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ME 280B
4/01/22
Assignment 5

Problem 1

The predictor corrector was implemented as in the notes

$$\int_R \xi_{n+1} \cdot \rho\left(\frac{\partial v^*_{n+1}}{\partial t} + \frac{\partial \bar{v}_n}{\partial x} v_n\right) dv = \int_R \left[-\left(\frac{\partial \bar{\xi}_{n+1}}{\partial x}\right)^S \cdot 2\mu \left(\frac{\partial \bar{v}_n}{\partial x}\right)^S + \xi_{n+1} \cdot \rho b_{n+1}\right] dv$$

$$\int \xi_{n+1} \cdot \rho \frac{v_{n+1} - v^*_{n+1}}{\Delta t_n} dv = \int_R \operatorname{div} \xi_{n+1} \cdot p_{n+1} dv + \int_{\Gamma_q} \xi_{n+1} \cdot \bar{t}_{n+1} dv$$

$$\int_R \sigma_{n+1} \operatorname{div} v_{n+1} dv = 0$$

where terms were discretized using the standard isoparametric shape functions

$$N_1 = \tfrac{1}{4}(1-\eta)(1-\xi)$$
$$N_2 = \tfrac{1}{4}(1-\eta)(1+\xi)$$
$$N_3 = \tfrac{1}{4}(1+\eta)(1+\xi)$$
$$N_4 = \tfrac{1}{4}(1+\eta)(1-\xi)$$

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

$$[v^*_{n+1}] = \hat{v}_n - \Delta t_n [M^{-1}]\left(A(\hat{v}_n)] + [K][v_n]\right) + \Delta t_n [M]^{-1} [F^b_{n+1}]$$

$$\frac{1}{\Delta t_n}[M](\hat{v}_{n+1} - \hat{v}^*_{n+1}) - [C][\hat{p}_{n+1}] = [F'_{n+1}]$$

$$[C]^T [\hat{v}_{n+1}] = [G_{n+1}]$$

$[K] = \mathbb{A}\, K_e \qquad K_e = (B^T)^S \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} B$ same for all elements

$[M] = \mathbb{A}\, M_e \qquad M_e = \left[\frac{h^2}{4}\right] I_4$

$[A] = \mathbb{A}\, A_e \qquad A_e = [N]^T \left[\frac{dN}{dx} v^e \quad \frac{dN}{dy} v^e\right] [N]\, v^e$

$[C] = \mathbb{A}\, C_e \qquad C_e = \begin{bmatrix} \frac{dN_1}{dx} \\ \frac{dN_1}{dy} \\ \vdots \\ \frac{dN_4}{dx} \\ \frac{dN_4}{dy} \end{bmatrix}$

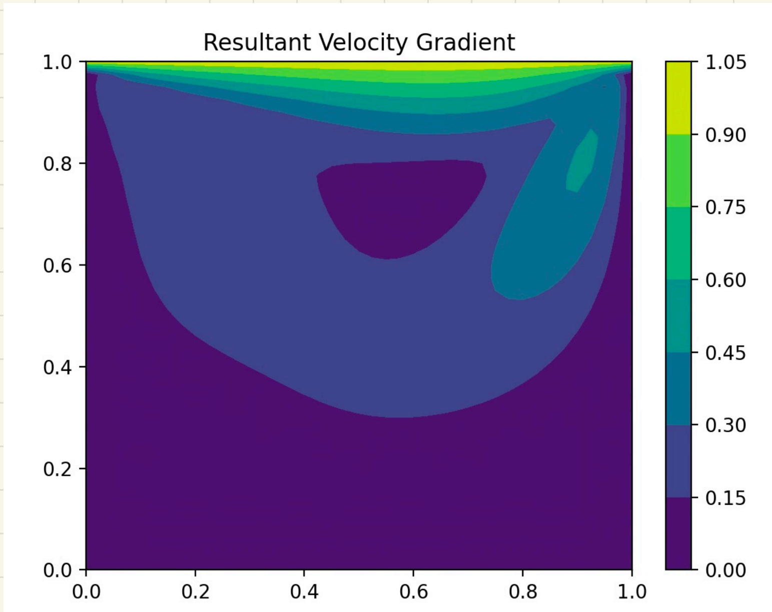
The G matrix is attained by rearranging matrices into

