Juan Meriles
ME 280B
Assignment 2

Problem 1

$$f(x) = \begin{cases} x & \text{for } x \le 1 \\ 2-x & \text{for } x > 1 \end{cases}$$

Does f possess a Gateaux differential at x = 1?

$$\lim_{\omega \to 0} \frac{f(1+\omega v) - f(1)}{\omega}$$

if $\omega v > 0$ $= \lim_{\omega \to 0} \frac{2-1-\omega v - 1}{\omega} = -v$

if $\omega v \le 0$ $= \lim_{\omega \to 0} \frac{1+\omega v - 1}{\omega} = v$

as approaching from both sides gives a different solution, there is no Gateaux differential at x=1

Problem 2

$$f(x_1,x_2) = \begin{cases} \frac{x_1^2 x_2}{x_1^4 + x_2^2} & \text{for } (x_1,x_2) \neq (0,0) \\ 0 & \text{for } (x_1,x_2) = (0,0) \end{cases}$$

a) show f posesses Gateaux diff in all directions at (0,0), but not a Gateaux derivative

$$D_G F(x_1,x_2)_{(0,0)} = \left[\frac{d}{d\omega} F(x_1+\omega v_1, x_2+\omega v_2)\right]_{\omega=0}$$

$$= \frac{d}{d\omega} \frac{\omega^3 v_1^2 v_2}{\omega^4 v_1^4 + \omega^2 v_2^2} = \frac{d}{d\omega} \frac{\omega v_1^2 v_2}{\omega^2 v_1^4 + v_2^2}$$

$$= \left[\frac{v_1^2 v_2}{\omega^2 v_1^4 + v_2^2} + \frac{\omega v_1^2 v_2 \cdot 2\omega v_1^4}{(\omega^2 v_1^4 + v_2^2)^2}\right]_{\omega=0}$$

$$= \frac{v_1^2 v_2}{v_2^2} = \frac{v_1^2}{v_2}$$

$$\rightarrow \quad D\, f(x_1,x_2)_{(0,0)} = \begin{cases} 0 & \text{if } v_2 = 0 \\ \frac{v_1^2}{v_2} & \text{if } v_2 \neq 0 \end{cases}$$

The Gateaux derivative requires a linear map on v, This cannot be done because v_1 is is non linear

$$D_G F(x_0, v) \neq \underbrace{[D_G F(x_0)]}_{\text{This would need to be a linear map}} (v)$$

b) does $f(x_1, x_2)$ posess a Frechet Differential at $(0,0)$

No because by definition

$D_F F(x_0): v \in U \rightarrow [D_F F(x_0)](v)$

which implies you can linearly map onto v, which cannot be done.

Furthermore, by definition if there is no Gateaux derivative at a point there will not be a Frechet differential

Problem 3

Stretch λ lies along direction M in ref config is related to Cauchy-Green def tensor C according to

$$\lambda^2 = M \cdot CM$$

a) determine linear part of λ in direction of tangent Δu with respect to $\bar{R}$

$$\mathcal{L}[F; \Delta u] = F(\bar{x}) + D_F F(x, v)$$

λ is our $f(x)$ defined as $\sqrt{M \cdot \bar{F}^T \bar{F} M}$
where M is a unit vector in the direction of X

$$\mathcal{L}[\lambda; \Delta u] = \sqrt{M \cdot \bar{F}^T \bar{F} M} + D_F \lambda(\bar{x}, \Delta u)$$

$$D_F \lambda(x, \Delta u) = \frac{d}{d\omega}\left[\frac{M^T F^T(\bar{x}+\omega\Delta u)\; F(\bar{x}+\omega\Delta u)\; M}{2(M^T C M)^{1/2}}\right]_{\omega=0}$$

$$= \left[\frac{M^T \frac{d F^T(\bar{x}+\omega\Delta u)}{d\omega} F(\bar{x}+\omega\Delta u) M + M^T F^T(\bar{x}+\omega\Delta u)\frac{dF(\bar{x}+\omega\Delta u)}{d\omega} M}{2(M^T C M)^{1/2}}\right]_{\omega=0}$$

