Juan Meriles
ME 280B
Assignment 4

Problem 1

$$f = \hat{f}(\underline{X}, t) = \hat{f}_M(X_M, t) = \tilde{f}(\underline{x}, t) = \tilde{f}_M(x_M, t)$$

Argue that $\hat{f}(Y, t) \neq \hat{f}_M(Y, t)$ and $\tilde{f}(\underline{y}, t) = \tilde{f}_M(\underline{y}, t)$

The first statement $\hat{f}(Y, t) \neq \hat{f}_M(Y, t)$ is true because

$\hat{f}(Y, t)$ is the scalar quantity at the material location that was at Y in the referential frame

$\hat{f}_M(Y, t)$ is the scalar quantity at the mesh location that was at Y in the referential frame.

Because $X \neq X_M$ generally, the point Y is mapped to two distinct locations. As such the mesh point that was at Y in the ref frame does not coincide with the material point that was at Y.

The second statement $\tilde{f}(\underline{y}, t) = \tilde{f}_M(\underline{y}, t)$ is true because in the spatial frame, the mesh point at $\underline{y}$ coincides w/ the material pnt at $\underline{y}$.

b) Show that

$$\frac{df}{dt} = \frac{d_M f}{dt} - \frac{\partial \bar{f}}{\partial x_M} \cdot (v_M - v)$$

localization $\int_P \phi \, dV = 0$

ϕ cont in x

$\phi = 0$ everywhere

$f = \tilde{f}_M(x_M, t) = f(x_M, t)$ because at x_M, the material pnt corresponds w/ the mesh point

$$\rightarrow \frac{d_M f}{dt} = \frac{\partial \tilde{f}_M(x_M,t)}{\partial t} + \frac{\partial \tilde{f}_M(x_M,t)}{\partial x_M} \cdot \frac{\partial x_M}{\partial t}$$

$$\frac{df}{dt} = \frac{\partial \tilde{f}(x,t)}{\partial t} + \frac{\partial \tilde{f}(x,t)}{\partial x} \cdot \frac{\partial x}{dt}$$

$$\frac{\partial \tilde{f}_M(x_M,t)}{\partial t} = \frac{\partial \tilde{f}_M(x,t)}{\partial t} = \frac{\partial \tilde{f}(x,t)}{\partial t}$$

$$\frac{\partial \tilde{f}_M(x_M,t)}{\partial t} = \frac{d_M f}{dt} - \frac{\partial \tilde{f}_M(x_M,t)}{\partial x_M} \cdot v_m$$

$$\frac{df}{dt} = \frac{d_M f}{dt} - \frac{\partial \tilde{f}_M(x_M,t)}{\partial x_M} \cdot v_m + \frac{\partial \tilde{f}(x,t)}{\partial x} \cdot v$$

but since $x_M = x$

$$\frac{df}{dt} = \frac{d_M f}{dt} - \frac{\partial \tilde{f}}{\partial t} \cdot (v_m - v)$$

c) Let v be a vector expressed in functional forms

$$\mathbf{V} = \hat{V}(X,t) = \hat{V}_M(X_M,t)$$

$$v = \tilde{v}(x,t) = \tilde{v}_M(x_m,t)$$

use (b) to establish

$$\frac{dv}{dt} = \frac{d_M v}{dt} - \frac{\partial \tilde{v}}{\partial x_M}(v_M - v)$$

$$\underline{V} = v_i \underline{e}_i \rightarrow \frac{d\underline{v}}{dt} = \frac{d}{dt}(v_i)\underline{e}_i$$

$$\frac{d\underline{v}}{dt} = \frac{dv_i}{dt}(\underline{e}_i) = \left(\frac{\partial \tilde{v}_i(x,t)}{\partial t} + \frac{\partial \tilde{v}_i(x,t)}{\partial x}\cdot\frac{\partial x}{\partial t}\right)(\underline{e}_i)$$

$$\frac{d_M v_i}{dt}(e_i) = \left(\frac{\partial \tilde{v}_{im}(x_m,t)}{\partial t} + \frac{\partial \tilde{v}_m(x_m,t)}{\partial x_m}\cdot\frac{\partial x_m}{\partial t}\right)(\underline{e}_i)$$

by the same logic as in (b) $\frac{\partial \tilde{v}_m(x_m,t)}{\partial t} = \frac{\partial \tilde{v}(x,t)}{\partial t}$

$$\frac{dv_i}{dt}(\underline{e}_i) = \frac{d_m v_i}{dt}(\underline{e}_i) - \frac{\partial \tilde{v}_m(x_m,t)}{\partial x_m}\underline{v_m}(\underline{e}_i) + \frac{\partial \tilde{v}(x,t)}{\partial x} v(\underline{e}_i)$$

$$\tilde{v}_{im}(x_m,t) = \tilde{v}_i(x,t)$$

$$\frac{d\underline{v}}{dt} = \frac{d_M\underline{v}}{dt} - \frac{\partial \tilde{\underline{v}}(x_m,t)}{\partial x_m}\cdot(\underline{v}_M - \underline{v})$$

d) derive locale ALE statements of mass + lin momentum balance

Mass balance

$$\dot{\rho} + \rho \operatorname{div} \underline{v} = 0 \longrightarrow -\rho \operatorname{div}_m \underline{v} = \frac{\partial_m \tilde{\rho}_m}{\partial t} - \frac{\partial \tilde{\rho}_m}{\partial x_m}(\underline{v}_m - \underline{v})$$

