

Simulations on Optimal Auctions in Multidimensional Settings

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Abstract

I explore the properties of optimal multi-dimensional auctions in a setting where a single object of multiple qualities is sold to several buyers. Using simulations, I test the hypothesis that the optimal mechanism is an *exclusive buyer mechanism*, where buyers compete to be the right to be the only buyer to choose between quality levels of a good. I find compelling evidence of the optimality of the exclusive buyer mechanism in multi-dimensional settings and explore other conjectures concerning the set of types excluded from the mechanism in equilibrium in optimal multidimensional auctions.

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1 Introduction

Since the seminal work of Roger Myerson (1981), revenue-maximizing auctions in the case of a single dimension (e.g., good) are well known. However, little is known about optimal auctions in settings with multiple dimensions of value. The problem is incredibly complex: despite sustained research efforts for decades, only in the past few years have we discovered how to optimally sell two goods to a single buyer (Daskalakis, Deckelbaum, and Tzamos 2017). In this chapter, I focus on the multidimensional setting of a single good with multiple quality levels and explore whether a specific mechanism—the *exclusive buyer mechanism*—is optimal. Building off existing work at the intersection of economics and computer science, I adopt a novel approach involving extensive simulations to explore the optimality of this particular mechanism. The approach adopted here makes use of a new approximation algorithm to uncover the qualitative features of optimal mechanisms in the particular multidimensional setting of a single good with multiple quality levels¹.

The setting of a single good with multiple quality levels can be understood as corresponding to the case of purchasing a single airplane ticket where the ‘quality level’ might be: economy class, business class, or first class. A buyer might prefer, say, business over economy class. Each quality level can be (and very often is) given a different price and might have different value to the buyer. Crucially, unlike the multidimensional setting where multiple goods are sold by a seller to an arbitrary number of buyers, here since only one good is sold so ‘bundling’—selling subsets of all offered goods for a discount—cannot occur. The absence of bundling renders this a much simpler setting and an ideal point of departure for investigating the qualitative features of optimal multidimensional auctions.

In this thesis chapter, I am particularly concerned with investigating the optimality of the exclusive buyer mechanism in the setting of selling a single good with multiple quality levels to multiple bidders. This mechanism can be understood as an (either a second price or ascending bid) auction with a reserve price for the exclusive right to be the buyer of one of the quality levels of the good. The intuition for why this mechanism might be optimal can be reduced to: when there is a single good for sale with multiple dimensions of value, it is enough to consider the bidder who values a single quality level more than any other bidder values any other quality level. The exclusive buyer mechanism has previously been explored in multidimensional settings (e.g., Brusco, Lopomo, and Marx 2011; Belloni, Lopomo, and Wang 2010) but this thesis chapter represents the first sustained exploration of its optimality in a wide range of settings. This chapter develops the exclusive buyer mechanism in much broader generality, facilitating an extended investigation into its optimality in a broad range of multidimensional settings.

The approach adopted in this chapter is to eschew analytic results in settings where few have been

¹The codebase for this project is available at <https://github.com/jmemich/optimal-auction-multidim>.

forthcoming in favor of exploring these complex, analytically intractable settings using simulations to approximate optimal mechanisms. The key idea is simple. Computer scientists and economists have made significant strides developing approximation algorithms that can arbitrarily well approximate optimal revenue using linear programming techniques (e.g., Cai, Daskalakis, and Weinberg 2012; Belloni, Lopomo, and Wang 2010). Taking these algorithmic developments as a point of departure, it is possible to explore a wide class of settings where a single good with multiple quality levels is sold to better understand important qualitative features of the optimal mechanism. Once an approximation algorithm is run to uncover the optimal mechanism, a qualitative analysis of the key features of the optimal mechanism then proceeds. Important questions explored in this thesis concern: Do these approximation algorithms yield deterministic optimal mechanisms or is randomization always required for revenue maximization? Is a positive measure of buyers always excluded from the allocation in equilibrium? Are there simple, intuitive mechanisms that might characterize the results from the approximation algorithms? These and related questions are explored in this chapter. Ultimately, the goal is to investigate the optimality of the exclusive buyer mechanism using simulations.

Across a broad range of settings, I find evidence for the optimality of the exclusive buyer mechanism and explore several related conjectures about the qualitative characteristics of optimal mechanisms using simulations. This mechanism effectively recovers the revenue achieved by the optimal mechanism in all settings considered here. Additionally, the exclusion region—the set of types excluded by the mechanism in equilibrium—is identical for all N considered in each setting. Finally, although there is considerable evidence that the interim allocations of the optimal mechanism yielded by the approximation algorithm are qualitatively similar to those of the exclusive buyer mechanism, there are settings where the optimal mechanism involves randomization and the allocations of the exclusive buyer mechanism fail to capture the behavior of the optimal mechanism. These results are designed to guide future theoretical research by way of conjecture: support for the conjectures presented in this chapter should help future research undertake more promising avenues of research in a complex and challenging field.

The structure of this chapter proceeds as follows. In section (2), we review the existing literature on multidimensional mechanism design, approximation algorithms, and specific works in the setting of a single good with multiple quality levels. The problem is formally introduced in section (3), and a description of the exclusive buyer mechanism can be found in section (3.2). Section (4) describes the approach of the chapter and explores several hypotheses concerning optimal multidimensional mechanisms. Finally, I explore and discuss these results in section (5) and conclude in section (6), highlighting the implications of these results for theoretical microeconomics. An Appendix with a complete description of the approximation algorithm developed for this thesis can be found in section (8).

2 Literature Review

The specific case considered here of selling a single good with multiple quality levels to an arbitrary number of buyers is a special case of the more general multidimensional mechanism design problem of selling an arbitrary number of goods to an arbitrary number of buyers. Since the groundbreaking work of Roger Myerson (1981) who solved the optimal auction design problem in the case of single-dimensional types, economists have sought to characterize optimal auctions in the more general multidimensional setting, with limited success. At this juncture, it is widely accepted that “[e]ssentially, nothing has been known about optimal auctions in this [multidimensional] setting” (Kolesnikov et al. 2022, p1). This literature review covers historical and recent developments by economists and computer scientists who have sought to uncover characteristics of optimal mechanisms in multidimensional settings, with a particular focus on the case of a single good with multiple quality levels.

In the case of single-dimensional types, early work on optimal mechanism design demonstrated the optimality of deterministic mechanisms (i.e., reserve or ‘take-it-or-leave-it’ prices) (Myerson 1981; Riley and Zeckhauser 1983). These approaches leveraged the approach of integration-by-parts (as used in Mussa and Rosen 1978) to solve the relaxed optimal mechanism design problem without directly considering incentive-compatibility constraints. These early results cannot be generalized to multi-dimensional settings because the integral solution to the optimization problem is path-dependent; any two points in the multi-dimensional typespace can be connected by a continuum of paths. Thus, a major breakthrough in multidimensional auction design came with the use of duality-based approaches to multidimensional screening developed by (Rochet and Choné 1998) which circumvents this problem.

The duality-based approach (Rochet and Choné 1998) builds on the single-dimensional nonlinear pricing framework of (Mussa and Rosen 1978), which was given its canonical formulation in multidimensional settings in (Wilson 1993) and (Armstrong 1996). This work takes as its point of departure the approach of (Mirrlees 1971) on optimal taxation and relies on results that establish the implementability of a decision rule in multidimensional settings (Rochet 1987). In this multidimensional screening problem, a few key findings emerge. The first is that ‘bunching’—a situation where multiple types are treated identically in the optimal solution—is a “robust” feature of multidimensional screening (Rochet and Stole 2003; Rochet and Choné 1998). There are two types of bunching: in the first case, a set of types of positive measure are excluded from purchasing the goods in the optimal solution (this is commonly known as the ‘exclusion region’ of the type space); in the second case, a non-negligible set of types outside the exclusion region receive the same product although they have different tastes. In addition, the work of (Rochet and Choné 1998) illustrates that the optimal solution to multidimensional screening problems may involve “bundling” the goods, which

involves selling multiple goods together.

In multi-item settings, authors have long sought to characterize when bundling multiple goods in a single contract is optimal for the seller. Bundling strategies available to a seller include ‘pure’ bundling, where only the bundle of all goods is offered to sellers, and ‘mixed’ bundling, where each different bundle of items is priced separately. Early results showed that offering mixed bundles strictly dominates offering pure bundles to the buyers (Adams and Yellen 1976; McAfee, McMillan, and Whinston 1989) and more recent results have demonstrated that randomized bundles may dominate mixed bundles (Thanassoulis 2004; Daskalakis, Deckelbaum, and Tzamos 2017). In these settings, the optimal menu of contracts may include infinitely many randomized bundles (Manelli and Vincent 2007; Hart and Nisan 2019). Additionally, recent work has demonstrated settings where simply offering only the grand bundle of all goods is optimal (Haghpanah and Hartline 2021).

In the past few years, major breakthroughs in optimal multidimensional mechanism design have come from the use of the methods of optimal transport applied to the optimization problems of microeconomic theory (see Ekeland 2010). These results (Daskalakis, Deckelbaum, and Tzamos 2017; Kolesnikov et al. 2022) greatly aid the *certification* of optimality: the techniques of optimal transport facilitate the identification of the dual of the seller’s optimization problem from which a given mechanism’s optimality can be verified. Thus, previously existing results that characterize optimal mechanisms in specific settings (for example, where valuations for two goods are i.i.d on $U[0, 1]^2$ (Pavlov 2011; Manelli and Vincent 2006)) can be shown to be optimal using a novel, more general approach. The success of the tools of optimal transport in mechanism design is due to the success of a ‘guess-and-verify’ approach where one guesses a solution to the primal problem and then the dual solution plays the role of a certificate of optimality for the initial guess.