$$= \frac{M^T F^T(\Delta\hat{u}) F(\bar{x}) M + M^T F^T(\bar{x}) F(\Delta\hat{u}) M}{2(M^T C M)^{1/2}}$$

$$= \frac{M^T\left(\left(\frac{\partial \Delta\hat{u}}{\partial X}\right)^T \bar{F} + \bar{F}\left(\frac{\partial \Delta\hat{u}}{\partial X}\right)\right) M}{2(M^T C M)^{1/2}}$$

$$= \frac{M^T\left(\left(\frac{\partial \Delta\bar{u}}{\partial \bar{x}}\bar{F}\right)^T \bar{F} + \bar{F}\frac{\partial \Delta\bar{u}}{\partial \bar{x}}\bar{F}\right) M}{(M^T C M)^{1/2}}$$

$$= M^T \bar{F}^T \left(\frac{\partial \Delta\bar{u}}{\partial \bar{x}}\right)^S \bar{F} M \,/\, (M^T C M)^{1/2}$$

$$\mathcal{L}(\lambda; \Delta u) = \sqrt{m^T \bar{C} m} + \frac{m^T \bar{F}^T \left(\frac{\partial \Delta \bar{u}}{\partial \bar{x}}\right)^S \bar{F} m}{(m^T C m)^{1/2}}$$

where $\left(\frac{\partial \Delta u}{\partial \bar{x}}\right)^S = \frac{1}{2}\left(\left(\frac{\partial \Delta u}{\partial \bar{x}}\right)^T + \left(\frac{\partial \Delta u}{\partial \bar{x}}\right)\right)$

b) for case where $\bar{R}$ coincides w/ reference configuration R_0

$\bar{x} = X \quad \Delta u = u \quad \bar{F} = I$

$$\mathcal{L}(\lambda; \Delta u) = \sqrt{m^T m} + \frac{m^T \left(\frac{\partial u}{\partial X}\right)^S m}{(m^T m)^{1/2}}$$

Problem 4

$F = RU$ $\quad$ R proper orthogonal rot tensor
$\quad$ U sym pos-def right stretch tensor

$DU(\bar{x};\Delta u)$ obtained by several authors

a) Derive linear parts of U and R along Δu w/ respect to reference config R_0. Which fields of classical infinitesimal theory are recovered?

$U^2 = F^T F = I$ in ref config $\quad$ R must also then be I

$$DU(X, \Delta U) = \frac{d}{d\omega}\left(F^T(X+\omega\Delta U)\,F(X+\omega\Delta U)\right)^{1/2}\Big|_{\omega=0}$$

ref R_0

$$= \frac{d}{d\omega}\left(\left(I + \left(\frac{d\,\omega\Delta u}{dX}\right)^T\right)\left(I + \frac{d\,\omega\Delta u}{dX}\right)\right)_{\omega=0}$$

$$= \frac{\frac{1}{2}\left(\frac{d\Delta u}{dx}\right)^T\left(I + \frac{d\,\omega\Delta u}{dx}\right)_{\omega=0} + \left(I + \frac{d\,\omega\Delta u}{dX}\right)_{\omega=0}\frac{d\Delta u}{dX}}{I}$$

$$= \frac{1}{2}\left(\frac{d\Delta u}{dX}^T + \frac{d\Delta u}{dX}\right)$$

in ref config $\quad = \frac{1}{2}\left(\frac{du}{dx}^T + \frac{du}{dX}\right)$

$\quad$ = infinitesimal strain tensor

$$\mathcal{L}(U;\Delta U) = I + \frac{1}{2}\left(\frac{du}{dX}^T + \frac{du}{dX}\right)$$

