# Is The ‘Exclusive Buyer Mechanism’ Optimal?

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**Abstract**

We explore the properties of optimal multi-dimensional auctions in a setting where a single object of multiple qualities is sold to several buyers. Using simulations, we test the hypothesis that the optimal mechanism is an *exclusive buyer mechanism*, where buyers compete to be the right to be the only buyer to choose between quality levels of a multi-dimensional good. We find compelling evidence of the optimality of the exclusive buyer mechanism in multi-dimensional settings and, as part of this work, we provide the first open-source library for multidimensional auction simulations written entirely in Python.

## Contents

# 1 Introduction

# 2 Literature Review

TODO vincent work.

The specific case considered here of selling a single good with multiple quality levels to an arbitrary number of buyers is a special case of the more general multidimensional mechanism design problem of selling an arbitrary number of goods to an arbitrary number of buyers. Since the groundbreaking work of Roger Myerson (1981) who solved the optimal auction design problem in the case of single-dimensional types, economists have sought to characterize optimal auctions in the more general multidimensional setting, with limited success. At this juncture, it is widely accepted that "[e]ssentially, nothing has been known about optimal auctions in this [multidimensional] setting" (Kolesnikov et al. 2022, p1). This literature review covers historical and recent developments by economists and computer scientists who have sought to uncover characteristics of optimal mechanisms in multidimensional settings, with a particular focus on the case of a single good with multiple quality levels.

In the case of single-dimensional types, early work on optimal mechanism design demonstrated the optimality of deterministic mechanisms (i.e., reserve or 'take-it-or-leave-it' prices) (Myerson 1981; Riley and Zeckhauser 1983). These approaches leveraged the approach of integration-by-parts (as used in Mussa and Rosen 1978) to solve the relaxed optimal mechanism design problem without directly considering incentive-compatibility constraints. These early results cannot be generalized to multi-dimensional settings because the integral solution to the optimization problem is path-dependent; any two points in the multi-dimensional typespace can be connected by a continuum of paths. Thus, a major breakthrough in multidimensional auction design came with the use of duality-based approaches to multidimensional screening developed by (Rochet and Choné 1998) which circumvents this problem.

The duality-based approach (Rochet and Choné 1998) builds on the single-dimensional nonlinear pricing framework of (Mussa and Rosen 1978), which was given its canonical formulation in multidimensional settings in (Wilson 1993) and (Armstrong 1996). This work takes as its point of departure the approach of (Mirrlees 1971) on optimal taxation and relies on results that establish the implementability of a decision rule in multidimensional settings (Rochet 1987). In this multidimensional screening problem, a few key findings emerge. The first is that 'bunching'—a situation where multiple types are treated identically in the optimal solution—is a "robust" feature of multidimensional screening (Rochet and Stole 2003; Rochet and Choné 1998). There are two types of bunching: in the first case, a set of types of positive measure are excluded from purchasing the goods in the optimal solution (this is commonly known as the 'exclusion region'); in the second case, a non-negligible set of types outside the exclusion region receive the same product although they have different tastes. In addition, the work of (Rochet and Choné 1998) illustrates that the optimal solution to multidimensional screening problems may involve "bundling" the goods, which involves selling multiple goods together.

In multi-item settings, authors have long sought to characterize when bundling multiple goods in a single contract is optimal for the seller. Bundling strategies available to a seller include 'pure' bundling, where only the bundle of all goods is offered to sellers, and 'mixed' bundling, where each different bundle of items is priced separately. Early results showed that offering mixed bundles strictly dominates offering pure bundles to the buyers (Adams and Yellen 1976; McAfee, McMillan, and Whinston 1989) and more recent results have demonstrated that randomized bundles may dominate mixed bundles (Thanassoulis 2004; Daskalakis, Deckelbaum, and Tzamos 2017). In these settings, the optimal menu of contracts may include infinitely many randomized bundles (Manelli and Vincent 2007; Hart and Nisan 2019). Additionally, recent work has demonstrated settings where simply offering only the grand bundle of all goods is optimal (Haghpanah and Hartline 2021).

In the past few years, major breakthroughs in optimal multidimensional mechanism design have come from the use of the methods of optimal transport applied to the optimization problems of microeconomic theory (see Ekeland 2010). These results (Daskalakis, Deckelbaum, and Tzamos 2017; Kolesnikov et al. 2022) greatly aid the *certification* of optimality: the techniques of optimal transport facilitate the identification of the dual of the seller's optimization problem from which a given mechanisms' optimality can be verified. Thus, previously existing results that characterize optimal mechanisms in specific settings (for example, where valuations for two goods are i.i.d on $U[0,1]^2$ (Pavlov 2011; Manelli and Vincent 2006)) can be shown to be optimal using a novel, more general approach. The success of the tools of optimal transport in mechanism design is due to the success of a 'guess-and-verify' approach where one guesses a solution to the primal problem and then the dual solution plays the role of a certificate of optimality for the initial guess.

