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ME 280B
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Hw 1

Problem 1

planar motion χ of $\mathcal{B}$

$$x_1 = X_1$$
$$x_2 = \chi_2(X_2, X_3, t), \qquad (\dagger)$$
$$x_3 = \chi_3(X_2, X_3, t)$$

reference basis $\{e_1, e_2, e_3\}$

$$Q = p \otimes p + \cos\theta(q \otimes q + r \otimes r) - \sin\theta(q \otimes r - r \otimes q),$$

p = unit eigenvector associated with eigenvalue of Q

$Qp = p$ $\quad$ $\{p, q, r\}$ form right hand orthonormal basis

a) Establish that for planar motion as in $(\dagger)$ components of R at point X and time t can be written as

$$R_{iA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$F = \frac{\partial x}{\partial X} = RU$ where R is proper orthogonal

if we take R and apply Rodrigues formula

$$R = e_1 \otimes e_1 + \cos\theta(e_2 \otimes e_2 + e_3 \otimes e_3) - \sin(e_2 \otimes e_3 - e_3 \otimes e_2)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

b) Recalling symmetry of right stretch U, show that

$$\tan\theta = \frac{F_{32}-F_{23}}{F_{22}+F_{33}}$$

$$F = RU$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{12} & U_{22} & U_{23} \\ U_{13} & U_{23} & U_{33} \end{bmatrix}$$

$$F_{32} = U_{22}\sin\theta + U_{23}\cos\theta$$

$$F_{23} = U_{23}\cos\theta - U_{33}\sin\theta$$

$$F_{22} = U_{22}\cos\theta - U_{23}\cos\theta$$

$$F_{33} = U_{23}\sin\theta + U_{33}\cos\theta$$

$$\frac{U_{22}\sin\theta + \cancel{U_{23}\cos\theta} - \cancel{U_{23}\cos\theta} + U_{33}\sin\theta}{U_{22}\cos\theta - \cancel{U_{23}\sin\theta} + \cancel{U_{23}\sin\theta} + U_{33}\cos\theta}$$

$$\frac{(U_{22}+U_{33})\sin\theta}{(U_{22}+U_{33})\cos\theta} = \tan\theta \checkmark$$

c)

$$[U_{AB}] = \frac{1}{F}\begin{bmatrix} F & 0 & 0 \\ 0 & J + F_{22}^2 + F_{32}^2 & F_{22}F_{23} + F_{32}F_{33} \\ 0 & F_{22}F_{23} + F_{32}F_{33} & J + F_{23}^2 + F_{33}^2 \end{bmatrix}$$

$J = \det F \qquad F = \sqrt{(F_{22}+F_{33})^2 + (F_{32}-F_{23})^2}$

$F = RU \rightarrow U = R^{-}F$

proper orthogonal $\rightarrow$ $R^{-} = R^{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}$$

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21}\cos\theta + F_{31}\sin\theta & F_{22}\cos\theta + F_{32}\sin\theta & F_{23}\cos\theta + F_{33}\sin\theta \\ -F_{21}\sin\theta + F_{31}\cos\theta & -F_{22}\sin\theta + F_{32}\cos\theta & -F_{23}\sin\theta + F_{33}\cos\theta \end{bmatrix}$$

$F_{11}=1 \quad F_{12}=0 \quad F_{13}=0 \quad F_{21}=0 \quad F_{31}=0$ from X

$\det F = F_{22}F_{33} - F_{32}F_{23}$

$\cos\theta = \dfrac{F_{22}+F_{33}}{F} \qquad \sin\theta = \dfrac{F_{32}-F_{23}}{F}$

$$F_{22} \cdot \frac{(F_{22}+F_{33})}{F} + \frac{F_{32}(F_{32}-F_{23})}{F}$$

$$= \frac{F_{22}^2 + F_{22}F_{33} + F_{32}^2 - F_{32}F_{23}}{F} = \frac{J + F_{22}^2 + F_{32}^2}{F} \checkmark$$

$$F_{23}\frac{(F_{22}+F_{33})}{F} + \frac{F_{33}(F_{32}-F_{23})}{F}$$

$$= F_{23}F_{22} + \cancel{F_{23}F_{33}} + F_{33}F_{32} - \cancel{F_{33}F_{23}}$$

$$= F_{23}F_{22} + F_{33}F_{32} \checkmark$$

$$-F_{22}\frac{(F_{32}-F_{23})}{F} + \frac{F_{32}(F_{22}+F_{33})}{F}$$

$$= \cancel{-F_{22}F_{32}} + F_{22}F_{23} + \cancel{F_{32}F_{22}} + F_{32}F_{33}$$

