Lectures on symmetries and particle physics

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Abstract: Notes for lectures that introduce students of physics to symmetries and particle physics. Prepared for a course at Heidelberg university in the summer term 2017.

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Suggested literature

The application of group theory in physics is a well established mathematical subject and there are many good books available. A selection of references that will be particularly useful for this course is as follows.

- P. Ramond, *Group Theory, A Physicist's Survey*
- A. Zee, *Group Theory in a Nutshell for Physicists*
- J. Fuchs and C. Schweigert, *Symmetries, Lie algebras and Representations*
- H. Georgi, *Lie algebras in Particle Physics*
- H. F. Jones, *Groups, Representations and Physics*

1 Introduction and motivation

1.1 Symmetry transformations

Studying different kinds of symmetries and their consequences is one of the most fruitful ideas in all branches of physics. This holds especially in high energy and particle physics but not only there. To make this work, we first define the notion of a symmetry transformation and relate it to the mathematical concept of a group.

It is natural to demand for symmetry transformations that

- A symmetry transformation followed by another should be a symmetry transformation itself.
- Symmetry transformations should be associative.
- There should be a unique trivial symmetry transformation doing nothing.
- For each symmetry transformation there needs to be a unique symmetry transformation reversing it.

With these properties, the set of all symmetry transformations G forms a group in the mathematical sense. More formally, a group G has the properties:

1) Closure: For all elements $f, g \in G$ the composition $f \cdot g \in G$.
2) Associativity: $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.
3) Identity element: There exists a unique $e \in G$ such that $e \cdot f = f = f \cdot e$ for all $g \in G$.
4) Inverse element: For all elements $f \in G$ there is a unique $f^{-1} \in G$ such that $f \cdot f^{-1} = f^{-1} \cdot f = e$.

1.2 Infinitesimal symmetries

In physics the group elements $g \in G$ which describe a symmetry can often be parametrised by a continuous parameter on which the group elements depend in a differentiable way, e.g.

$$\begin{aligned} \mathbb{R} &\to G \\ \alpha &\mapsto g(\alpha) \ . \end{aligned}$$

In this situation it is possible to study infinitesimal symmetries which are characterised by their action close to the identity element e of the group. Without loss of generality we can choose $g(0) = e$. Moreover, if the parametrisation obeys

$$g(\alpha_1) \cdot g(\alpha_2) = g(\alpha_1 + \alpha_2)$$

for any two group elements $g(\alpha_1)$ and $g(\alpha_2)$ one speaks of a one-parameter subgroup. Let us now restrict to very small parameters α_1, α_2 such that the corresponding group elements are close to the identity element. In this case one speaks of an infinitesimal symmetry which is described as a local one-parameter subgroup, and characterised by the derivative

$$\frac{d}{d\alpha} g(\alpha)\Big|_{\alpha=0} .$$

1.3 Classical mechanics

Lagrangian description. In classical mechanics, the equations of motions of a physical system with action

$$S = \int dt\; L(q(t), \dot{q}(t), t) \ ,$$

where the Lagrangian L is a function of position q and velocity $\dot{q}$, can be derived by the principle of least action

$$\begin{aligned} \delta S &= \int dt\; \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right\} \\ &= \int dt\; \left\{ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right\} \delta q = 0 \ . \end{aligned}$$

This gives the Euler-Lagrange equation

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0 \ .$$

Suppose there is a mapping

$$h_s : q \mapsto h_s(q) \ , \tag{1.1}$$

for $s \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$ such that

$$h_{s=0}(q) = q \ ,$$

for any position q, and an induced map

$$\hat{h}_s : \dot{q} \mapsto \hat{h}_s(\dot{q}) = \frac{\partial h_s(q)}{\partial q} \dot{q} \ , \tag{1.2}$$

for the corresponding velocity $\dot{q}$. The Lagrangian is then said to be invariant under (1.1) and (1.2) if there is a differentiable function $F_s(q, \dot{q}, t)$ such that

$$L(h_s(q), \hat{h}_s(\dot{q}), t) = L(q, \dot{q}, t) + \frac{d}{dt} F_s(q, \dot{q}, t) \ . \tag{1.3}$$

The invariance (1.3) gives rise to a conservation law. For any solution to the equations of motion

$$\phi : t \mapsto q = \phi(t)$$

we define

$$\Phi(s, t) = h_s \circ \phi(t)$$

and find from (1.3)

$$
\begin{aligned}
0 &= \frac{\partial}{\partial s}\left(L(\Phi, \partial_t \Phi) - \frac{d}{dt}F_s\right) \\
&= \frac{\partial L}{\partial q}\frac{\partial \Phi}{\partial s} + \frac{\partial L}{\partial \dot{q}}\frac{\partial^2 \Phi}{\partial s \partial t} - \frac{d}{dt}\frac{\partial F_s}{\partial s} \\
&= \frac{d}{dt}\underbrace{\left(\frac{\partial L}{\partial \dot{q}}\frac{\partial \Phi}{\partial s} - \frac{\partial F_s}{\partial s}\right)}_{=Q}
\end{aligned}
$$

where Q is a conserved quantity called Noether charge.

Consider for example as a point particle of mass m with the Lagrangian

$$
L = \frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x}) \ ,
$$

where $\mathbf{x} \in \mathbb{R}^3$ it the position. Suppose the potential $V(\mathbf{x})$ is translational invariant such that L is invariant under

$$
\begin{aligned}
h_s : \mathbb{R}^3 &\to \mathbb{R}^3 \\
\mathbf{x} &\mapsto \mathbf{x} + s\mathbf{a}
\end{aligned}
$$

for $\mathbf{a} \in \mathbb{R}^3$ and $s \in (-\varepsilon, \varepsilon) \subset \mathbb{R}$ while $\hat{h}_s$ is the identity map and $F_s = 0$. Then

$$
\frac{\partial \Phi}{\partial s} = \frac{\partial h_s}{\partial s} = \mathbf{a} \ ,
$$

and therefore

$$
Q = m\dot{\mathbf{x}} \cdot \mathbf{a} \ .
$$

We have found momentum conservation in direction $\mathbf{a}$. The close connection between symmetries and conservation law is truly remarkable.

Hamiltonian description. In the Hamiltonian formulation of classical mechanics the physical information of a system in encoded in the Hamiltonian H which is a function of position q, momenta p and time t. The equations of motion are

$$
\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p} \ .
$$

For systems which can also be described in the Lagrangian framework, the Hamiltonian is given by the Legendre transformation

$$
H(q, p, t) = p\dot{q} - L(q, \dot{q}, t) \ ,
$$

where $\dot{q}$ is defined implicitly by

$$
p = \frac{\partial L}{\partial \dot{q}} \ .
$$

The fundamental tool in Hamiltonian mechanics is the Poisson bracket. It is a map from the space of pairs of differentiable functions of dynamical variables q and p to a single differential function. For such functions f, g we define

$$\{\cdot,\cdot\} : \Big(f(q,p), g(q,p)\Big) \mapsto \{f,g\}(q,p) \ ,$$

where

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} \ .$$

The Poisson bracket possesses three properties:

1) Bilinear: $\{\lambda f + \mu g, h\} = \lambda\{f,h\} + \mu\{g,h\}$,

2) Antisymmetric: $\{f,g\} = -\{g,f\}$,

3) Jacobi identity: $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$,

for differentiable functions f, g, h and for $\lambda, \mu \in \mathbb{R}$. The bilinearity holds in both arguments. By these properties the Poisson bracket turns the space of functions of q and p into a Lie algebra.