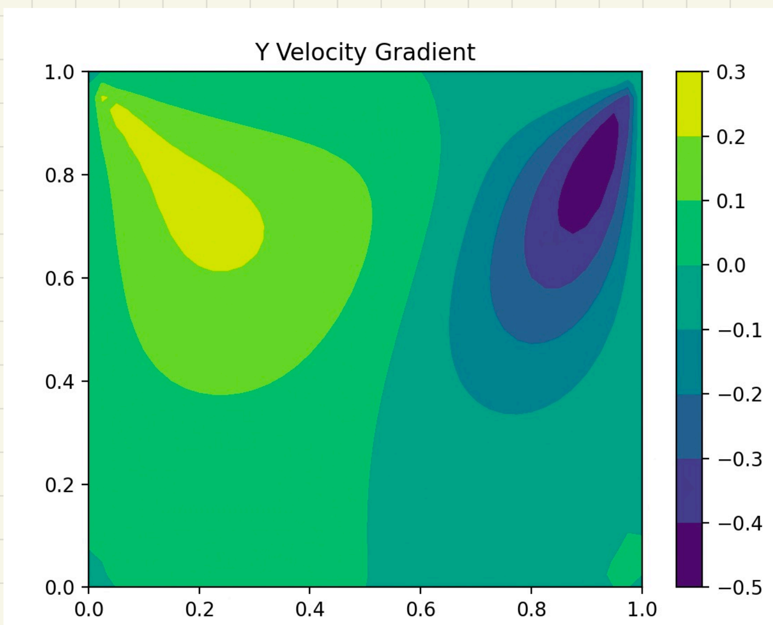
$$Ax = B \Rightarrow \begin{bmatrix} A_{free} & A_{freebound} \\ A_{boundfree} & A_{boundbound} \end{bmatrix} \begin{bmatrix} x_{free} \\ x_{bound} \end{bmatrix} = \begin{bmatrix} B_{free} \\ B_{bound} \end{bmatrix}$$

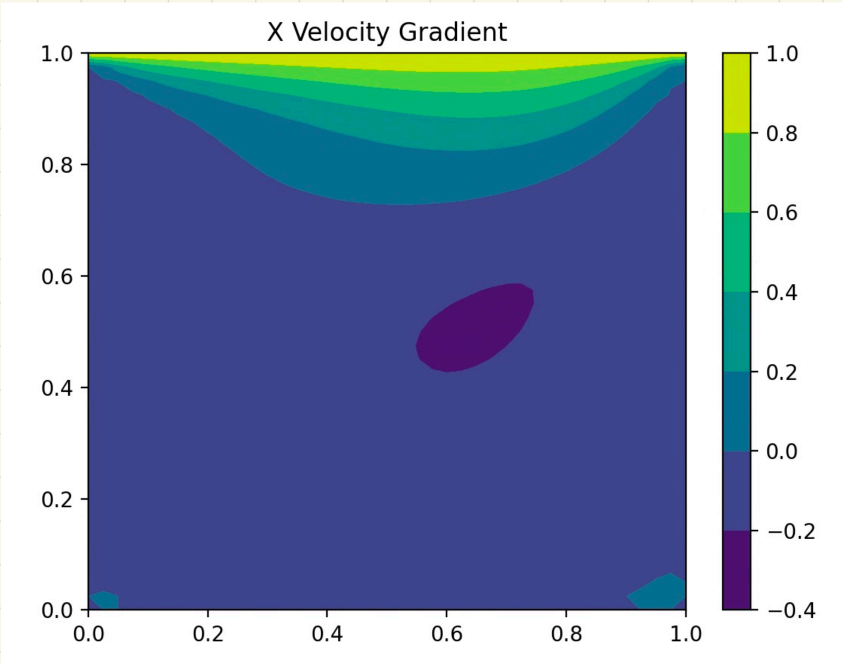
Where x_{bound} are known, and using this to generate $A_{freebound}\, x_{bound}$ which becomes the b vector

