Mechanical Oscillators and Resonance

Advanced Lab 2023

Seth White, John Meneghini

August 29, 2023

Saint Vincent College
Department of Physics

Abstract

The harmonic oscillator is a second order linear differential equation which describes many physical systems throughout physics and engineering. In this experiment we constructed a forced, damped harmonic oscillator and a coupled forced, damped harmonic oscillator and determined the resonance frequency of each system. The measured resonance frequencies were statistically different from the theoretical resonance frequency, which suggested the presence of damping in the systems.

Introduction

The harmonic oscillator is a system that becomes displaced from its equilibrium position and experiences a restoring force proportional to its displacement. This system can be seen widely throughout the natural world in such examples pendulums, acoustics, masses attached to springs, and any small vibrational systems.[1] The forced, damped harmonic oscillator is a system that has a force driving the oscillations and a dissipative force acting on the object preventing some motion. The damping force is found in almost all physical systems as forces such as air-resistance, viscosity, and friction can all be found in systems that are not idealized under laboratory settings such as in a vacuum.

Mathematically, the harmonic oscillator is a solution to a second order differential equation. Therefore we can describe the motion of an oscillating object with a mathematical equation. The solutions to these equations create three cases classifying the damping of the system. The first case, overdamped, is a system where the square of the damping factor is greater than four times the mass multiplied by the spring constant of the system and returns the system to equilibrium in a short period. The second, underdamped, is a system where the square of the damping factor is less than four times the mass multiplied by the spring constant and the system oscillates until it returns to equilibrium. The third and final case is the critically damped case where the square of the damping factor is equal to four times the mass multiplied by the spring constant. This case is a unique solution that returns the system to equilibrium the fastest.[2]

A forced, damped harmonic oscillator has two separate solutions. The *transient* solution is the solution to the damped oscillator. As time increases, this term approaches zero. The second solution is the *steady-state* solution which is produced from the force applied on the harmonic oscillating system. This oscillation is similar to the oscillation of the natural frequency for free oscillations. The system reaches resonance frequency when the amplitude is the greatest in oscillation. In fact, we can compare the amplitude at the resonance frequency of this steady-state solution to the amplitude at the natural frequency of an unforced, undamped system with the same mass and spring constant.

Comparing amplitude resonance is vital in understanding various physical situations for real world applications. Engineers apply this method to buildings that shift and sway in the wind. They compare the resonant frequency of the building to the natural frequeny, the natural unforced vibration of the object, and design the building to not have resonance when wind is incident on the building. This means that the building does not reach maximum amplitude while swaying due to the wind.

In this experiment, we compared the resonance frequency of this steady-state solution in a spring mass system as damping approaches zero to the natural frequency and compared the amplitudes of the oscillations at these frequencies. The amplitudes of a coupled two-mass system were also compared at these two frequencies.

Methods

In this experiment, we attempted to measure the natural resonance frequencies of various systems of forced, damped harmonic oscillators and compare these measured values to theoretical values. Our first system that was used in the experimentation involved a single mass attached to a precision sine drive motor and a static object by springs. This was done on an air track to decrease damping due to friction on the system. Figure 1 displays this set up.

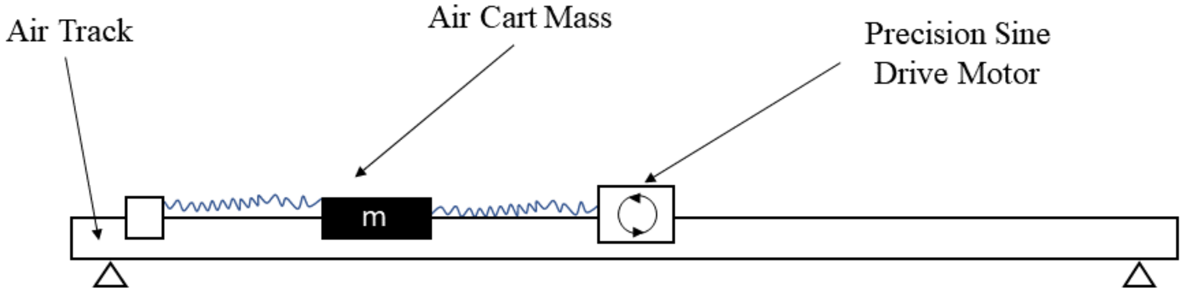

Figure 1: Air track configuration for one mass driven by a precision sine drive.

