Question 1

Two 1 MHz radio antennae emitting in phase are separated by 600 m along a north-south line. A radio receiver placed 2 km east is equidistant from both transmitting antennae and picks up a fairly strong signal. How far north should the receiver be moved if it is again to detect a signal nearly as strong?

Assuming these two antennae are emitting spherical wavefronts, we can use our results from the double slit. We found that maxima in brightness at a point with vertical distance y on some screen a horizontal distance x away, both with respect to the midpoint of the two antennae, is given by $\sin\theta = \frac{m\lambda}{a}$, where a is the spacing between the antennae, $m \in \mathbb{Z}$, and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$.

In our case, $a = 600\,\mathrm{m}$, $\lambda = \frac{c}{\nu} = \frac{3\times10^8\,\mathrm{m\,s^{-1}}}{1\times10^6\,\mathrm{Hz}} = 299.792\,\mathrm{m}$, and $x = 2\times10^3\,\mathrm{m}$. However, since our interference pattern will be periodic with θ, we can find the distance between the maxima corresponding to $m = 0$ and $m = 1$, which will correspond to points $y = 0$ and $y = y_1$, respectively, and this will approximately be the distance to a similar strength position.

$$\frac{y_1}{\sqrt{x^2+y_1^2}} = \frac{m\lambda}{a} = \frac{\lambda}{a} \to y^2\left(1-\frac{\lambda^2}{a^2}\right) = \frac{\lambda^2x^2}{a^2}$$

$$y_1 = \frac{x}{\sqrt{\frac{a^2}{\lambda^2}-1}} = \frac{2\times10^3\,\mathrm{m}}{\sqrt{\frac{(600\,\mathrm{m})^2}{(299.792\,\mathrm{m})^2}-1}} = 1153.64\,\mathrm{m}$$

So, the receiver should be moved $\boxed{1153.64\,\mathrm{m}\text{ north.}}$

Question 2

An expanded beam of red light from a HeNe laser ($\lambda_0 = 632.8\,\mathrm{nm}$) is incident on a screen containing two very narrow horizontal slits separated by 0.2 mm. A fringe pattern appears on a white screen held 1 m away.

(a) How far (in radians and millimeters) above and below the central axis are the first zeros of irradiance?

This is a double slit, so as before, we can use our results from class. In this case, we want to use the condition for minima in irradiance, which is

$$\sin\theta = \frac{(m+\frac{1}{2})\lambda}{a}$$

The first minima above and below the central axis correspond to $m = 0$ and $m = -1$, respectively. We can use the same formula derived in Question 1, but with $(m+\frac{1}{2})$ present, it will show up in the root.

$$y_m = \frac{x}{\sqrt{\frac{a^2}{(m+\frac{1}{2})^2\lambda^2}-1}}$$

Since $(m+\frac{1}{2})$ is squared,

$$y_0 = y_{-1} = \frac{x}{\sqrt{\frac{4a^2}{\lambda^2} - 1}} = \frac{1\,\mathrm{m}}{\sqrt{\frac{4(0.2\times10^{-3}\,\mathrm{m})^2}{(632.8\times10^{-9}\,\mathrm{m})^2} - 1}} = \boxed{1.582\,\mathrm{mm}}$$

and,

$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{1.582}{1e3}\right) = \boxed{1.58\times10^{-3}\,\mathrm{rad}}$$

(b) How far (in millimeters) from the axis is the fifth bright band?

As before, but $(m+\frac{1}{2}) \to m$ since we are looking for the bright bands.

$$y_m = \frac{x}{\sqrt{\frac{a^2}{m^2\lambda^2} - 1}}$$

$$y_5 = \frac{x}{\sqrt{\frac{a^2}{25\lambda^2} - 1}} = \frac{1\,\mathrm{m}}{\sqrt{\frac{(0.2\times10^{-3}\,\mathrm{m})^2}{25(632.8\times10^{-9}\,\mathrm{m})^2} - 1}} = \boxed{15.82\,\mathrm{mm}}$$

Question 3

A stream of electrons, each having an energy of 0.5 eV, impinges on a pair of extremely thin slits separated by 1×10^{-2} mm. What is the distance between adjacent minima on a screen 20 m behind the slits?

Since $x >> a$, we are in the far-field region and can apply the small angle approximation $\sin\theta \approx \tan\theta \approx \theta$:

$$\sin\theta \approx \tan\theta = \frac{y}{x} = \frac{(m+\frac{1}{2})\lambda}{a} \to y = \frac{(m+\frac{1}{2})\lambda x}{a}$$

Let's define the distance between adjacent minima as

$$\delta = y_{m+1} - y_m = \frac{(m+\frac{3}{2})\lambda x}{a} - \frac{(m+\frac{1}{2})\lambda x}{a} = \frac{\lambda x}{a} = \frac{\lambda x}{a}$$

Treating these electrons as wave's with a de Broglie wavelength

$$\lambda = \frac{hc}{E} = 2.48\,\mathrm{nm}$$

So,

$$\delta = \frac{2.48\times10^{-6}\,\mathrm{m}20\,\mathrm{m}}{1\times10^{-5}\,\mathrm{m}} = \boxed{4.959\,\mathrm{m}}$$

Question 5

Determine the general solution to the equation

$$\nabla^2 \varphi = 0$$

if the universe is restricted to one dimension. (I.e. $\varphi = \varphi(x)$)

In one dimension, the Laplacian becomes $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$

$$\frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2} = 0$$

One approach to solving this is by finding the characteristic equation to the second order homogenous linear differential equation:

$$\lambda^2 = 0$$

Which has the repeated root $\lambda = 0$. This case of a repeated real root of order 2 gives the following general solution:

$$\varphi(x) = e^{\lambda x}(A + Bx) = \boxed{\text{A + Bx}}$$

Question 10

Solve the differential equation

$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 4x = 0$$

This second order homogenous linear differential equation has the following characteristic equation:

$$\lambda^2 - 6\lambda + 4 = 0 \rightarrow \lambda = \frac{6 \pm \sqrt{36 - 16}}{2} = 3 \pm \sqrt{5}$$

For the case where the characteristic equation has unique real roots λ_1 and λ_2, the general solution takes the form of

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} = Ae^{(3-\sqrt{5})t} + Be^{(3+\sqrt{5})t}$$