Question 1

Two 1 MHz radio antennae emitting in phase are separated by 600 m along a north-south line. A radio receiver placed 2 km east is equidistant from both transmitting antennae and picks up a fairly strong signal. How far north should the receiver be moved if it is again to detect a signal nearly as strong?

Assuming these two antennae are emitting spherical wavefronts, we can use our results from the double slit. We found that maxima in brightness at a point with vertical distance y on some screen a horizontal distance x away, both with respect to the midpoint of the two antennae, is given by $\sin\theta = \frac{m\lambda}{a}$, where a is the spacing between the antennae, $m \in \mathbb{Z}$, and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$.

In our case, $a = 600\,\mathrm{m}$, $\lambda = \frac{c}{\nu} = \frac{3\times10^8\,\mathrm{m\,s^{-1}}}{1\times10^6\,\mathrm{Hz}} = 299.792\,\mathrm{m}$, and $x = 2\times10^3\,\mathrm{m}$. However, since our interference pattern will be periodic with θ, we can find the distance between the maxima corresponding to $m = 0$ and $m = 1$, which will correspond to points $y = 0$ and $y = y_1$, respectively, and this will approximately be the distance to a similar strength position.

$$\frac{y_1}{\sqrt{x^2+y_1^2}} = \frac{m\lambda}{a} = \frac{\lambda}{a} \to y^2\left(1 - \frac{\lambda^2}{a^2}\right) = \frac{\lambda^2 x^2}{a^2}$$

$$y_1 = \frac{x}{\sqrt{\frac{a^2}{\lambda^2} - 1}} = \frac{2\times10^3\,\mathrm{m}}{\sqrt{\frac{(600\,\mathrm{m})^2}{(299.792\,\mathrm{m})^2} - 1}} = 1153.64\,\mathrm{m}$$

So, the receiver should be moved $\boxed{1153.64\,\mathrm{m}\text{ north.}}$

Question 2

An expanded beam of red light from a HeNe laser ($\lambda_0 = 632.8\,\mathrm{nm}$) is incident on a screen containing two very narrow horizontal slits separated by 0.2 mm. A fringe pattern appears on a white screen held 1 m away.

(a) How far (in radians and millimeters) above and below the central axis are the first zeros of irradiance?

This is a double slit, so as before, we can use our results from class. In this case, we want to use the condition for minima in irradiance, which is

$$\sin\theta = \frac{(m+\frac{1}{2})\lambda}{a}$$

The first minima above and below the central axis correspond to $m = 0$ and $m = -1$, respectively. We can use the same formula derived in Question 1, but with $(m+\frac{1}{2})$ present, it will show up in the root.

$$y_m = \frac{x}{\sqrt{\frac{a^2}{(m+\frac{1}{2})^2\lambda^2} - 1}}$$

Since $(m+\frac{1}{2})$ is squared,

$$y_0 = y_{-1} = \frac{x}{\sqrt{\frac{4a^2}{\lambda^2} - 1}} = \frac{1\,\mathrm{m}}{\sqrt{\frac{4(0.2\times10^{-3}\,\mathrm{m})^2}{(632.8\times10^{-9}\,\mathrm{m})^2} - 1}} = \boxed{1.582\,\mathrm{mm}}$$

and,

$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{1.582}{10^3}\right) = \boxed{1.58\times10^{-3}\,\mathrm{rad}}$$

(b) How far (in millimeters) from the axis is the fifth bright band?

As before, but $(m + \frac{1}{2}) \to m$ since we are looking for the bright bands.

$$y_m = \frac{x}{\sqrt{\frac{a^2}{m^2\lambda^2} - 1}}$$

$$y_5 = \frac{x}{\sqrt{\frac{a^2}{25\lambda^2} - 1}} = \frac{1\,\mathrm{m}}{\sqrt{\frac{(0.2\times10^{-3}\,\mathrm{m})^2}{25(632.8\times10^{-9}\,\mathrm{m})^2} - 1}} = \boxed{15.82\,\mathrm{mm}}$$

Question 3

A stream of electrons, each having an energy of 0.5 eV, impinges on a pair of extremely thin slits separated by 1×10^{-2} mm. What is the distance between adjacent minima on a screen 20 m behind the slits?

