Question 1

Write an expression for $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields that constitute a plane harmonic wave traveling in the $+z$-direction. The wave is linearly polarized with its plane of polarization at $45°$ to the xy-plane. Note that this indicates the direction of the $\vec{\boldsymbol{E}}_0$ vector when writing the waves.

Since the wave is travelling in the $+z$-direction, $\vec{\boldsymbol{k}} \cdot \vec{\boldsymbol{r}} = kz$. Additionally, since $\vec{\boldsymbol{E}}_0$ is at $45°$ to the xy-plane, $\vec{\boldsymbol{E}}_0 = \frac{E_0}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$. So,

$$\begin{aligned}
\vec{\boldsymbol{E}} &= \frac{E_0}{\sqrt{2}} \cos(kz - \omega t)(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) \\
\vec{\boldsymbol{B}} &= \frac{E_0}{c\sqrt{2}} \cos(kz - \omega t)(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})
\end{aligned}$$

Question 2

A 550 nm harmonic EM-wave whose electric field is in the z-direction is traveling in the y-direction in vacuum. (a) What is the frequency of the wave? (b) Determine both ω and k for this wave. (c) If the electric field amplitude is 600 V/m, what is the amplitude of the magnetic field? (d) Write an expression for both $\vec{\boldsymbol{E}}(t)$ and $\vec{\boldsymbol{B}}(t)$ given that each is zero at $y = 0\,\text{m}$ and $t = 0\,\text{s}$.

(a)

$$\nu = \frac{c}{\lambda} = \frac{3 \times 10^8\,\text{m/s}}{550\,\text{nm}} = 5.45 \times 10^{14}\,\text{Hz}$$

(b)

$$\begin{aligned}
\omega &= 2\pi\nu = 3.43 \times 10^{15}\,\text{rad/s} \\
k &= \frac{2\pi}{\lambda} = 1.14 \times 10^{7}\,\text{m}^{-1}
\end{aligned}$$

(c)

$$B_0 = \frac{E_0}{c} = \frac{600\,\text{V/m}}{3 \times 10^8\,\text{m/s}} = 2.00\,\mu\text{T}$$

(d) $\sin(ky - \omega t)$ is zero at $y = 0\,\text{m}$ and $t = 0\,\text{s}$:

$$\begin{aligned}
\vec{\boldsymbol{E}} &= (600\,\text{V/m}) \sin\left[(1.14 \times 10^7\,\text{m}^{-1})y - (3.43 \times 10^{15}\,\text{rad/s})t\right] \hat{\boldsymbol{k}} \\
\vec{\boldsymbol{B}} &= (2.00\,\mu\text{T}) \sin\left[(1.14 \times 10^7\,\text{m}^{-1})y - (3.43 \times 10^{15}\,\text{rad/s})t\right] \hat{\boldsymbol{\imath}}
\end{aligned}$$

Question 3

A light bulb puts out 20 W of radiant energy. Assume it to be a point source and calculate the irradiance at a distance of 1 m.

For a point source:

$$I = \frac{P}{4\pi r^2} = \frac{20\,\mathrm{W}}{4\pi(1\,\mathrm{m})^2} = 1.59\,\mathrm{W/m^2}$$

Question 4

A completely absorbing screen receives 300 W of light for 100 s. Compute the total linear momentum transferred to the screen.

$$E = Pt = pc \rightarrow p = \frac{Pt}{c} = \frac{(300\,\mathrm{W})(100\,\mathrm{s})}{3\times10^{8}\,\mathrm{m/s}} = 1.00\times10^{-4}\,\mathrm{kg\,m/s^2}$$

Question 5

If $\vec{\boldsymbol{r}}$ is the vector from the origin to the point (x, y, z), and $\vec{\boldsymbol{u}}$ is any vector, prove: (a) $\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{r}} = 3$, (b) $\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}$, and (c) $(\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{\nabla}})\vec{\boldsymbol{r}} = \vec{\boldsymbol{u}}$.

$$r = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$$

(a)

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{r}} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z = 1+1+1 = 3$$

(b)

$$\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{r}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \hat{\boldsymbol{\imath}}(0-0) - \hat{\boldsymbol{\jmath}}(0-0) + \hat{\boldsymbol{k}}(0-0) = \vec{\boldsymbol{0}}$$