To measure the resonance frequency, we first need to use the equation for forced, damped harmonic oscillation and solve for the theoretical natural resonance frequency. The force being applied to our system by the motor is

$$\mathbf{F}_{driving} = F_0 cos(\omega t + \phi) \tag{1}$$

where F_0 is maximum force applied from the motor, ω is the varied frequency of the motor rotation, t is time, and ϕ is an arbitrary phase adjustment. This gives a differential equation of the system described by

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m} cos(\omega t + \phi), \tag{2}$$

where $\ddot{x}$ is the acceleration, b is the damping, $\dot{x}$ is the velocity, k is the spring constant, and x is the displacement. This equation can be derived from the general solution for a forced, damped harmonic oscillator and using equation 1 as our force being applied. Futher solving equation (2) gives us the solution

$$x(t) = Ae^{-\gamma t} cos(\omega_1 t) + \frac{\frac{F_0}{m}}{(\omega_0 - \omega)^2 + 4\gamma^2\omega^2} cos(\omega t + \phi) \tag{3}$$

where ω_1 is the natural frequency, $\gamma = \frac{b}{2m}$, and $\omega_0 = \sqrt{\frac{k}{m}}$. The first term in equation (3) goes to zero after a period of time and is denoted the "transient solution". The second term remains after this period of time and is called the "steady-state term". This term is at its greatest amplitude when $\omega = \omega_0$. When this occurs, the system has achieved its resonance frequency. This resonance frequency is what we determine throughout the experiment.

To measure the resonance frequency for one mass, we first nudge the air-cart while the air track is on and measure the period of oscillation without the sinusoidal wave drive. This gives us the natural period and natural frequency. We then calculate the angular frequency from these values using the equation

$$\omega_1 = ({\omega_0}^2 - \gamma^2)^{\frac{1}{2}}. \tag{4}$$

Equation (4) is maximized when the damping coefficient β goes to zero, such is the case on the air track. Then the natural frequency is equal to the resonance frequency. Using the precision sine drive motor, we varied the frequency of motor oscillations and plotted the measured cart displacements resulting from the oscillations and determine the experimental resonance frequency. This was done in increments of 0.1 rad/s, and once we were close to within 0.05 rad/s from our theoretical resonance frequency we incremented by 0.01 rad/s until we demonstrated resonance frequency. The experiment was then repeated with a 20g weight applied to the air-cart to demonstrate a change in natural and resonance frequencies.

A similar approach was done using two air-carts to demonstrate and measure the two natural frequencies of the system. This is known as the coupled, forced, damped harmonic oscillator. This system has two possible resonance frequencies known as the symmetric and the asymmetric solutions. For convenience, the symmetric solution will be called the *slow* case and the asymmetric solution will be called the *fast* case. The two resonant frequencies from these cases are as follows:

$$\omega_s = \omega_0 \sqrt{\frac{1}{2}} \tag{5}$$

$$\omega_f = \omega_0 \sqrt{\frac{3}{2}} \tag{6}$$

The experiment was done with the same increments as the single air-cart experiment, 0.1 rad/s and once close to resonance frequency varied on 0.01 rad/s. This was measured for both slow and fast cases.

Data and Results

To determine the spring constant of the springs used in the harmonic oscillator system, each spring was connected to a metal stand and mass was added to the end of the spring, increasing the spring's displacement from equilibrium. This displacement was measured with a meter stick while increasing the attached mass from 50 g to 130 g in steps of 20 g. The associated spring constant was then determined by plotting the gravitational force of the mass against the displacement of the spring, and then fitting the data, located in Table 1, using a linear fit. This process was repeated for each of the three springs.

x_1 ($\pm$0.05 cm)	x_2 ($\pm$0.05 cm)	x_2 ($\pm$0.05 cm)	Mass (g)	F (N)
14.0	14.5	13.0	50.0	0.490
19.5	20.0	18.5	70.0	0.686
25.0	25.7	24.0	90.0	0.883
30.5	31.5	29.5	110.0	1.08
36.3	37.0	35.2	130.0	1.28

Table 1: The displacements of springs from equilibrium x (cm), the mass of the attached mass (g), and the gravitational force F (N) of the attached mass for determining the spring constants of the three springs used in the harmonic oscillator system. Each displacement was measured using a meter stick with an uncertainty of 0.05 cm. The mass was given by the labelling on the mass and the gravitational force was calculated using the mass and the acceleration due to gravity.