These breakthroughs which facilitate the certification of optimality are particularly helpful when viewed in light of the growth in work at the intersection of economics and computer science. One line of work (Chawla, Hartline, and Kleinberg 2007; Cai, Daskalakis, and Weinberg 2012; Cai, Devanur, and Weinberg 2016; Belloni, Lopomo, and Wang 2010; Alaei et al. 2019) provides an algorithmic approximation of optimal mechanisms in multidimensional settings. Here, the buyer's typespace is discretized, and linear programming techniques are used to approximate the optimal solution, often using simple mechanisms like posted prices. Work in this area aims to achieve a constant factor of the optimal revenue achievable by a Bayesian incentive-compatible mechanism through an approximation. Other work at the intersection of computer science and economics offers insights into the nature of the optimal mechanisms in multidimensional settings. These works show that in specific settings, optimal mechanisms contain only a few contract points (Wang and Tang 2014) or that menus with only a finite number of items cannot ensure any positive fraction of optimal revenue (Hart and Nisan 2019).

Returning to the specific context multidimensional mechanism design context of a single good with multiple quality levels, the work of (Belloni, Lopomo, and Wang 2010) provides insight into the character of the optimal mechanism in this particular setting. Applying their algorithm to concrete cases, they find a number of surprising results from their simulations. First, there is clear evidence that in the optimal solution, a measure-zero set of buyers is excluded from the allocation in equilibrium. This stands in marked contrast to results in the multi-item case which show that the optimal solution requires exclusion (Rochet and Choné 1998; Armstrong 1996). Second, their results indicate the optimality an *exclusive-buyer mechanism*: it performs "quite well" relative to the numerical optimal solutions and that it "shares many of its defining features with its one-dimensional counterparts" (Belloni, Lopomo, and Wang 2010, p1085-6), including being implementable in dominant strategies. This mechanism involves an auction (with a reserve price) among buyers for who gets to be the sole recipient of the good. A premium can then be paid for whichever quality grade the winning buyer desires. Interestingly, this mechanism is entirely deterministic in the single-bidder case. This is particularly surprising because, in the neighboring multi-item case, randomized allocations are widely considered necessary for revenue maximization (Daskalakis 2015).

The theoretical study of exclusive-buyer mechanisms originates from the phenomenon of 'contingent re-auctions' where sellers will modify objects sold to benefit themselves or the general public (Brusco, Lopomo, and Marx 2011). For example, in the context of the US Spectrum License Auction 73 held in 2008[1], the US government adopted a contingent re-auction format where it offered restricted spectrum licenses first, and committed to re-auction the licenses without many of the restrictions in the case the reserve prices were not met. Brusco, Lopomo, and Marx (2011) show that an exclusive-buyer mechanism can always be parameterized such that the mechanism induces the efficient outcome in dominant strategies. However, outside of a restrictive context where all bidders' valuations for the restricted object are a fixed percentage of the unrestricted object, no general results concerning the optimality of the mechanism are provided.

Analytic results concerning optimal multidimensional auctions for a single good with multiple quality levels and a single bidder are scarce. Notably, (Pavlov 2011) investigates the case where the bidder's valuations for the object are uniformly distributed on the unit square $[c, c+1]^2$. Pavlov finds that the optimal mechanism varies considerably with $c$ and sometimes requires randomization for revenue maximization. This approach was further generalized in the work of (Thirumulanathan, Sundaresan, and Narahari 2019a)

[1]For more details see (Brusco, Lopomo, and Marx 2009).

who study an almost identical case where a bidder's valuations are distributed uniformly on the rectangle $[c, c+b_1] \times [c, c+b_2]$. Similarly to (Pavlov 2011), the solution to the optimal mechanism design problem entails both deterministic and stochastic contracts. Surprisingly, however, (Thirumulanathan, Sundaresan, and Narahari 2019a) find evidence of settings where optimal mechanisms do not exclude a position measure of buyers. Additionally, a working paper by (Haghpanah and Hartline 2014) gives sufficient conditions for the optimality of posting a single, uniform price for all quality levels of a good, albeit in a restricted class of settings.

Analytic results for optimally selling substitute goods have also been given in the Hotelling model (Hotelling 1929) where two horizontally differentiated goods are located at the endpoints of a segment. In this setting, (Balestrieri, Izmalkov, and Leao 2020) find that stochastic contracts are part of the optimal mechanism. The economic intuition that arises from this body of research is clear: by offering a lottery over which good the bidder receives, a seller can offer a discount to entice marginal buyers who would otherwise choose the outside option. Similarly, (Loertscher and Muir 2023), find that in this setting randomization is required by the seller to maximize revenue. These results support earlier work (Thanassoulis 2004) which shows that in the standard auction design problem for two substitute goods, the seller can always increase revenue by including stochastic contracts alongside take-it-or-leave-it prices in the optimal mechanism. As noted above, these result supports the view that in multidimensional settings randomization is required to maximize revenue (see Daskalakis 2015).