$F = RU$

$DF = (DR)U + RDU$

in ref config $U = I \quad R = I$

$DF = DR + DU$

$$
\begin{aligned}
DR &= DF - DU \\
&= \frac{\partial u}{\partial X} - \left(\frac{\partial u}{\partial X}\right)^S \\
&= \frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial u}{\partial X} - \frac{1}{2}\left(\frac{\partial u}{\partial X}\right)^T \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial X} - \left(\frac{\partial u}{\partial X}\right)^T\right) = \text{infinitesimal rotation tensor}
\end{aligned}
$$

$$\mathcal{L}(R;\Delta u) = I + \frac{1}{2}\left(\frac{\partial u}{\partial X} - \left(\frac{\partial u}{\partial X}\right)^T\right)$$

b) From Paper

$$DH(C)[T] = \frac{4}{\Delta U}\{I_U\, CTC - I_U^2(CTU + UTC) + (I_U II_U - III_U)(CT + TC) + (I_U^3 + III_U)UTU - I_U^2 II_U(UT + TU) + [I_U^2\, III_U + (I_U II_U - III_U) II_U]T$$

$I_U = 3$

$II_U = \frac{1}{2}(\cancel{9} - 3) = 3$

$III_U = 1$

because $U = I$ in ref config

$T = \Delta C = [\Delta F^T F + F^T \Delta F]$

$\quad = (\Delta F + \Delta F^T)$

$C = I$

$$\rightarrow \frac{4}{\Delta U}\{3T - 9(2T) + 8(2T) + 28T - 27(2T) + [9 + 24]T\}$$

$\Delta U = 8(I_U II_U - III_U) III_U$

$$= \frac{4}{\Delta U}\{8T\}$$

$$= \frac{\cancel{4}}{\cancel{8}\cdot\cancel{8}}\{\cancel{8}T\} = \frac{T}{2}$$

With this information

$$DU(\bar{x}, \Delta U)_x = \frac{DC}{2} = \frac{2\varepsilon}{2} = \varepsilon$$

Problems

Nansons Formula

$$n\,da = J\,F^{-T} N\,dA$$

$J = \det F$

a) determine linear part of n in direction of tangent Δu with respect to $\bar{R}$

$$n \frac{da}{dA} = J F^{-T} N$$

$$D(n)\frac{da}{dA} + n\,D\left(\frac{da}{dA}\right) = D(J) F^{-T} N + J\,D(F^{-T})\,N$$

$D(N) = 0$
$\downarrow$
N not fn of $\bar{x}$

$$D(n) = \frac{dA}{da}\left(D(J) F^{-T} N + J\,D(F^{-T}) N - n\,D\left(\frac{da}{dA}\right)\right)$$

$$= \frac{dA}{da}\left(\bar{J}\,\mathrm{div}\,\Delta u\, F^{-T} N + J\left[\frac{d}{d\omega}\left(\frac{\partial(\bar{x}-\omega\Delta u)}{\partial x}\right)^{-T}\right]_{\omega=0} N - n\left(\mathrm{tr}\,\varepsilon(u) - N\cdot\varepsilon(u)N\right)\right.$$

$$= \frac{dA}{da}\left(\bar{J}\,\mathrm{div}\,\Delta u\, F^{-T} N + J\left(\frac{\partial \Delta u}{\partial x}\right)^{-T} N - n\left(\mathrm{tr}\,\varepsilon(u) - N\cdot\varepsilon(u)N\right)\right)$$

b) When $\bar{R} = R_0$ $\quad \frac{dA}{da} = 1 \quad \bar{J} = 1 \quad \Delta u = u \quad F = I \quad n = N$

$$D(n) = \left(\mathrm{div}\,uF + \left(\frac{\partial u}{\partial X}\right)^{-T}\right) N - N\left(\mathrm{tr}\,\varepsilon(u) - N\cdot\varepsilon(u)N\right.$$