Since $x >> a$, we are in the far-field region and can apply the small angle approximation $\sin\theta \approx \tan\theta \approx \theta$:

$$\sin\theta \approx \tan\theta = \frac{y}{x} = \frac{(m+\frac{1}{2})\lambda}{a} \to y = \frac{(m+\frac{1}{2})\lambda x}{a}$$

Let's define the distance between adjacent minima as

$$\delta = y_{m+1} - y_m = \frac{(m+\frac{3}{2})\lambda x}{a} - \frac{(m+\frac{1}{2})\lambda x}{a} = \frac{\lambda x}{a} = \frac{\lambda x}{a}$$

Treating these electrons as wave's with a de Broglie wavelength

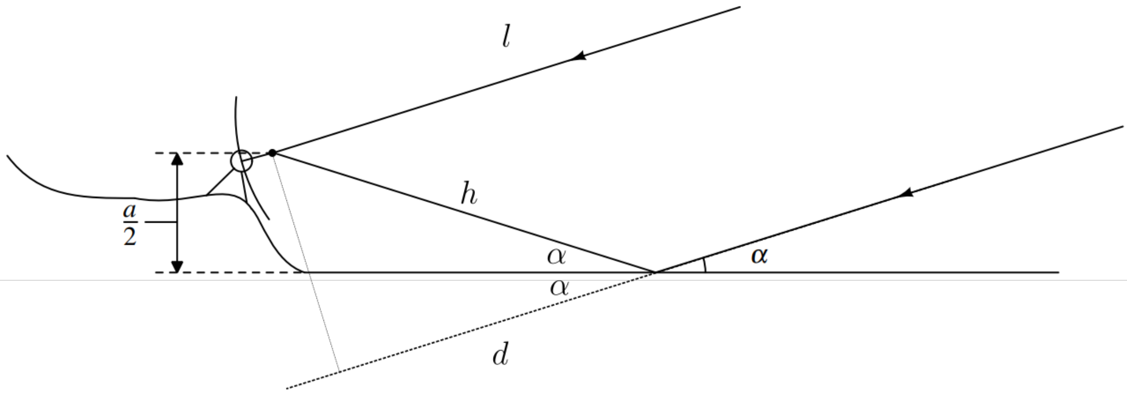
$$\lambda = \frac{hc}{E} = 2.48\,\mathrm{nm}$$

So,

$$\delta = \frac{(2.48\times10^{-6}\,\mathrm{m})(20\,\mathrm{m})}{1\times10^{-5}\,\mathrm{m}} = \boxed{4.959\,\mathrm{m}}$$

Question 4

Imagine that we have an antenna at the edge of a lake picking up a signal from a distant radio star, which is just coming up above the horizon so that light from the star makes angle α with the lake surface. Write expressions for δ and for the angular position of the star when the antenna detects its first maximum.



Finding h:

$$\sin\alpha = \frac{a}{2h} \to h = \frac{a}{2\sin\alpha}$$

Now d:

$$d = h\cos 2\alpha = \frac{a\cos 2\alpha}{2\sin\alpha}$$

Finding the OPL of the lower ray (2nd ray)

$$OPL_2 = l - d + h = l - \frac{a\cos 2\alpha}{2\sin\alpha} + \frac{a}{2\sin\alpha} = l + \frac{a}{2\sin\alpha}(1-\cos 2\alpha) = l + a\sin\alpha$$

Now ΔOPL

$$\Delta OPL = OPL_2 - OPL_1 = l + a\sin\alpha - l = a\sin\alpha$$

For the case of multiple beams with ray 2 externally reflecting

$$\delta = k\Delta OPL + \pi = ka\sin\alpha + \pi$$

For multiple beams, we'll have a maximum intensity when $\cos\delta = 1$, so, this occurs when $\delta = 2\pi n$, where $n \in \mathbb{Z}$:

$$ka\sin\alpha + \pi = 2\pi n$$

Setting $n = 1$ for the first maximum ($n = 0$ would be impossible since this would result in $\alpha = 0$):

$$ka \sin \alpha = \pi \rightarrow \alpha = \sin^{-1}\left(\frac{\pi}{ka}\right) = \sin^{-1}\left(\frac{\lambda}{2a}\right)$$