The approaches of (Pavlov 2011; Thirumulanathan, Sundaresan, and Narahari 2019a) to solving the mechanism design problem for the case of a single good with multiple quality levels follows the work of (Guesnerie and Laffont 1984), where optimal control theory is used to address the fact the measure of participating types endogenously depends on the mechanism. The optimal control theory approach[2] has also been successfully applied to single-dimensional settings when the participation constraints are endogenously determined by the mechanism (Jullien 2000). Here, the bidder's reservation utility depends on their type. This approach generalizes to accommodate the fact that the measure of participating types in a given mechanism is endogenously determined (for example, in the multidimensional case of a single good with two quality levels and a single buyer considered by Pavlov 2011; Thirumulanathan, Sundaresan, and Narahari 2019a).

Finally, although it has long been believed that it is always profitable for the seller to exclude some measure of bidders (Rochet and Choné 1998; Armstrong 1996) in multidimensional settings, recent theoretical and computational work auction design in these settings has challenged these conclusions. The original result

[2]See (Basov 2005, §7) for an extended discussion of the different approaches to multidimensional mechanism design and their respective strengths and weaknesses.

of (Armstrong 1996) demonstrated that in multi-product settings with a single bidder, the seller benefits from always excluding a positive measure of bidder types. By relaxing Armstrong's strong assumptions about the bidder's utility function and the convexity of the type space this result has been extended and it has been shown that "exclusion is generically optimal in a large class of models" (Barelli et al. 2014, p. 75). The intuition is as follows: in a multidimensional screening problem of dimension $m$, when the seller raises the price by $\epsilon > 0$ then they earn extra profits of order $O(\epsilon)$ from the remaining bidder types but the measure types excluded from the mechanism is of order $O(\epsilon^m)$. However, simulation results from (Belloni, Lopomo, and Wang 2010) suggest that in certain asymmetric settings, this intuition fails and it is optimal for the seller not to exclude any bidder types. This finding is corroborated by the theoretical work of (Thirumulanathan, Sundaresan, and Narahari 2019a) where the optimal mechanism for a single bidder with valuations distributed uniformly on a rectangle will include settings without exclusion.

# 3 Model & Setup

TODO set up the whole model; the EBM is just the final piece...

## 3.1 Exclusive Buyer Mechanism

Suppose there are $i = 1, \dots, N$ bidders in an auction for a good with $j = 1, 2$ quality levels. Suppose each bidder's valuation for the good is given by $x^i = (x_1^i, x_2^i) \in [\underline{x}_1^i, \overline{x}_1^i] \times [\underline{x}_2^i, \overline{x}_2^i]$. We assume bidders' valuations are identical and independently distributed but we allow an individual bidder's valuations for quality grades to be arbitrarily correlated. Similarly to (Belloni, Lopomo, and Wang 2010, p. 1085) we construct an approximation for the optimal auction design problem which can be implemented by an *exclusive-buyer mechanism.*

Our approximation is a little different[3] than (Belloni, Lopomo, and Wang 2010)'s. Since bidder valuations are symmetric, we omit the superscript $i$ (i.e., $X = X^i$). For any given *reserve price* $p = (p_1, p_2)$, let

$$\beta_1 = x_1 - p_1 \quad \text{and} \quad \beta_2 = x_2 - p_2 \tag{1}$$

[3]Our approximation does not rely on a "add-on price" in addition to a "reserve price". Instead, we assume a price is given for each quality grade.

and define the interim allocations[4] as:

$$Q_1(x;p) = \mathbb{1}\{\beta_1 > \beta_2 \text{ and } \beta_1 \geq 0\}F^{N-1}(x_1, \min\{\overline{x}_2, p_2 + \beta_1\}) \tag{2}$$

$$Q_2(x;p) = \mathbb{1}\{\beta_2 > \beta_1 \text{ and } \beta_2 \geq 0\}F^{N-1}(\min\{\overline{x}_1, p_1 + \beta_2\}, x_2) \tag{3}$$

where, recall, $F^{N-1}(x_1, x_2) = F(x_1, x_2)^{N-1}$ since bidder's valuations are independent and identically distributed. Furthermore, the interim expected utility for each bidder can be calculated as:

$$U(x;p) = \max\left\{\int_{\underline{x}_1}^{x_1} Q_1(t, x_2; p)dt, \int_{\underline{x}_2}^{x_2} Q_2(x_1, t; p)dt\right\} \tag{4}$$

The expected revenue from price $p$ is given by:

$$R(p) = \int_X (x - c) \cdot Q(x;p) - U(x;p)dF(x) \tag{5}$$

where costs are given by $c = (c_1, c_2)$. In the context of a discretized approximation $T$, the expected revenue is approximated as follows:

$$R(p) \approx N\left(\sum_{x \in X_T}\left[\sum_{j=1,2}(x_j - c_j)Q_j(x;p) - U(x;p)\right]f(x)\right) \tag{6}$$

# 4 Conjectures and Simulations

In this section we test whether the exclusive buyer mechanism approximates the optimal revenue in settings where analytic results are known or those where they can approximated arbitrarily well algorithmically. The strategy we adopt to investigate the optimality of the exclusive buyer mechanism is:

1. Determine the optimal mechanism either (1) from analytic results (e.g., Pavlov 2011) or (2) from the results of running the approximation algorithm outline in Appendix 6.

2. Once the optimal allocation is found, we construct the exclusive buyer mechanism from optimal reserve prices (including lotteries).