Problem 6

$$DJ(\bar{x}, \Delta u) = \bar{J}\, tr(\bar{F}^{-1} G) \; ; \; G = \frac{\partial \Delta \hat{u}}{\partial X}$$

a) obtain exact formula for $\frac{\partial \bar{J}}{\partial F}(\bar{x})$

$$\frac{\partial J}{\partial F}(\bar{x}) \cdot G = \bar{J}\, tr(\bar{F}^{-1} G)$$

$$= \det(\bar{F})\left(tr(\bar{F}^{-1} G)\right) \begin{bmatrix} \bar{F}_{11} & \bar{F}_{12} & \bar{F}_{13} \\ \bar{F}_{21} & \bar{F}_{22} & \bar{F}_{23} \\ \bar{F}_{31} & \bar{F}_{32} & \bar{F}_{33} \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{F}_{11} G_{11} + \bar{F}_{12} G_{21} + \bar{F}_{13} G_{31} & - & - \\ - & \bar{F}_{21} G_{12} + \bar{F}_{22} G_{22} + \bar{F}_{23} G_{32} & - \\ - & - & \bar{F}_{31} G_{13} + \bar{F}_{32} G_{23} + \bar{F}_{33} G_{33} \end{bmatrix}$$

$$tr(\bar{F}^{-1} G)$$

$$= \bar{F}_{11} G_{11} + \bar{F}_{11} G_{21} + \bar{F}_{13} G_{31} + \bar{F}_{21} G_{12} + \bar{F}_{22} G_{22} + \bar{F}_{23} G_{32} + \bar{F}_{31} G_{13} + \bar{F}_{32} G_{23} + \bar{F}_{33} G_{33}$$

$$\rightarrow \left\langle \frac{\partial J}{\partial F}, G \right\rangle = Tr\left(\left(\frac{\partial J}{\partial F}\right)^T G\right)$$

$$\Rightarrow \left(\frac{\partial \bar{J}}{\partial F}\right)^T = \bar{J}\, \bar{F}^{-1}$$

$$\left(\frac{\partial \bar{J}}{\partial F}\right) = \bar{J}\, \bar{F}^{-T}$$

b) estimate $\frac{\partial J}{\partial F}(\bar{x})$ using $DJ(\bar{x}, \Delta u) \doteq \frac{1}{\omega}\left[J(\bar{x} + \omega \Delta u) - J(\bar{x})\right]$

$$\frac{\partial J}{\partial F} \cdot \frac{\partial \Delta \hat{u}}{\partial X} = DJ(\bar{x}, \Delta u)$$

$$\frac{\partial J}{\partial F} \cdot \frac{\partial \Delta \hat{u}}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial X} = DJ(\bar{x}, \Delta u) \Rightarrow \frac{\partial J}{\partial F} F^T \cdot \frac{\partial \Delta \hat{u}}{\partial \bar{x}} = DJ(\bar{x}, \Delta u)$$

if we choose $\Delta u = [x_i, 0, 0] \rightarrow \frac{\partial \Delta u}{\partial \bar{x}} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$

thus directions should be cycling through xs in each pos

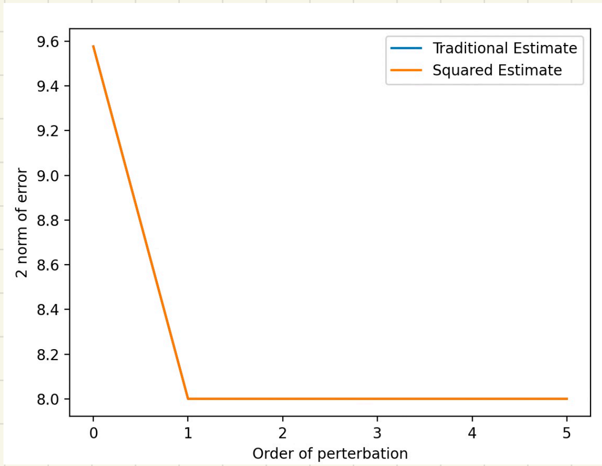
$i = 1 \rightarrow 3 \quad [x_i \;\; 0 \;\; 0]$