If H does not explicitly depend on time, and for any differentiable function $f(q(t), p(t))$ which is on a trajectory that is a solution to the equations of motion, the time derivative can be written

$$\frac{d}{dt} f(q(t), p(t)) = \{H, f\} \ .$$

For a conserved quantity we therefore get

$$\{H, f\} = 0 \ .$$

Quantum mechanics. Starting from the Hamiltonian description it is most convenient to work in the Heisenberg picture. The canonical quantisation maps the Poisson bracket of differentiable functions in q and p to the commutator of the associated operators $\hat{q}$ and $\hat{p}$ in some suitable Hilbert space $\mathcal{H}$,

$$\{\cdot,\cdot\} \mapsto \frac{i}{\hbar}[\cdot,\cdot] \ ,$$

where i is the imaginary unit, $\hbar$ denotes Planck's constant and the commutator is defined by

$$[A, B] = A \cdot B - B \cdot A \ .$$

In particular one has

$$\{p, q\} = 1 \mapsto \frac{i}{\hbar}[\hat{p}, \hat{q}] = 1 \ ,$$

which is the Heisenberg commutation relation. The commutator equips $\mathcal{H}$ with a Lie algebra in the same way the Poisson bracket does in the Lagrangian description. In particular we have the properties

1) Bilinear: $[\lambda A + \mu B, C] = \lambda[A, C] + \mu[B, C]$,

2) Antisymmetric: $[A, B] = -[B, A]$,

3) Jacobi identity: $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$,

for operators A, B, C and $\lambda, \mu \in \mathbb{R}$. Analogous to before the time dependance of the operators is described by

$$\frac{d}{dt}A(t) = \frac{i}{\hbar}[H, A] \ .$$

Observables which describe conservations laws are characterised by

$$[H, A] = 0 \ .$$

By the Jacobi identity, the space of all conserved quantities forms a closed Lie subalgebra.

As an example we consider angular momentum conservation of the electron in the quantised hydrogen atom. Let the eigenstates of the Hamiltonian H be labelled by the principal quantum number n, total angular momentum l and the projection on the z-axis, m. Since angular momentum is conserved the operator commutes with the Hamiltonian,

$$[L_z, H] = 0$$

and therefore for two eigenstates $|n', l', m'\rangle$, $|n, l, m\rangle$,

$$0 = \langle n', l', m'|[L_z, H]|n, l, m\rangle = (m' - m)\,\langle n', l', m'|H|n, l, m\rangle \ .$$

Accordingly,

$$\langle n', l', m'|H|n, l, m\rangle$$

can only be non-zero for $m \neq m'$, which is the selection rule.

One may also start from a conserved operator and construct the corresponding symmetry transformation. For a self-adjoint operator $A \in \mathcal{H}$ with

$$[H, A] = 0 \ ,$$

we define the unitary operator

$$U_A(t) = e^{itA} = \sum_{n=0}^{\infty} \frac{(itA)^n}{n!} \ ,$$

for $t \in \mathbb{R}$. They obey the multiplication law

$$U_A(t_1) \cdot U_A(t_2) = U_A(t_1 + t_2) \ ,$$

for all $t_1, t_2 \in \mathbb{R}$ and form a unitary one-parameter group. As an example consider the momentum operator in position space defined as

$$\hat{p} = \frac{\hbar}{i}\frac{d}{dx} \ ,$$

which generates infinitesimal translations. For $a \in \mathbb{R}$ the operator

$$\exp\left(ip\frac{a}{\hbar}\right) = e^{a\frac{d}{dx}}$$

generates finite translations, e.g.

$$e^{a\frac{d}{dx}} f(x) = f(x+a) \ ,$$

for a suitable function f.

2 Finite groups

Now that we discussed the close relation between symmetries and the mathematical concept of a group it seems natural to start our analysis with simple cases, i.e. finite groups of low order. The order ord(G) of a group G is defined to be the number of group elements.

2.1 Order 2

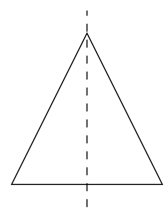
The simplest symmetry transformation is a reflection,

$$P : x \to -x \tag{2.1}$$

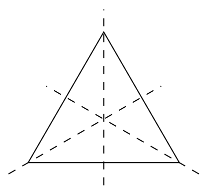
which applied twice is the identity, i.e. $PP = e$. The parity transformation (2.1) has the simplest finite group structure involving only two elements. The group is known as the cyclic group Z_2 with the multiplication table

Z_2	e	P
e	e	P
P	P	e

The group Z_2 has many manifestations for example in terms of reflections about the symmetry axis of an isosceles triangle.



Studying higher order group will naturally involve more elements which can manifest in more symmetries, e.g. the equilateral triangle is symmetric under reflections about any of its three medians as well as rotations of $\frac{2\pi}{3}$ around its center,



which is described by a group of order six.

In general we denote the elements of a group of order n by $\{e, a_1, a_2, ..., a_{n-1}\}$ where e denotes the identity element.

2.2 Order 3

There is only one group of order three which is Z_3 with the multiplication table

Z_3	e	a_1	a_2
e	$e \cdot e = e$	$e \cdot a_1 = a_1$	$e \cdot a_2 = a_2$
a_1	$a_1 \cdot e = a_1$	$a_1 \cdot a_1 = a_2$	$a_1 \cdot a_2 = e$
a_2	$a_2 \cdot e = a_2$	$a_2 \cdot a_1 = e$	$a_2 \cdot a_2 = a_1$

written compactly for clarity

Z_3	e	a_1	a_2
e	e	a_1	a_2
a_1	a_1	a_2	e
a_2	a_2	e	a_1

The group elements can be written as $\{e, a_1 = a, a_2 = a^2\}$, with the law that $a^3 = e$. One says that a is an order three element. The structure of the group is completely fixed by the multiplication table. However, this get rather cumbersome, in particular for larger groups and it is convenient to use another characterisation of the group elements and their multiplication laws, the so-called *presentation.* For example Z_3 has a presentation

$$\langle a \mid a^3 = e \rangle \ .$$

The first part of the presentation lists the group elements from which all others can be constructed – the so-called generators – while the second part gives the rules needed to construct the multiplication table.
This generalises to the cyclic group of order $n \in \mathbb{N}$ which has the group elements $\{e, a, a^2, ..., a^{n-1}\}$ and a presentation is given by

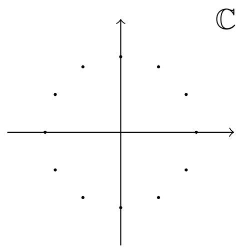
$$\langle a \mid a^n = e \rangle \ . \tag{2.2}$$

One distinguishes between a group G which is an abstract entity defined by the set of its group elements and their multiplication table (or, equivalently, a presentation) and a *representation* of a group.

A representation can be seen as a manifestation of the group multiplication laws in a concrete system or, in other words, a kind of incarnation of the group.
For example, the group Z_3 has a representation in terms of rotations by $\frac{2\pi}{3}$ around the center of an equilateral triangle. Z_n has a one dimensional representation

$$\{1, e^{i\frac{2\pi}{n}}, e^{2i\frac{2\pi}{n}}, ..., e^{(n-1)i\frac{2\pi}{n}}\} \tag{2.3}$$

where the group elements are represented equidistantly on the unit circle in the complex plane and the action of the generator a is represented by a rotation of $\frac{2\pi}{n}$ around the centre.



