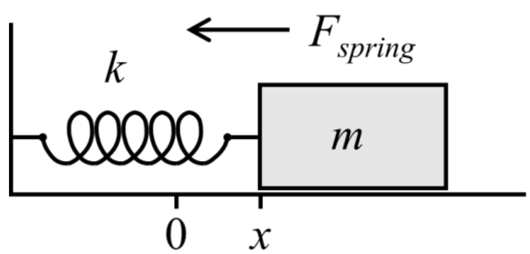
Mechanical Oscillators and Resonance

Introduction

In this laboratory the behavior of the forced, damped harmonic oscillator will be investigated. Using an air track with standard accessories and a motor capable of delivering a periodic force at various frequencies, a forced, damped harmonic oscillator will be constructed. Students will compare the predicted behavior of the system with their observation of the actual behavior. The behavior of both a single oscillator and a coupled oscillator will be explored. The concept of mechanical resonance will be emphasized.

The forced, damped harmonic oscillator can be used to describe the motion of many systems in physics and engineering. Because this system is very common in a number of physical situations it is one of the systems emphasized in courses in classical mechanics and differential equations. The motion of a pendulum, elastic springs, beams, vibrating strings, resonance of air cavities such as organ pipes, and the motion of charges in electric circuits and cavities are all examples of these types of systems.

Fig. 1 – A simple harmonic oscillator. A spring attached to a wall and a movable mass on a frictionless surface is a simple situation in which mechanical harmonic motion ensues.



To begin a discussion of the forced, damped harmonic oscillator let us consider the simple linear harmonic oscillator. As show in Fig. 1 the linear harmonic oscillator can be represented as a mass attached to the spring. The spring in this case obeys Hooke's law delivering a force proportional to the displacement of the mass from the equilibrium position. Equation 1 shows this relationship.

$$F_{spring} = - k\, x \tag{1}$$

Here k represents the spring constant. In this case, Newton's Second law yields equation 2.

$$x'' = -(k/m)\, x \tag{2}$$

The solution to this simple differential equation is shown in equation 3 where ω_0 is the angular frequency defined by equation 4. The angular frequency is also proportional to the linear frequency *f*. Equation 5 shows this relationship. The linear frequency is the inverse of the period of oscillation as shown in equation 6.

$$x = A \sin(\omega_0 t + \phi) \tag{3}$$
$$\omega_0 = (k/m)^{1/2} \tag{4}$$
$$\omega_0 = 2\pi f \tag{5}$$
$$f = 1/Period \tag{6}$$

The simple harmonic oscillator represents one of the simplest and most useful mechanical systems. As a result this system is explored in detail in all beginning physics courses. However, the simple harmonic oscillator represents an ideal situation called *free oscillation*. That is, a simple harmonic oscillator put in motion will stay in motion forever.

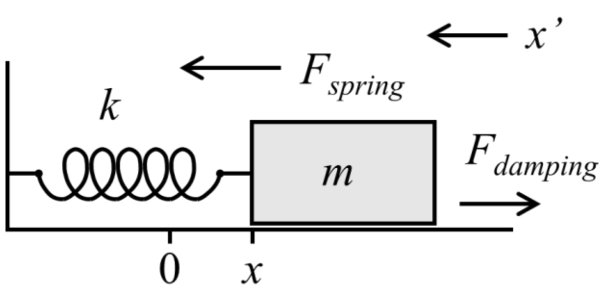


Fig. 2 – Simple harmonic oscillator with a damping force. The damping force is always directed oppositely to the direction of motion of the mass.

In all physical situations dissipative forces cause the resulting motion to be damped. Figure 2 illustrates the physical situation. To describe the damping force mathematically we assume a viscous force – one whose magnitude is proportional to velocity. This is common for any object moving through a fluid medium as long as turbulence in negligible.   Equation 7 shows the mathematical representation of the damping force. The quantity b is a positive damping constant. Velocity is of course given by x'.

$$F_{damping} = - b\, x' \tag{7}$$

In this case the total force acting on the mass is described in equation 8.

$$F_{total} = - k\, x – b\, x' \tag{8}$$

Again, using Newton's Second Law the differential equation describing the motion is given by equation (9).