$i = 1 \rightarrow 3 \quad [0 \;\; x_i \;\; 0]$

$i = 1 \rightarrow 3 \quad [0 \;\; 0 \;\; x_i]$

$\rightarrow$ we can build the $\frac{\partial J}{\partial F} F^T$ matrix and then

```python
import numpy as np
import matplotlib.pyplot as plt
x = [1,1,0]
F = np.array([[1+x[0],(x[0]**2)*x[1],0],
              [-2*x[1],1+2*x[0]*x[1],0],
              [0,0,1]])
FT = F.transpose()
J = np.linalg.det(F)
DJDF1 = np.zeros((6,3,3))
DJDF2 = np.zeros((6,3,3))
eps = np.finfo(float).eps
w = np.array([10**0,10**1,10**2,10**3,10**4,10**5])*eps

dirs = np.array([[x[0],0,0],
                 [x[1],0,0],
                 [x[2],0,0],
                 [0,x[0],0],
                 [0,x[1],0],
                 [0,x[2],0],
                 [0,0,x[0]],
                 [0,0,x[1]],
                 [0,0,x[2]]])

for j in range(len(w)):
    count1 = 0
    count2 = 0
    for i in range(len(dirs)):
        xad = x + w[j]*dirs[i];
        Fad = np.array([[1+xad[0],(xad[0]**2)*xad[1],0],
              [-2*xad[1],1+2*xad[0]*xad[1],0],
              [0,0,1]])
        Jad1 = np.linalg.det(Fad)
        DJDF1[j,count1,count2] = (1/w[j])*(Jad1-J)
        DJDF2[j,count1,count2] = np.sqrt(2*J*(1/w[j])*(Jad1-J))
        count1 = count1+1
        if count1 == 3:
            count1 = 0;
            count2 = count2+1
    DJDF1[j,:,:] = DJDF1[j,:,:] @ np.linalg.inv(FT)
    DJDF2[j,:,:] = DJDF2[j,:,:] @ np.linalg.inv(FT)

DJDF = np.linalg.inv(FT)*J
error1 = np.zeros(len(w))
error2 = np.zeros(len(w))

for i in range(len(w)):
    error1[i] = np.linalg.norm(DJDF-DJDF1[i,:,:],2)
    error2[i] = np.linalg.norm(DJDF-DJDF1[i,:,:],2)

plt.figure(2)
plt.plot(error1)
plt.plot(error2)
plt.xlabel("Order of perterbation")
plt.ylabel("2 norm of error")
```



As can be seen from the error plot, both methods of estimation perform equally poorly, primarily because they are incapable of estimating the 33 position due to x3 being equal to 0. I am unsure of how to adjust for this problem and have not been able to resolve it.

Problem 7

$\mathrm{Div}\, P + \rho_0 b = \rho_0 a$

P is Piola-Kirchoff stress ρ_0 is mass density in ref config, b is body force per unit mass, a is acc

a) linearize in direction Δu w/ respect to $\bar{R}$. Simplify by assuming linear momentum balance holds in $\bar{R}$

$$\mathcal{L}[\mathrm{div}\, P; \Delta u]_{\bar{x}} + \mathcal{L}[\rho_0 b; \Delta u]_{\bar{x}} = \mathcal{L}[\rho a; \Delta u]_{\bar{x}}$$

$$D[\mathrm{Div}\, P](\bar{x}, \Delta u) = \left[\frac{d}{d\omega}\{\mathrm{div}\, P(\bar{x} + \omega \Delta u)\}\right]_{\omega=0}$$

$$= \left[\frac{d}{d\omega} \frac{\partial P_{ij}(\bar{\underset{\sim}{x}} + \omega \Delta \underset{\sim}{u})}{\partial (X_j)} \underset{\sim}{e}_i\right]_{\omega=0}$$