Lin momentum balance

$$\operatorname{div} \underline{T} + \rho \underline{b} = \rho \underline{a} \to \rho \frac{d_m \tilde{v}}{dt} = \rho \underline{b} + \operatorname{div} \underline{T} + \frac{\partial \tilde{v}}{\partial \underline{x}_m} \rho(\underline{v}_m - \underline{v})$$

Mass

$$\dot{\rho} = \frac{d\rho}{dt} = \frac{d_m \rho}{dt} - \frac{\partial \rho}{\partial \underline{x}_m} \cdot (v_m - v)$$

$$\rightarrow -\rho \operatorname{div} \underline{v} = \frac{d_m \rho}{dt} - \frac{\partial \rho}{\partial x_m} \cdot (v_m - v)$$

$\operatorname{div}_m = \operatorname{div}_x$ in this case so ALE mass balance is recovered

Lin momentum

$$\operatorname{div} \underline{T} + \rho \underline{b} = \rho \frac{dv}{dt} \to \frac{d\underline{v}}{dt} = \frac{d_m \underline{v}}{dt} - \frac{\partial \tilde{v}}{dx_m}(\underline{v}_m - \underline{v})$$

$$\operatorname{div} \underline{T} + \rho b = \rho \frac{d_m \underline{v}}{dt} - \rho \frac{\partial \tilde{\underline{v}}}{dx_m}(\underline{v}_m - \underline{v})$$

ALE lin momentum recovered

Problem 2

$(*)$ $\dot{T} = WT - TW + \alpha D + \beta D^2$

T cauchy stress
$\dot{T}$ is mat time derivative
D rate of def tensor
W vorticity
α, β are constants

a) Show $(*)$ is objective

$T^+ = QTQ^T \qquad W^+ = QWQ^T + \Omega \qquad D^+ = QDQ^T$

show that $\dot{T}^+$ is fcn of only + variables

$\rightarrow \quad \dot{T}^+ = W^+T^+ - T^+W^+ + \alpha D^+ + \beta D^+$

$$\dot{T}^+ = (QWQ^T + \Omega)QTQ^T - QTQ^T(QWQ^T + \Omega) + \alpha QDQ^T + \beta QDQ^T(QDQ^T)^T$$

$$= QWTQ^T + \Omega(QTQ^T) - QTWQ^T - QTQ^T\Omega + \alpha QDQ^T + \beta QDD^TQ^T$$

$$= Q(WT - TW + \alpha D + \beta D^2)Q^T + \dot{Q}Q^TQTQ^T - QTQ^T\dot{Q}Q^T$$

if $QQ^T = I \quad \frac{d}{dt}QQ^T = 0 = \dot{Q}Q^T + Q\dot{Q}^T = 0$

$\Rightarrow \dot{Q}Q^T = -Q\dot{Q}^T$

$$\rightarrow Q(WT - TW + \alpha D + \beta D^2)Q^T + \dot{Q}TQ^T + QT\dot{Q}^T$$

$$= Q\dot{T}Q^T + \dot{Q}TQ^T + QT\dot{Q}^T$$

$$= \frac{d}{dt}(QTQ^T) = \frac{d}{dt}(T^+) \rightarrow * \text{ is objective}$$

b) ALE employed to model body that follows (*). Argue why this cannot be directly used in ALE expressed as (*

The constitutive can as written cannot be used directly as it does not take the difference in mesh and material motion into account. This must be explicitly taken into account for the ALE formulation to be complete as the Lin derivative with respect to the mesh is different than with respect to the material.

c) show that (*) can be written as

$$\frac{d_m \underline{T}}{dt} = \frac{\partial \underline{T}}{\partial x_m} \cdot (v_m - v) + W\underline{T} - \underline{T}\underline{W} + \alpha \underline{D} + \beta \underline{D}^2$$

$$\frac{d\underline{T}}{dt} = \frac{d_m \underline{T}}{dt} - \frac{\partial \underline{T}}{\partial x_m} \cdot (\underline{v}_m - \underline{v})$$ by Extension of 1c

$$\rightarrow \frac{d_m \underline{T}}{dt} = \frac{\partial \underline{T}}{\partial x_m} \cdot (\underline{v}_m - \underline{v})) + W\underline{T} - \underline{T}\underline{W} + \alpha \underline{D} + \beta \underline{D}^2$$