These breakthroughs which facilitate the certification of optimality are particularly helpful when viewed in light of the growth in work at the intersection of economics and computer science². One line of work (Chawla, Hartline, and Kleinberg 2007; Cai, Daskalakis, and Weinberg 2012; Cai, Devanur, and Weinberg 2016; Belloni, Lopomo, and Wang 2010; Alaei et al. 2019) provides an algorithmic approximation of optimal mechanisms in multidimensional settings. Here, the buyer’s typespace is discretized, and linear programming techniques are used to approximate the optimal solution, often using simple mechanisms like posted prices. Work in this area aims to achieve a constant factor of the optimal revenue achievable by a Bayesian incentive-compatible mechanism through an approximation. Other work at the intersection of computer science and economics offers insights into the nature of the optimal mechanisms in multidimensional settings. These works show that in specific settings, optimal mechanisms contain only a few contract points (Wang and Tang 2014) or that menus with only a finite number of items cannot ensure any positive fraction of optimal

²TODO vincent work somewhere here.

revenue (Hart and Nisan 2019).

Returning to the specific context of multidimensional mechanism design in the case of a single good with multiple quality levels, the work of (Belloni, Lopomo, and Wang 2010) provides insight into the character of the optimal mechanism in this particular setting. Applying their algorithm to concrete cases, they find a number of surprising results from their simulations. First, there is clear evidence that in the optimal solution, a measure-zero set of buyers is excluded from the allocation in equilibrium. This stands in marked contrast to results in the multi-item case which show that the optimal solution requires exclusion (Rochet and Choné 1998; Armstrong 1996). Second, their results indicate the optimality of an *exclusive-buyer mechanism*: it performs “quite well” relative to the numerical optimal solutions and that it “shares many of its defining features with its one-dimensional counterparts” (Belloni, Lopomo, and Wang 2010, p1085-6), including being implementable in dominant strategies. This mechanism involves an auction (with a reserve price) among buyers for who gets to be the sole recipient of the good. A premium can then be paid for whichever quality grade the winning buyer desires. Interestingly, this mechanism is entirely deterministic in the single-bidder case. This is particularly surprising because, in the neighboring multi-item case, randomized allocations are widely considered necessary for revenue maximization (Daskalakis 2015).

The theoretical study of exclusive-buyer mechanisms originates from the phenomenon of ‘contingent re-auctions’ where sellers will modify objects sold to benefit themselves or the general public (Brusco, Lopomo, and Marx 2011). For example, in the context of the US Spectrum License Auction 73 held in 2008³, the US government adopted a contingent re-auction format where it offered restricted spectrum licenses first, and committed to re-auction the licenses without many of the restrictions in the case the reserve prices were not met. Brusco, Lopomo, and Marx (2011) show that an exclusive-buyer mechanism can always be parameterized such that the mechanism induces the efficient outcome in dominant strategies. However, outside of a restrictive context where all bidders’ valuations for the restricted object are a fixed percentage of the unrestricted object, no general results concerning the optimality of the mechanism are provided.

Analytic results concerning optimal multidimensional auctions for a single good with multiple quality levels and a single bidder are scarce. Notably, (Pavlov 2011) investigates the case where the bidder’s valuations for the object are uniformly distributed on the unit square $[c, c + 1]^2$. Pavlov finds that the optimal mechanism varies considerably with c and sometimes requires randomization for revenue maximization. This approach was further generalized in the work of (Thirumulanathan, Sundaresan, and Narahari 2019a) who study an almost identical case where a bidder’s valuations are distributed uniformly on the rectangle $[c_1, c_1 + b_1] \times [c_2, c_2 + b_2]$. Similarly to (Pavlov 2011), the solution to the optimal mechanism design problem entails both deterministic and stochastic contracts. Surprisingly, however, (Thirumulanathan, Sundaresan,

³For more details see (Brusco, Lopomo, and Marx 2009).

and Narahari 2019a) find evidence of settings where optimal mechanisms do not exclude a position measure of buyers. Additionally, a working paper by (Haghpanah and Hartline 2014) gives sufficient conditions for the optimality of posting a single, uniform price for all quality levels of a good, albeit in a restricted class of settings.

Analytic results for optimally selling substitute goods have also been given in the Hotelling model (Hotelling 1929) where two horizontally differentiated goods are located at the endpoints of a segment. In this setting, (Balestrieri, Izmalkov, and Leao 2020) find that stochastic contracts are part of the optimal mechanism. The economic intuition that arises from this body of research is clear: by offering a lottery over which good the bidder receives, a seller can offer a discount to entice marginal buyers who would otherwise choose the outside option. Similarly, (Loertscher and Muir 2023), find that in this setting randomization is required by the seller to maximize revenue. These results support earlier work (Thanassoulis 2004) which shows that in the standard auction design problem for two substitute goods, the seller can always increase revenue by including stochastic contracts alongside take-it-or-leave-it prices in the optimal mechanism. As noted above, these results support the view that in multidimensional settings randomization is required to maximize revenue (see Daskalakis 2015).

The approaches of (Pavlov 2011; Thirumulanathan, Sundaresan, and Narahari 2019a) to solving the mechanism design problem for the case of a single good with multiple quality levels follows the work of (Guesnerie and Laffont 1984), where optimal control theory is used to address the fact the measure of participating types endogenously depends on the mechanism. The optimal control theory approach⁴ has also been successfully applied to single-dimensional settings when the participation constraints are endogenously determined by the mechanism (Jullien 2000). Here, the bidder’s reservation utility depends on their type. This approach generalizes to accommodate the fact that the measure of participating types in a given mechanism is endogenously determined (for example, in the multidimensional case of a single good with two quality levels and a single buyer considered by Pavlov 2011; Thirumulanathan, Sundaresan, and Narahari 2019a).

Finally, although it has long been believed that it is always profitable for the seller to exclude some measure of bidders (Rochet and Choné 1998; Armstrong 1996) in multidimensional settings, recent theoretical and computational work in auction design in these settings has challenged these conclusions. The original result of (Armstrong 1996) demonstrated that in multi-product settings with a single bidder, the seller benefits from always excluding a positive measure of bidder types. By relaxing Armstrong’s strong assumptions about the bidder’s utility function and the convexity of the type space this result has been

⁴See (Basov 2005, §7) for an extended discussion of the different approaches to multidimensional mechanism design and their respective strengths and weaknesses.

extended and it has been shown that “exclusion is generically optimal in a large class of models” (Barelli et al. 2014, p. 75). The intuition is as follows: in a multidimensional screening problem of dimension m , when the seller raises the price by $\epsilon > 0$ then they earn extra profits of order $O(\epsilon)$ from the remaining bidder types but the measure types excluded from the mechanism is of order $O(\epsilon^m)$. However, simulation results from (Belloni, Lopomo, and Wang 2010) suggest that in certain asymmetric settings, this intuition fails and it is optimal for the seller not to exclude any bidder types. This finding is corroborated by the theoretical work of (Thirumulanathan, Sundaresan, and Narahari 2019a) where the optimal mechanism for a single bidder with valuations distributed uniformly on a rectangle will include settings without exclusion.

3 Model & Setup

In this section, I introduce the optimal multidimensional auction design problem for a single with multiple quality levels in addition to the specific *exclusive buyer mechanism*. There is one seller wishing to sell one item with $j = 1, \dots, K$ quality levels to $i = 1, \dots, N$ bidders. Bidder i ’s valuation (their type) of quality level j is denoted $X_j^i = [\underline{x}_j^i, \bar{x}_j^i] \subset \mathbb{R}_+$. Each bidder’s vector of valuations is given by $X^i = \prod_j X_j^i$ and I will denote by $X = \prod_i X^i$. I will denote by X^{-i} the types of all bidders except for i . Bidder i ’s type is private information and is known only to themselves.

Bidder i ’s valuation for quality level j is distributed according to the cumulative density function F_j^i . The joint density of all bidders’ valuations of all quality grades is denoted F , and, again, denote by F^{-i} the distribution of types of all bidders except bidder i . The joint density is known to the seller. It is assumed that F is continuously differentiable. Furthermore, as is common in the setting of (Myerson 1981), it is assumed that the distributions of bidders’ valuations are independent and identical. However, a bidder’s valuations across quality grades may be correlated.

A crucial step to solving the optimal auction design problem was the use of the *revelation principle* which simplifies the search space for optimal mechanisms (see Myerson 1981, Lemma 1). The revelation principle allows the auction designer to restrict their attention to a class of mechanisms called *direct mechanisms*. Direct mechanisms are those where the bidders simultaneously and confidentially reveal their types to the seller and the seller decides who gets the object and how much each bidder must pay, as a function of their types.

Thus, a direct mechanism is described by a pair of functions (q, p) . The *allocation function* $q : X \rightarrow [0, 1]^{KN}$ specifies the probability $q_j^i(x)$ for some $x \in X$ that bidder i receives the good with quality level j . Note that in deterministic mechanisms $q_j^i(x) \in \{0, 1\}$ for all $x \in X$. The *price function* $p : X \rightarrow \mathbb{R}^N$ specifies the amount each bidder pays (bidders might be required to pay even if they do not receive the good, as

occurs in an ‘all-pay’ auction).

The utility functions of the seller and bidders are risk-neutral and additively separable. The bidders’ utilities are given by

$$u^i(x) = \sum_j x_j^i q_j^i(x) - p^i(x) \quad (1)$$

for all $x \in X$. Denote bidder i ’s expected utility as

$$U^i(x^i) = \int_{X^{-i}} u^i(x^i, x^{-i}) dF^{-i}(x^{-i}) \quad (2)$$

for all $x^i \in X^i$. I assume for simplicity of presentation that costs are zero⁵. The seller’s utility function is given by

$$u^0(x) = \sum_i p^i(x) - \sum_i \sum_j r_j q_j^i(x) \quad (3)$$

where r is the seller’s value estimate for the object, which is most commonly interpreted as the reserve price. Thus, the seller’s expected utility is given by

$$\int_X u^0(x) dF(x) \quad (4)$$

However, not every pair of functions (q, p) represents a *feasible* auction mechanism. There are three types of constraints⁶ that must be imposed on (q, p) .