$$= F_{22}F_{23} + F_{32}F_{33} \checkmark$$

$$-F_{23}\cos\theta + F_{33}\sin\theta$$

$$-F_{23}\frac{(F_{32}-F_{23})}{F} + \frac{F_{33}(F_{22}+F_{33})}{F}$$

$$\frac{-F_{23}F_{23} + F_{23}^2 + F_{33}F_{22} + F_{33}^2}{F} = \frac{J + F_{23}^2 + F_{33}^2}{F} \checkmark$$

$$\frac{1}{F}\begin{bmatrix} F & 0 & 0 \\ 0 & J + F_{22}^2 + F_{32}^2 & F_{22}F_{23} + F_{32}F_{33} \\ 0 & F_{22}F_{23} + F_{32}F_{33} & J + F_{23}^2 + F_{33}^2 \end{bmatrix}$$

Problem 2

```python
import numpy as np
import time


def cube(x):
    if 0<=x: return x**(1./3.)
    return -(-x)**(1./3.)

t0 = time.time()
#Iterative formula
eps = np.finfo(float).eps
#F = np.array([[1,0,0],[0,2,0],[0,0,3]])
F = np.array([[1,2,0],[0,1,0],[0,0,1]])
#F = np.array([[2,1,0],[2,3,-1],[-1,2,2]])
U1 = np.eye(3)

C = F.T@F
error = 1

# part a

while error > 100*eps:
    Ulast = U1
    U1 = 1/2*(U1+C@np.linalg.inv(U1))
    error = (abs(U1-Ulast)).max()
t1 = time.time()


#part b closed form
IC = np.trace(C)
IIC = 1/2 *((np.trace(C)**2)-np.trace(C.T@C))
IIIC = np.linalg.det(C)
e = ((2**5)/27)*(2*(IC**3)-9*IC*IIC+27*IIIC)
n = ((2**10)/27)*(4*(IIC**3)-(IC**2)*(IIC**2)+4*(IC**3)*(IIIC)-18*(IC*IIC*IIIC)+27*(IIIC**2))
if n<0:
    n = -n
sq = -(2/3)*IC+cube(e+np.sqrt(n))+cube(e-np.sqrt(n))
if sq != -2*IC:
    I = 1/2*(np.sqrt(2*IC+sq)+np.sqrt(2*IC-sq+(16*np.sqrt(IIIC)/(np.sqrt(2*IC+sq)))))
else:
    I = np.sqrt(IC+2*np.sqrt(IIC))
II = np.sqrt(IIC+2*np.sqrt(IIIC)*I)
III = np.sqrt(IIIC)
U2 = np.linalg.inv(C+II*np.eye(3))@(I*C+III*np.eye(3))

t2 = time.time()
```

```python
#part c eigenvalue

[lam2,M] = np.linalg.eig(C)
lam = np.sqrt(lam2)

U3 = lam[0]*np.outer(M[:,0],M[:,0])+lam[1]*np.outer(M[:,1],M[:,1])+lam[2]*np.outer(M[:,2],M[:,2])

t3 = time.time()

T = [t1-t0,t2-t1,t3-t2]

print(U1)
print('')
print(U2)
print('')
print(U3)
print('')
print(T)
```

```
[[1. 0. 0.]
 [0. 2. 0.]
 [0. 0. 3.]]

[[1.00127131 0.         0.        ]
 [0.         2.00650661 0.        ]
 [0.         0.         3.01235459]]

[[1. 0. 0.]
 [0. 2. 0.]
 [0. 0. 3.]]

[0.00033283233642578125, 0.0001819133758544922,
0.00011324882507324219]
```

```
[[0.70710678 0.70710678 0.        ]
 [0.70710678 2.12132034 0.        ]
 [0.         0.         1.        ]]

[[0.7092744  0.71144174 0.        ]
 [0.71144174 2.13215789 0.        ]
 [0.         0.         1.00358963]]

[[0.70710678 0.70710678 0.        ]
 [0.70710678 2.12132034 0.        ]
 [0.         0.         1.        ]]

[0.00046706199645996094, 0.0002570152282714844,
0.00025773048400878906]
```

```
[[ 2.66604404  1.01160681 -0.93212706]
 [ 1.01160681  3.58549283  0.34769644]
 [-0.93212706  0.34769644  2.00255995]]

[[ 2.67165176  1.01489029 -0.93497894]
 [ 1.01489029  3.59403341  0.34870434]
 [-0.93497894  0.34870434  2.00600677]]

[[ 2.66604404  1.01160681 -0.93212706]
 [ 1.01160681  3.58549283  0.34769644]
 [-0.93212706  0.34769644  2.00255995]]

[0.0002989768981933594, 0.0001399517059326172,
0.00014638900756835938]
```

It does appear that the eigenvalue problem and the analytical method are consistently the fastest. With that said, it does appear that there is some amount of error present in the analytical solution.