Problem 3

Recall local statement of energy balance in Eulerian form

$$\rho \dot{\varepsilon} = \rho r - \operatorname{div} \underline{q} + \underline{T} \cdot \underline{D}$$

ρ mass density
ε internal energy per unit mass
r heat supply per unit mass
$\underline{q}$ heat flux vector
$\underline{T}$ Cauchy stress tensor
$\underline{D}$ rate of deformation tensor

Show ALE formulation

$$\rho \frac{d_m \varepsilon}{dt} = \rho r - \operatorname{div} \underline{q} + T \cdot D + \rho \frac{\partial \tilde{\varepsilon}}{\partial x_m} \cdot (v_m - \underline{v})$$

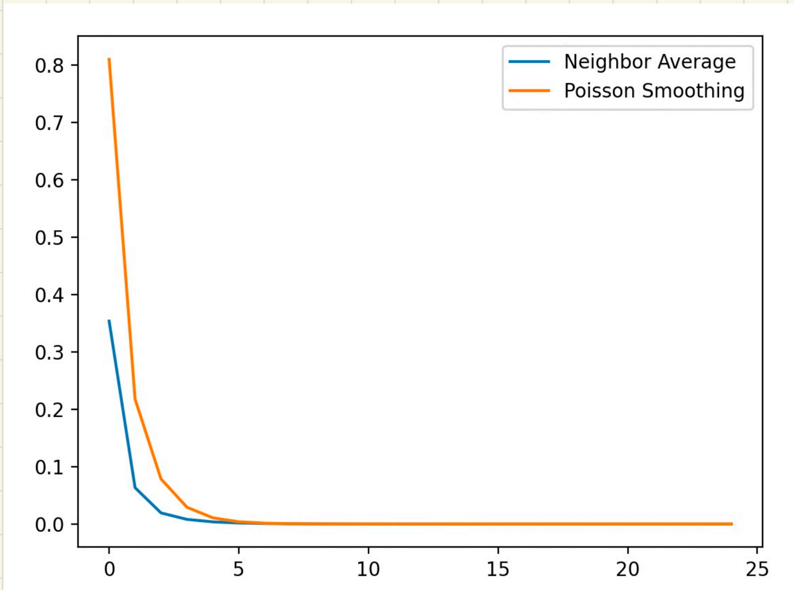
spatial derivatives taken w/ respect to mesh coords x_m

$$\frac{d\varepsilon}{dt} = \frac{d_m \varepsilon}{dt} - \frac{\partial \varepsilon}{\partial x_m}(v_m - v)$$

$$\rightarrow \quad \rho \frac{d_m \varepsilon}{dt} = \rho r - \operatorname{div} \underline{q} + T \cdot D + \rho \frac{\partial \tilde{\varepsilon}}{\partial x_m} \cdot (\underline{v_m} - \underline{v})$$