First, since there is only one object to be allocated, the allocation function must satisfy the following feasibility conditions (F):

$$\sum_i \sum_j q_j^i(x) \leq 1 \text{ and } q_j^i(x) \geq 0 \quad (F)$$

for all $i = 1, \dots, N$, $j = 1, \dots, K$ and $x \in X$. Note that, in contrast to the multidimensional setting of a single good with multiple quality levels, in the multi-item case where the seller has K goods to sell the probability conditions are given by $\sum_i q_j^i(x) \leq 1$ and $q_j^i(x) \geq 0$ for all $i = 1, \dots, N$, $j = 1, \dots, K$ and $x \in X$.

Second, the mechanism (p, q) must be *individually rational* (IR) in the sense that every bidder has non-negative expected utility from participating in the mechanism. More formally,

$$U^i(x^i) \geq 0 \quad (IR)$$

for all bidders $i = 1, \dots, N$ and all $x^i \in X^i$.

⁵TODO add costs (or note these are simply another way of handling reserve prices...)

⁶Here, we outline (IR) and (IC) constraints when the solution concept is a Bayesian Nash equilibrium.

Third, the revelation mechanism can only be implemented if no bidder can expect to gain from lying about their type. If bidder i misrepresents their true type x^i with the lie \hat{x}^i their expected utility would be

$$\int_{X^{-i}} \sum_j x_j^i q_j^i(\hat{x}^i, x^{-i}) - p^i(\hat{x}^i, x^{-i}) dF^{-i}(x^{-i}) \quad (5)$$

Thus, in a direct mechanism it is necessary to ensure

$$U^i(x^i) \geq \int_{X^{-i}} \sum_j x_j^i q_j^i(\hat{x}^i, x^{-i}) - p^i(\hat{x}^i, x^{-i}) dF^{-i}(x^{-i}) \quad (\text{BIC})$$

for all $i = 1, \dots, N$ and $x^i, \hat{x}^i \in X^i$. This final condition is known as *Bayesian incentive compatibility* (BIC).

The revenue maximization problem faced by the seller is therefore

$$\begin{aligned} \max_{p, q} \int_X \left(\sum_i p^i(x) - \sum_i \sum_j r_j q_j^i(x) \right) dF(x) \\ \text{subject to } (F), (IR), (BIC) \end{aligned} \quad (\text{OPT})$$

which I shall refer to as OPT throughout.

3.1 Interim Allocation Formulation

In the case where the buyers have identical valuation distributions for the quality levels of the good, it is possible to reformulate the objective function to remove the dependence of the dimensionality N . This is particularly helpful for designing approximation algorithms (see, for example, Belloni, Lopomo, and Wang 2010). This is done using *interim* variables (Q, U) , obtained by integrating out all but one type of buyer. The interim probability that a buyer is awarded the object quality grade j is

$$Q_j(x) = \int_{X^{-i}} q_j^i(x, x^{-i}) dF^{-i}(x^{-i}) \quad (6)$$

and the interim expected utility of each buyer can therefore be defined as

$$U(x) = \sum_j x_j Q_j(x) - \int_{X^{-i}} p^i(x, x^{-i}) dF^{-i}(x^{-i}) \quad (7)$$

It is then possible to rewrite the above constraints that used the *ex-post* allocations and transfer functions

(q, p) in terms of interim variables (Q, U) . The incentive-compatibility constraint BIC becomes:

$$U(x) - U(\hat{x}) \geq \sum_j Q_j(x)(x_j - \hat{x}_j) \quad \forall x, \hat{x} \in X \times X \quad (\text{IIC})$$

Additionally, the interim individual rationality constraint becomes:

$$U(x) \geq 0 \quad \forall x \in X \quad (\text{IIR})$$

Thus, the objective function OPT can be rewritten in terms of interim variables (Q, U) :

$$\begin{aligned} \max_{Q, U} \int_X \left(\sum_j (x_j - r_j) Q_j(x) - U(x) \right) dF(x) \\ \text{subject to } (F), (IIR), (IIC) \end{aligned} \quad (\text{OPT}_*)$$

Furthermore, the feasibility constraints (F) can be rewritten using (**border 1991 implementation**)⁷.

3.2 Exclusive Buyer Mechanism

The exclusive buyer mechanism was introduced in (Brusco, Lopomo, and Marx 2011) for the case of two quality levels. This can be understood as an auction where the buyers compete in a second price or ascending-bid auction (with reserve prices) for the right to be the only buyer and choose which quality grade to purchase. Here, I generalize the exclusive buyer mechanism to the case with an arbitrary number of bidders and then show how to recover an analog of the interim allocations from the objective function (OPT_*) which facilitate the qualitative comparison of the approximately optimal mechanism yielded by algorithm with a description of the exclusive buyer mechanism. This section is definitional: the objective is to express formal properties of the exclusive buyer mechanism clearly and generally to enable our exploration of its optimality.

We can develop a formal description of the exclusive buyer mechanism by considering an analog of the single dimensional case. Following (Myerson 1981), each quality grade j of the good has a *virtual value* for each bidder i , which we denote $\beta_j^i(x_j^i)$. This virtual value is monotone increasing in x_j^i and depends only on this argument. (Hereafter, we simply refer to each virtual value as β_j^i without its argument). For each bidder, we define $\beta^i = \max_j \beta_j^i$ as the maximum of the quality grade-specific virtual values. (Note, although the virtual values depend on the reserve price, we omit the notational dependence on r for clarity.)

From this, we can determine the winner of the auction as the highest β^i . Allowing for ties, we can denote

⁷See Appendix 8 for details about how this affects computational performance.

the set of winners as follows:

$$M(x) = \{i \mid \beta^i > \beta^{i'} \forall i' \neq i \text{ and } \beta^i \geq 0\} \quad (8)$$

Then the allocation q is defined as:

$$q_j^i(x) = \begin{cases} \frac{1}{|M(x)|} & i \in M(x) \text{ and } \beta_j^i = \max_{j'} \beta_{j'}^i \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

To find the price $p(x)$, consider the simple problem of $N = 1$ bidder with $J = 2$ quality grades. Since we are constrained such that all paths of integration used to find the price $p(x)$ for $x \in X \subset \mathbb{R}^{NJ}$ must be equal, we can find the price by integrating along a simple path consisting of $\ell_1 : (0, 0) \rightarrow (x_1, 0)$ and $\ell_2 : (x_1, 0) \rightarrow (x_1, x_2)$. Without loss of generality let $x_2 - r_2 \geq x_1 - r_1$ and assume that all bids are above the reserve price (otherwise there would be no winner):

$$p(x) = p(0) + \int_0^x q(t) dt \quad (10)$$

$$= p(0) + \int_{\ell_1(t)} q(\ell_1(t)) \cdot \ell_1'(t) dt + \int_{\ell_2(t)} q(\ell_2(t)) \cdot \ell_2'(t) dt \quad (11)$$

$$= p(0) + \int_0^{x_1} q_1(t, 0) dt + \int_0^{x_2} q_2(x_1, t) dt \quad (12)$$

$$= p(0) + \int_0^{x_1} \mathbb{1}\{t - r_1 \geq 0\} dt + \int_0^{x_2} \mathbb{1}\{t - r_2 \geq x_1 - r_1 \text{ and } t - r_2 \geq 0\} dt \quad (13)$$

$$= p(0) + \beta_2 \quad (14)$$

Where $p(0)$ denotes the smallest reserve price⁸.

We can also work with the interim formulations of the allocations for the exclusive buyer mechanism. As defined above,

$$Q_j(x) = \int_{X^{-i}} q_j^i(x, x^{-i}) dF^{-i}(x^{-i}) \quad (15)$$

$$= \int_{\ell(x^{-i})} q_j^i(x, \ell(x^{-i})) \cdot \ell'(x^{-i}) dF^{-i}(x^{-i}) \quad (16)$$

Which, for an arbitrary number of quality levels J , yields⁹:

$$Q_j(x) = \mathbb{1}\{\beta_j \geq \beta_{j'} \forall j' \neq j \text{ and } \beta_j \geq 0\} F^{N-1}(\max\{\bar{x}_1, r_1 + \beta_j\}, \dots, x_j, \dots, \max\{\bar{x}_J, r_J + \beta_j\}) \quad (17)$$

⁸TODO what is going on here...

⁹In the event of ties where $\beta_1 = \beta_2$ then both allocations Q_1, Q_2 are equal and are half the value of $F^{N-1}(x_1, x_2)$.

Where, recall, $F^{N-1}(x_1, x_2) = F(x_1, x_2)^{N-1}$ since bidder's valuations are independent and identically distributed. Furthermore, the interim expected utility for each bidder can be calculated as:

$$U(x) = \sum_j x_j Q_j(x) - \int_{X^{-i}} p^i(x, x^{-i}) dF^{-i}(x^{-i}) \quad (18)$$

$$= \max_j \int_{\underline{x}_j}^{x_j} Q_j(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_J) dt \quad (19)$$

What to do here...

Lastly, note the expected revenue for a given reserve price r is

$$R(r) = \int_X p(x; r) dF(x) \quad (20)$$

where p is defined above. Additionally, this can be numerical approximated as

$$R(r) \approx \sum_{x \in X_T} p(x; r) f(x) \quad (21)$$

This expression is used below for the calculations of the expected revenue from the exclusive buyer mechanism.

4 Conjectures and Simulations

In this section, I explore whether the exclusive buyer mechanism (exclusive buyer mechanism) approximates the optimal revenue in settings where analytic results are known or those where they can be approximated arbitrarily well algorithmically. The goal is to uncover the qualitative features of the optimal mechanism in the multidimensional setting of a single good with multiple quality levels. The structure of this section is as follows. First, we introduce the strategy adopted to investigate the qualitative features of optimal mechanisms in section (4.1). Next, in section (4.2), we outline the specific conjectures investigated in this chapter. Finally, we investigate the optimal mechanisms that result from our approximation algorithms in a wide range of settings in section (4.3).

