2dt_1\Big)(10.1 - 5)f(x_1, x_2)dx_2dx_1 \tag{12}$$

$$= \frac{(6 - .9)}{8}\int_6^{6.9}\int_9^{10.1 + x_1 - 6}\Big(\int_6^{x_1}\int_9^{10.1 + t_1 - 6} dt_2dt_1\Big)dx_2dx_1 \tag{13}$$

$$+ \frac{(6 - .9)}{8}\int_{6.9}^8\int_9^{11}\Big(\int_6^{x_1}\int_9^{\min\{11, 10.1 + t_1 - 6\}} dt_2dt_1\Big)dx_2dx_1 \tag{14}$$

$$+ \frac{(10.1 - 5)}{8}\int_6^{6.9}\int_{10.1 + x_1 - 6}^{11}\Big(\int_{10.1}^{x_2}\int_6^{6 + t_2 - 10.1} dt_1dt_2\Big)dx_2dx_1 \tag{15}$$

$$= 4.17\ldots \tag{16}$$

3 References

Belloni, Alexandre, Giuseppe Lopomo, and Shouqiang Wang (2010). "Multidimensional mechanism design: Finite-dimensional approximations and efficient computation". In: *Operations Research* 58.4-part-2, pp. 1079–1089.

Pavlov, Gregory (2011). "Optimal mechanism for selling two goods". In: *The BE Journal of Theoretical Economics* 11.1.