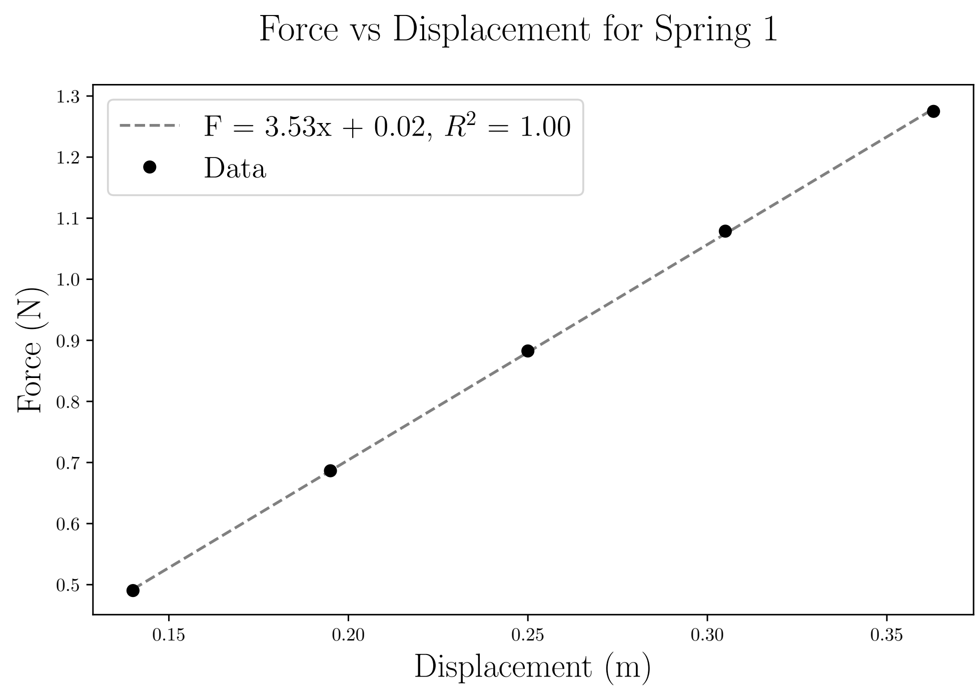

Figure 2: Force (N) vs. Displacement (m) for spring 1 used in the harmonic oscillator system. Mass and, therefore, force was increased while the displacement of the spring from equilibrium was measured using a meter stick with an uncertainty of 0.05 cm; this uncertainty was not considered in calculation of the spring constant. The data was then linearly fit using Hooke's Law, $F = kx$, resulting in $k = 3.53 \pm 0.02$ N/m where the uncertainty was calculated from the fit's covariance matrix. Values obtained from Table 1.