4.1 Methodology

The strategy we adopt to investigate the qualitative properties of optimal mechanisms in multidimensional settings is:

1. For a given setting, determine the optimal mechanism from the results of running the approximation algorithm (outlined in Appendix 8).

2. Once the optimal mechanism is found, examine the discretized interim allocations yielded by the approximation algorithm. The idea is to visually represent the discretized allocation for a quality level of the good (e.g., Figure 1) and find a mathematical expression that corresponds to the approximately optimal mechanism.

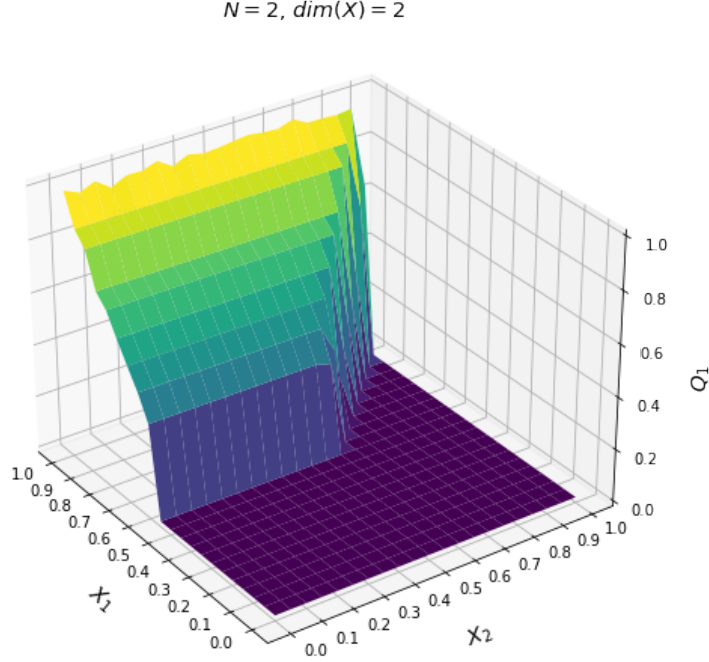


FIGURE 1: The interim allocation Q_1 for the first quality grade in the setting of symmetric, independent uniform setting of (Pavlov 2011) when $X \sim U[0, 1]^2$. Notice, for example, the allocation is monotonic in values of X_1 and that the reserve price $p \in [.5, .6)$.

3. Based on the description of the exclusive buyer mechanism in Section 3.2, check if it matches the interim allocations yielded by the approximation algorithm. In Figure (1), the interim allocation can be approximated by the interim allocation for the exclusive buyer mechanism (above):

$$Q_1(x; p) = \mathbb{1}\{\beta_1 > \beta_2 \text{ and } \beta_1 \geq 0\} F^{N-1}(x_1, \min\{\bar{x}_2, r_2 + \beta_1\}) \quad (22)$$

Examination of the interim allocations of mechanisms yielded by the approximation algorithms is the basis for hypotheses concerning the optimal mechanisms.

By considering a wide enough range of cases and leveraging results from existing research in multidimensional mechanism design, we can develop some economic intuitions about the qualitative features of optimal mech-

anisms in a comparatively simpler multidimensional setting without worrying about the problem of bundling in the setting of multiple bidders *and* multiple goods. Since this approach is broadly susceptible to problems of discretization when approximating the optimal mechanism as well as precision issues with numerical computing, conclusions are, at best, a promising guide to developing further theoretical results. Thus, results are best interpreted as conjectures concerning the character of optimal mechanisms in multidimensional settings.

4.2 Conjectures

The principal conjecture investigated in this thesis chapter concerns the optimality of the exclusive buyer mechanism in the multidimensional setting of a single good with multiple quality levels:

Conjecture 1 (Revenue). *The revenue of the exclusive buyer mechanism well-approximates the revenue of the optimal mechanism.*

By measuring the discrepancy between revenue from the exclusive buyer mechanism and that returned by the approximation algorithm it is possible to confirm or reject this conjecture.

It is also important to explore the interim allocations yielded by the approximation algorithm and compare them to those of the exclusive buyer mechanism. There should be visual confirmation that the optimal mechanism yielded by the algorithm is qualitatively similar to the exclusive buyer mechanism:

Conjecture 2 (Allocations). *The allocation of the exclusive buyer mechanism well-approximates the allocation of the optimal mechanism yielded by the approximation algorithm.*

Note, if conjecture 2 holds, it automatically implies 1, since the allocations include the reserve price from which the revenue is calculated. However, since it is possible that the exclusive buyer mechanism approximates the revenue of the optimal mechanism but fails to share qualitative features of its allocation, it is helpful to distinguish between both conjectures.

Additionally, a surprising feature of some optimal mechanisms in the setting of a single good with multiple quality levels noted by several economists is that the set of types excluded by the allocation—the *exclusion region*—in equilibrium sometimes has measure zero (e.g., Thirumulanathan, Sundaresan, and Narahari 2019b; Belloni, Lopomo, and Wang 2010). This surprising finding stands in opposition to the result of (Armstrong 1996), where it was shown that in the case of a multiproduct monopolist that it is always optimal to exclude a positive measure of buyers. Thus, we explore under what circumstances the exclusion region is measure zero. Specifically, we conjecture:

Conjecture 3 (Measure Zero Exclusion Region). *There exist multidimensional settings where a single good with multiple quality levels is sold to multiple bidders with a measure zero exclusion region.*

Finally, we conjecture that the exclusion region does not change with the number of bidders.

Conjecture 4 (Same Exclusion Region for all N). *The exclusion region of the optimal mechanism in the multidimensional setting of a single good with multiple quality levels remains the same for $N = 1, 2, 3, \dots$ bidders.*

4.3 Simulations

In what follows we explore conjectures 1, 2, 4 in the following contexts:

1. The **symmetric, independent, and uniform setting**, where buyers' valuations are independent across quality grades, uniformly and symmetrically distributed. I analyze the case where two bidders have identical valuations for a good with two quality grades, where each valuation is assumed to be distributed $X_1, X_2 \sim U[0, 1]$. **Additionally, I analyze the case where $X_1, X_2 \sim U[2, 3]$ since, in contrast to the previous setting, it is known that in these settings the optimal mechanism involves randomization when $N = 1$.** Analytic results in the single buyer case when $X \sim U[c, c + 1]^2$ are known (Pavlov 2011) and serve as a benchmark.
2. The **symmetric, independent, and non-uniform setting**. Here, it is desirable to see if the conclusions reached in the first setting extend to non-uniform distributions. In particular, we consider the case of the $Beta(\alpha, \beta)$ distribution (where $\alpha = 1, \beta = 2$), which was explored in (Daskalakis, Deckelbaum, and Tzamos 2017) in the context of multiple-goods.
3. The **symmetric, correlated setting**. It is unknown how arbitrary correlations between a buyer's dimensions of value affect the revenue-maximization problem faced by the auction designer in multidimensional settings. This setting aims to shed light on this problem by considering buyers with valuations drawn from $X_1 = X_2 = [0, 1]$ where the distribution of valuations is $f(x_1, x_2) = x_1 + x_2$.
4. The **asymmetric, independent, and uniform setting**. No analytic results are known in this similar setting; however, a very provisional analysis of the optimality of the exclusive buyer mechanism in this setting can be found in (Belloni, Lopomo, and Wang 2010), where $X_1 \sim U[6, 8], X_2 \sim U[9, 11]$ with costs $c_1 = .9, c_2 = 5$. I replicate their analyses and extend their results in the context of the three conjectures proposed above.
5. The **asymmetric, independent, and non-uniform setting**, a direct extension of the above setting where the valuations for each quality level of the good are drawn from two different truncated normal distributions. In particular, I consider the case where $X_1 \sim truncnorm(\mu = 2.5, \sigma = 1, \underline{x}_1 = 2, \bar{x}_1 = 3)$ and $X_2 \sim truncnorm(\mu = 2.8, \sigma = .2, \underline{x}_2 = 2, \bar{x}_2 = 3)$.

4.3.1 Symmetric, independent, and uniform

Analytic results in the case of a single buyer exist in the symmetric, independent, and uniform setting considered here. (Pavlov 2011) studied the case of two substitute goods¹⁰ independently and uniformly distributed on $U[c, c + 1]^2$. In the specific case of a single buyer with valuations distributed according to $X \sim U[0, 1]^2$ with zero costs, it is known that the optimal mechanism is deterministic and involves setting reserve price $p^* = \frac{1}{\sqrt{3}}$ for both goods (since the valuations are symmetric). Thus, the optimal allocation is given in Figure 2 and the auctioneer's revenue is simply $p^*(1 - p^{*2}) = 0.3849\dots$

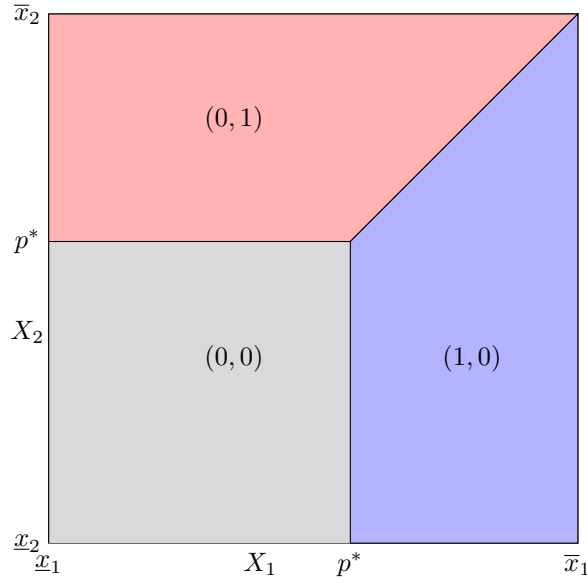


FIGURE 2: The optimal allocation of a single good with two quality symmetric levels to a single buyer with valuations $X \sim U[0, 1]^2$ (Pavlov 2011). The area denoted $(0, 0)$ is the ‘exclusion region’, where the good is not allocated. Note, in the setting of (Pavlov 2011), $p^* = \sqrt{1/3}$.