3. Using the reserve prices, we then compare the revenue gained from the exclusive buyer mechanism and compare the qualitative features of the exclusive buyer mechanism with those of the optimal mechanism, specifically examining the allocations to determine if optimal mechanism is well-approximated by the exclusive buyer mechanism.

[4] In the event of ties where $\beta_1 = \beta_2$ then both allocations $Q_1, Q_2$ are equal and are half the value of $F^{N-1}(x_1, x_2)$.

Thus, there are two conjectures which are systematically investigated in this thesis:

**Conjecture 1** (Revenue)**.** *The revenue of the exclusive buyer mechanism well-approximates*[5] *the revenue of the optimal mechanisml.*

**Conjecture 2** (Allocations)**.** *The allocation of exclusive buyer mechanism capture the same qualitative behavior of the allocation of the optimal mechanism.*

## 4.1 Settings Without Randomization

### 4.1.1 $N = 1, X = U[0,1]^2, c = [0,0]$ (Pavlov 2011, Example 1)

We begin with the well studied multi-dimensional setting of selling a single good with two quality levels to a single buyer, where the buyer's valuations are uniformly and symmetrically distributed on the unit square $X \sim U[0,1]^2$. A more general investigation of this setting can be found in (Pavlov 2011), where the optimal mechanism is analytically derived for a single buyer who has valuations distributed on $U[c, c+1]^2$ for $c \geq 0$. In this setting, it is known that for $0 \leq c \leq 1$, the optimal mechanism does not involve randomization, instead the reserve price is given by:

$$p_{\text{PAVLOV}} = \frac{2}{3}c + \frac{1}{3}\sqrt{c^2 + 3} \tag{7}$$

which, when $c = 0$, reduces to $p_{\text{PAVLOV}} = \sqrt{\frac{1}{3}}$. When $x_1 > x_2$, then the first quality level of the good is allocated and when $x_2 > x_1$, the second quality level of the good is allocated. Thus, the optimal allocation is in Figure 1[6] and the auctioneer's revenue is simply $p_{\text{PAVLOV}}(1 - p_{\text{PAVLOV}}^2) = 0.3849...$

[5] TODO do we need to be more specific about this?

[6] TODO make this diagram in TikZ.

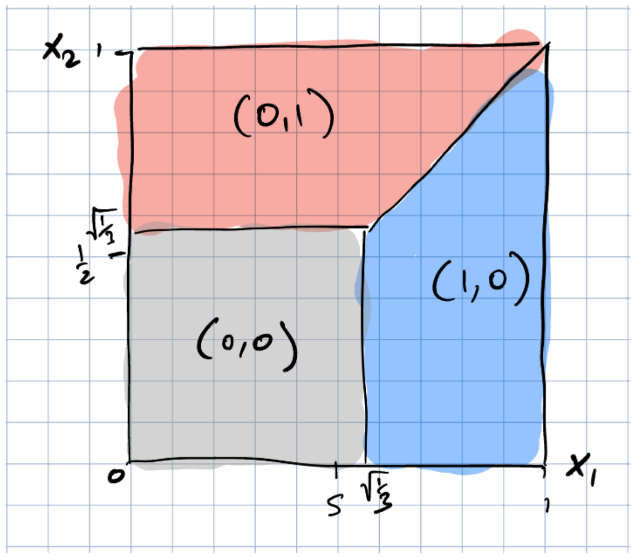


FIGURE 1: The optimal allocation of a single good with two quality symmetric levels to a single buyer with valuations $X = U[0,1]^2$. The area denoted $(0,0)$ is the 'exclusion region', where the good is not allocated.

The exlusive buyer mechanism yields the same revenue in this setting (see Appendix 7 for calculations).

#### 4.1.2 $N = 1, X = U[6,8] \times U[9,11], c = [.9,5]$ (Belloni, Lopomo, and Wang 2010)

Analytic results in single-buyer setting considered by (Belloni, Lopomo, and Wang 2010) have only become available in the past few years (see Thirumulanathan, Sundaresan, and Narahari 2019b)[7]. We begin our analysis of this setting with an examination of the results of the approximation algorithm described in Appendix 6. First, note that the revenue is sensitive to the coarseness of the discretization grid; however, it approaches stable values for larger $T$[8] (see Figure 2).

[7] TODO calculate optimal mechanism from (Thirumulanathan, Sundaresan, and Narahari 2019b)?
[8] TODO kick off script on server to do this...

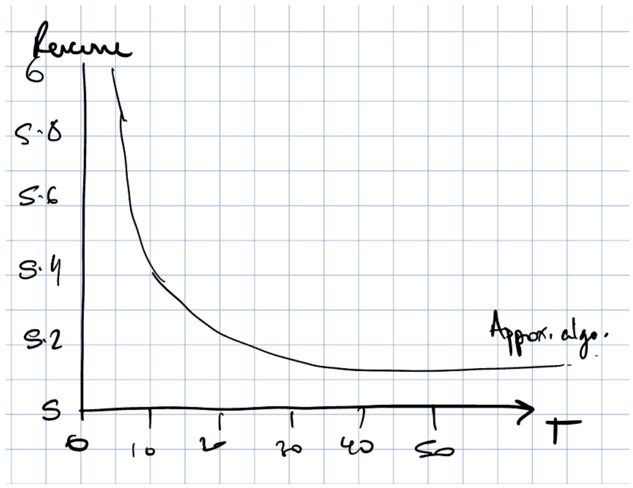


FIGURE 2: The revenue from the approximated optimal mechanism as a function of increasing grid discretization $T$.