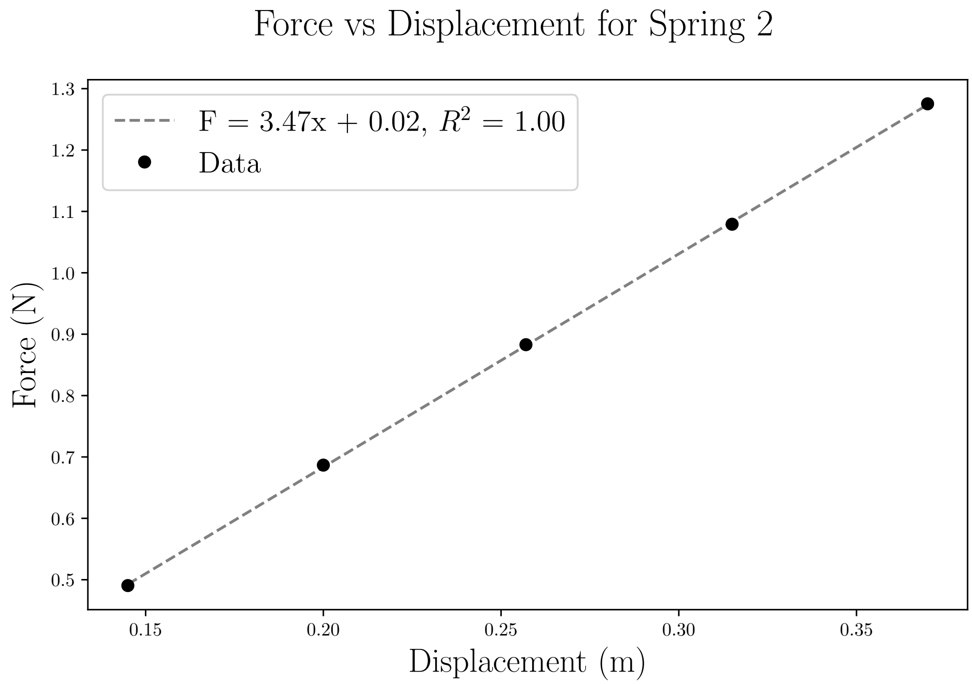

Figure 3: Force (N) vs. Displacement (m) for spring 2 used in the harmonic oscillator system. Mass and, therefore, force was increased while the displacement of the spring from equilibrium was measured using a meter stick with an uncertainty of 0.05 cm; this uncertainty was not considered in calculation of the spring constant. The data was then linearly fit using Hooke's Law, $F = kx$, resulting in $k = 3.47 \pm 0.02$ N/m where the uncertainty was calculated from the fit's covariance matrix. Values obtained from Table 1.

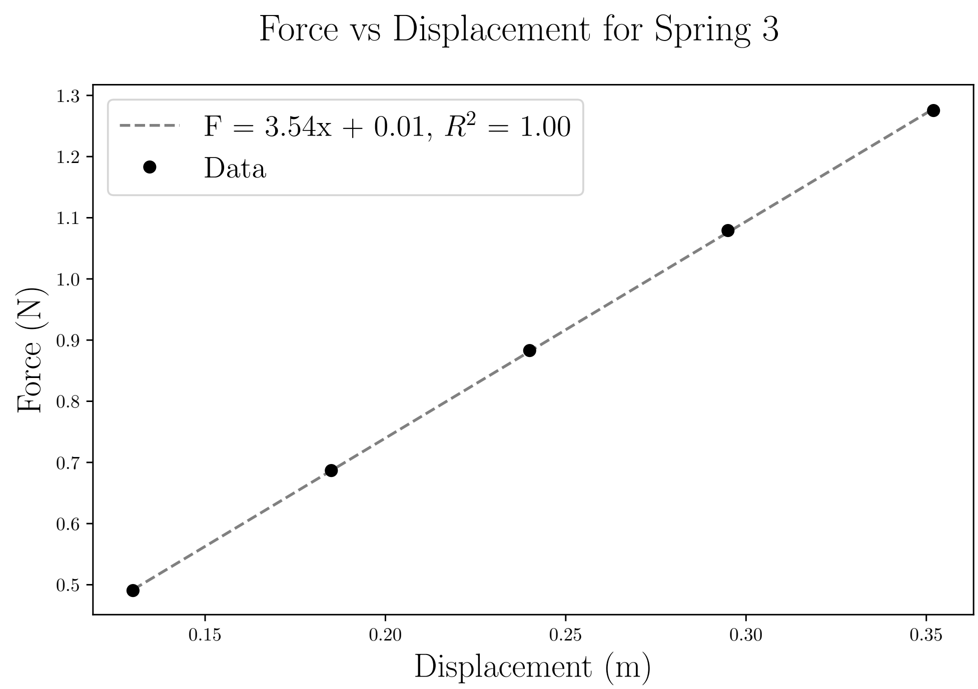

Figure 4: Force (N) vs. Displacement (m) for spring 3 used in the harmonic oscillator system. Mass and, therefore, force was increased while the displacement of the spring from equilibrium was measured using a meter stick with an uncertainty of 0.05 cm; this uncertainty was not considered in calculation of the spring constant. The data was then linearly fit using Hooke's Law, $F = kx$, resulting in $k = 3.54 \pm 0.01$ N/m where the uncertainty was calculated from the fit's covariance matrix. Values obtained from Table 1.

