Question 1

Two thin lenses having focal lengths of +15.0 cm and −15.0 cm respectively are positioned 60.0 cm apart. A page of print is held 25.0 cm in front of the positive lens. Describe, in detail, the image of the print (i.e. insofar as it's paraxial).

$$s_1 = 25.0 \text{ cm},\ f_1 = 15.0 \text{ cm},\ f_2 = -15.0 \text{ cm, and } d = 60.0 \text{ cm.}$$

$$\frac{1}{f_1} = \frac{1}{s_1} + \frac{1}{p_1} \rightarrow \frac{1}{p_1} = \frac{1}{15.0} - \frac{1}{25.0}$$
$$p_1 = 37.5 \text{ cm}$$
$$\frac{1}{f_2} = \frac{1}{d-p_1} + \frac{1}{p_2} \rightarrow \frac{1}{p_2} = -\frac{1}{15.0} - \frac{1}{60-37.5}$$
$$p_2 = -9.0 \text{ cm}$$

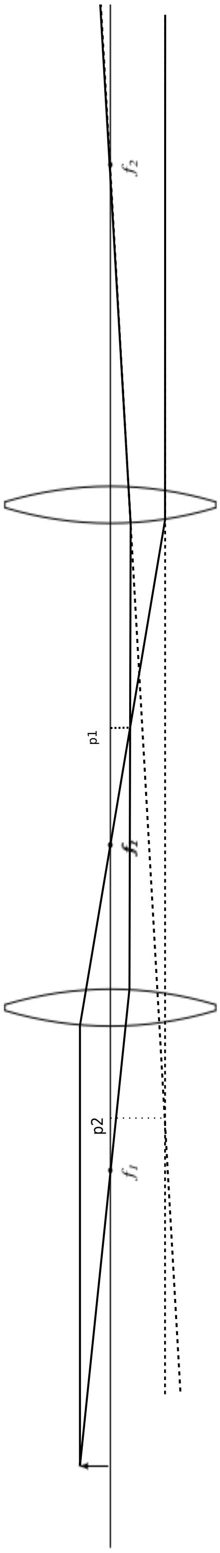
What about magnification?

$$m = m_1 m_2 = \left(-\frac{p_1}{s_1}\right)\left(-\frac{p_2}{s_2}\right)$$
$$m = \left(-\frac{37.5}{25.0}\right)\left(\frac{9}{60-37.5}\right) = -0.5984$$

So, the image will be virtual, 9.0 cm in front of the negative lens (between the lenses), 0.5984 times smaller, and inverted.

Question 2

Draw a ray diagram for the combination of two positive lenses wherein their separation equals the sum of their respective focal lengths. Do this carefully, with a ruler. You may do your sketch on the arrangement shown on the last page of this assignment



Question 3

Consider the case of two positive thin lenses, L_1 and L_2, separated by 5 cm. Their diameters are 6 cm and 4 cm, respectively, and their focal lengths are $f_1 = 9$ cm and $f_2 = 3$ cm. If a diaphragm with a hole 1 cm in diameter is located between them, 2 cm from L_2, find the aperture stop and the locations and sizes of the pupils for an axial point, S, 12 cm in front of (to the left of) L_1.

In order to determine the aperture stop, we can trace a separate ray from the axial point to the margin of each potential aperture stop. We then can back out the initial angle of the rays, and the potential stop that resulted in the smallest initial angle will be the aperture stop. This will determine the initial angle of the marginal ray.