Note that revenue results from the approximation algorithm displayed in Table 3 match those from the analytic calculations of the exclusive buyer mechanism (see Appendix 7).

| Result Type | $T$ | Revenue |
|---|---|---|
| Approximation | 10 | 5.124793388429744 |
| Approximation | 20 | 5.11904761904701 |
| Approximation | 50 | |
| EBM | — | TODO: redo with correct price! |

TABLE 3: Revenue comparisons from the approximation algorithm and the exclusive buyer mechanism.

The reserve prices can be calculated by observing the allocations from the approximation algorithm in Figure 4. Thus, $p^* = (6, 10.4)$.

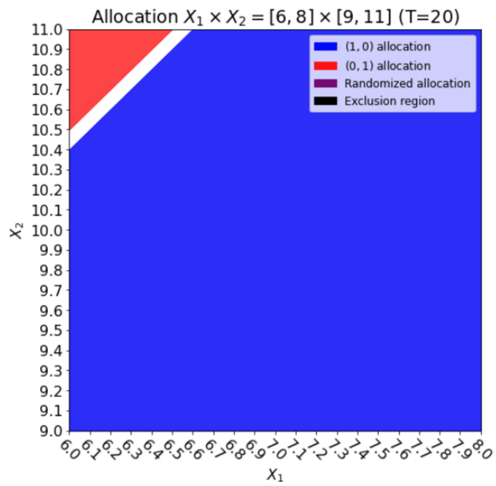


FIGURE 4: The optimal allocation in the setting of (Belloni, Lopomo, and Wang 2010) when $N = 1$. Note, the measure of the exclusion region is zero.

#### 4.1.3 $N = 2, X = U[6, 8] \times U[9, 11], c = [.9, 5]$ (Belloni, Lopomo, and Wang 2010)

TODO cannot use simple integral approximation for exclusive buyer mechanism. Redo calculations. TODO x2 check prices...

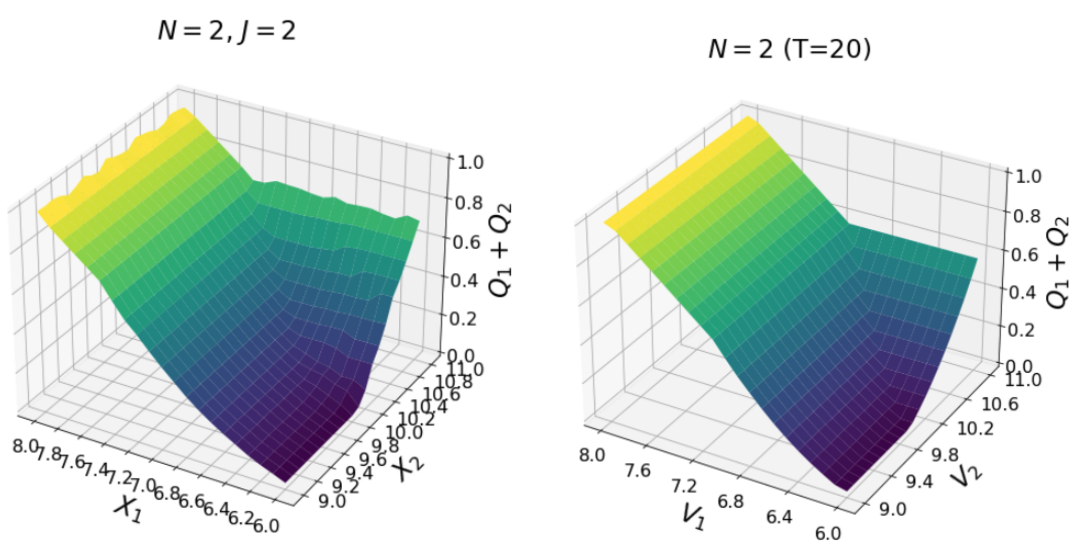


FIGURE 5: The allocations produced by the approximation algorithm (left) and the exclusive buyer mechanism (right).

## 4.2 Settings With Randomization

### 4.2.1 $N = 1, X = U[2,3]^2, c = [0,0]$ (Pavlov 2011, Example 1)

## 4.3 Extensions

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# 6 Appendix: Simulation Description

TODO fix references+links

We adopt and improve the original finite-dimensional approximation algorithm of (Belloni, Lopomo, and Wang 2010) by focusing on local and downward-sloping incentive-compatibility constraint (ICC) violations. Although these local constraints are often violated in this approximate setting, we drastically reduce the number of times *all* incentive-compatible constraints need to be checked.