With the R^2 of each fit being 1.00 and each value having a low uncertainty, these values were deemed to be both accurate and precise. Furthermore, due to the similarity of the three spring constants, the average of the three, $k = 3.51 \pm 0.02$ N/m (assuming Gaussian) was used for the harmonic oscillator calculations.

For the single air cart, forced, damped harmonic oscillator, the measured oscillation amplitude was plotted against the set driving frequency of the motor.

Driving frequency (± 0.0005 Hz)	Oscillation amplitude (± 0.05 cm)
0.410	0.3
0.513	0.4
0.607	0.5
0.708	0.5
0.805	0.9
0.874	3.5
0.883	4.5
0.895	11.0
0.911	11.5
0.920	11.5
0.924	5.5
0.937	2.5
0.949	1.5
0.992	0.9
1.016	0.6
1.118	0.4
1.216	0.3
1.333	0.2

Table 2: The driving frequency (Hz) and oscillation amplitude (cm) for the single air cart, forced, damped harmonic oscillator with an air cart mass of 207.9 ± 0.05 g as measured by an electronic scale, with an uncertainty of ± 0.05 g, and two springs with spring constants of 3.54 ± 0.01 N/m. The driving force was set manually on the motor interface, which had an uncertainty of ± 0.0005 Hz, and the displacement was recorded using a ruler attached to the air track, which had an uncertainty of ± 0.05 cm.

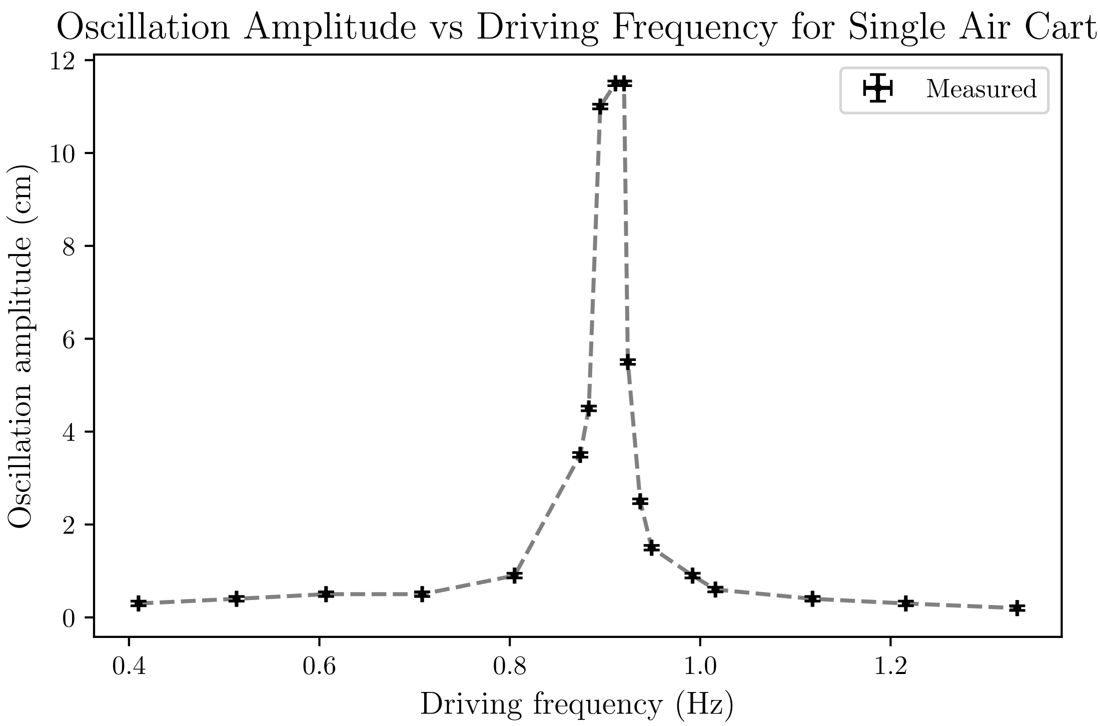

Figure 5: Oscillation amplitude (cm) vs. driving frequency (Hz) for the single air cart, forced, damped harmonic oscillator with an air cart mass of 207.9 ± 0.05 g as measured by an electronic scale and two springs with spring constants of 3.54 ± 0.01 N/m. Values obtained from Table 2.