In the case where there is more than one bidder we need to rely on the approximation algorithm to study the qualitative features of the optimal mechanism. First, we can confirm the approximation algorithm yields similar revenue to that calculated by the appropriate exclusive buyer mechanism in this setting. These results are presented in Table 3.

¹⁰Note that when there is a single buyer an equivalent interpretation of the setting with one good and multiple quality levels is that there are multiple goods but the buyer has unit demand.

| Result Type | T | Revenue |
|---------------------------|-----|-------------|
| approximation | 5 | 0.66094... |
| approximation | 10 | 0.625929... |
| approximation | 15 | 0.612877... |
| approximation | 20 | 0.606033... |
| exclusive buyer mechanism | 20 | 0.596034 |

TABLE 3: comparison of revenue generated by approximation algorithm with that of the exclusive buyer mechanism. T represents the number of intervals used to discretize each dimension of the buyers' valuations.

Note that the revenue generated by the exclusive buyer mechanism was computed using the *ex-post* description of the auction. Additionally, the exclusive buyer mechanism's revenue was computed by numerical integration on the same-sized discretization grid as that used by the approximation algorithm. The trend for different T implies that the approximation algorithm is converging to the result provided by the exclusive buyer mechanism. This supports Conjecture 1.

Using the *interim* allocation of the exclusive buyer mechanism described in section (3.2), we can plot the allocations against those returned by the approximation algorithm. These are presented side-by-side in Figure 4.

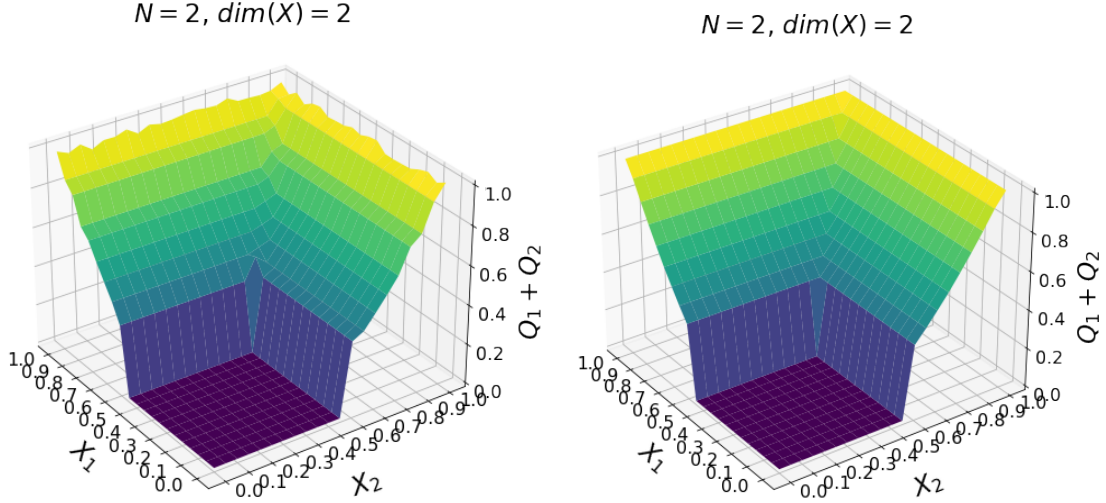


FIGURE 4: The allocations produced by the approximation algorithm (left) and exclusive buyer mechanism (right).

Conjecture 2 requires qualitatively evaluating the allocations returned from the approximation algorithm and those calculated from the exclusive buyer mechanism, without an obvious means to conclusively determine whether the conjecture is supported. However, the results in Figure 4 imply that exclusive buyer mechanism

captures the behavior of the (approximately) optimal mechanism. Thus, these results offer support for Conjecture 2.

Notice that the exclusion region in Figure 4 is similar to that discovered in (Pavlov 2011). This is evidence against Conjecture 3 concerning the existence of measure zero exclusion regions. The price $p^* = \frac{1}{\sqrt{3}} = 0.577\dots$ that maximizes revenue in the case when $N = 1$ is consistent with the exclusion region defined by $p^* = 0.6$ returned by the approximation algorithm for the case of $N = 2$. (Note that when $[0, 1]$ is discretized into 20 intervals, the price is $p \in \{\dots, .5, .55, .6, \dots\}$ so the choice by the algorithm reflects its approximate optimality). Indeed, when $T = 20$, the approximation algorithm yields the same exclusion region for all of $N = 1, 2, 3$. This supports Conjecture 4.

4.3.2 Symmetric, independent, and non-uniform setting

I examine the four conjectures in the context of a symmetric, independent, and non-uniform setting. Following the multi-unit example in (Daskalakis, Deckelbaum, and Tzamos 2017), I consider the case where $X_1, X_2 \sim \text{Beta}(\alpha, \beta)$ where $\alpha = 1, \beta = 2$. Note, to the best of my knowledge, no prior work on analytic solutions to the optimal auction design problem exists in this setting. Therefore, we proceed by running the optimization algorithm and comparing the output of the algorithm to that provided by the exclusive buyer mechanism described above.

First, we compare the revenue generated by the approximation algorithm and the exclusive buyer mechanism. The results are presented in Table 5. Again, note the revenue from the exclusive buyer mechanism was calculated using *ex-post* from a second-price auction, and the discretized grid was the same as that used by the approximation algorithm. The similarity of the revenues generated by the algorithm and the exclusive buyer mechanism provides support for Conjecture 1.

| Result Type | T | Revenue |
|---------------------------|-----|-------------|
| approximation | 5 | 0.448709... |
| approximation | 10 | 0.418815... |
| approximation | 15 | 0.406948... |
| approximation | 20 | 0.400615... |
| exclusive buyer mechanism | 20 | 0.391506... |

TABLE 5: comparison of revenue generated by the approximation algorithm with that of the exclusive buyer mechanism when $X_1, X_2 \sim \text{Beta}(1, 2)$.

Next, we can compare the interim allocations from the approximation algorithm with those from the exclusive buyer mechanism. These are displayed graphically in Figure 6. There is a clear similarity between

both allocations, supporting Conjecture 2.

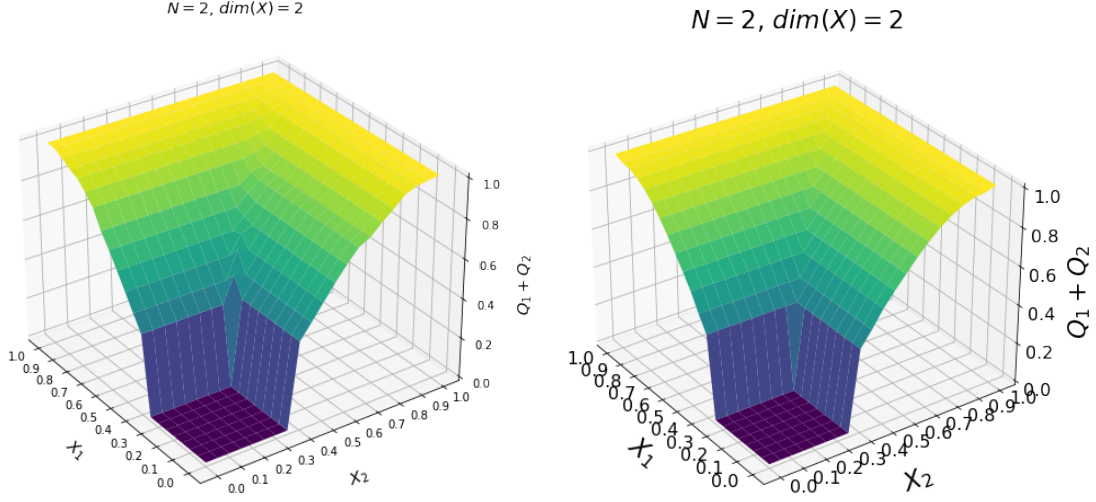


FIGURE 6: The allocations produced by the approximation algorithm (left) and exclusive buyer mechanism (right).

Note, these allocations offer a lack of support of Conjecture 3.

Finally, running the approximation algorithm for $N = 1, 2, 3$ confirms Conjecture 4. The exclusion region is the same for all N tested: the value of $p^* = 0.4$. (Recall, discretization of the grid into $T = 20$ intervals per quality level requires that $p^* \in \{\dots, 0.35, 0.4, 0.45, \dots\}$).

Thus, in conclusion, all conjectures are supported in the setting of symmetric, independent, and non-uniform distributions except for Conjecture 3 concerning the existence of a measure zero exclusion region.

4.3.3 Symmetric, correlated setting

In this setting, I allow for the valuations X_1 and X_2 to be correlated. Here, $X \sim F, f(x_1, x_2) = x_1 + x_2$, where $X_1 = X_2 = [0, 1]$. This extension of the previous settings on the unit square facilitates a deeper understanding of correlated valuations in a familiar setting. As above, I proceed by running the approximation algorithm and comparing the results with those from the exclusive buyer mechanism.

With regard to revenue, we can see in Table 7 that the algorithm's revenue is well-approximated by the exclusive buyer mechanism. This supports Conjecture 1.

| Result Type | T | Revenue |
|---------------------------|-----|-------------|
| approximation | 5 | 0.75159... |
| approximation | 10 | 0.718683... |
| approximation | 15 | TODO |
| approximation | 20 | 0.698962... |
| exclusive buyer mechanism | 20 | 0.686437... |

TABLE 7: comparison of revenue generated by the approximation algorithm with that of the exclusive buyer mechanism when $f(x_1, x_2) = x_1 + x_2$.