In order to approximate an optimal solution to **??**, we discretize the type space $V$. Let $T$ denote a positive integer that controls the granularity of the discretization. For each $j \in J$, let $V_T(j)$ denote the discretization of the interval $[\underline{v}_j, \overline{v}_j]$ given by $V_T(j) = \{\underline{v}_j, \underline{v}_j + \epsilon, \underline{v}_j + 2\epsilon, \ldots, \overline{v}_j\}$ where $\epsilon = \min_{j \in J}\{(\overline{v}_j - \underline{v}_j)/T\}$. Our discretized version of the type space $V$ is given by $V_T := \prod_{j \in J} V_T(j)$. Furthermore, we define a probability density function on $V_T$ by setting $\hat{f}(v) = f(v)/(\sum_{t \in V_T} f(t))$. We thus obtain a linear program which is a finite-dimensional approximation of **??** for each $T > 0$ by replacing $V$ with $V_T$.

Belloni et al. (2010) use a plane-putting algorithm which works with a randomly chosen subset of incentive-compatibility (ICC) and Border (B) constraints at each iteration. They provide an efficient reduction in the growth in $T$ of the Border constraints (B) from $O(2^{T^J})$ to $O(T^J \log(T^J))$ (Belloni, Lopomo, and Wang 2010, Lemma 10). We adopt their solution to checking (B) constraints; however, our approach to checking (ICC) constraints involves iteratively growing the 'local' region of the type space around each point

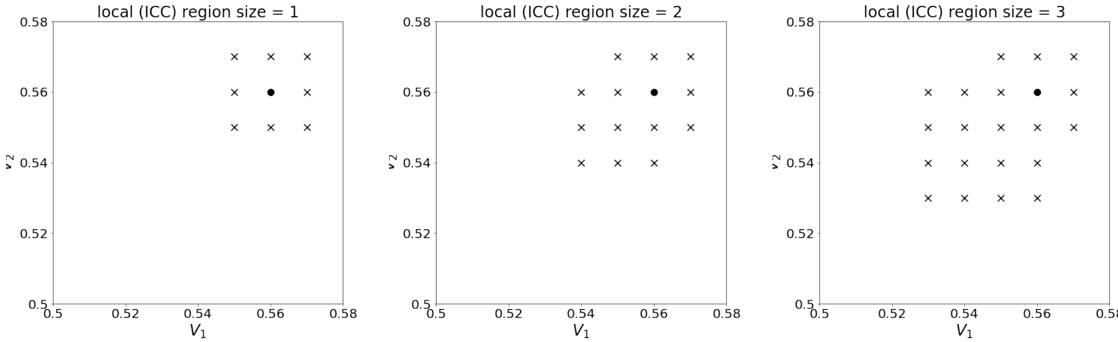


FIGURE 6: We iteratively grow the local region of the discretized type space checked for downwards-sloping constraint violations. Notice that immediately adjacent (ICC) constraints are always checked ($\times$) when the local region increases in size around a point ($\bullet$).

$v$ in the discretized set of types $V_T$. We do two things. First, all the immediately adjacent points in the discretized type space are always checked for incentive compatibility. Secondly, downwards-sloping points in the discretized type space are also checked. Furthermore, the downwards-sloping region of the type space grows until all (ICC) constraints are ultimately satisfied. This procedure is illustrated visually in Figure 6. Thus, for a fixed-size local region around each point in the discretized type space, we first satisfy *local* (ICC) and (B) constraints as in the iterative plane-cutting algorithm of (Belloni, Lopomo, and Wang 2010). Then we run the separation oracle with *all* (ICC) and (B) constraints. We then restart the solver with any previously violated constraints, this time increasing the size of the local region around each point in the discretized type space. This procedure iterates until no constraints are violated. This modified version of (Belloni, Lopomo, and Wang 2010)'s algorithm is described in Algorithm 1.

Our algorithm[9] is written in Python 3.10 and uses Google's open source linear programming solver 'GLOP' available in their `or-tools` package (Perron and Furnon 2023).

[9]For more details see: `https://github.com/jmemich/optimal-auction-multidim`

$L = 1, S = \emptyset, A = \emptyset, \overline{OPT} = \infty$;
`violated_any_icc` $\leftarrow$ TRUE;
**while** *violated_any_icc* **do**
    `violated_local_icc` $\leftarrow$ TRUE;
    **while** *violated_local_icc* **do**
        $k = 1, A^k = A, S^k = S$;
        Solve the linear program associated with $S^k$. Let $OPT^k$ denote the optimal value.;
        Solve the separation oracle using only local (ICC) constraints in region $L$. Let $A^k$ donate all violated local (ICC) and (B) constraints.;
        **if** $A^k = \emptyset$ **then**
            `violated_local_icc` $\leftarrow$ FALSE;
            Break;
        **end**
        Select a subset $I^k \subset S^k$ of inactive (ICC) and (B) constraints;
        **if** $OPT^k < \overline{OPT}$ **then**
            $S^{k+1} \leftarrow (S^k \setminus I^k) \cup A^k \cup A$;
            $\overline{OPT} \leftarrow OPT^k$;
        **else**
            $S^{k+1} \leftarrow S^k \cup A^k \cup A$;
        **end**
        $k \leftarrow k + 1$;
    **end**
    Solve the separation oracle using all (ICC) constraints. Let $A^*$ donate all violated (ICC) constraints.;
    **if** $A^* = \emptyset$ **then**
        `violated_any_icc` $\leftarrow$ FALSE;
        Break;
    **end**
    $A \leftarrow A \cup A^*$;
    $S \leftarrow S^k$;
    $L \leftarrow L + 1$;
**end**