Often one works with representations as operations in a vector space:

A matrix representation $\mathcal{R}$ of a group G on a vector space V is a group homomorphism

$$\mathcal{R} : G \to GL(V)$$

onto the general linear group $GL(V)$ on V, i.e. a map from a group element g to a matrix $\mathcal{R}(g)$ such that

$$\mathcal{R}(g_1 \cdot g_2) = \mathcal{R}(g_1) \cdot \mathcal{R}(g_2)$$

for all $g_1, g_2 \in G$. We define the dimension of the representation $\mathcal{R}$ by

$$\dim(\mathcal{R}) = \dim(V)$$

and denote the identity element of a group by e and of a representation by $\mathbb{1}$.

2.3 Order 4

At order four there are two different groups. Again there is Z_4 which in the representations (2.3) reads $\{1, i, -1, -i\}$. The other group is the dihedral group D_2 with the multiplication table

D_2	e	a_1	a_2	a_3
e	e	a_1	a_2	a_3
a_1	a_1	e	a_3	a_2
a_2	a_2	a_3	e	a_1
a_3	a_3	a_2	a_1	e

The fact that the multiplication table is symmetric shows that D_2 is Abelian, $a_i \cdot a_j = a_j \cdot a_i$. Two representations of D_2 are:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} ,$$

$$\left\{ f_1(x) = x,\ f_2(x) = -x,\ f_3(x) = \frac{1}{x},\ f_4(x) = -\frac{1}{x} \right\} .$$

For both groups of order four elements form a subgroup, namely Z_2. It is straight forward to see from the presentation (2.2) for Z_4,

$$\langle a \mid a^4 = e \rangle \ ,$$

that the group generated by a^2 is Z_2. The same goes for the group generated by $a_i \in D_2$ for $i \in \{1, 2, 3\}$. For D_2 one can actually go on and write it as a direct product of two factors Z_2,

$$D_2 = Z_2 \otimes Z_2 \ .$$

We use here the notion of a *direct product*.

Let $(G, \circ)$ and $(K, *)$ be two groups with elements $g_a \in G$, $a \in \{1, ..., n_G\}$ and $k_i \in$

$K,\ i \in \{1, ..., n_K\}$ respectively. The direct product is a group $(G \otimes K, \star)$ of order $n_G n_K$ with elements (g_a, k_i) and the multiplication rule

$$(g_a, k_i) \star (g_b, k_j) = (g_a \circ g_b, k_i * k_j) \ .$$

Since $\circ$ and $*$ act on different sets, they can be taken to be commuting subgroups and we simply write $g_a k_i = k_i g_a$ for the elements of $G \otimes K$.

As long as it is clear we denote for notational simplicity the group $(G, \circ)$ by G and the group operation as $g_1 \circ g_2 = g_1 g_2$.

Exercise: *Show that $D_2 = Z_2 \otimes Z_2$ but $Z_4 \neq Z_2 \otimes Z_2$.*

2.4 Lagrange's theorem

If a group G of order N has a subgroup H of order n, then N is necessarily an integer multiple of n.

Consider a group $G = \{g_a \mid a \in \{1, ..., N\}\}$ with subgroup $H = \{h_i \mid i \in \{1, ..., n\}\} \subset G$. Take a group element $g_a \in G$ but not in the subgroup, $g_a \notin H$. Then it follows that also $g_a h_i$ is not in H, $g_a h_i \notin H$, because if there would exist $h_j \in H$ such that $g_a h_i = h_j$ then $g_a = h_j h_i^{-1} \in H$ which would object our assumption. Continuing in this way we can construct the disjoint cosets

$$g_a H = \{g_a h_i \mid i \in \{1, ..., n\}\}$$

and write G as a (right) coset decomposition

$$G = H \cup g_1 H \cup ... \cup g_k H \ ,$$

with $k \in \mathbb{N}$. Therefore the order of G can only be a multiple of its subgroup's order: $N = nk$.

As an immediate consequence, groups of prime order cannot have subgroups of smaller order. Hence for a group of prime order p all elements are of order one or order p and thus $G = Z_p$.

2.5 Order 5

By Lagrange's theorem there is only Z_5.

2.6 Order 6

We now know that there are at least two groups of order six: Z_6 and $Z_2 \otimes Z_3$ but the question is whether they actually differ. The generator a of Z_3 is an order-three element, the generator b of Z_2 an order-two element and thus $a^3 = e = b^2$ and $ab = ba$. Now $(ab)^6 = a^6 b^6 = e$ is a order-six element und therefore

$$Z_6 = Z_2 \otimes Z_3 \ .$$

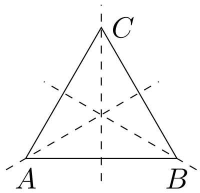
To construct another order six group we again take an order-three element a and an order-two element b. The set

$$\{e, a, a^2, b, ab, a^2b\}$$

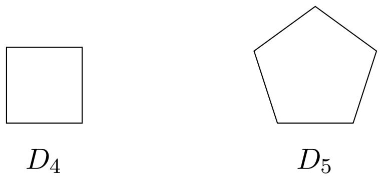
together with the relation $ba = a^2b \neq ab$ form the dihedral group D_3 which is non-Abelian. A presentation is given by

$$\langle a, b \mid a^3 = e, b^2 = e, bab^{-1} = a^{-1} \rangle \ .$$

D_3 is the symmetry group of the equilateral triangle



where a is a rotation by $\frac{2\pi}{3}$ $(ABC \to BCA)$ and b is a reflection $(ABC \to BAC)$. Higher dihedral groups have higher polygon symmetries:



The group can also be represented by permutations as indicated above. The element a corresponds to a three-cycle $A \to B \to C \to A$

$$\begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix} \tag{2.4}$$

whereas the element b is a two-cycle: $A \to B \to A$

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \ . \tag{2.5}$$

In general the $n!$ permutations of n objects form the symmetric group S_n. From (2.4) and (2.5) we see

$$D_3 = S_3 \ .$$

The group operation for a k-cycle can be represented by $k \times k$ matrices, e.g. a three-cycle can be represented by (3×3) matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} B \\ C \\ A \end{pmatrix} \ .$$

2.7 Order 8

From what we know already we can immediately write down four order eight groups which are not isomorphic to each other since they contain elements of different order: Z_8, $Z_2 \otimes Z_2 \otimes Z_2$, $Z_4 \otimes Z_2$. Additionally there is the dihedral group D_4 and a new group Q called the quaternion group.

An element q of the deduced quaternion vector space $\mathbb{H}$ generalises the complex numbers,

$$q = x_0 + e_1 x_1 + e_2 x_2 + e_3 x_3$$
$$\bar{q} = x_0 - e_1 x_1 - e_2 x_2 - e_3 x_3$$

with $x_i \in \mathbb{R},\ i \in \{1, 2, 3\}$ and the relation

$$e_i^2 = -1$$

for the imaginary units e_i as well as

$$e_1 e_2 = -e_2 e_1 = e_3$$

plus cyclic permutations. For $q \in \mathbb{H}$

$$N(q) = \sqrt{q\bar{q}}$$

defines a norm with

$$N(qq') = N(q)N(q') \ .$$

A finite group of order eight can be taken to be the set

$$\{1, e_1, e_2, e_3, -1, -e_1, -e_2, -e_3\}$$

and is called the quaternion group Q.