With the resonance frequencies of a harmonic oscillator being the particular frequencies that result in a local oscillation amplitude maximum, the single resonance frequency of the single mass system was 0.920 ± 0.0005 Hz, due to it resulting in a maximum oscillation amplitude of 11.5 ± 0.05 cm. With the theoretical resonance frequency of the system being 0.926 ± 0.002 Hz, the experimental and theoretical values are not statistically equivalent; however, they are within three standard deviations of each other.

Driving frequency (± 0.0005 Hz)	Oscillation amplitude (± 0.05 cm)
0.187	2.4
0.269	1.2
0.380	0.8
0.486	0.8
0.593	0.9
0.696	1.2
0.770	2.2
0.795	2.3
0.809	1.8
0.820	2.1
0.840	4.1
0.855	5.2
0.869	14.9
0.885	12.2
0.908	7.7
0.924	4.2
0.955	3.2
0.973	2.2
1.053	1.4
1.205	0.8
1.304	0.7

Table 3: The driving frequency (Hz) and oscillation amplitude (cm) for the single air cart, forced, damped harmonic oscillator with an increased air cart mass of 227.9 ± 0.05 g as measured by an electronic scale, with an uncertainty of ± 0.05 cm, and two springs with spring constants of 3.54 ± 0.01 N/m. The driving force was set manually on the motor interface, which had an uncertainty of ± 0.0005 Hz, and the displacement was recorded using a ruler attached to the air track, which had an uncertainty of ± 0.05 cm.