The allocations are also similar. The interim allocations from the approximation algorithm and the exclusive buyer mechanism are presented in Figure 8. This supports Conjecture 2 concerning the qualitative similarity of the optimal mechanism yielded by the approximation algorithm and that of the exclusive buyer mechanism.

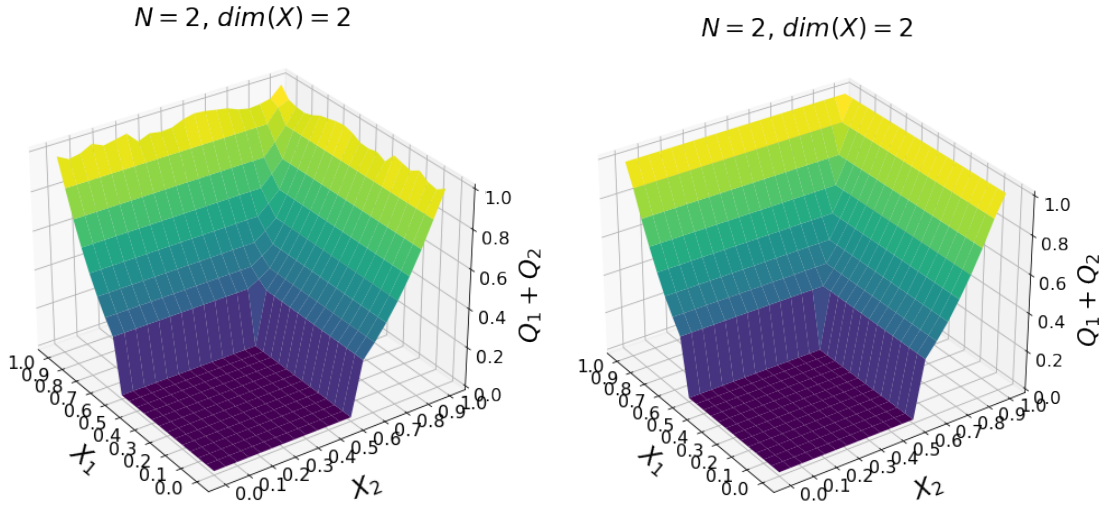


FIGURE 8: The allocations produced by the approximation algorithm (left) and the exclusive buyer mechanism (right).

Again, note the presence of an exclusion region undermines support for Conjecture 3 concerning the existence of settings without an exclusion region.

Finally, the exclusion regions in the symmetric, correlated, uniform setting where $X_1 = X_2 = [0, 1]$ and $f(x_1, x_2) = x_1 + x_2$ are the same for all $N = 1, 2, 3$. Note, the reserve price $p^* = .65$ in this setting. This supports Conjecture 4.

As in the previous setting, all conjectures except for Conjecture 3 concerning the existence of a measure

zero exclusion region are supported.

4.3.4 Asymmetric, independent, and uniform setting

In this setting, we consider the case where $X_1 \sim U[6, 8]$ and $X_2 \sim U[9, 11]$ as considered in (Belloni, Lopomo, and Wang 2010), who first studied the optimality of the exclusive buyer mechanism in multidimensional settings. Additionally, the costs associated with selling the first quality grade of the good are $c_1 = .9$ and the second quality grade are $c_2 = 5$. Since prior computational work exists assessing the optimality of the exclusive buyer mechanism, where possible, I can compare my findings here with those in (Belloni, Lopomo, and Wang 2010).

The revenue yielded by the approximation algorithm is similar to that of the exclusive buyer mechanism. The data from are displayed in Table 9. These revenue numbers are consistent with those in (Belloni, Lopomo, and Wang 2010, Table 3), supporting Conjecture 1, namely, that the revenue of the optimal mechanism is well-approximated by the exclusive buyer mechanism.

| Result Type | T | Revenue |
|---------------------------|-----|-------------|
| approximation | 5 | 6.02496... |
| approximation | 10 | 5.941549... |
| approximation | 15 | TODO |
| approximation | 20 | 5.893113... |
| exclusive buyer mechanism | 20 | 5.814852... |

TABLE 9: comparison of revenue generated by the approximation algorithm with that of the exclusive buyer mechanism in the setting of (Belloni, Lopomo, and Wang 2010).

With regard to the allocations, we can see a superficial similarity between the interim allocations produced by the approximation algorithm and those of the exclusive buyer mechanism. These are presented in Figure 10. However, unlike previous settings, there is a clear discrepancy between the interim allocation of the exclusive buyer mechanism and that yielded by the algorithm: the exclusive buyer mechanism's interim allocation for $Q_1 + Q_2$ is much lower for large values of X_2 than that of the approximation algorithm.

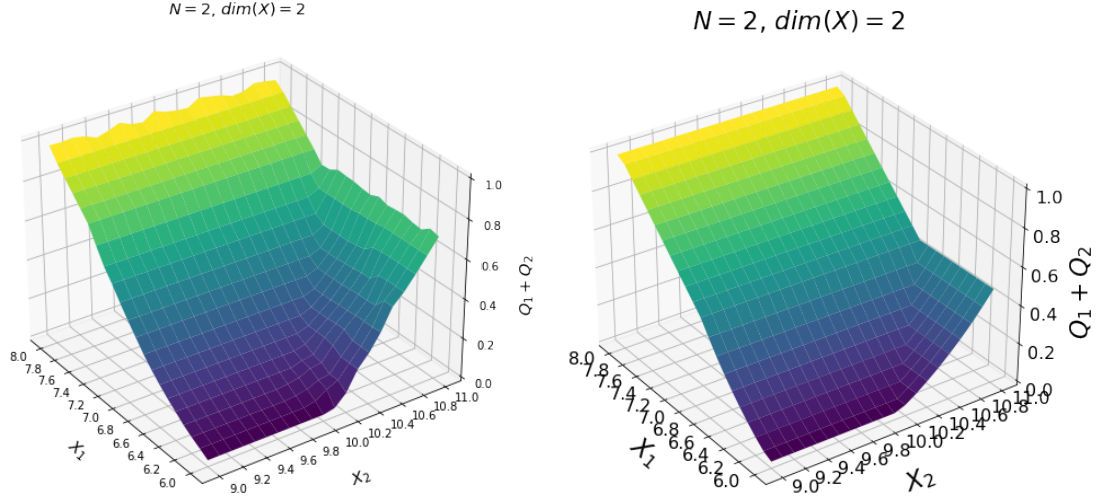


FIGURE 10: The allocations produced by the approximation algorithm (left) and the exclusive buyer mechanism (right).

This surprising feature of the exclusive buyer mechanism's interim allocation suggests that Conjecture 2 is *not* supported in the asymmetric, independent, and uniform setting. Here, although the optimal revenue is well approximated by the exclusive buyer mechanism, the qualitative features of the optimal mechanism are sufficiently different from those conjectured by the exclusive buyer mechanism. We will explore this result in more detail in the discussion section (5) below.

Notice, however, in this setting Conjecture 3 is supported. This is visible in Figure 11 below¹¹. This was also noted in the simulations (Belloni, Lopomo, and Wang 2010), suggesting it is robust to computational or approximation error. Additionally, Conjecture 4 is supported for all $N = 1, 2, 3$; however, it is important to note a number of surprising features of the optimal mechanism yielded by the approximation algorithm in this setting. Firstly, although the exclusion region is the same for all N , the *allocation* itself varies with the number of buyers. Secondly, there is evidence of randomization in the optimal mechanism yielded by the approximation algorithm.

¹¹TODO fix axis labels.

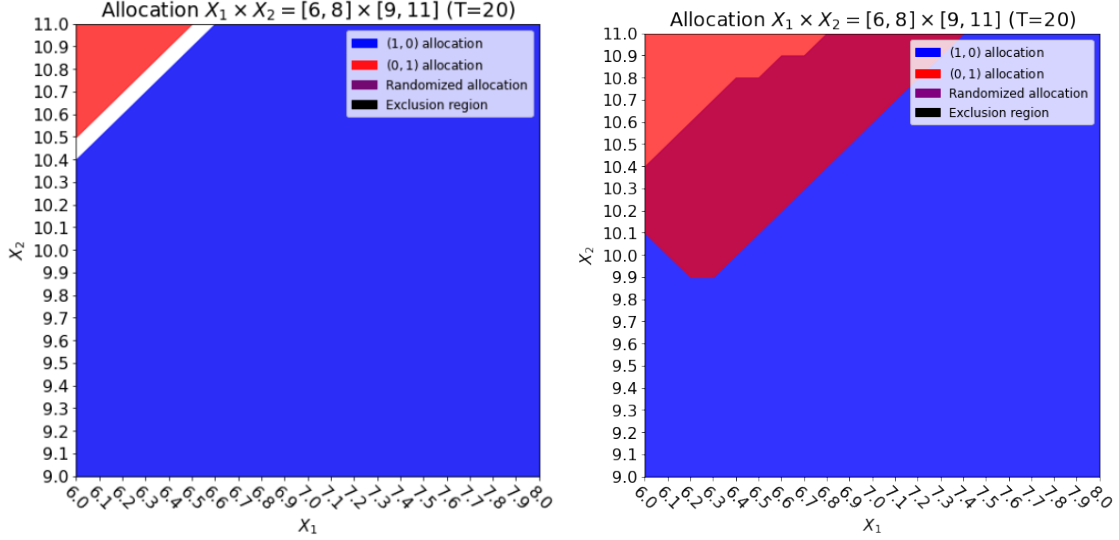


FIGURE 11: Graphs showing where the allocations for the first (Q_1) and second (Q_2) quality level of the good are non-zero for the optimal mechanism yielded by the approximation algorithm for $N = 1$ (left) and $N = 2$ (right).