Exercise: *Convince yourself that Q is a closed group, indeed.*

A matrix representation of the quaternion group Q is given by the Pauli matrices

$$e_j = -i\sigma_j, \quad \sigma_j \sigma_k = \delta_{jk} + i\varepsilon_{jkl}\sigma_l \ ,$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

2.8 Permutations

The permutations of n objects can be represented by the symbol

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ a_1 & a_2 & a_3 & \ldots & a_n \end{pmatrix}$$

and form the group S_n. Every permutation can uniquely be resolved into cycles, e.g.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \sim (1234)$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \sim (12)(3)(4)$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \sim (132)(4) \ .$$

Any permutation can also be decomposed into a product of two-cycles,

$$(a_1 a_2 a_3 ... a_n) \sim (a_1 a_2)(a_1 a_3)...(a_1 a_n) \ .$$

The symmetric group S_n has a subgroup of *even* permutations $A_n \subset S_n$ of order $\frac{n!}{2}$. Notice that the odd permutations do not form a subgroup since there is no unit element.

2.9 Cayley's theorem

Every group of finite order n is isomorphic to a subgroup of S_n.

Let $G = \{g_a \mid a \in \{1, ..., n\}\}$ be a group and associate the group elements with the permutations

$$g_a \mapsto P_a = \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_1 g_a & g_2 g_a & \cdots & g_n g_a \end{pmatrix} \ .$$

This construction leaves the multiplication table invariant,

$$g_a g_b = g_c \mapsto P_a P_b = P_c$$

and $\{P_a \mid a \in \{1, ..., n\}\}$ is called the regular representation of G and it is by construction a subgroup of S_n.

2.10 Concepts

Conjugacy. For a group $G = \{g_a \mid a \in \{1, ..., n\}\}$ we define the conjugate to $g_a \in G$ with respect to the element $g \in G$ as

$$\tilde{g}_a = g g_a g^{-1} \ .$$

Then

$$g_a g_b = g_c \mapsto \tilde{g}_a \tilde{g}_b = \tilde{g}_c$$

since

$$g g_a \underbrace{g^{-1} g}_{e} g_b g^{-1} = g g_a g_b g^{-1} = g g_c g^{-1} \ .$$

Note if $[g, g_a] = 0$ then g_a and g are self-conjugate with respect to each other. For a permutation $P = \big(k \ l \ m \ p \ q\big)$ conjugacy leaves the cycle structure invariant, that is

$$g P g^{-1} = \Big(g(k) \ g(l) \ g(m) \ g(p) \ g(q)\Big) \ ,$$

since $g P g^{-1} g(k) = g P(k) = g(l)$.

Classes. For a group G we define the class C_a to consist of all $\tilde{g}_a$ that are conjugate to g_a,

$$C_a = \{\tilde{g}_a = g g_a g^{-1} \mid g \in G\} \ .$$

Now take another group element $g_b \in G$ but such that $g_b \notin C_a$. The set of all conjugates forms the class C_b and so on,

$$\begin{aligned} C_b &= \{\tilde{g}_b = g g_b g^{-1} \mid g \in G\} \\ &\vdots \end{aligned}$$

Note that the classes are disjoint, $C_i \cap C_j = \emptyset$, for $i \neq j$. In this way we can decompose G in classes

$$G = C_1 \cup C_2 \cup ... \cup C_k \ .$$

Normal subgroup. Let G be an group and $H \subset G$ a subgroup such that for all $g \in G$ and $h_i \in H$ the conjugates stay in H, that is $g h_i g^{-1} \in H$. In this case H is called a *normal subgroup.*

Quotient group. Let G, K be groups and consider a map:

$$\begin{aligned} G &\to K \\ g_a &\mapsto k_a \end{aligned}$$

such that $g_c = g_a g_b \mapsto k_a k_b = k_c$. Set H to be the kernel of that map,

$$H = \{g_a \in G \mid g_a \mapsto e\} \subset G \ .$$

H then is a normal subgroup because for all $h_i \in H$

$$g h_i g^{-1} \mapsto k e k^{-1} = e \ .$$

We now use H to build set of group elements and define a multiplication between these sets. More specific, let $g_a H$, $g_b H$ be two cosets and H the trivial coset. We define the multiplication of cosets,

$$\begin{aligned} \underbrace{(g_a h_i)}_{\in g_a H} \underbrace{(g_b h_j)}_{\in g_b H} &= g_a g_b \underbrace{g_b^{-1} h_i g_b}_{\tilde{h}_i} h_j \\ &= g_a g_b \underbrace{\tilde{h}_i h_j}_{\in H} \in g_a g_b H \ . \end{aligned}$$

This shows that the cosets can be multiplied in the same way as the original group elements g_a and g_b.

One then easily verifies that the cosets $\{g_a H\}$ have a group structure. This group is called the quotient group G/H. One can show that the quotient group G/H has no normal subgroup if H is the maximal normal subgroup. The systematic classification of groups in form of a decomposition into simple groups is based on this.

Simple group. A simple group is a group without (non-trivial) normal subgroups. One can classify simple groups:

- Z_p for p prime
- $A_n, n \geq 5$
- Infinite families of groups of Lie type
- 26 sporadic groups

However, a detailed discussion of this goes beyond the scope of these lectures.

2.11 Representations

Let V be a N dimensional vector space with an orthonormal basis $|i\rangle$,

$$\sum_{i=1}^{N} |i\rangle \langle i| = \mathbb{1} \ , \quad \langle i|j\rangle = \delta_{ij} \ ,$$

where $\mathbb{1}$ is the identity element in the space of N-dimensional matrices $GL(V)$. Let G be a order n group with representation $\mathcal{R}$ on V such that the action of $g \in G$ is represented by

$$|i\rangle \mapsto |i(g)\rangle = M_{ij}(g)\,|j\rangle$$

where $M(g) \in GL(V)$. Then for $g_a, g_b, g_c, g \in G$ and $g_a g_b = g_c$,

$$\begin{aligned} M(g_a) \cdot M(g_b) &= M(g_c) \\ M(g^{-1})_{ij} &= (M(g)^{-1})_{ij} \ . \end{aligned}$$