Let's call the initial ray at the axial point S, $\vec{\mathbf{r}}_S = \begin{bmatrix} 0 \\ \alpha_0 \end{bmatrix}$, where α_0 is the initial angle we're trying to determine. The translation matrix that will bring $\vec{\mathbf{r}}_S$ to L_1 is

$$\boldsymbol{T_1} = \begin{bmatrix} 1 & 12 \\ 0 & 1 \end{bmatrix},$$

and we want to find the α_0 which satisfies the following matrix equation

$$\begin{bmatrix} 1 & 12 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} 3 \\ \alpha_0 \end{bmatrix},$$

where 3 cm is the height of L_1. Solving this matrix equation gives

$$\alpha_0 = 0.25.$$

Now for the diaphragm, the matrix that will bring $\vec{\mathbf{r}}_S$ to the margin of the diaphragm is given by the following matrix

$$\boldsymbol{T_d R_1 T_1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{9} & 1 \end{bmatrix} \begin{bmatrix} 1 & 12 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & 11 \\ -\frac{1}{9} & -\frac{1}{3} \end{bmatrix},$$

where the refraction matrix for a thin lens

$$\boldsymbol{R}(f) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix},$$

was used.

So, the matrix equation is

$$\begin{bmatrix} \frac{2}{3} & 11 \\ -\frac{1}{9} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \alpha_{new} \end{bmatrix},$$

where $\frac{1}{2}$ cm is the height of the diaphragm.

Solving,

$$\alpha_0 = 0.045.$$

Next, find the matrix that will bring $\vec{\mathbf{r}}_S$ to the margin of the second lens:

$$\boldsymbol{T_2 T_d R_1 T_1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & 11 \\ -\frac{1}{9} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & \frac{31}{3} \\ -\frac{1}{9} & -\frac{1}{3} \end{bmatrix}.$$

Then, our matrix equation is

$$\begin{bmatrix} \frac{4}{9} & \frac{31}{3} \\ -\frac{1}{9} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} 2 \\ \alpha_{new} \end{bmatrix},$$

where 2 cm is the height of L_2. So,

$$\alpha_0 = 0.1935.$$

Since the initial angle made for the ray marginal to the diaphragm is the smallest, THE DIAPHRAGM IS THE APERTURE STOP.

Now to find the exit pupil and entrance pupil, we need to trace a chief ray. We'll let it be 1 cm above the axial point:

$$\boldsymbol{T_d R_1 T_1} \begin{bmatrix} 1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & 11 \\ -\frac{1}{9} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha_{new} \end{bmatrix}.$$

Resulting in

$$\alpha_0 = -0.061.$$

If we then forward project this ray by translating until the height of ray is zero, this will give the location of entrance pupil:

$$\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.061 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.061 \end{bmatrix},$$

$$L = 16.5.$$

So, the entrance pupil will be located 16.5 cm from S.

The height of the entrance pupil can then be found by forward projecting the marginal ray through L_1 a distance equal to 16.5 cm:

$$\begin{bmatrix} 1 & 16.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.045 \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.045 \end{bmatrix}.$$

So, the entrance pupil will have a diameter of 1.5 cm.

Now for the exit pupil, we need the angle of the marginal ray after it's been refracted through L_2:

$$R_2 \begin{bmatrix} \frac{4}{9} & \frac{31}{3} \\ -\frac{1}{9} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{9} & \frac{31}{3} \\ -\frac{1}{9} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & \frac{31}{3} \\ -\frac{7}{27} & -\frac{34}{9} \end{bmatrix},$$

$$\begin{bmatrix} \frac{4}{9} & \frac{31}{3} \\ -\frac{7}{27} & -\frac{34}{9} \end{bmatrix} \begin{bmatrix} 0 \\ 0.045 \end{bmatrix} = \begin{bmatrix} 0.47 \\ -0.17 \end{bmatrix}.$$

We also need the refracted angle after the chief ray passed through L_2:

$$\begin{bmatrix} \frac{4}{9} & \frac{31}{3} \\ -\frac{7}{27} & -\frac{34}{9} \end{bmatrix} \begin{bmatrix} 1 \\ -0.061 \end{bmatrix} = \begin{bmatrix} -0.18 \\ -0.03 \end{bmatrix}.$$

Now we backtrace this ray until the height is zero:

$$\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -0.18 \\ -0.03 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.03 \end{bmatrix},$$

giving $L = -6.0$ cm. Making the exit pupil $(12 + 5) - 6 = 11$ cm in front of the axial point.