$$x'' + (b/m)\, x' + (k/m)\, x = 0 \tag{9}$$

The solution to this differential equation is given in equation 10 below. Here γ and ω_0 are defined by equations 11 and 12 respectively and A_1 and A_2 are arbitrary constants

$$x = e^{-\gamma t}\left(A_1 e^{t\sqrt{\gamma^2 - \omega_0^2}} + A_2 e^{-t\sqrt{\gamma^2 - \omega_0^2}}\right) \tag{10}$$

$$\gamma = (b / 2m) \tag{11}$$

$$\omega_0 = (k / m)^{1/2} \tag{12}$$

The lead term in the general solution always gets smaller as time increases. (Remember that γ is always positive.) Thus the displacement from the equilibrium position (x) will always approach zero as time increases. The amount of dampening force simply determines how long this will take.

Figure 3 shows the behavior of a damped oscillator for two values of the damping coefficient. Here the position of the oscillating mass is plotted as a function of time. The position is simply the x coordinate distance $x(t)$. In each case, the mass and spring constant are the same. For each value of the damping constant, there are three different initial velocities. In one case the damping constant is set at the "critically damped" value – when $\gamma = \omega_0$. In the other case the motion is "overdamped", meaning that $\gamma > \omega_0$. In both cases the exponentials in the parentheses of equation 10 are real – this implies a non-oscillatory solution for x. In fact the amplitude of oscillation in the overdamped case is never negative. In other words the solution only asymptotically approaches the equilibrium position of the system.

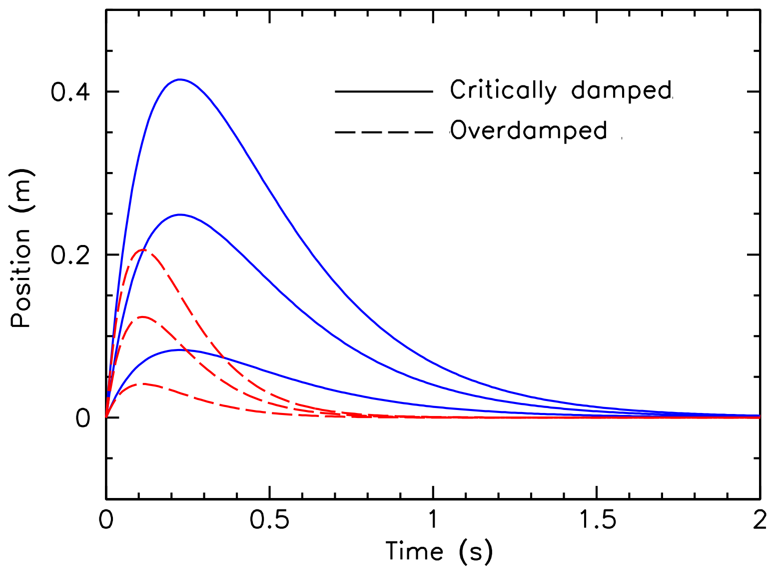


Fig. 3 – Solutions to the damped harmonic oscillator equation for various values of initial velocity and the spring constant. This figure illustrates solutions for the critically damped (solid) and overdamped (dashed) cases.

On the other hand, the damping coefficient could be less than the natural frequency of oscillation ($\gamma < \omega_0$), which is called the underdamped case. In this case the exponentials in the parentheses of equation 10 are imaginary – this implies an oscillatory solution for x. The position of the oscillating mass does cross the equilibrium (x=0) point. This is illustrated in Figs. 4-6.