The representation is called trivial if $M(g) = \mathbb{1}$ for all $g \in G$. The representation $\mathcal{R}$ is called reducible if one can arrange $M(g)$ in the form

$$M(g) = \begin{pmatrix} M^{[1]}(g) & 0 \\ N(g) & M^{[\perp]}(g) \end{pmatrix} \ ,$$

where $M^{[1]}(g) \in GL(V_1)$ acts on the subspace $V_1 \subset V$ of dimension d_1, $M^{[\perp]}(g) \in GL(V_1^{\perp})$ on its orthogonal complement $V_1^{\perp} \subset V$ of dimension $N - d_1$ and $N(g)$ is a $(N - d_1) \times d_1$ matrix. Then for $g, g' \in G$

$$\begin{aligned} M^{[1]}(gg') &= M^{[1]}(g) \cdot M^{[1]}(g') \\ M^{[\perp]}(gg') &= M^{[\perp]}(g) \cdot M^{[\perp]}(g') \end{aligned}$$

as well as

$$N(gg') = N(g) \cdot M^{[1]}(g') + M^{[\perp]}(g) \cdot N(g') \ .$$

One can in fact simplify this further. With

$$W = \frac{1}{\mathrm{ord}(G)} \sum_{g \in G} M^{[\perp]}(g^{-1}) \cdot N(g)$$

we diagonalise the representation

$$\begin{pmatrix} 1 & 0 \\ W & 1 \end{pmatrix} \cdot \begin{pmatrix} M^{[1]} & 0 \\ N & M^{[\perp]} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -W & 1 \end{pmatrix} = \begin{pmatrix} M^{[1]} & 0 \\ 0 & M^{[\perp]} \end{pmatrix}$$

by

$$\begin{aligned} W \cdot M^{[1]}(g_1) &= \frac{1}{\mathrm{ord}(G)} \sum_{g \in G} M^{[\perp]}(g^{-1}) \cdot N(g) \cdot M^{[1]}(g_1) \\ &= -N(g_1) + \frac{1}{\mathrm{ord}(G)} \sum_{g' \in G} M^{[\perp]}(g_1 g'^{-1}) \cdot N(g') \\ &= -N(g_1) + M^{[\perp]}(g_1) \cdot W \end{aligned}$$

with $g' = gg_1$. In this basis we say $\mathcal{R}$ is completely reducible,

$$\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_1^{\perp} \ .$$

If $\mathcal{R}_1^{\perp}$ is reducible we can further reduce until there are no subspaces left,

$$\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus ... \oplus \mathcal{R}_k \ ,$$

with $k \in \mathbb{N}$.

2.12 Schur's lemmas

From the notation established before we write the action of a $g \in G$ in the subspace $V_1 \subset V$ as

$$|a\rangle \mapsto |a(g)\rangle = M^{[1]}_{ab}(g) \, |b\rangle$$

where $|a\rangle$ is a orthonormal basis of V_1. Since V_1 is a subspace of V we can write

$$|a\rangle = S_{ai} \, |i\rangle$$

with S a $d_1 \times N$ matrix. Now we write the group action

$$|a\rangle \mapsto |a(g)\rangle = M^{[1]}_{ab}(g) \, |b\rangle = M^{[1]}_{ab}(g) \ S_{bi} \, |i\rangle$$

or equivalently

$$|a\rangle = S_{ai} \, |i\rangle \mapsto S_{ai} \, |i(g)\rangle = S_{ai} \ M_{ij}(g) \, |j\rangle$$

Therefore

$$\mathcal{R} \text{ reducible} \Rightarrow S_{ai} \ M_{ij}(g) = M^{[1]}_{ab}(g) \ S_{bj} \text{ for all } g \in G \ . \tag{2.6}$$

Schur's first lemma: *If matrices of two irreducible representations of different dimension can*

be related as in (2.6), then $S = 0$.

If now $d_1 = N, S$ is a $N \times N$ matrix and if $|a\rangle$ and $|j\rangle$ span the same space the representations $\mathcal{R}$ and $\mathcal{R}_1$ are related by a similarity transformation

$$M^{[1]} = S \cdot M \cdot S^{-1} \ .$$

Thus for $S \neq 0 \Rightarrow \mathcal{R}$ is reducible or there is a similarity relation.

Schur's second lemma: *Let $\mathcal{R}$ be an irreducible representation. Any matrix S with*

$$M(g) \cdot S = S \cdot M(g)$$

for all $g \in G$ is proportional to $\mathbb{1}$.

If $|i\rangle$ is an eigenket of S then $|i(g)\rangle$ is also an eigenket:

$$\underbrace{M(g)\, \underbrace{S\,|i\rangle}_{\lambda|i\rangle}}_{\lambda|i(g)\rangle} = S\, \underbrace{M(g)\,|i\rangle}_{|i(g)\rangle}$$

$$\Rightarrow S = \lambda \mathbb{1} \ .$$

Let $\mathcal{R}_a, \mathcal{R}_b$ be two irreducible representations of dimension d_a and d_b respectively. Construct the mapping

$$S = \sum_{g \in G} M^{[a]}(g) \cdot N \cdot M^{[b]}(g^{-1}) \ ,$$

where N is any $d_a \times d_b$ matrix. Then for $g \in G$

$$M^{[a]}(g) \cdot S = S \cdot M^{[b]}(g) \ .$$

For $\mathcal{R}_a \neq \mathcal{R}_b$ by Schur's first lemma

$$\frac{1}{\mathrm{ord}(G)} \sum_{g \in G} M^{[a]}_{ij}(g)\ M^{[b]}_{pq}(g^{-1}) = 0 \ .$$

On the other hand for $\mathcal{R}_a = \mathcal{R}_b$ by Schur's second lemma

$$\frac{1}{\mathrm{ord}(G)} \sum_{g \in G} M^{[a]}_{ij}(g)\ M^{[a]}_{pq}(g^{-1}) = \frac{1}{d_a} \delta_{iq} \delta_{jp} \ .$$

Combining these two results leaves us with

$$\frac{1}{\mathrm{ord}(G)} \sum_{g \in G} M^{[a]}_{ij}(g)\ M^{[b]}_{pq}(g^{-1}) = \frac{1}{d_a} \delta_{iq} \delta_{jp} \delta^{ab} \ .$$

Let us define the scalar product

$$(i, j) = \frac{1}{\mathrm{ord}(G)} \sum_{g \in G} \langle i(g) | j(g) \rangle \ \ ,$$

where $|j(g)\rangle = M(g)\,|j\rangle$. It is invariant under the group action

$$\begin{aligned}
(i(g_a), j(g_a)) &= \frac{1}{\mathrm{ord}(G)} \sum_{g \in G} \langle i(g_a g) | j(g_a g) \rangle \\
&= \frac{1}{\mathrm{ord}(G)} \sum_{g' \in G} \langle i(g') | j(g') \rangle \\
&= (i, j)
\end{aligned}$$

where $g' = g_a g$ runs over all group elements. Therefore for $\mathbf{v}, \mathbf{u} \in V$

$$(M(g^{-1})\mathbf{v}, \mathbf{u}) = (\mathbf{v}, M(g)\mathbf{u}) \ .$$

On the other hand

$$(M^\dagger(g)\mathbf{v}, \mathbf{u}) = (\mathbf{v}, M(g)\mathbf{u}) \ ,$$

where $M^\dagger$ is the Hermitian conjugate of M and therefore

$$M^\dagger(g) = M(g^{-1}) = M^{-1}(g) \ .$$

Therefore all representations of finite groups are unitary with respect to the scalar product $(\cdot, \cdot)$.