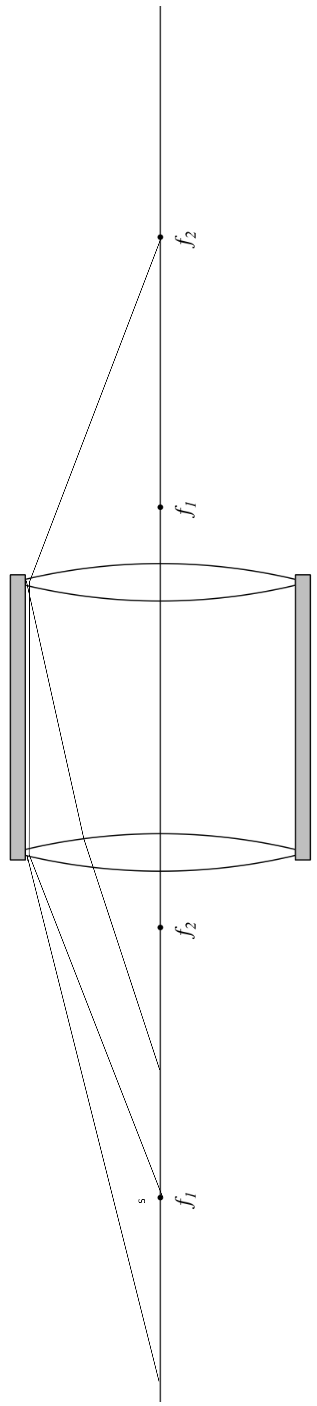
We then can back trace the marginal ray after it's been refracted through L_2:

$$\begin{bmatrix} 1 & -6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.47 \\ -0.17 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -0.17 \end{bmatrix},$$

making the exit pupil have a diameter of 3 cm.

Question 4

This problem requires the axial point to specified, since its position would determine the aperture stop. If it was located between f_1 and the left lens, then the aperture stop would be the right lens. If it was to the left of f_1, then the left lens would be the aperture stop. The exit and entrance pupil's location and size would be dependent on the exact position of the axial point.



Question 5

Show that the triple scalar product $(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot \vec{\mathbf{C}}$ can be written as

$$(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot \vec{\mathbf{C}} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Write the vector's in term's of their components:

$$\vec{\mathbf{A}} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

and similarly for $\vec{\mathbf{B}}$ and $\vec{\mathbf{C}}$.

So,

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} \times \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} = \hat{\mathbf{i}}(A_2B_3 - A_3B_2) + \hat{\mathbf{j}}(A_3B_1 - A_1B_3) + \hat{\mathbf{k}}(A_1B_2 - A_2B_1)$$

by cofactor expansion of the determinant. Then,

$$(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot \vec{\mathbf{C}} = C_1(A_2B_3 - A_3B_2) + C_2(A_3B_1 - A_1B_3) + C_3(A_1B_2 - A_2B_1)$$

We can reverse the cofactor expansion, giving

$$(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot \vec{\mathbf{C}} = \begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Switching a row of the determinant switches the sign of the result:

$$-(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot \vec{\mathbf{C}} = \begin{vmatrix} A_1 & A_2 & A_3 \\ C_1 & C_2 & C_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

Switching again, the sign once again becomes positive:

$$(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \cdot \vec{\mathbf{C}} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Question 6

Suppose we have a positive meniscus lens of radii 6 and 10 cm and a thickness of 3 cm, made with a material of index of refraction $n = 1.5$. Determine its focal length and the locations of its principle points.