(c)

$$\vec{\boldsymbol{u}}\cdot\vec{\boldsymbol{\nabla}} = u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}$$

$$(\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{\nabla}})\vec{\boldsymbol{r}} = \left(u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + u_z \frac{\partial}{\partial z}\right)\left(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}\right)$$
$$= \hat{\boldsymbol{\imath}}\left(u_x \frac{\partial}{\partial x}x + u_y \frac{\partial}{\partial y}x + u_z \frac{\partial}{\partial z}x\right) + \hat{\boldsymbol{\jmath}}\left(u_x \frac{\partial}{\partial x}y + u_y \frac{\partial}{\partial y}y + u_z \frac{\partial}{\partial z}y\right) + \hat{\boldsymbol{k}}\left(u_x \frac{\partial}{\partial x}z + u_y \frac{\partial}{\partial y}z + u_z \frac{\partial}{\partial z}z\right)$$
$$= u_x\hat{\boldsymbol{\imath}} + u_y\hat{\boldsymbol{\jmath}} + u_z\hat{\boldsymbol{k}} = \vec{\boldsymbol{u}}$$

Question 6

What force on average will be exerted on the $40\,\mathrm{m} \times 50\,\mathrm{m}$ flat, highly reflecting side of a space station wall if it is facing the Sun while in Earth orbit?

Question 7

A plane, harmonic, linearly polarized light wave has an electric field given by

$$E_z = E_0 \cos\left[\pi 10^{15}\left(t - \frac{x}{0.65c}\right)\right]$$

while traveling through a piece of glass. (a) Find the frequency of the light. (b) What is the wavelength of this wave? (c) Determine the index of refraction of the glass.

(a)

$$\nu = \frac{\omega}{2\pi} = \frac{\pi 10^{15}}{2\pi} = 5.00 \times 10^{14}\,\mathrm{Hz}$$

(b)

$$\lambda = \frac{2\pi}{k} = 2\pi\frac{0.65c}{\pi 10^{15}} = 3.90 \times 10^{-1}\,\mu\mathrm{m}$$

(c)

$$v = \lambda\nu = (5.00 \times 10^{14}\,\mathrm{Hz})(3.90 \times 10^{-1}\,\mu\mathrm{m}) = 1.95 \times 10^{8}\,\mathrm{m/s} = 0.65c$$
$$n = \frac{c}{v} = \frac{1}{0.65} = 1.54$$

Question 8

Pulses of UV lasting 2 ns each are emitted from a laser that has a beam of diameter 2.5 mm. Given that each burst carries an energy of 6 J, (a) determine the length in space of each wave pulse, and (b) find the average energy per unit volume for such a pulse.

(a)

$$l = ct = (3 \times 10^{8}\,\mathrm{m/s})(2\,\mathrm{ns}) = 6.00 \times 10^{-1}\,\mathrm{m}$$

(b)

$$\frac{E_{\text{avg}}}{V} = \frac{6\,\text{J}}{0.25\pi d^2 l} = \frac{6\,\text{J}}{0.25\pi(2.5\,\text{mm})^2(6.00\times 10^{-1}\,\text{m})} = 2.04\,\text{MJ/m}^3$$

Question 9

Imagine an electromagnetic wave that is traveling in the x-direction that has its electric field in the y-direction, given by

$$\vec{\boldsymbol{E}} = \vec{\boldsymbol{E}}_0 \cos(kx-\omega t)$$
$$\vec{\boldsymbol{B}} = \vec{\boldsymbol{B}}_0 \cos(kx-\omega t)$$

SHOW that an application of the relation

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

gives the relation $E_0 = cB_0$.

$$\frac{\partial E}{\partial x} = \frac{\partial}{\partial x} E_0 \cos(kx-\omega t) = -kE_0 \sin(kx-\omega t)$$
$$\frac{\partial B}{\partial t} = \frac{\partial}{\partial t} B_0 \cos(kx-\omega t) = \omega B_0 \sin(kx-\omega t)$$

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \to -kE_0 \sin(kx-\omega t) = -\omega B_0 \sin(kx-\omega t) \to E_0 = \frac{\omega}{k} B_0 = cB_0$$

Question 10

Suppose you are solving some complicated problem. You have boiled the mathematics down to two equations,

$$X(r,\theta) = A\frac{\cos\theta}{r} + B\frac{\sin\theta}{r}$$
$$Y(r,\theta) = Cr\cos\theta + Dr\sin\theta$$

In addition, you know that at r = R (a particular value of r), the condition

$$X(r,\theta) = Y(r,\theta) = K_0 \sin\theta$$

must be satisfied, where K_0 is some (otherwise known) constant. Please determine the values of coefficients A, B, C, and D for which this condition is met by both functions listed above, keeping in mind that the condition must be met for any and all value(s) of θ.

$$A\frac{\cos\theta}{R} + B\frac{\sin\theta}{R} = CR\cos\theta + DR\sin\theta = K_0\sin\theta$$

For this to hold for all θ, $A = C = 0$.

$$B\frac{\sin\theta}{R} = K_0\sin\theta \to B = RK_0$$
$$DR\sin\theta = K_0\sin\theta \to D = \frac{K_0}{R}$$