2.13 Crystals

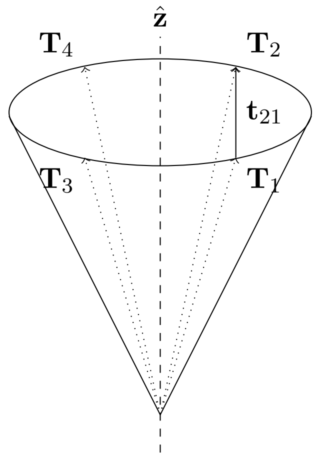
A crystal is defined as a lattice which is invariant under translations,

$$\mathbf{T} = n_1\mathbf{u}_1 + n_2\mathbf{u}_2 + n_3\mathbf{u}_3 \ , \tag{2.7}$$

with $\mathbf{u}_i \in \mathbb{R}^3, \ n_i \in \mathbb{N}, \ i \in \{1, 2, 3\}$. Specifying additional symmetries of the crystal one can consider two symmetry groups:

- Space group: translations, rotations, reflections and possibly inversion.
- Point group: rotations, reflections and possibly inversion, e.g. Z_n cyclic group, D_n dihedral group.

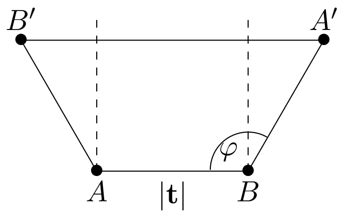
Crystallographic restriction theorem: *Consider a crystal invariant under rotations through* $\frac{2\pi}{n}$ *around an axis. Then n is restricted to $n \in \{1, 2, 3, 4, 6\}$.*



Consider a translation vector $\mathbf{T}_1$. By rotations it gets transformed to $\mathbf{T}_2, \mathbf{T}_3, ..., \mathbf{T}_n$. By the group property of the space group, also the differences $\mathbf{t}_{ij} = \mathbf{T}_i - \mathbf{T}_j$ are translation vectors and by construction they are orthogonal to the rotations axis, which we may take to coincide with the z-axis. Take now the minimum of the differences,

$$|\mathbf{t}| = \min_{i,j}(|\mathbf{t}_{ij}|)$$

and without loss of generality normalise $|\mathbf{t}| = 1$. Take now a point A on the rotational symmetry axis which by $\mathbf{t}$ gets translated to another symmetry point B. Rotation by an angle φ around A brings B to B'. Similar, rotation by φ around B brings A to A'. Because A' and B' are also symmetry points the difference $A'B'$ must be a translation vector and so we can infer that $\overline{A'B'} = p \in \mathbb{N}_0$.



With

$$\begin{aligned} \overline{A'B'} &= 1 + 2\sin\left(\varphi - \frac{\pi}{2}\right) \\ &= 1 - 2\cos(\varphi) \end{aligned}$$

we conclude

$$\cos(\varphi) = \frac{1-p}{2}$$

and thus

$$\begin{aligned} p &\in \{0, 1, 2, 3\} \\ \Rightarrow \varphi &\in \left\{\frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi\right\} \\ \Rightarrow n &\in \{6, 4, 3, 2\} \ . \end{aligned}$$

This closes the prove of the restriction theorem above.

Quasicrystals can show five-fold symmetries but have no periodic structure (2.7), e.g. Penrose tiling.

3 Lie groups and Lie algebras

3.1 Lie groups

Lets assume that for the group G the group elements $g \in G$ depend smoothly on a set of N continuous parameters $\alpha_A \in \mathbb{R}$ and $A \in \{1, ..., N\}$ such that

$$g(\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}=0} = e \ ,$$

is the identity element of the group. Thus for a representation of the group

$$M(\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}=0} = \mathbb{1} \ , \tag{3.1}$$

where $M(\boldsymbol{\alpha})$ is the action of $g(\boldsymbol{\alpha})$ represented on the vector space V and $\mathbb{1}$ the identity element of the representation. Expanding (3.1) in a small neighbourhood of the identity element up to first order yields

$$M(\boldsymbol{\alpha}) = \mathbb{1} + i\alpha_A T^A \ ,$$

with the so-called generators of the representation of the group

$$T^A = -i\frac{\partial}{\partial \alpha_A} M(\boldsymbol{\alpha})\Big|_{\boldsymbol{\alpha}=0} \ . \tag{3.2}$$

By (3.2) the group generators are hermitian for unitary representations. Groups of this type are called Lie groups. Because of the closure of the group

$$M(\boldsymbol{\alpha}) = \lim_{k\to\infty}\left(\mathbb{1} + \frac{i\alpha_A T^A}{k}\right)^k = e^{i\alpha_A T^A} \tag{3.3}$$

is well-defined and called the exponential parametrisation of the representation. It can reach all elements at least in some neighbourhood of the unit element.

3.2 Lie algebras

In the exponential parametrisation there is a one-parameter subgroup of the form

$$U(\lambda) = e^{i\lambda\alpha_A T^A}, \quad \lambda \in \mathbb{R} \ ,$$

for which for $\lambda_1, \lambda_2 \in \mathbb{R}$

$$U(\lambda_1) \cdot U(\lambda_2) = U(\lambda_1 + \lambda_2) \ .$$

In general for two different generators

$$e^{i\alpha_A T^A} e^{i\beta_B T^B} \neq e^{i(\alpha_A + \beta_A)T^A} \ ,$$

for continuous parameters $\alpha_A, \beta_A \in \mathbb{R}, \ A \in \{1, ..., N\}$, but because the exponential parametrisation forms a representation of the group close to the identity element, there needs to be $\delta_A \in \mathbb{R}, \ A \in \{1, ..., N\}$ such that

$$e^{i\alpha_A T^A} e^{i\beta_B T^B} = e^{i\delta_C T^C} \ .$$

To see which conditions have to be met we write

$$i\delta_C T^C = \ln(\mathbb{1} + \underbrace{e^{i\alpha_A T^A} e^{i\beta_B T^B} - \mathbb{1}}_{K}) \ ,$$

and expand K to leading order

$$\begin{aligned} K &= \Big(\mathbb{1} + i\alpha_A T^A - \frac{1}{2}(\alpha_A T^A)^2 + ...\Big)\Big(\mathbb{1} + i\beta_B T^B - \frac{1}{2}(\beta_B T^B)^2 + ...\Big) - \mathbb{1} \\ &= i\alpha_A T^A + i\beta_B T^B - \alpha_A T^A \beta_B T^B - \frac{1}{2}(\alpha_A T^A)^2 - \frac{1}{2}(\beta_B T^B)^2 + ... \ , \end{aligned}$$

as well as the logarithm

$$\ln(\mathbb{1} + K) = K - \frac{1}{2}K^2 + ... \ .$$

Then

$$\begin{aligned} i\delta_C T^C &= K - \frac{1}{2}K^2 + ... \\ &= i\alpha_A T^A + i\beta_B T^B - \alpha_A T^A \beta_B T^B - \frac{1}{2}(\alpha_A T^A)^2 - \frac{1}{2}(\beta_B T^B)^2 + \frac{1}{2}(\alpha_A T^A + \beta_B T^B)^2 + ... \\ &= i\alpha_A T^A + i\beta_B T^B - \frac{1}{2}[\alpha_A T^A, \beta_B T^B] + ... \ , \end{aligned}$$

and therefore up to second order

$$[\alpha_A T^A, \beta_B T^B] = -2i(\delta_C - \alpha_C - \beta_C)T^C = i\gamma_C T^C \ , \tag{3.4}$$

for continuous parameters $\gamma_A \in \mathbb{R},\ A \in \{1, ..., N\}$. Therefore there is a relation

$$\gamma_C = \alpha_A \beta_B f^{AB}{}_C \ ,$$

where the $f^{AB}{}_C \in \mathbb{R}$ and $A, B, C \in \{1, ..., N\}$ are called structure constants. Thus from (3.4)