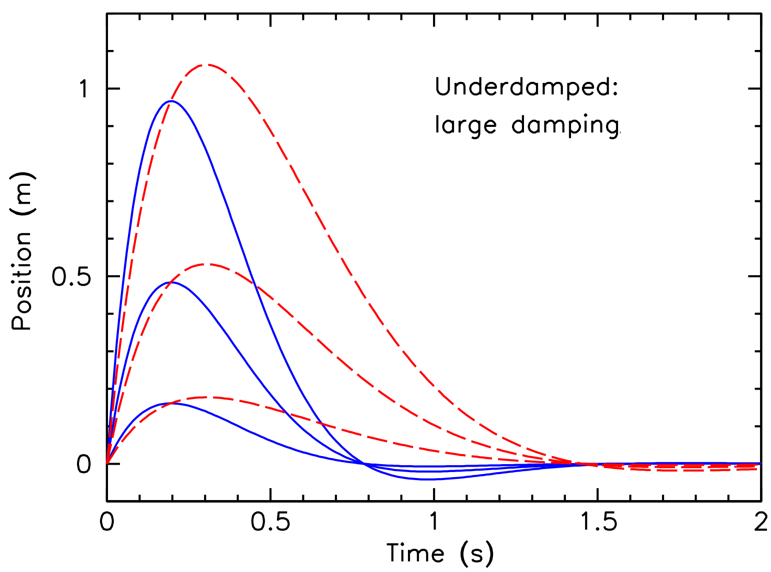


Fig. 4 – Solutions to the damped harmonic oscillator equation for various values of initial velocity and the spring constant. Solid – stronger spring constant; dashed – weaker spring constant. This figure illustrates solutions for the underdamped case when the damping constant is slightly lower than the critical value.

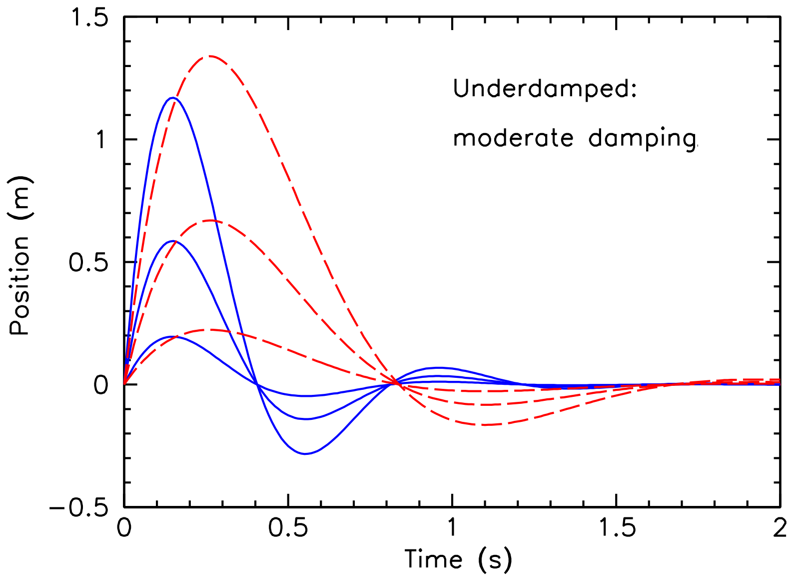


Fig. 5 – Solutions to the damped harmonic oscillator equation for various values of initial velocity and the spring constant. Solid – stronger spring constant; dashed – weaker spring constant. This figure illustrates solutions for the underdamped case when the damping constant is substantially lower than the critical value, but significantly larger than zero.

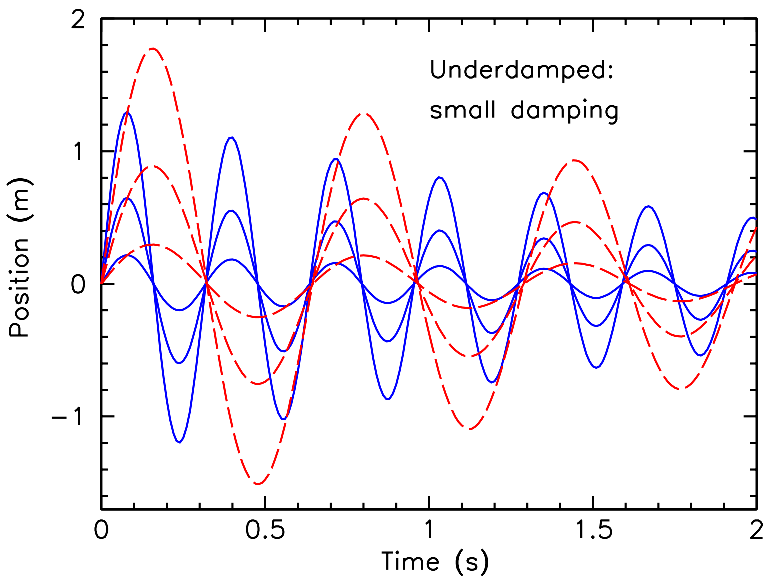


Fig. 6 – Solutions to the damped harmonic oscillator equation for various values of initial velocity and the spring constant. Solid – stronger spring constant; dashed – weaker spring constant. This figure illustrates solutions for the underdamped case when the damping constant is quite small compared to the critical value.