Positive lens, so

$$R_1 = 6 \text{ cm}, \ R_2 = 10 \text{ cm}, \ d = 3 \text{ cm}, \text{ and } n = 1.5$$

First, we need f

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}\right) = (1.5-1)\left(\frac{1}{6} - \frac{1}{10} + \frac{(1.5-1)(3)}{(1.5)(6)(10)}\right)$$

$$f = 24 \text{ cm}$$

$$h_1 = -\frac{f(n-1)d}{nR_2} = -\frac{(24)(1.5-1)(3)}{(1.5)(10)} = -2.4 \text{ cm}$$

$$h_2 = -\frac{f(n-1)d}{nR_1} = -\frac{(24)(1.5-1)(3)}{(1.5)(6)} = -4.0 \text{ cm}$$

Question 7

A spherical glass bottle 20 cm in diameter with walls that are negligibly thin is filled with water. The bottle is sitting on the back seat of a car on a nice, sunny day. What is the focal length of the "lens?"

$$R_1 = 10 \text{ cm}, \ R_2 = -10 \text{ cm}, \ d = 20 \text{ cm}, \text{ and } n = 1.333$$

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}\right) = (1.333-1)\left(\frac{1}{10} + \frac{1}{10} + \frac{(1.333-1)(20)}{(1.333)(10)(-10)}\right)$$

$$f = 20 \text{ cm}$$

Question 8

It is found that sunlight is focused to a spot 29.6 cm from the back face of a thick lens, which has principle points $h_1 = 0.2$ cm and $h_2 = -0.4$ cm. Determine the location of the image of a candle that is placed 49.8 cm in front of this lens.

Assuming the rays of light from the sun are parallel, this means the $s \to \infty$ and $p = f$, but f and p are measured from h_2, making $p = 29.6 + 0.4 = 30.0 \text{ cm} = f$. So,

$$\frac{1}{s} + \frac{1}{p} = \frac{1}{f} \to \frac{1}{p} = \frac{1}{30} - \frac{1}{s}$$

But, s is measured from the h_1, so $s = 49.8 + 0.2 = 50.0$ cm.

$$\frac{1}{p} = \frac{1}{30} - \frac{1}{50.0} \to p = 75.0 \text{ cm}$$

So, the image will be located $75.0 - 0.4 = 74.6$ cm from the back of the lens.

Question 9

A crown glass double-convex lens that is 4.0 cm thick has an index of refraction of 3/2. Given that its radii are 4.0 cm and 15 cm, locate its principle points and compute its focal length. If a television screen is placed 1.0 m from the front of the lens, where will the real image of the picture appear?

Since the lens is double-convex, R_2 is negative.

$$R_1 = 4.0 \text{ cm},\ R_2 = -15 \text{ cm},\ d = 4.0 \text{ cm, and } n = 3/2$$

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}\right) = (1.5-1)\left(\frac{1}{4} + \frac{1}{15} + \frac{(1.5-1)(4)}{(1.5)(4)(-15)}\right)$$

$$f = 6.79$$

$$h_1 = -\frac{f(n-1)d}{nR_2} = -\frac{(6.79)(1.5-1)(4)}{(1.5)(-15)} = 0.604 \text{ cm}$$

$$h_2 = -\frac{f(n-1)d}{nR_1} = -\frac{(6.79)(1.5-1)(4)}{(1.5)(4)} = -2.264 \text{ cm}$$

$$\frac{1}{s} + \frac{1}{p} = \frac{1}{f} \to \frac{1}{p} = \frac{1}{6.79} - \frac{1}{100 + 0.604}$$

$$p = 7.28 \text{ cm}$$

But p is measured from h_2, so the image is $7.28 - 2.1 = 5.02$ cm past the back of the lens.

Question 10

Given the following matrices,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix},\ \boldsymbol{B} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

work out the product $\boldsymbol{AB}$.

$$\boldsymbol{AB} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (1)(2)+(2)(0)+(-1)(1) & (1)(1)+(2)(-1)+(-1)(1) & (1)(0)+(2)(2)+(-1)(3) \\ (0)(2)+(3)(0)+(1)(1) & (0)(1)+(3)(-1)+(1)(1) & (0)(0)+(3)(2)+(1)(3) \\ (2)(2)+(0)(0)+(1)(1) & (2)(1)+(0)(-1)+(1)(1) & (2)(0)+(0)(2)+(1)(3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 9 \\ 5 & 3 & 3 \end{bmatrix}$$