Thus, in the symmetric correlated setting, although Conjectures 1, 3, 4 are supported, Conjecture 2 is not. The complexities of this result will be further explored below.

4.3.5 Asymmetric, independent, and non-uniform setting

I extend the investigation of asymmetric settings by considering the cases where buyers' valuations are distributed according to two different distributions. In particular, I consider the case where the valuations are drawn from $X_1 = X_2 = [2, 3]$ and the distribution of valuations is asymmetric, where $X_1 \sim \text{truncnorm}(\mu = 2.5, \sigma = 1)$ and $X_2 \sim \text{truncnorm}(\mu = 2.8, \sigma = .2)^{12}$. Note, in contrast to the previous asymmetric setting considered above, in this setting the sets from which the valuations are drawn are equal (i.e., $X_1 = X_2$) but the distributions are not (i.e., $f_1(x) \neq f_2(x)$). Again, no prior analytic results exist in this setting and therefore I proceed by running the approximation algorithm for the optimal auction and comparing the result to that of the exclusive buyer mechanism.

First, I compare the revenue generated by the approximation algorithm and the exclusive buyer mechanism. This is presented in Table 12. The similarity of the revenues generated by the approximation algorithm and the exclusive buyer mechanism lends support to Conjecture 1.

¹²TODO make this more asymmetric with $\mu_1 = 2.3$ and $\mu_2 = 2.9$

| Result Type | T | Revenue |
|---------------------------|-----|-------------|
| approximation | 5 | 2.784335... |
| approximation | 10 | 2.753352... |
| approximation | 15 | 2.740046... |
| approximation | 20 | 2.733195... |
| exclusive buyer mechanism | 20 | 2.727316... |

TABLE 12: comparison of revenue generated by the approximation algorithm with that of the exclusive buyer mechanism when $X_1 \sim \text{truncnorm}(\mu = 2.5, \sigma = 1, \underline{x}_1 = 2, \bar{x}_1 = 3)$ and $X_2 \sim \text{truncnorm}(\mu = 2.8, \sigma = .2, \underline{x}_2 = 2, \bar{x}_2 = 3)$.

When we compare the interim allocations generated by the approximation algorithm to those of the exclusive buyer mechanism in Figure 13, we can qualitatively see that the allocations are similar. This supports Conjecture 2.

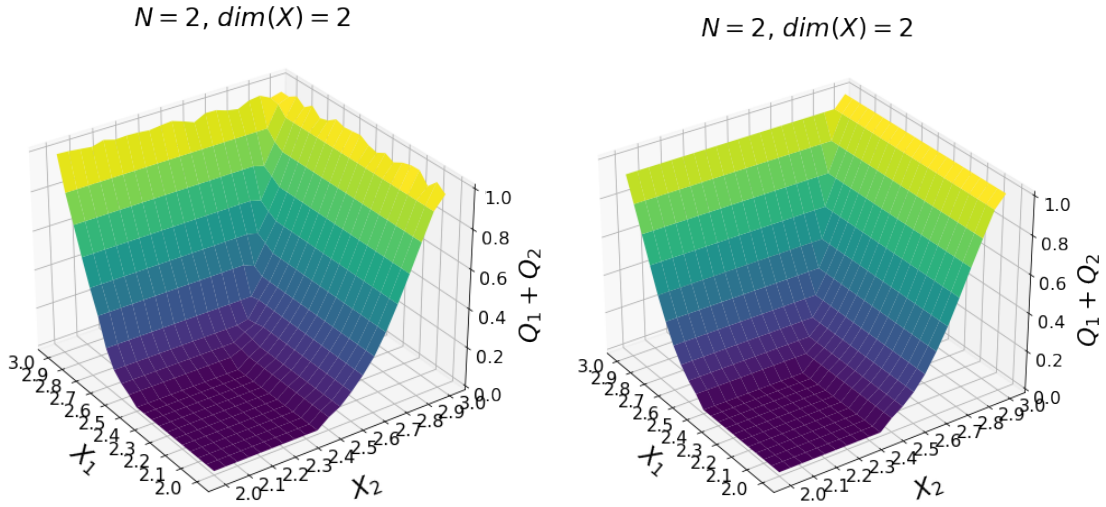


FIGURE 13: The allocations produced by the approximation algorithm (left) and exclusive buyer mechanism (right).

Additionally, note that the existence of an exclusion region in Figure 13 above undermines Conjecture 3.

Finally, running the approximation algorithm for all $N = 1, 2, 3$ confirms Conjecture 4. In this asymmetric setting $p_1^* = 2.55$ and $p_2^* = 2.5$ and the exclusion region remains the same for all N .

Thus, in conclusion, all conjectures are supported in the setting of asymmetric, independent, and non-uniform distributions except for Conjecture 3 concerning the existence of a measure zero exclusion region.

5 Discussion

In this section, I consider the results presented across a variety of settings in Section 4.3 and assess the support for each conjecture in turn.

5.1 Conjecture 1

This conjecture asserted that “the revenue of the exclusive buyer mechanism well-approximates the revenue of the optimal mechanism”. Across all settings considered above, this conjecture is supported. However, support for this conjecture alone is far from sufficient to demonstrate the optimality of the exclusive buyer mechanism. It is well known (see, for example, Cai, Daskalakis, and Weinberg 2012) that ‘simple’ mechanisms can approximate optimal (stochastic) mechanisms up to some constant fraction of their revenue. In practice; however, simple mechanisms can often do much better. Thus, these results indicate that the exclusive buyer mechanism might be a suitable candidate mechanism to use the multidimensional setting of a single good with multiple quality levels when stochastic mechanisms might be optimal. Ultimately, the results presented here are consistent with existing evidence from simulations concerning the optimality of the exclusive buyer mechanism (Belloni, Lopomo, and Wang 2010) and support Conjecture 1.

5.2 Conjecture 2

Across all settings considered here, the interim allocations of the optimal mechanism yielded by the approximation algorithm are qualitatively similar to those of the exclusive buyer mechanism. This similarity is hard to quantify and, in some cases, there is evidence that the two allocations are noticeably different. This is particularly true for the asymmetric, independent, and uniform case of (Belloni, Lopomo, and Wang 2010), which I will discuss shortly. For all symmetric settings considered here as well as the asymmetric, independent, and non-uniform setting, Conjecture 2 is supported. Indeed, the similarity in many cases is clear and highly suggestive: the exclusive buyer mechanism captures the same qualitative behavior of the optimal mechanism yielded by the approximation algorithm.

However, this similarity is not apparent in the asymmetric, uniform case of (Belloni, Lopomo, and Wang 2010). In this setting, there is evidence of randomization; which suggests the exclusive buyer mechanism does capture the qualitative behavior of the optimal mechanism. This randomization can be seen in Figure 14 where the allocations for each quality grade are shown separately. As can be seen in the graph of Q_1 on the left, a “fold” in the allocation occurs around the line $x_1 - c_1 = x_2 - c_2$ (recall, $c_1 = .9, c_2 = 5$). In this region, both $Q_1(x) > 0$ and $Q_2(x) > 0$.

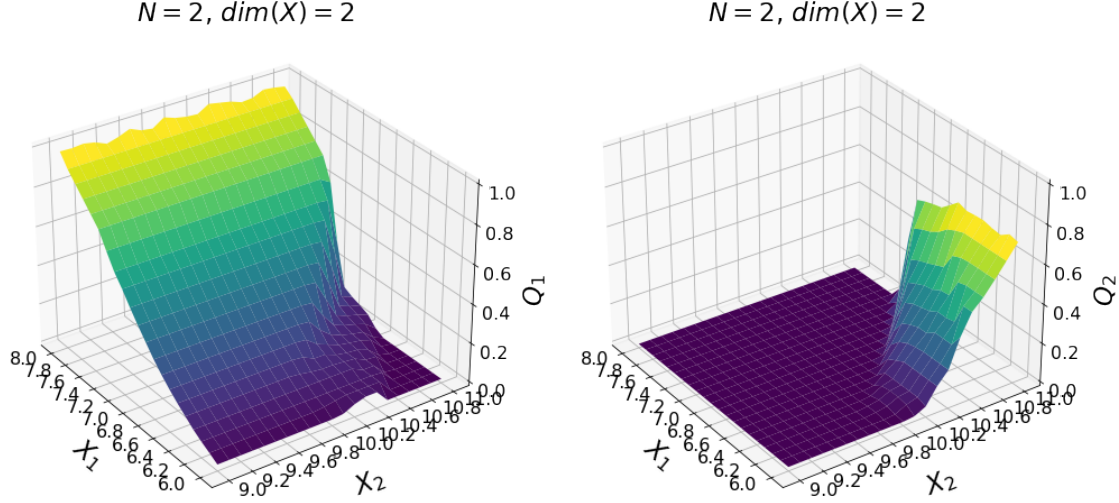


FIGURE 14: The allocation Q_1 for the first quality level (left) and Q_2 for the second quality level (right) of the good produced by the approximation algorithm when $N = 2$.

The lack of support for Conjecture 2 in this instance suggests that, where the optimal mechanism includes stochastic contracts, the exclusive buyer mechanism is not qualitatively similar to it. It is possible that this result is a computational artifact: the optimal mechanism might nonetheless be ‘simple’ in the sense of deterministic but the approximation algorithm yields an approximately optimal stochastic mechanism.