$$[T^A, T^B] = i f^{AB}{}_C T^C \ , \tag{3.5}$$

is again a generator and (3.5) is the Lie algebra. From the commutator property in (3.5) we see that the structure constants are anti-symmetric,

$$f^{AB}{}_C = -f^{BA}{}_C \ .$$

Moreover, for unitary representations with $T^A = (T^A)^\dagger$ one has

$$-i\big(f^{AB}{}_C\big)^* T^C = [T^A, T^B]^\dagger = [T^B, T^A] = i f^{BA}{}_C T^C = -i f^{AB}{}_C T^C$$

and thus the structure constants are real,

$$f^{AB}{}_C = \big(f^{AB}{}_C\big)^* \ .$$

The generators also satisfy the Jacobi identity

$$[T^A, [T^B, T^C]] + [T^B, [T^C, T^A]] + [T^C, [T^A, T^B]] = 0 \ .$$

4 SU(2)

4.1 Algebras

The Lie algebra of $SU(2)$ is the smallest non-trivial Lie algebra. It plays an important role in physics not only because it is isomorphic to the Lie algebra of rotations $SO(3)$ but also because we are interested in the group $SU(2)$. We will study representations of the Lie algebra of $SU(2)$ in Hilbert spaces since a lot of physics can be described there.

In the simplest non-trivial case the Lie algebra of $SU(2)$ is represented in a two-dimensional Hilbert space with orthonormal basis $|i\rangle$, $i \in \{1, 2\}$ by the three generators

$$T^+ = |1\rangle\langle 2| \ , \qquad T^- = |2\rangle\langle 1| \ , \qquad T^3 = \frac{1}{2}\Big(|1\rangle\langle 1| - |2\rangle\langle 2| \Big) \ .$$

They satisfy the algebra

$$[T^+, T^-] = 2T^3 \ , \qquad [T^3, T^\pm] = \pm T^\pm \ , \tag{4.1}$$

and by defining the hermitian operators

$$T^1 = \frac{1}{2}\Big(T^- + T^+\Big) \ , \qquad T^2 = \frac{i}{2}\Big(T^- - T^+\Big) \ ,$$

the commutator algebra reads

$$[T^A, T^B] = i\epsilon^{ABC} T^C \ , \tag{4.2}$$

for $A, B, C \in \{1, 2, 3\}$. The algebra (4.2) is closed under commutation and satisfies the Jacobi identity and is therefore a Lie algebra. This explicit construction is an irreducible representation of the Lie algebra of $SU(2)$ of dimension two, denoted **2**. It is called the fundamental or spinor representation.

To study other representations it is useful to introduce Casimir operators. These are operators that commute with the Lie algebra and in the case of $SU(2)$ there is only one,

$$C_2 = (T^1)^2 + (T^2)^2 + (T^3)^2 \ , \qquad [C_2, T^A] = 0 \ .$$

Since $T^A = (T^A)^\dagger$ by construction we also have $C_2 = C_2^\dagger$.

The states of the Hilbert space can be labelled by the eigenvalues of the maximal number of commuting operators. That is C_2 and by choice T^3 and thus the algebra will be represented on eigenstates of these two operators,

$$C_2 |c, m\rangle = c\, |c, m\rangle \ , \qquad T^3 |c, m\rangle = m\, |c, m\rangle \ ,$$

with $c, m \in \mathbb{R}$ since the operators are hermitian. Because the Casimir operator C_2 is positive definite, we know the spectrum of T^3 is bounded and therefore expect the maximal and minimal values of T^3 to be of the order $\pm\sqrt{C_2}$. From the commutation relations (4.1) we infer

$$T^+ |c, m\rangle \propto |c, m+1\rangle \ ,$$

because the states are uniquely labelled by the eigenvalues of C_2 and T^3. Then

$$\begin{aligned} T^3\, T^+\, |c,m\rangle &= (m+1)\; T^+\, |c,m\rangle \\ &= (m+1)\; d_m^{(+)}\, |c,m+1\rangle \end{aligned}$$

for some $d_m^{(+)} \in \mathbb{R}$, $m \in \mathbb{R}$ and

$$\begin{aligned} C_2\; T^+\, |c,m\rangle &= c\; T^+\, |c,m\rangle \\ &= c\; d_m^{(+)}\, |c,m+1\rangle \ . \end{aligned}$$

Since we know T^3 is bounded, there needs to be a so-called highest weight state $|c,j\rangle$ of the representation for which $j \in \mathbb{R}$ is the maximal value of T^3 subject to,

$$T^+\, |c,j\rangle = 0 \ , \qquad T^3\, |c,j\rangle = j\, |c,j\rangle \ .$$

Similarly we can infer

$$T^-\, |c,m\rangle = d_m^{(-)}\, |c,m-1\rangle$$

from the commutation relations (4.1) for some $d_m^{(-)} \in \mathbb{R}$, $m \in \mathbb{R}$ and find a lowest weight state $|c,k\rangle$, $k \in \mathbb{R}$,

$$T^-\, |c,k\rangle = 0 \ , \qquad T^3\, |c,k\rangle = k\, |c,k\rangle \ .$$

Then the Casimir operator is determined by the highest weight state,

$$\begin{aligned} C_2\, |c,j\rangle &= \left((T^3)^2 + \frac{1}{2}\Big(T^+T^- + T^-T^+\Big) \right) |c,j\rangle \\ &= \left((T^3)^2 + \frac{1}{2}\big[T^+, T^-\big] \right) |c,j\rangle \\ &= \left((T^3)^2 + T^3 \right) |c,j\rangle \\ &= (j^2 + j)\, |c,j\rangle \ , \end{aligned}$$

and therefore $c = j(j+1)$. Analogously for the lowest weight state

$$C_2\, |c,k\rangle = (k^2 - k)\, |c,k\rangle \ ,$$

and thus

$$k(k-1) = j(j+1) \ . \tag{4.3}$$

Under the assumption that j is the maximal value of T^3 there is only the solution $k = -j$ to (4.3). Thus there are $(2j+1)$ states,

$$|j,k\rangle \ , \qquad k \in \{-j, ..., j\} \ ,$$

such that $2j \in \mathbb{N}$ and where we have relabelled c by j since it is determined by it. We denote the representation by the number of states, $\mathbf{2j+1}$. Mathematicians usually use $2j$ for the classification, the so-called Dynkin label. Using

$$T^\pm\, |j,m\rangle = d_m^{(\pm)}\, |j,m\pm 1\rangle \ ,$$

we find from

$$[T^+, T^-] \left| j, m \right\rangle = 2T^3 \left| j, m \right\rangle \ ,$$

that

$$d^{(+)}_{m-1} d^{(-)}_{m} - d^{(+)}_{m} d^{(-)}_{m+1} = 2m \ .$$

One then finds that

$$\begin{aligned} d^{(+)}_m &= \sqrt{(j-m)(j+m+1)} \ , \\ d^{(-)}_m &= \sqrt{(j+m)(j-m+1)} \ , \end{aligned}$$

must hold and the eigenstates form an orthonormal set

$$\langle j, m | j, m' \rangle = \delta_{mm'} \ , \qquad \sum_{m=-j}^{j} \left| j, m \right\rangle \left\langle j, m \right| = \mathbb{1} \ .$$