**Algorithm 1:** Iterative plane-cutting algorithm with local and downwards-sloping (ICC) constraints

# 7 Appendix: Calculations

## 7.1 $N = 1, X = U[0,1]^2, c = [0,0]$ (Pavlov 2011)

From (Pavlov 2011, Example 1), we know that the optimal reserve price is $p^* = (\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})$. The revenue according to the exclusive buyer mechanism is given by:

$$
\begin{aligned}
Rev(p) &= \int_{x\in A} Q_1(x;p)(p^* - c_1)dF(x) + \int_{x\in B} Q_2(x;p)(p^* - c_2)dF(x) && (8)\\
&= \int_{p^*}^{1}\int_0^x (p^* - c_1)f(x_1,x_2)dx_2dx_1 + \int_{p^*}^{1}\int_0^y (p^* - c_2)f(x_1,x_2)dx_1dx_2 && (9)\\
&= p^*\left(\int_{p^*}^{1}\int_0^x dx_2dx_1 + \int_{p^*}^{1}\int_0^y dx_1dx_2\right) && (10)\\
&= p^*(1 - p^{*2}) && (11)
\end{aligned}
$$

## 7.2 $N = 1, X = U[6,8] \times U[9,11], c = [.9, 5]$ (Belloni, Lopomo, and Wang 2010)

Note, for all calculations that follow we compute the revenue from a mechanism defined by reserve prices $p^* = (6, 9.9)$.

### 7.2.1 $\int_X Q(x;p) \cdot (p - c)dF(x)$

For ease of calculation, we divide up the type space according to which quality level of the good is preferred:

$$A := \{x \in X | x_1 - p_1 > x_2 - p_2 \text{ and } x_1 - p_1 \geq 0\} \quad (12)$$

$$B := \{x \in X | x_2 - p_2 > x_1 - p_1 \text{ and } x_2 - p_2 \geq 0\} \quad (13)$$

Thus, the objective function becomes

$$
\begin{aligned}
Rev_1(p) &= \int_{x\in A} Q_1(x;p)(x_1 - c_1)dF(x) + \int_{x\in B} Q_2(x;p)(x_2 - c_2)dF(x) && (14)\\
&= \int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\min\{p_2^*+x_1-p_1^*, \overline{x}_2\}} (p_1 - c_1)f(x_1,x_2)dx_2dx_2 + \int_{\underline{x}_1}^{\underline{x}_1+\overline{x}_2-p_2^*}\int_{p_2^*+x_1-p_1^*}^{\overline{x}_2} (p_2 - c_2)f(x_1,x_2)dx_2dx_2 && (15)\\
&= \frac{1}{4}\left(\int_6^8\int_9^{\min\{9.9+x_1-6, 11\}} (6 - .9)dx_2dx_2 + \int_6^{7.1}\int_{9.9+x_1-6}^{11} (9.9 - 5)dx_2dx_2\right) && (16)\\
&= \frac{1}{4}\left(3.395(6 - .9) + 0.605(9.9 - 5)\right) && (17)\\
&= 5.06975 && (18)
\end{aligned}
$$

#### 7.2.2 $\int_X Q(x;p)\cdot(x-c) - U(x;p)dF(x)$

Alternatively, using the definition of $A, B$, note that for all $x \in A$, $U(x;p) = \int_{\underline{x}_1}^{x_1} \mathbb{1}\{t \geq p_1^*\}dt$ and similarly, for $x \in B$, $U(x;p) = \int_{\underline{x}_2}^{x_2} \mathbb{1}\{t \geq p_2^*\}dt$. Therefore,

$$\begin{aligned}
Rev_2(p) &= \int_{x\in A} Q_1(x;p)(x_1 - c_1) - \left(\int_{\underline{x}_1}^{x_1} \mathbb{1}\{t \geq p_1^*\}dt\right)dF(x) && (19)\\
&\quad + \int_{x\in B} Q_2(x;p)(x_2 - c_2) - \left(\int_{\underline{x}_2}^{x_2} \mathbb{1}\{t \geq p_2^*\}dt\right)dF(x) && (20)\\
&= \int_{\underline{x}_1}^{\overline{x}_1} \int_{\underline{x}_2}^{\min\{p_2^*+x_1-p_1^*,\overline{x}_2\}} (x_1 - c_1) - \mathbb{1}\{x_1 \geq p_1^*\}(x_1 - p_1^*)f(x_1, x_2)dx_2dx_1 + && (21)\\
&\quad + \int_{\underline{x}_1}^{\underline{x}_1+\overline{x}_2-p_2^*} \int_{p_2^*+x_1-p_1^*}^{\overline{x}_2} (x_2 - c_2) - \mathbb{1}\{x_2 \geq p_2^*\}(x_2 - p_2^*)f(x_1, x_2)dx_2dx_2 && (22)\\
&= \frac{1}{4}\left(\int_6^8 \int_9^{\min\{9.9+x_1-6,11\}} (6 - .9)dx_2dx_1 + \int_6^{7.1} \int_{9.9+x_1-6}^{11} (9.9 - 5)dx_2dx_2\right) && (23)\\
&= 5.06975 && (24)
\end{aligned}$$

### 7.3 $N = 2, X = U[6,8] \times U[9,11], c = [.9,5]$ (Belloni, Lopomo, and Wang 2010)

Again, we use the reserve price $p* = (6, 9.9)$.