With every oscillation, the maximum displacement from the equilibrium position ($x=0$) gets smaller and smaller due to the leading exponential term ($e^{-\gamma t}$). As the dampening coefficient becomes lower, the maximum displacement takes longer to approach the equilibrium position. However, as long as the damping is greater than zero the amplitude will approach zero eventually.

As stated above, all realistic physical situations have dissipative forces that result in damping. In order for harmonic motion to be sustained, energy must be added to the system to compensate for the losses due to damping. When this is done we call the motion of the system *forced oscillation* or *driven oscillation*. For the purpose of this laboratory we will investigate the effects of a periodic driving force that takes the form shown in equation (13).

$$F_{drive} = F_0 \cos(\omega t + \phi) \tag{13}$$

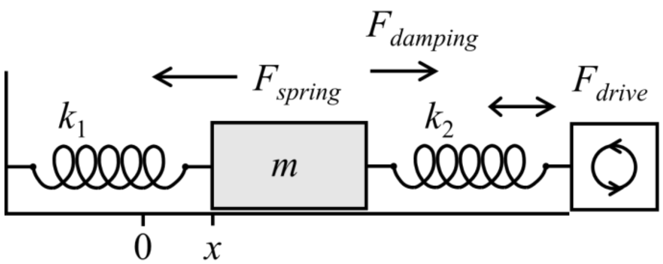


Fig. 7 – A forced, damped, harmonic oscillator. In addition to a damping force, a periodic driving force is applied to the oscillating mass. In this diagram, there is also a second spring attached to the mass.

Considering the physical system shown in Fig. 7, the total force on the box is now the force of the springs, the damping force and the driving force. Equation (14) shows the equation for the total force.

$$F_{total} = -k\,x - b\,x' + F_0 \cos(\omega t + \phi) \qquad (14)$$

Here k is the *effective* spring constant. Since two springs are being used as shown in Fig. 7, the effective spring constant is the sum of the two spring constants. Using Newton's second law, equation (14) yields the differential equation shown in equation (15).

$$x'' + (b/m)\,x' + (k/m)\,x = (F_0/m) \cos(\omega t + \phi) \qquad (15)$$

Notice that the homogeneous equation is simply equation (9) and its solution is equation (10). The general solution is the sum of the solution to the homogeneous equation and the solution to the inhomogeneous equation. Equation (16) below shows the general solution to the differential equation (15).

$$(16) \qquad x = A_h e^{-\gamma t} \cos(\omega_1 t) + \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \cos(\omega t - \Phi)$$

In equation (16) ω_0 is defined by equation (12). The natural frequency of oscillation for a damped oscillator is ω_1, and is always less than the natural frequency of oscillation for a free oscillator ω_0. Equation (17) shows the relationship between the two frequencies.

$$\omega_1 = (\omega_0^2 - \gamma^2)^{1/2} \qquad (17)$$

The first term in the solution (equation (16)) is the solution to the homogeneous equation (the damped oscillator). This term approaches zero as time increases. Thus, this term is known as the *transient solution*; its contribution to the general solution is negligible after a certain amount of time. For example, Fig. 8 shows the first 20 seconds of the motion of a forced damped harmonic oscillator. After about 8 seconds, the system has settled into simple harmonic motion at the frequency of the driving force. The transient solution no longer plays a significant role in the solution. The initial conditions influence only the first term, so the system is independent of the initial conditions after the transient term has vanished.