Problem 3

$$E^{(m)} = \begin{cases} \frac{1}{m}\left(C^{m/2} - I\right) & \text{if } m \neq 0 \\ \frac{1}{2}\ln C & \text{if } m = 0 \end{cases}$$

$$C^{m/2} = \sum_{I=1}^{3} \lambda_I^m \, M_I \otimes M_I \qquad \ln C = \sum_{I=1}^{3} (\ln \lambda_I^2)\, M_I \otimes M_I$$

λ_I $I=1\text{-}3$ are principal stretches M_I lie along principle directions and form orthonormal basis in E^3

a) $E^{(2)} = \frac{1}{2}\left(C^{1} - I\right)$

$= E$

b) $E^{(-2)} = -\frac{1}{2}\left(C^{-1} - I\right)$

$= \frac{1}{2}\left(I - C^{-1}\right)$

This is similar to the Eulerian strain tensor, but in a referential sense

c) $\lim_{m \to 0} E^{(m)} = E^{(0)}$

$\lim_{m \to 0} \frac{1}{m}\left(C^{m/2} - I\right)$ of the form $\frac{\left(C^{m/2} - I\right)}{m}$

$\lim_{\Delta x \to 0}\left(\frac{a^{\Delta x} - 1}{\Delta x}\right) = \ln(a)$

from this we get

$$\lim_{m \to 0} \frac{1}{m}\left(C^{m/2} - I\right) = \frac{1}{2}\ln C = E^{(0)}$$

d) for

$$F_{iA} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{plot } E_{11}^{(m)} \text{ as fcn of } m \text{ for } m[-10,10]$$

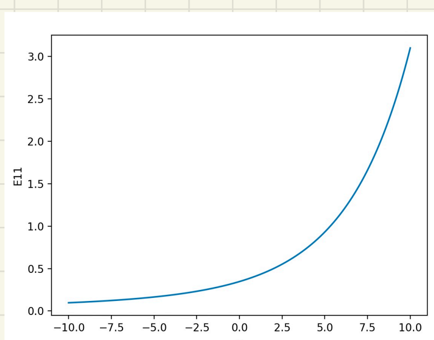
```python
#!/usr/bin/env python3
# -*- coding: utf-8 -*-
"""
Created on Fri Jan 28 14:26:45 2022

@author: juanmeriles
"""
import numpy as np
import matplotlib.pyplot as plt
F = np.array([[2,0,0],[0,1,0],[0,0,1]])

C= F.T@F
[lam2,M] = np.linalg.eig(C)
lam = np.sqrt(lam2)



m = np.linspace(-10,10,10000)
E = np.zeros(len(m))
for i in range(len(m)):
    if m[i] != 0:
        Cm2 = (lam[0]**(m[i]/2))*np.outer(M[:,0],M[:,0])+(lam[1]**(m[i]/2))*\
            np.outer(M[:,1],M[:,1])+(lam[2]**(m[i]/2))*np.outer(M[:,2],M[:,2])
        E[i] = (1/m[i]*(Cm2-np.eye(3)))[0][0]
    else:
        lnC = (np.log(lam[0]**2))*np.outer(M[:,0],M[:,0])+(np.log(lam[1]**2))*\
            np.outer(M[:,1],M[:,1])+(np.log(lam[2]**2))*np.outer(M[:,2],M[:,2])
        E[i] = (1/2)*lnC[0][0]
plt.plot(m,E)
plt.xlabel('m')
plt.ylabel('E11')
```



Problem 4

$\operatorname{div} T + \rho b = \rho a$

T - cauchy stress tensor
b - body force per unit mass
a - acc vector
ρ - mass density

$\operatorname{div} T \cdot v + \rho b \cdot v = \rho a \cdot v$

Cons of mass $\int_P (\dot{\rho} + \rho \operatorname{div} v)\, dv = 0$

$\rho_0 = \rho J$ $\qquad \dot{\rho} + \rho \operatorname{div} v = 0$

balance of angular momentum $\qquad$ div Thm

$$P F^T = F P^T$$

$\rho a \cdot v = \operatorname{div} T \cdot v + \rho b \cdot v$ $\qquad$ by product rule

$= \rho \frac{dv}{dt} \cdot v = \operatorname{div} T \cdot v + \rho b \cdot v$

$= \frac{1}{2} \rho \frac{d}{dt}(v \cdot v) = \operatorname{div} T \cdot v + \rho b \cdot v$