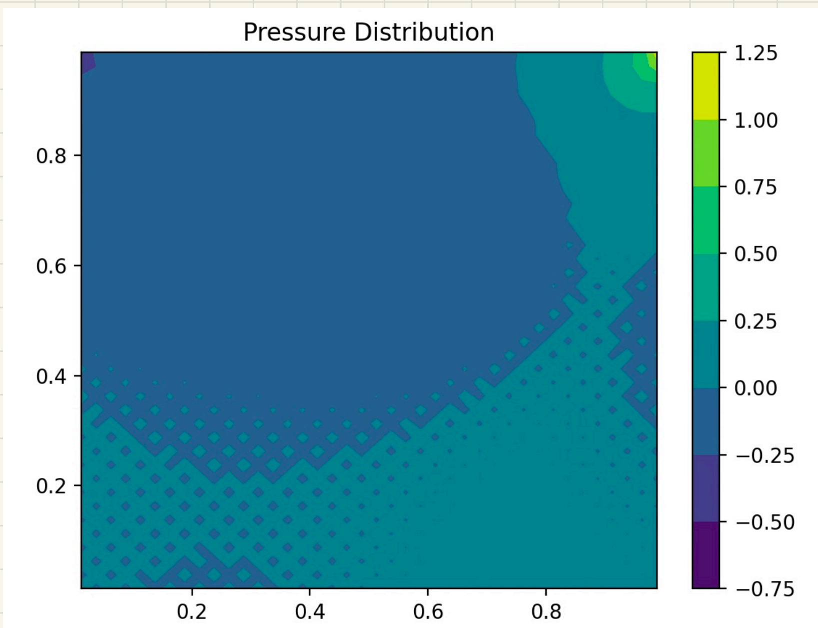
Using this, the predictor corrector was implemented in the attached python code w/ the following results

40 x 40 mesh was used with Δt of 0.001 up to 10 s


Resultant Velocity Gradient
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.05
0.90
0.75
0.60
0.45
0.30
0.15
0.00


Y Velocity Gradient
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
0.3
0.2
0.1
0.0
−0.1
−0.2
−0.3
−0.4
−0.5


X Velocity Gradient
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4


Pressure Distribution
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.25
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75

Problem 2

$$\int_{\Omega^e} \xi \cdot \rho \ddot{u} \, dv + \int_{\Omega^e} \frac{\partial \xi}{\partial x} : T \, dv = \int_{\Omega^e} \xi \cdot \rho b \, dv + \int_{\partial\Omega \cap \Gamma_a} \xi \cdot \bar{t} \, da + \int_{\partial\Omega^e - \Gamma_a \cap \partial\Omega^e} \xi \cdot t \, da$$

$$\Phi(x) = \frac{1}{2} k \bar{x} \cdot \bar{x} \qquad b = -\frac{\partial \bar{\Phi}}{\partial \bar{x}}$$

position vector

a) determine differential of body force along direction of disp increment Δu

$$D \int_{\Omega^e} \xi \cdot \rho \left(-\frac{\partial \bar{\Phi}}{\partial x}\right) dv \, (u, \Delta u)$$

$$= \int_{\Omega^e} \xi \cdot \rho D \left(-\frac{1}{2} k \bar{x} \cdot \bar{x}\right)(u, \Delta u) \qquad x = u + X$$

$$= \int_{\Omega^e} \xi \cdot \rho \lim_{\Delta\omega \to 0} -\frac{1}{2} \frac{b(u + \Delta\omega \Delta u) - b(u)}{\Delta \omega}$$

$$\int_{\Omega^e} \xi \cdot \rho \lim_{\Delta\omega \to 0} \frac{-\frac{1}{2} k (u \cdot u + 2u \cdot \Delta\omega \Delta u + \Delta\omega^2 u^2) + \frac{1}{2} u \cdot u}{\Delta \omega}$$

$$= \int_{\Omega^e} \xi \cdot \rho (-k \Delta u) \, dv$$

b) Suppose motion is approximated by isopar fem $\bar{x}$

$$x^e = N^e \hat{x}^e \qquad \xi^e = N^e \hat{\xi}^e$$

Write contribution k_b of body forces to tangent stiffness by linearizing body force along direction $\Delta \hat{u}^e$. Show k_b is sym

$$\mathcal{L}\, k_b = k_b + D k_b$$

$$= \int_{\Omega^e} \xi \cdot \rho (-k x) \, dv + \int_{\Omega^e} \xi \cdot \rho (-k \Delta u) \, dV$$

$$= \int_{\Omega^e} N \hat{\xi}^e \cdot \rho (-k N \hat{x}) \, dv + \int_{\Omega^e} N \hat{\xi}^e \cdot \rho (-k N \Delta \hat{u}^e) \, dv$$

$$= -\int_{\Omega^e} \hat{\xi}^e \cdot N^T \rho k N \hat{x} \, dv - \int_{\Omega^e} \hat{\xi}^e \cdot N^T \rho k N \Delta \hat{u}^e \, dv$$