#### 7.3.1 $\int_X Q(x;p)\cdot(p-c)dF(x)$

$$
\begin{aligned}
Rev_1(p) &= N\int_{x\in A} Q_1(x;p)(p_1-c_1)dF(x) + \int_{x\in B} Q_2(x;p)(p_2-c_2)dF(x) && (25)\\
&= N\Bigg(\int_{\underline{x}_1}^{\overline{x}_1}\int_{\underline{x}_2}^{\min\{p_2^*+x_1-p_1^*,\overline{x}_2\}} F(x_1,\min\{p_2^*+x_1-p_1,\overline{x}_2\})(p_1-c_1)f(x_1,x_2)dx_2dx_2 && (26)\\
&\quad+\int_{\underline{x}_1}^{\underline{x}_1+\overline{x}_2-p_2^*}\int_{p_2^*+x_1-p_1^*}^{\overline{x}_2} F(\min\{\overline{x}_1,p_1^*+x_2-p_2^*\},x_2)(p_2-c_2)f(x_1,x_2)dx_2dx_2\Bigg) && (27)\\
&= \frac{N}{4}\Bigg(\int_6^8\int_9^{\min\{9.9+x_1-6,11\}}\left(\frac{x_1-6}{2}\right)\left(\frac{\min\{9.9+x_1-6,11\}-9}{2}\right)(6-.9)dx_2dx_2 && (28)\\
&\quad+\int_6^{7.1}\int_{9.9+x_1-6}^{11}\left(\frac{x_2-9.9}{2}\right)\left(\frac{x_2-9}{2}\right)(9.9-5)dx_2dx_2\Bigg) && (29)\\
&= \frac{N}{4}\Bigg(\int_6^{7.1}\int_9^{9.9+x_1-6}\left(\frac{x_1-6}{2}\right)\left(\frac{9.9+x_1-6-9}{2}\right)(6-.9)dx_2dx_2 && (30)\\
&\quad+\int_{7.1}^8\int_9^{11}\left(\frac{x_1-6}{2}\right)(6-.9)dx_2dx_2 && (31)\\
&\quad+\int_6^{7.1}\int_{9.9+x_1-6}^{11}\left(\frac{x_2-9.9}{2}\right)\left(\frac{x_2-9}{2}\right)(9.9-5)dx_2dx_2\Bigg) && (32)\\
&= \frac{2}{4}\Big(2.10971+7.1145+.937523\Big) && (33)\\
&= 5.0808665 && (34)
\end{aligned}
$$

#### 7.3.2 $\int_X Q(x;p)\cdot(x-c)-U(x;p)dF(x)$

Now, consider the revenue in $B$:

$$
\begin{aligned}
&N\int_{x\in B} Q(x;p)\cdot(x-c)-U(x;p)dF(x) && (35)\\
&= \int_{x\in B} Q_2(x;p)(x_2-c_2)-\left(\int_{\underline{x}_2}^{x_2} Q_2(x_1,t)dt\right)dF(x) && (36)\\
&= N\int_{x\in B} F(\min\{\overline{x}_1,p_1^*+x_2-p_2^*\},x_2)(x_2-c_2)-\left(\int_{\underline{x}_2}^{x_2}\mathbb{1}\{t\geq p_2^*\}F(\min\{\overline{x}_1,p_1^*+x_2-p_2^*\},x_2)dt\right)dF(x) && (37)\\
&= N\int_6^{7.1}\int_{9.9+x_1-6}^{11}\left(\frac{x_2-9.9}{2}\right)\left(\frac{x_2-9}{2}\right)(x_2-5)-\left(\int_{9.9}^{x_2}\left(\frac{t-9.9}{2}\right)\left(\frac{t-9}{2}\right)dt\right)f(x_1,x_2)dx_2dx_1 && (38)\\
&= \frac{N}{4}\int_6^{7.1}\int_{9.9+x_1-6}^{11}\left(\frac{x_2-9.9}{2}\right)\left(\frac{x_2-9}{2}\right)(x_2-5)-\left(\frac{1}{24}(x_2-9.9)^2(2x_2+9.9-27)\right)dx_2dx_1 && (39)\\
&= 0.516192 && (40)
\end{aligned}
$$

This is not equivalent[10] to the revenue from $N \int_{x \in B} Q(x;p) \cdot (p-c) dF(x)$.

[10] See here for computation.