$\operatorname{div} T \cdot v = \operatorname{div}(T^T v) - T \cdot \operatorname{grad} v$
$= \operatorname{div}(Tv) - T \cdot (D + W)$
$= \operatorname{div}(Tv) - T \cdot D$

$\Rightarrow \frac{1}{2} \rho \frac{d}{dt}(v \cdot v) = \operatorname{div}(Tv) - T \cdot D + \rho b \cdot v$

$\Rightarrow \frac{1}{2} \rho \frac{d}{dt}(v \cdot v) + T \cdot D = \operatorname{div}(Tv) + \rho b \cdot v$

$\Rightarrow \frac{d}{dt} \int_P \frac{1}{2} \rho (v \cdot v)\, dv + \int_P T \cdot D\, dv = \int_P \operatorname{div}(Tv)\, dv + \int_P \rho b \cdot v\, dv$

$\Rightarrow \frac{d}{dt} \int_P \frac{1}{2} \rho (v \cdot v)\, dv + \int_P T \cdot D\, dv = \int_{\partial P} Tv \cdot n\, da + \int_P \rho b \cdot v\, dv$

Problem 5

$T \cdot D$ stress power $\Rightarrow$ Show

$$\int_P T \cdot D \, dv = \int_{P_0} P \cdot \dot{F} \, dV = \int_{P_0} S \cdot \dot{E} \, dV = \int_{P_0} S^{(1)} \cdot \dot{U} \, dV$$

$$S^{(1)} = \frac{1}{2}(R^T P + P^T R)$$

Things that may be useful $\quad \dot{E} = F^T D F \quad P = J T F^{-T} \quad \dot{U} =$

$L = D + W \quad S = F^{-1} P$

$\dot{F} = L F \quad F = R U$

$$P \cdot \dot{F} = P \cdot (D + W) F$$
$$= J T F^{-T} \cdot (D + W) F$$

$$\int_P T \cdot D \, dv = \int_{P_0} T \cdot D J \, dV$$
$$= \int_{P_0} J T \cdot L$$

Because $T W = 0$?

$$= \int_{P_0} J T F^{-T} \cdot L F \, dV$$
$$= \int_{P_0} P \cdot \dot{F} \, dV \checkmark$$

$$\int_P T \cdot D \, dv = \int J T \cdot D \, dv$$
$$= \int J F F^{-1} T \cdot D F F^{-1} \, dV$$
$$= \int J F^{-1} T F^{-T} \cdot F^T D F \, dV$$
$$= \int S \cdot \dot{E} \, dV$$

$$\frac{1}{2}(R^T P + P^T R) \cdot \dot{U}$$

$$= \frac{1}{2}(R^T P \cdot \dot{U} + P^T R \cdot \dot{U})$$

$$= \frac{1}{2}(P \cdot R\dot{U} + P^T \cdot \dot{U} R^T)$$

$$= \frac{J}{2}(T F^{-T} \cdot R\dot{U} + F^{-1} T^T \cdot \dot{U} R^T) \qquad T = T^T$$

$$\int_{P_0} J\, T \cdot \frac{1}{2}(L + L^T)$$

$$= \int_{P_0} \frac{J}{2} T \cdot \frac{1}{2}(L + L^T) F F^{-1} dV$$

$$= \int_{P_0} \frac{J}{2}(T \cdot L F F^{-1} + F F^{-1} T \cdot L^T) \qquad \dot{F} = LF \quad \dot{F}^T = F^T L^T$$

$$(T \cdot \dot{F} F^{-1} + F F^{-1} T \cdot L^T)$$

$$(T \cdot \dot{F} F^{-1} + F^{-1} T \cdot \dot{F}^T)$$

$$(T F^{-T} \cdot \dot{F} + F^{-1} T^T \cdot \dot{F}^T)$$

$$\int_{P_0} \frac{1}{2}(P \cdot \dot{F} + P^T \cdot \dot{F}^T) dV \qquad \dot{F} = \dot{(RU)} = \dot{R}U + R\dot{U}$$

$$\int_{P_0} \frac{1}{2}(P \cdot (\dot{R}U + R\dot{U}) + P^T \cdot (\dot{R}U + R\dot{U})^T dV$$

$$\int_{P_0} \frac{1}{2}(P \cdot \dot{R}U + P \cdot R\dot{U} + P^T \cdot U^T \dot{R}^T + P^T \cdot \dot{U}^T R^T) dV$$