5.3 Conjecture 3

Only in the asymmetric, independent, and uniform setting of (Belloni, Lopomo, and Wang 2010) considered here is there support for Conjecture 3. This results was also been remarked on by (Thirumulanathan, Sundaresan, and Narahari 2019a), who study the multidimensional setting of a single good with multiple quality levels when bidder valuations are uniformly distributed on an arbitrary rectangle $[a, b] \times [c, d]$. This finding stands in contrast to theoretical work in multidimensional mechanism design where it has been shown in the multi-unit case that it is always optimal to exclude a positive measure of bidder types (Armstrong 1996; Rochet and Choné 1998). Notice that the typespace $[6, 8] \times [9, 11]$ used in this example violates the assumption of strict convexity in (Armstrong 1996)’s result. In addition, Figure 10 suggests for the mechanism yielded by the approximation algorithm, there is an interval $6 \times [9, p^*]$ for some $p^* \in [9, 11]$ that receives zero utility in equilibrium. The measure of this interval is zero. However, contrary to (Armstrong 1996, Proposition 1), if one increases price by ϵ , one would exclude a mass of types proportional to ϵ not ϵ^2 . Furthermore, in the case of (Rochet and Choné 1998), they require the cost function to be smooth, which is violated in the setting above.

5.4 Conjecture 4

Conjecture 4 is supported in all settings considered here for $N = 1, 2, 3$. Computational intractability precludes the study of settings with more bidders but these initial results suggest a promising line of research where, for any given multidimensional setting of a single good with multiple quality levels and bidders with identical distributions of valuations, finding the set of values excluded from the mechanism in equilibrium when $N = 1$ can be used as a ‘stepping-stone’ to begin research for $N > 1$. This is particularly helpful for work at the intersection of optimal transport and auction theory, where, for example, (Kolesnikov et al. 2022) develop tools for the certification of (potentially) optimal mechanisms.

6 Conclusion

Analytic results concerning the qualitative characteristics of optimal mechanisms in multidimensional settings are scarce. In this chapter, I explored the particular multidimensional setting of a single good with multiple quality levels. In particular, I investigated whether the *exclusive buyer mechanism*—understood as an (either a second price or ascending bid) auction with a reserve price for the exclusive right to be the buyer of one of the quality levels of the good—is optimal in this setting. To do this, I developed an approximation algorithm that facilitates the investigation of the optimal mechanism in multidimensional settings and compared the performance of the exclusive buyer mechanism to that of the optimal mechanism yielded by the approximation algorithm across a number of questions.

I explored four conjectures concerning the optimality of the exclusive buyer mechanism in the multidimensional setting of a single good with multiple quality levels. In particular,

1. Does the exclusive buyer mechanism result in similar revenue as the approximately optimal mechanism?
(**Yes.**)
2. Are the interim allocations of the exclusive buyer mechanism qualitatively similar to those of the approximately optimal mechanism? (**Yes when there is no randomization.**)
3. Are there settings when the measure of excluded types is zero in equilibrium? (**Yes for asymmetric, independent, and uniform settings.**)
4. Is the set of types excluded from the mechanism the same for all N ? (**Yes.**)

There is strong support for Conjectures 1 and 4 in all settings explored here. While there is strong evidence for Conjecture 2 in some settings, there is also evidence that the exclusive buyer mechanism fails to capture the qualitative behavior of the optimal mechanism when the optimal mechanism involves randomization, as

in the case of (Belloni, Lopomo, and Wang 2010). Finally, only in the prior asymmetric, independent, and uniform settings is there support for Conjecture 3.

Unfortunately, all the settings considered here involve the case of bidders with identical valuation distributions. This limits the generality of the results in this chapter. Further research should extend these results to cases with a variety of different bidders. This is especially important because Conjecture 4 likely requires identical bidder distributions to hold.

Finally, it should be noted these results suggest a fruitful avenue of further theoretical research. Since there is strong evidence to support Conjecture 4, namely, that the exclusion region is the same for all N , it might potentially be productive to begin analytic investigations of optimal mechanisms in the multidimensional setting of a single good with multiple quality levels with a determination of the exclusion region when $N = 1$. This may shed light on the exclusion region when $N > 1$. Ultimately, it is my hope these computational results facilitate new theoretical developments in multidimensional auction design when analytic results are currently out of reach.

7 References

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8 Appendix: Approximation Algorithm

I¹³ adopt and improve the original finite-dimensional approximation algorithm of (Belloni, Lopomo, and Wang 2010) by focusing on local and downward-sloping incentive-compatibility constraint (ICC) violations. Although these local constraints are often violated in this approximate setting, I drastically reduce the number of times *all* incentive-compatible constraints need to be checked.

In order to approximate an optimal solution to OPT_* , I discretize the type space X . Let T denote a positive integer that controls the granularity of the discretization. For each $j \in J$, let $X_T(j)$ denote the discretization of the interval $[\underline{x}_j, \bar{x}_j]$ given by $X_T(j) = \{\underline{x}_j, \underline{x}_j + \epsilon, \underline{x}_j + 2\epsilon, \dots, \bar{x}_j\}$ where $\epsilon = \min_{j \in J} \{(\bar{x}_j - \underline{x}_j)/T\}$. Our discretized version of the type space X is given by $X_T := \prod_{j \in J} X_T(j)$. Furthermore, I define a probability density function on X_T by setting $\hat{f}(v) = f(v)/(\sum_{t \in X_T} f(t))$. I thus obtain a linear program which is a finite-dimensional approximation of OPT_* for each $T > 0$ by replacing X with X_T .

Belloni et al. (2010) use a plane-putting algorithm which works with a randomly chosen subset of incentive-compatibility (ICC) and Border (B) constraints at each iteration. They provide an efficient reduction in the growth in T of the Border constraints (B) from $O(2^{T^J})$ to $O(T^J \log(T^J))$ (Belloni, Lopomo,

¹³TODO fix references+links+notation

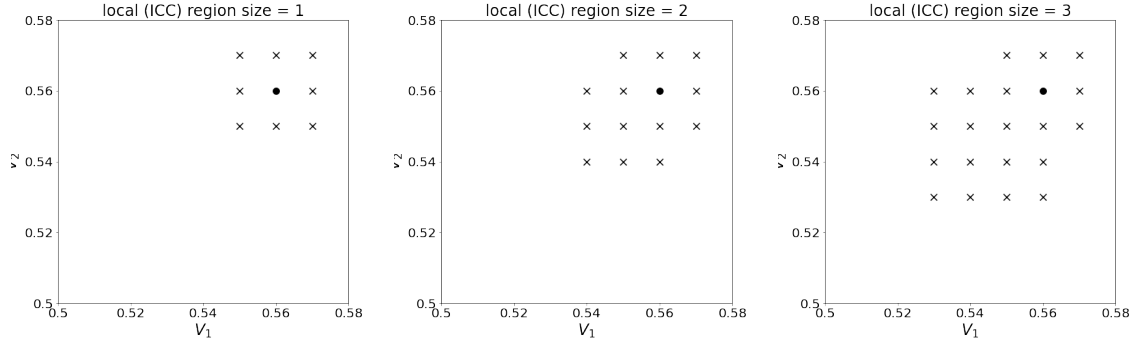


FIGURE 15: We iteratively grow the local region of the discretized type space checked for downwards-sloping constraint violations. Notice that immediately adjacent (ICC) constraints are always checked (\times) when the local region increases in size around a point (\bullet).

and Wang 2010, Lemma 10). We adopt their solution to checking (B) constraints; however, our approach to checking (ICC) constraints involves iteratively growing the ‘local’ region of the type space around each point v in the discretized set of types V_T . We do two things. First, all the immediately adjacent points in the discretized type space are always checked for incentive compatibility. Secondly, downwards-sloping points in the discretized type space are also checked. Furthermore, the downwards-sloping region of the type space grows until all (ICC) constraints are ultimately satisfied. This procedure is illustrated visually in Figure 15. Thus, for a fixed-size local region around each point in the discretized type space, we first satisfy *local* (ICC) and (B) constraints as in the iterative plane-cutting algorithm of (Belloni, Lopomo, and Wang 2010). Then we run the separation oracle with *all* (ICC) and (B) constraints. We then restart the solver with any previously violated constraints, this time increasing the size of the local region around each point in the discretized type space. This procedure iterates until no constraints are violated. This modified version of (Belloni, Lopomo, and Wang 2010)’s algorithm is described in Algorithm 1.

Our algorithm¹⁴ is written in Python 3.10 and uses Google’s open source linear programming solver ‘GLOP’ available in their `or-tools` package (Perron and Furnon 2023).

¹⁴For more details see: <https://github.com/jmemich/optimal-auction-multidim>

```

 $L = 1, S = \emptyset, A = \emptyset, \overline{OPT} = \infty;$ 
violated_any_icc  $\leftarrow$  TRUE;
while violated_any_icc do
    violated_local_icc  $\leftarrow$  TRUE;
    while violated_local_icc do
         $k = 1, A^k = A, S^k = S;$ 
        Solve the linear program associated with  $S^k$ . Let  $OPT^k$ 
            denote the optimal value.;
        Solve the separation oracle using only local (ICC) constraints
            in region  $L$ . Let  $A^k$  denote all violated local (ICC) and (B)
            constraints.;
        if  $A^k = \emptyset$  then
            violated_local_icc  $\leftarrow$  FALSE;
            Break;
        end
        Select a subset  $I^k \subset S^k$  of inactive (ICC) and (B)
            constraints;
        if  $OPT^k < \overline{OPT}$  then
             $S^{k+1} \leftarrow (S^k \setminus I^k) \cup A^k \cup A;$ 
             $\overline{OPT} \leftarrow OPT^k;$ 
        else
             $S^{k+1} \leftarrow S^k \cup A^k \cup A;$ 
        end
         $k \leftarrow k + 1;$ 
    end
    Solve the separation oracle using all (ICC) constraints. Let  $A^*$ 
        denote all violated (ICC) constraints.;
    if  $A^* = \emptyset$  then
        violated_any_icc  $\leftarrow$  FALSE;
        Break;
    end
     $A \leftarrow A \cup A^*;$ 
     $S \leftarrow S^k;$ 
     $L \leftarrow L + 1;$ 
end

```

Algorithm 1: Iterative plane-cutting algorithm with local and downwards-sloping (ICC) constraints