Question 5

Determine the general solution to the equation

$$\nabla^2 \varphi = 0$$

if the universe is restricted to one dimension. (I.e. $\varphi = \varphi(x)$)

In one dimension, the Laplacian becomes $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$

$$\frac{\mathrm{d}^2 \varphi}{\mathrm{d}x^2} = 0$$

One approach to solving this is by finding the characteristic equation to the second order homogenous linear differential equation:

$$\lambda^2 = 0$$

Which has the repeated root $\lambda = 0$. This case of a repeated real root of order 2 gives the following general solution:

$$\varphi(x) = e^{\lambda x}(A + Bx) = \boxed{\text{A + Bx}}$$

Question 6

A soap film surrounded by air has an index of refraction of 1.34. If a region of the film appears bright red ($\lambda_0 = 633\,\text{nm}$) in normally reflected light, what is its minimum thickness?

For normal incidence and when $n_0 < n_1 < n_2$ (which holds since the soap film is surrounded by air), bright spots satisfy

$$2n_1 d = (m - \frac{1}{2})\lambda$$

Taking $m = 1$ to find the minimum thickness:

$$2n_1 d = \frac{\lambda}{2} \rightarrow d = \frac{\lambda}{4n_1} = \frac{633\,\text{nm}}{4(1.34)} = \boxed{118.1\,\text{nm}}$$

Question 7

A Michelson Interferometer is illuminated with monochromatic light. One of its mirrors is then moved 2.53×10^{-5} m, and it is observed that 92 fringe-pairs, bright and dark, pass by in the process. Determine the wavelength of the incident beam.

For a Michelson interferometer surrounded by air,

$$2\Delta d = m\lambda \rightarrow \lambda = \frac{2\Delta d}{m} = \frac{22.53 \times 10^{-5}\,\mathrm{m}}{92} = 5.5 \times 10^{-7}\,\mathrm{m}$$

Question 8

Determine the refractive index and thickness of a film to be deposited on a glass surface ($n_g = 1.54$) such that no normally incident light of wavelength 540 nm is reflected.

For the consideration of normal incidence and the case of external reflection at both boundaries, destructive interference occurs when

$$2n_1 d = (m - \frac{1}{2})\lambda$$

Taking $m = 1$ to find the thinnest layer, and since we have 2 free variables, we'll choose $n_1 = 1.33$ so $n_g > n_1$

$$2n_1 d = \frac{\lambda}{2} \rightarrow d = \frac{\lambda}{4n_1} = \frac{540\,\mathrm{nm}}{4(1.33)} = \boxed{105.5\,\mathrm{nm}}$$

Question 9

A glass microscope lens having an index of 1.55 is to be coated with a magnesium fluoride film to increase the transmission of normally incident yellow light ($\lambda_0 = 550\,\mathrm{nm}$). What minimum thickness should be deposited on the lens?

According to the Refractiveindex.info database, $n_{MgF_2} = 1.3777 \approx 1.38$. In the case of $n_0 < n_1 < n_2$, the interference of transmitted light is constructive, so we'll consider the case of $m = 1$ destructive interference of the reflected light, since this will result in the minimum thickness.

$$d = \frac{\lambda}{4n_{MgF_2}} = \frac{550\,\mathrm{nm}}{4(1.38)} = 99.64\,\mathrm{nm}$$

Question 10

Solve the differential equation

$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 4x = 0$$

This second order homogenous linear differential equation has the following characteristic equation:

$$\lambda^2 - 6\lambda + 4 = 0 \rightarrow \lambda = \frac{6 \pm \sqrt{36-16}}{2} = 3 \pm \sqrt{5}$$

For the case where the characteristic equation has unique real roots λ_1 and λ_2, the general solution takes the form of

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} = Ae^{(3-\sqrt{5})t} + Be^{(3+\sqrt{5})t}$$