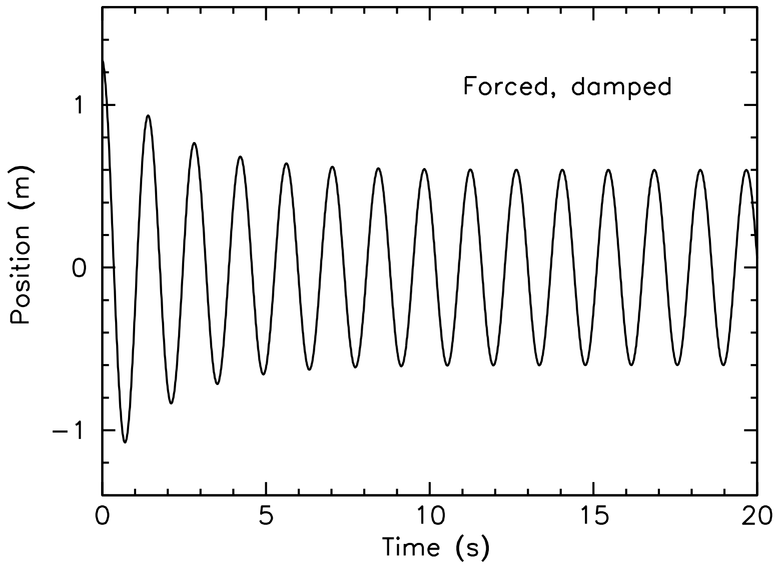


Fig. 8 – Solution to the periodically forced, damped harmonic oscillator equation for a particular set of parameters and initial conditions. The transient solution fades away to almost zero after several cycles of oscillation. The frequency of oscillation and amplitude settle into fixed values related to the driving force, and given by the steady-state solution.

The second term in equation (16) is called the *steady-state solution*. After the transient term has vanished the system will be described by the steady-state term. Notice that this term (the solution to the inhomogeneous equation) is very similar to the solution for free oscillations (the linear harmonic oscillator). The difference is in the frequency of the oscillation due to the frequency of the driving force (ω), and the amplitude depends on the natural frequency for free oscillations (ω_0), the damping coefficient (γ), and the driving force frequency (ω) as shown by equation (18).

$$Amplitude = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \qquad (18)$$

Inspection of equation (18) shows that when ω_0 is approximately equal to ω, the amplitude will be largest. A more careful inspection shows further that if the damping coefficient (γ) is small the maximum amplitude when ω_0 is equal to ω can be very large.

In fact, if $\gamma = 0$ then the amplitude will be infinite when the driving frequency (ω) is equal to the free oscillation frequency (ω_0). This is the known as *amplitude resonance*. The frequency that results in amplitude resonance is the resonance frequency. For the system shown in Fig. 7, the resonance frequency is the natural frequency of free oscillations (ω_0). Fig. 9 illustrates the case for a system with a natural frequency of 1.0 s^{-1}.

Amplitude resonance is a very important feature of several physical situations. For example tuning a radio relies on amplitude resonance. In many engineering situations it is very important to make sure that the natural frequency of oscillation of buildings, bridges and other structures is far from any periodic driving force (wind gusts or tremors, for example) that may result in an amplitude resonance. Another option is to make sure that these natural oscillations are sufficiently damped.

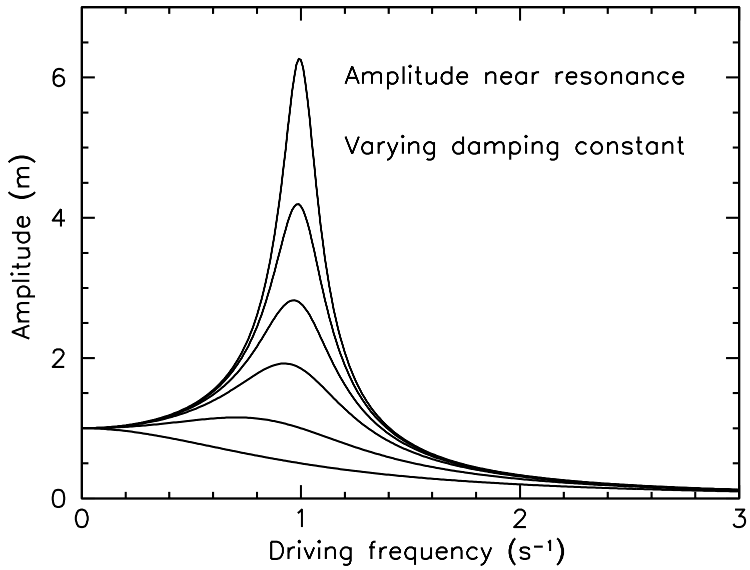


Fig. 9 – The dependence of the amplitude of a periodically forced damped harmonic oscillator on the driving frequency. Each curve represents the same system but with different values of the damping constant. The natural frequency of the system is 1 s^{-1}. When the driving frequency approaches the natural frequency, the amplitude can be very large, if the damping constant is low.