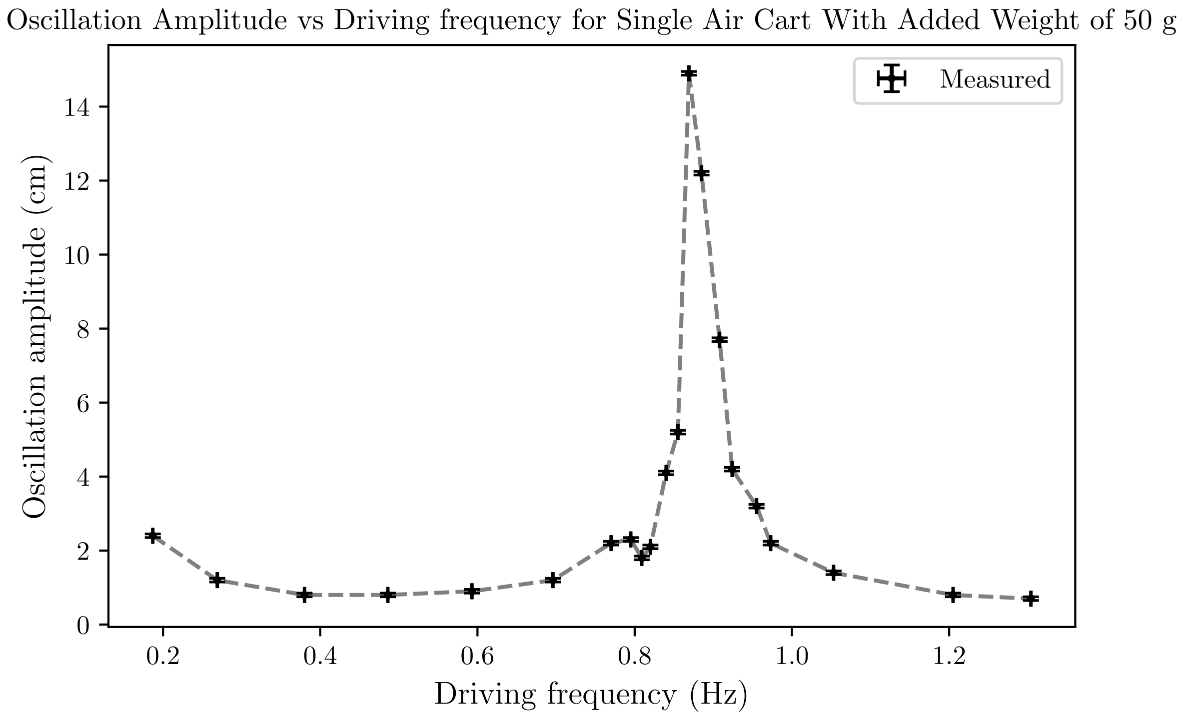

Figure 6: Oscillation amplitude (cm) vs. driving frequency (Hz) for the single air cart, forced, damped harmonic oscillator with an increased air cart mass of 227.9 ± 0.05 g as measured by an electronic scale, with an uncertainty of ± 0.05 g, and two springs with spring constants of 3.54 ± 0.01 N/m. Values obtained from Table 3.

Using the same method from the previous system, a resonance frequency of 0.869 ± 0.0005 Hz was obtained experimentally for the increased mass air cart. With the theoretical resonance frequency of the system being 0.884 ± 0.002 Hz, the experimental and theoretical values are not statistically equivalent.

Driving frequency ($\pm$0.0005 Hz)	Oscillation amplitude ($\pm$0.05 cm)
0.186	0.4
0.289	0.8
0.388	0.4
0.490	0.4
0.584	2.2
0.620	4.0
0.636	5.5
0.649	18.0
0.658	16.5
0.688	5.5
0.708	2.0
0.761	1.6
0.815	1.0
0.886	0.6
0.928	1.5
0.997	3.5
1.055	4.2
1.130	6.0
1.217	3.5
1.322	3.0
1.401	2.2
1.499	1.3
1.602	0.7
2.026	0.3
2.512	0.2

Table 4: The driving frequency (Hz) and oscillation amplitude (cm) for the double air cart, forced harmonic oscillator with two air carts of masses 207.9 ± 0.05 g and 206.4 ± 0.05 g as measured by an electronic scale, with an uncertainty of ± 0.05 g, and three springs with spring constants of 3.54 ± 0.01 N/m. The driving force was set manually on the motor interface, which had an uncertainty of ± 0.0005 Hz, and the displacement was recorded using a ruler attached to the air track, which had an uncertainty of ± 0.05 cm.