$$\int_{P_0} \frac{1}{2}(P \cdot \dot{R}U + R^T P \cdot \dot{U} + P^T \cdot U^T \dot{R}^T + P^T R \cdot \underbrace{\dot{U}}_{\text{sym so } =T}) dV$$

$$\int_{P_0} \frac{1}{2}\Big(\underbrace{\cancel{(P \cdot \dot{R}U + P^T \cdot U^T \dot{R}^T)}}_{0} + (R^T P + P^T R) \cdot \dot{U}\, dV$$

$$= \int_{P_0} \frac{1}{2}(R^T P + P^T R) \cdot \dot{U}\, dV$$

Problem 6

χ and χ^+

$x = \chi(X,t)$ $\quad x^+ = \chi^+(X,t)$ $\quad x^+ = \underset{\text{proper ortho}}{Q(t)}x + \underset{\text{vector in } E_3}{c(t)}$

$F^+ = QF$

a) Show $\dot{C}^+ = \dot{C}$ $\quad \dot{B}^+ = \Omega B^+ + (\Omega B^+)^T + Q\dot{B}Q^T$; $\Omega = \dot{Q}Q^T$

$$\dot{C} = \overline{F^TF}^{\,\cdot} = \dot{F}^TF + F^T\dot{F}$$

$\dot{F} = LF$ / $\dot{F}^+ = L^+F^+$
$= L^+QF$
$= QLQ^TQF + \Omega QF$
$= QLF + \dot{Q}F$

$$\dot{C}^+ = \dot{F}^{T+}F^+ + F^{T+}\dot{F}^+$$
$$= (QLF + \dot{Q}F)^TQF + F^TQ^T(QLF + \dot{Q}F)$$
$$= (F^TL^TQ^TQF + F^T\dot{Q}^TQF) + (F^TQ^TQLF + F^TQ^T\dot{Q}F)$$
$$= \dot{F}^TF + \cancel{F^T\dot{Q}^TQF} + F^T\dot{F} + \cancel{F^TQ^T\dot{Q}F}$$
$$= \dot{F}^TF + F^T\dot{F}$$

since $\dot{Q}Q^T$ is skew sym and Q is sym

$F^T\dot{Q}^TQF + F^TQ^T\dot{Q}F = 0$
$= F^T\dot{Q}Q^TF + F^TQ\dot{Q}^TF = 0$

$$\dot{B}^+ = \overline{FF^T}^{\,\cdot\,+} = \dot{F}^+F^{T+} + F^+\dot{F}^{T+}$$
$$= Q\dot{F}F^TQ^T + QF\dot{F}^TQ^T + \dot{Q}FF^TQ^T + QFF^T\dot{Q}^T$$
$$= Q\dot{B}Q + \dot{Q}Q^TQFF^TQ^T + QFF^TQ^TQ\dot{Q}^T$$
$$= Q\dot{B}Q + \Omega\dot{B} + (\Omega\dot{B})^T$$

b) $T^+ = QTQ^T \Rightarrow T$ is an objective eulerian tensor

$$\dot{T}^+ = (\dot{T}^+) = \overline{QTQ^T}^{\,\cdot} = \dot{Q}TQ^T + Q\dot{T}Q^T + QT\dot{Q}^T$$
$$= \Omega QTQ^T + Q\dot{T}Q^T + QT(\Omega Q)^T$$
$$= \Omega T^+ + Q\dot{T}Q^T - T^+\Omega$$

$\rightarrow$ not an objective tensor

c) $\overset{\square}{T} = \dot{T} - \dot{R}R^T T + T\dot{R}R^T$

R = rotation tensor show $\overset{\square}{T}$ is objective

$$\overset{\square}{T}^+ = \dot{T}^+ - (\dot{R}R^T T)^+ + (T\dot{R}R^T)^+$$

$$= \cancel{\Omega T^+} + Q\dot{T}Q^T - \cancel{T^+\Omega}$$
$$- \cancel{\dot{Q}QT^+} - Q\dot{R}R^TQT^T$$
$$+ \cancel{T^+\dot{Q}Q} + TQ\dot{R}R^TQ^T$$

$$= Q\dot{T}Q^T - Q\dot{R}R^TT^TQ + QT\dot{R}R^TQ^T$$

$$= Q\overset{\square}{T}Q^T$$

$\rightarrow$ $\overset{\square}{T}$ is objective.

$\dot{R}^+ = \dot{(QR)}$
$= \dot{Q}R + Q\dot{R}$

$R^+ = QR$

$(\dot{R}R^T)^+ = \dot{Q}\underbrace{\cancel{RR^T}}_{I}Q$
$+ Q\dot{R}R^TQ^T$