Even more complicated systems can be easily constructed. One such system that we will investigate is the coupled, forced, damped harmonic oscillator. As shown in Fig. 10, this system has two masses connected by springs to each other, a fixed point, and a driving motor. This system will behave like the forced, damped harmonic oscillator except there will be two resonance frequencies. These will be related to the natural frequency of free oscillation (ω_0). The amplitude resonance is predicted to occur at the frequencies shown in equations 19 and 20.

$$\omega_1 = \omega_0\,(1/2)^{1/2} \tag{19}$$

$$\omega_2 = \omega_0\,(3/2)^{1/2} \tag{20}$$

Because of its many applications to a wide variety of problems in physics and engineering the harmonic oscillator is one of the most important physical systems students study. In the laboratory that follows we will investigate several of the properties of the motion of the harmonic oscillator and the phenomena of amplitude resonance.

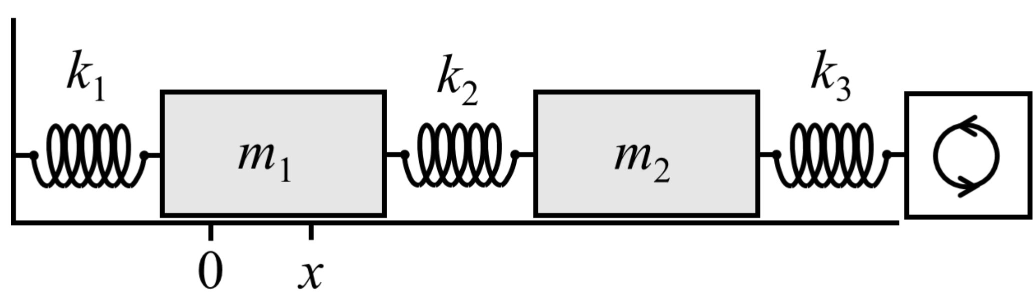


Fig. 10 – A coupled, periodically forced, damped harmonic oscillator. Two masses are connected together by a spring, and the pair of masses is connected to a wall and a periodic driving force by other springs.

Procedure

Preliminary measurements

Begin by measuring the spring constants for the three springs you will use in this laboratory. Ring stands, scales, and hangers are available for this measurement. I suggest taking *at least* five measurements of the length of the spring supporting various masses (ranging from about 5 g to 50 g). Graph the force and the length of the spring, and calculate the slope to find the spring constant. You should find that the spring constants are all *approximately* equal. Consider carefully how you will assess the **uncertainty** in the spring constants.

Measure the mass of the air carts to be used and the extra masses that will be used.

The forced, damped, harmonic oscillator

Construct the forced, damped, harmonic oscillator shown in Fig. 7 on an air track.

With the drive motor off, measure the period of the natural oscillation with the air track turned on. From the period, calculate the natural frequency (f_0) and the natural angular frequency (ω_0). Calculate the predicted natural frequency, and the natural angular frequency from the mass and the spring constants. Since two springs are used, the effective spring constant is the sum of the two spring constants.

Now, turn the drive motor on. Adjusting the knob on the motor control box can vary the driving force frequency (f). The display shows the frequency in Hz. You will notice that control knob is very sensitive, but careful adjustment allows the frequency to be set to less than a tenth of a Hz. Given your calculations above you have two predictions for the resonance frequency (f_0). Begin your observation by setting the driving frequency approximately 0.5 Hz from the predicted resonance frequency. At each frequency measure the amplitude of the oscillations. To do this, you will have to wait for the transient term (remember this term is sensitive to the initial conditions) to vanish. You will know this occurs when the oscillations become regular at the frequency of the drive motor. When the motion becomes regular, measure the amplitude using the scale on the air track. The amplitude will be the difference in the displacement of the cart. This will seem difficult to measure at first but after several trials you will be able to estimate the amplitude to within 1 or 2 mm. Vary the drive motor frequency by 0.1 Hz at first measuring the amplitude each time. As you get near the predicted resonance frequency (f_0) [within 0.05 Hz] attempt to vary the drive motor frequency by 0.01 Hz. Once you have passed the observed resonance frequency by about 0.05 Hz, continue to take four or five measurements increasing the drive motor frequency by 0.1 Hz.