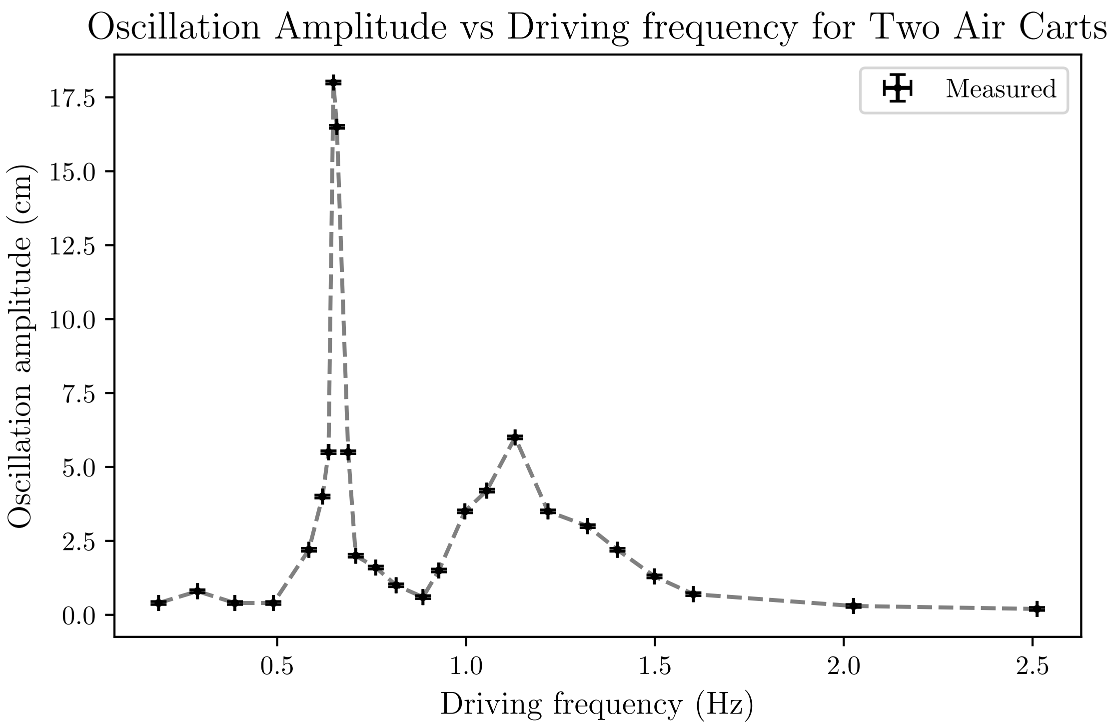

Figure 7: Oscillation amplitude (cm) vs. driving frequency (Hz) for the double air cart, forced harmonic oscillator with two air carts of masses 207.9 ± 0.05 g and 206.4 ± 0.05 g as measured by an electronic scale, with an uncertainty of ± 0.05 g, and three springs with spring constants of 3.54 ± 0.01 N/m. Values obtained from Table 4.

For the double air cart system, as expected, we observed two resonance frequencies, represented by the two local maxima. In particular, the experimental *slow* and *fast* resonance frequencies were 0.649 ± 0.0005 Hz and 1.130 ± 0.0005 Hz, respectively. With the theoretical *slow* and *fast* resonance frequencies being 0.655 ± 0.002 Hz and 1.134 ± 0.003 Hz, respectively, neither of our experimental values were statistically equivalent to their theoretical expectations.

Discussion

In all three experiments, the amplitude resonance frequencies measured were statistically different from the theoretical values. This difference could have been caused due to the relatively large step size taken in varying the driving force. For example, analyzing Table 4 shows that the next recorded oscillation frequency after our measured $\omega_2 = 1.130 \pm 0.0005$ Hz was 1.217 ± 0.0005 Hz. It is entirely possible that a larger amplitude may have been measured if the driving frequency had been set closer to our theoretical expectation of 1.134 ± 0.003 Hz; however, since this value was skipped over, it is impossible to know without further experimentation. Additionally, since all of our measured frequencies were less than their theoretical counterparts, this would suggest that a damping factor was still observed on the air track during the experiment. In all theoretical calculations, the increase of damping factor would reduce the natural frequency as can be seen in the derivation of the single cart resonance frequency in equation (4).

This damping could possibly have been caused by some friction on the air track still present in the experiment. The experiment where we added the weight to the single air cart system showed the greatest deviation from theoretical value. For the track to account for this gain in mass, the air track would have to produce a stronger output of air; this was impossible due to the experiment being run on the highest setting for the pump throughout. Another possibility could be the presence of turbulence under the carts. Because the system is moving and changing the direction constantly, the air has difficulty escaping and may get trapped and cause small changes in the air cart as it oscillates on the track.

Due to the possibility of damping in the system, the observed maximum amplitudes might not have been the theoretical maximum possible in the system. From this we could estimate the damping factor γ by relating the difference of theoretical resonance frequency to the measured resonance frequency of the system. This would be slightly unreliable because of the possibility that the maximums arent truly being measured as suggested. Therefore multiple factors of uncertainty would render this estimate difficult.

Though the maximum amplitudes were not achieved, the coupled, damped harmonic oscillator still demonstrated both resonance frequencies. Another common reference of these resonance frequencies are the symmetric and the asymmetric resonance. This is due to the way the carts oscillate in reference to each other. The carts moved for one frequency back and forth together in the same direction (asymmetric), and then opposite direction for the other frequency (symmetric). This nomenclature makes sense with reference to a center axis between the two carts.

References

[1] Blinder, S. Harmonic oscillator (2023).

[2] Meth, J. Harmonic oscillators (2023).