$$= \mathrm{Div}(D P(\bar{x}, \Delta u))$$

$$D[\rho_0 b](\bar{x}, \Delta u) = \rho_0 D b(\bar{x}, \Delta u)$$

$$D[\rho_0 a](\bar{x}, \Delta u) = \rho_0 \Delta \ddot{\underset{\sim}{u}}$$

$$\mathcal{L}[\mathrm{Div}\, P; \Delta u]_{\bar{x}} + \mathcal{L}[\rho_0 b; \Delta u]_{\bar{x}} = \mathcal{L}[\rho a; \Delta u]_{\bar{x}}$$

$$\cancel{\mathrm{Div}\, \bar{P}} + \mathrm{Div}(D\bar{P}(\bar{x}, \Delta u)) \cancel{+ \rho_0 b} + \rho_0 D b(\bar{x}, \Delta u) = \cancel{\rho_0 \ddot{u}} + \rho_0 \Delta \ddot{u}$$

By linear momentum balance

$$\mathrm{Div}(DP(\bar{x}, \Delta u)) + \rho_0 D b(\bar{x}, \Delta u) = \rho_0 \Delta \ddot{u}$$

b) Show this is consistent w/ what was derived in class

$$\mathrm{div}[D\,T[\bar{x},\Delta u]] - \mathrm{grad}\,\bar{T}\,\mathrm{grad}^T\Delta u + \{-\bar{\rho}\,\mathrm{div}\,\Delta \underline{u}\,\bar{b} + \bar{\rho}\,D\underline{b}(\bar{x},\Delta u)\}$$
$$= \{-\bar{\rho}\,\mathrm{div}\Delta\underline{u}\,\ddot{\bar{u}} + \rho\,\Delta\ddot{\underline{u}}\}$$

$$P = J\,T\,F^{-T}$$

$$\mathrm{Div}(D(J\,T\,F^{-T}(\bar{x},\Delta u)))$$

$$= \mathrm{Div}(DJ(\bar{x},\Delta u)\,T\,F^{-T} + J\,D\,T(\bar{x},\Delta u)\,F^{-T} + J\,T\,D\,F^{-T}(\bar{x},\Delta u))$$

$$= \mathrm{Div}(\bar{J}\,\mathrm{div}\,\Delta u\,\bar{T}\bar{F}^{-T} + J\,D\,T(\bar{x},\Delta u)\,F^{-T} - J\,\bar{T}\,\mathrm{grad}\,\Delta u^T\,\bar{F}^{-T}$$

$$= \bar{J}\,\mathrm{div}(\mathrm{div}\,\Delta u\,\bar{T} + D\,T(x,\Delta u) - \bar{T}\,\mathrm{grad}\,\Delta u^T)$$

$$= \bar{J}\,(\mathrm{grad}\,\mathrm{div}\,\Delta u \cdot \bar{T} + \mathrm{div}\,\Delta u\,\mathrm{div}\,\bar{T} + \mathrm{div}(D\,T(\bar{x},\Delta u))$$
$$- (\mathrm{grad}\,T\,\mathrm{grad}\,\Delta u - \bar{T}\,\mathrm{grad}\,\mathrm{div}\,\Delta u)$$

$$= \bar{J}\,(\mathrm{div}\,\Delta u\,\mathrm{div}\,T + \mathrm{div}(D\,T(\bar{x},\Delta u)) - \mathrm{grad}\,\bar{T}\,\mathrm{grad}\,\Delta u$$

This along with $\rho_0 = \bar{J}\bar{\rho}$ leads to the equation

$$\mathrm{div}[D\,T[\bar{x},\Delta u]] - \mathrm{grad}\,\bar{T}\,\mathrm{grad}\,\Delta u + \underbrace{\mathrm{div}\,\Delta u\,\mathrm{div}\,T}_{= \bar{\rho}\,\mathrm{div}\,\Delta u(\bar{b} - \ddot{\bar{u}})} + \bar{\rho}\,D b(\bar{x},\Delta u)$$
$$+ \bar{\rho}\,D\,\Delta\ddot{u}(\bar{x},\Delta u)$$

↳ thus confirming formula from class