$\rightarrow k_b = \int_{\Omega^e} N^T \rho k N \, dv$

This is symetric because

$$(N^T \rho k N)^T = N^T \rho k N$$

c) argue k_b is sym for a body force derivable from any smooth potential Φ

for a smooth potential $\bar{b} = -\frac{\partial \Phi}{\partial x}$ ~~exists~~ exists

when we take $\frac{d}{d\omega} b(x + \Delta\omega \Delta u)$

this can be written as $\frac{b(x+\Delta\omega\Delta u)}{d(x+\Delta\omega\Delta u)} \frac{d(x+\Delta\omega\Delta u)}{d\omega}$

$$= \frac{b(x+\Delta\omega\Delta u)}{d(x+\Delta\omega\Delta u)} \Delta u$$

leaving $\frac{b(x+\Delta\omega\Delta u)}{d(x+\Delta\omega\Delta u)}$ in the stiffness matrix. so long as this is symmetric, so is the matrix

$$\frac{db}{d(x+\Delta\omega\Delta u)} = \frac{d^2\Phi}{d(x+\Delta\omega\Delta u)^2}$$

This can be shown to be symmetric by

$$\bar{\Phi}_{,ij} e_i \otimes e_j = \bar{\Phi}_{,ji} e_j \otimes e_i$$

and thus $k_b = \rho N \frac{d^2\Phi}{d(x+\Delta\omega\Delta u)^2} N^T$ is symmetric

Problem 3

Given $\bar{t} = -p\bar{n}$ show differential of boundary traction term along Δu can be written as

$$-p\int_{\partial\Omega^e \cap \Gamma_q} \xi \cdot \left[\{ \mathrm{div}(\Delta\bar{u}) \} \bar{n} - \left(\frac{\partial \Delta\bar{u}}{\partial x}\right)^T \bar{n} \right] da$$

$$D\left[\int_{\partial\Omega^e \cap \Gamma_q} \xi \cdot \bar{t}\, da\right](\bar{u}, \Delta\bar{u}) = D\int_{\partial\Omega^e \cap \Gamma_q} \xi \cdot (-p\bar{n})\, da\, (\bar{u}, \Delta\bar{u})$$

$$= D\int_{\partial\Omega^e \cap \Gamma_q} \xi \cdot -p J F^{-T} N dA\, (\bar{u}, \Delta\bar{u})$$

$$= \int_{\partial\Omega^e \cap \Gamma_q} \xi \cdot -p(DJ\, F^{-T} N dA + J\, DF^{-T} N dA)\, (\bar{u}, \Delta\bar{u})$$

$$= -p\int_{\partial\Omega^e \cap \Gamma_q} \xi \cdot (J\, \mathrm{div}\, \Delta u\, F^{-T} N dA + J DF^{-T} N dA)\, (\bar{u}, \Delta\bar{u})$$

$$DF = \frac{\partial \Delta u}{\partial X} \qquad FF^{-} = I \qquad D(FF^{-}) = 0 = DF\, F^{-} + F DF^{-}$$

$$DF^{-} = -F^{-} DF\, F^{-}$$

$$= -F^{-} \underbrace{\frac{\partial \Delta u}{\partial X} F^{-}}$$

$$-F^{-} \frac{\partial \Delta u}{\partial x}$$

$$\rightarrow DF^{-T} = -\frac{\partial \Delta u}{\partial x}^T F^{-T}$$

$$-p\int_{\partial\Omega^e \cap \Gamma_a} \xi \cdot \left(J\, \mathrm{div}\, \Delta u\, F^{-T} N dA + J - \frac{\partial \Delta u}{\partial x}^T F^{-T} N dA \right)$$

$$= -p\int_{\partial\Omega^e \cap \Gamma_a} \xi \cdot \left(\mathrm{div}\, \Delta u - \frac{\partial \Delta u}{\partial x}^T\right)\bar{n}\, da$$