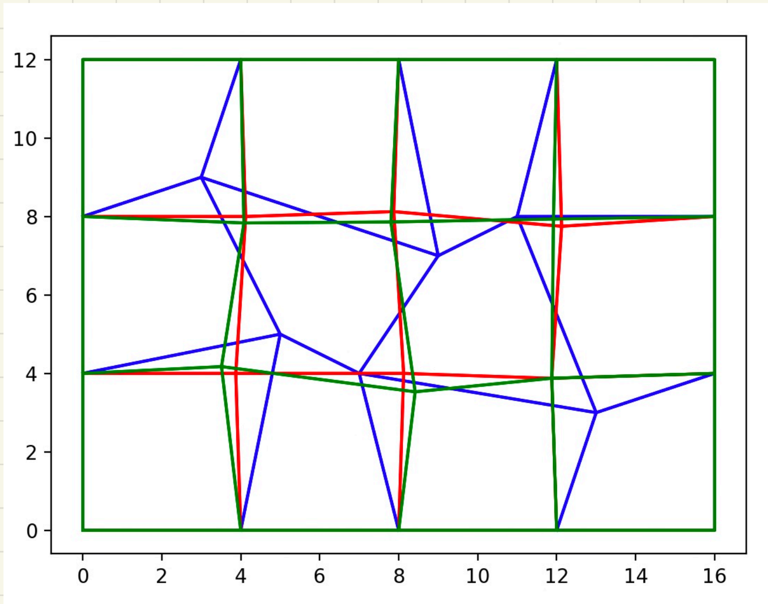
Problem 4

Comparison of smoothing

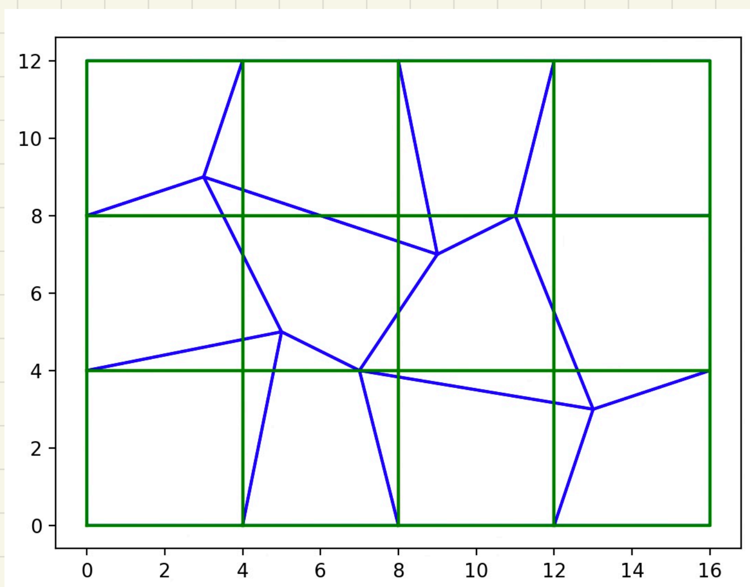


Was not able to find issue in code that caused Poisson to converge slower than Neighbor green – Poisson red – neigbor

1 step



25 Steps



Code Attached

Problem 5

	0	1	2	3	4	5	6	7	8	9	10	11
0	1.77778	0.944444	0	0.833333	0.444444	0	0	0	0	0	0	0
1	0.944444	4.22222	1.05556	0.444444	2.11111	0.555556	0	0	0	0	0	0
2	0	1.05556	2	0	0.555556	1.05556	0	0	0	0	0	0
3	0.833333	0.444444	0	3.05556	1.55556	0	0.861111	0.416667	0	0	0	0
4	0.444444	2.11111	0.555556	1.55556	7.77778	2.22222	0.416667	1.77778	0.472222	0	0	0
5	0	0.555556	1.05556	0	2.22222	4.27778	0	0.472222	0.916667	0	0	0
6	0	0	0	0.861111	0.416667	0	3.94444	1.88889	0	0.944444	0.444444	0
7	0	0	0	0.416667	1.77778	0.472222	1.88889	6.44444	1.44444	0.444444	1.44444	0.333333
8	0	0	0	0	0.472222	0.916667	0	1.44444	2.94444	0	0.333333	0.722222
9	0	0	0	0	0	0	0.944444	0.444444	0	1.77778	0.833333	0
10	0	0	0	0	0	0	0.444444	1.44444	0.333333	0.833333	2.88889	0.722222
11	0	0	0	0	0	0	0	0.333333	0.722222	0	0.722222	1.55556

A for a) b) and c)

	0
0	4
1	9.33333
2	4.66667
3	7.16667
4	17.3333
5	9.5
6	8.5
7	14.6667
8	6.83333
9	4
10	6.66667
11	3.33333

b for a)

	0
0	15.7778
1	86.7778
2	67.6667
3	47.6944
4	210.444
5	162.306
6	95.6944
7	246.444
8	148.528
9	54
10	127
11	81.6667

b for b)

	0
0	15.503
1	28.6288
2	-4.38814
3	24.2008
4	16.3944
5	2.12687
6	4.85105
7	-7.61475
8	9.574
9	-8.2975
10	-12.6199
11	0.199306

b for c)

Projection Method

	0
0	1
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	1
11	1

z proj a)

5.42820020031759e-15

2norm proj a)

	0
0	-4.44089e-16
1	8
2	16
3	4
4	11
5	20
6	8
7	19
8	24
9	12
10	20
11	28

z proj b)

[136.33467019]

2norm proj b)

	0
0	1.82593
1	7.74117
2	-6.45611
3	6.53836
4	-1.13128
5	0.92969
6	2.17995
7	-2.05679
8	4.52643
9	-3.88227
10	-3.04847
11	-0.117329

z proj c)

[[31.52052507]]

2norm proj c)

Collocation method

	0
0	1
1	1
2	1
3	1
4	1
5	1
6	1
7	1
8	1
9	1
10	1
11	1

z col a)

	0
0	0
1	8
2	16
3	4
4	11
5	20
6	8
7	19
8	24
9	12
10	20
11	28

z col b)

	0
0	2
1	8
2	-7
3	6
4	-5.75696
5	1
6	3
7	0.0735383
8	4
9	-4
10	-3
11	0

z col c)

```
0.0
```

2 norm col a)

```
[136.33467019]
```

2 norm col b)

```
[[35.5002747]]
```

2 norm col c)

It's clear that the collocation method can do as well as projections when the function is smooth, but when the function isn't smooth, the projection method has a much smaller 2norm.

Code Attached