Repeat the experiment, but vary the mass of the cart by adding weights. This should change both the predicted and the observed natural frequencies.

For both experiments compare the observed and predicted resonance frequencies. Compare the observed resonance frequencies for the two carts of different masses. Does the ratio of the observed resonance frequencies match your expectations?

The coupled, forced, damped harmonic oscillator

Construct the coupled, forced, damped harmonic oscillator depicted in Fig. 10. This will definitely entail moving the drive motor to make more room. With the drive motor off, measure the natural frequency of oscillation of the system. Calculate the predicted natural angular frequency (ω_0) and predicted natural frequency (f_0). These frequencies should be calculated using the same effective spring constant used in the previous experiment and the mass of one cart. Given this predicted natural angular frequency (ω_0) calculate the two predicted resonance angular frequencies (ω_1 and ω_2) using equations 19 and 20. Finally, calculate the two predicted resonance frequencies (f_1 and f_2).

Repeat the experimental procedure carried out above for the single cart. This time you expect two instances of amplitude resonance. You should vary the frequency in small increments around these resonances in order to obtain a good estimate of the shape of the amplitude resonance curve.

Compare the two measured resonance frequencies with the predictions.

Carefully observe the oscillation of the carts during the experiment. Is there a difference in the way the carts oscillate at either resonance? Describe this difference.

<u>Independent Study</u>

Time permitting, do one of the following experiments.

1) Attempt to increase the damping of the single-cart forced, damped harmonic oscillator. (Make sure you do not damage the air track.) Examine the effect of changing the damping force.

2) Vary the mass of the carts (either keep the mass of the cart the same and/or change the mass of only one cart) in the coupled, forced, damped harmonic oscillator. Make predictions of how this will change the amplitude resonance. Confirm or refute these predictions by observing the amplitude resonance.

3) Use a different spring with a vastly different spring constant and observe the effect on the amplitude resonance of either (or both) the forced, damped harmonic oscillator or the coupled, forced, damped harmonic oscillator.

4) Add a third cart to the coupled, forced, damped harmonic oscillator. Predict how this will affect the amplitude resonance. Confirm or refute these predictions by observing the amplitude resonance.

Analysis

Your laboratory report should follow the conventions discussed in class, and should contain the following (at the appropriate places):

- Graphs used to measure the spring constants. You may use any graphing program you want.
- Graphs of the amplitude versus frequency for every experiment conducted.
- Predicted and observed resonance frequencies for each experiment and a comparison between them.
- A discussion of the errors and approximations associated with the experiment that may influence the difference between the predicted and observed resonance frequencies.
- A detailed comparison of the observed oscillation of the forced, damped harmonic oscillator, and the predicted motion from the steady-state solution of the differential equation (15).
- A description of the qualitative features of the motion of each of the other experiments conducted.
- A discussion specifically focusing on answering the questions in the section below.

Questions for Discussion

1) Given your observations of the oscillations of the cart do you think the maximum amplitude theoretically possible was obtained? That is, did the cart move back and forth as much as it possibly could or did the experimental setup in some way limit the maximum amplitude of oscillation?

2) The purpose of the air track is to remove friction. Why is a damping force present? In other words, what is causing the damping?

3) How does the damping manifest itself with regard to the amplitude resonance? Specifically what does dampening do to the system?

4) Can you think of any way to estimate the damping (γ)? Discuss the problems associated with any estimate that you may choose to make. (In some cases the answer to question 1 may be important.)

5) The two resonances of the coupled, forced, damped harmonic oscillator are often called the symmetric and the asymmetric resonance. Given your observation of the oscillations at resonance explain why these labels make sense.