Fundamental representation 2: In the smallest representation **2** with $j = \frac{1}{2}$, the generators are represented by the Pauli matrices,

$$T^A = \frac{\sigma^A}{2} \ ,$$

which satisfy

$$\sigma^A \sigma^B = \delta^{AB} \mathbb{1} + i\epsilon^{ABC} \sigma^C \ , \qquad \sigma^{A*} = -\sigma^2 \sigma^A \sigma^2 \ .$$

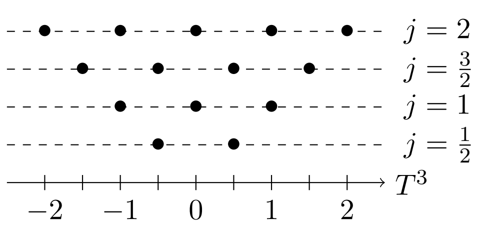
Adjoint representation 3: For $j = 1$ the generators can be represented by the three hermitian matrices

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \ , \qquad T^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \ , \qquad T^3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ .$$

They can be written using the Levi-Civita-symbol,

$$(T^A)_{bc} = -i\epsilon^A{}_{bc} \ .$$

This generalises to an infinite number of irreducible representations such that if the values of T^3 are on an axis



then a point represents an element of a representation, with different representations aligned parallel to the T^3-axis. As shown above, the Lie algebra of $SU(2)$ has one Casimir operator and the states are uniquely determined by one label. In general Lie algebras will have as many labels as Casimir operators. This is called the rank of the Lie algebra and hence the Lie algebra of $SU(2)$ has rank one.

Another way of generating all irreducible representations is by taking direct products of the smallest representation. Lets suppose $T^A_{(1)}$ and $T^A_{(2)}$ are two copies of the Lie algebra of $SU(2)$ each satisfying

$$\left[T^A_{(a)}, T^B_{(a)}\right] = i\varepsilon^{ABC} T^C_{(a)} \ ,$$

for $a \in \{1,2\}$, $A, B, C \in \{1,2,3\}$ and

$$\left[T^A_{(1)}, T^B_{(2)}\right] = 0 \ .$$

Then the sum of the generators

$$\left[T^A_{(1)} + T^A_{(2)}, T^B_{(1)} + T^B_{(2)}\right] = i\varepsilon^{ABC}(T^C_{(1)} + T^C_{(2)}) \ ,$$

generate the same algebra acting on the direct product states, denoted $|\cdot\rangle_{(1)} |\cdot\rangle_{(2)}$. Since the sum of the generators satisfies the same commutation relations one is able to represent their action in terms of the previously derived representations. Consider the direct product of representation $\mathbf{2j+1} \otimes \mathbf{2k+1}$. The highest weight state $|j\rangle_{(1)} |k\rangle_{(2)}$ is uniquely determined by its values of $T^3_{(a)}$, $a \in \{1,2\}$ since the Lie algebra of $SU(2)$ is of rank one and therefore the highest weight state satisfies

$$T^3 |j\rangle_{(1)} |k\rangle_{(2)} = \left(T^3_{(1)} + T^3_{(2)}\right) |j\rangle_{(1)} |k\rangle_{(2)} = (j+k) |j\rangle_{(1)} |k\rangle_{(2)} \ ,$$

which must also be the highest weight state of the representation $\mathbf{2(j+k)+1}$. To generate the rest of the states we apply the sum of the lowering operators $T^- = T^-_{(1)} + T^-_{(2)}$ to the highest weight state,

$$T^- |j\rangle_{(1)} |k\rangle_{(2)} \propto |j-1\rangle_{(1)} |k\rangle_{(2)} + |j\rangle_{(1)} |k-1\rangle_{(2)} \ .$$

The orthogonal combination

$$|j-1\rangle_{(1)} |k\rangle_{(2)} - |j\rangle_{(1)} |k-1\rangle_{(2)} \ ,$$

is the highest weight state of the $\mathbf{2(j+k-1)+1}$ representation because

$$T^3\Big(|j-1\rangle_{(1)} |k\rangle_{(2)} - |j\rangle_{(1)} |k-1\rangle_{(2)} \Big) = (j+k-1)\Big(|j-1\rangle_{(1)} |k\rangle_{(2)} - |j\rangle_{(1)} |k-1\rangle_{(2)} \Big) \ ,$$

and

$$T^+\Big(|j-1\rangle_{(1)} |k\rangle_{(2)} - |j\rangle_{(1)} |k-1\rangle_{(2)} \Big) = 0 \ .$$

Subsequently applying the sum of lowering operators decomposes the direct product representation,

$$\left[\mathbf{2j+1}\right] \otimes \left[\mathbf{2k+1}\right] = \left[\mathbf{2(j+k)+1}\right] \oplus \left[\mathbf{2(j+k-1)+1}\right] \oplus ... \oplus \left[\mathbf{2(j-k)+1}\right]$$

where we assumed without loss of generality $j \geq k$.

As an example consider the direct product $\mathbf{2} \otimes \mathbf{2}$ of two spinor representations of the Lie algebra of $SU(2)$. Denote the highest weight state by

$$|\uparrow\uparrow\rangle = |\frac{1}{2}, \frac{1}{2}\rangle_{(1)} |\frac{1}{2}, \frac{1}{2}\rangle_{(2)} \ ,$$

and generate the state

$$|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle = |\frac{1}{2}; -\frac{1}{2}\rangle_{(1)} |\frac{1}{2}; \frac{1}{2}\rangle_{(2)} + |\frac{1}{2}; \frac{1}{2}\rangle_{(1)} |\frac{1}{2}; -\frac{1}{2}\rangle_{(2)}$$

by applying the sum of lowering operators. Doing so again will give the lowest weight state

$$|\downarrow\downarrow\rangle = |\frac{1}{2}; -\frac{1}{2}\rangle_{(1)} |\frac{1}{2}; -\frac{1}{2}\rangle_{(2)} \ .$$

These three states form the three-dimensional representation $\mathbf{3}$ of the Lie algebra. The linear combination

$$|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle = |\frac{1}{2}; -\frac{1}{2}\rangle_{(1)} |\frac{1}{2}; \frac{1}{2}\rangle_{(2)} - |\frac{1}{2}; \frac{1}{2}\rangle_{(1)} |\frac{1}{2}; -\frac{1}{2}\rangle_{(2)}$$

is a singlet state as it is annihilated by either the sum of lowering or raising operators and therefore forms the representation $\mathbf{1}$. In summary we have confirmed $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$.

Similarly consider the direct product $\mathbf{2} \otimes \mathbf{3}$ of the a spinor and adjoint representation of the Lie algebra of $SU(2)$. The highest weight state

$$|\frac{3}{2}; \frac{3}{2}\rangle = |\frac{1}{2}, \frac{1}{2}\rangle_{(1)} |1, 1\rangle_{(2)}$$

is uniquely determined by its values of T^3. Applying the sum of lowering operators generates the other states, e.g.

$$\begin{aligned} |\frac{3}{2}; \frac{1}{2}\rangle &= \frac{1}{\sqrt{3}} \, T^- \, |\frac{3}{2}; \frac{3}{2}\rangle \\ &= \sqrt{\frac{1}{3}} \, |\frac{1}{2}, -\frac{1}{2}\rangle \, |1, 1\rangle + \sqrt{\frac{2}{3}} \, |\frac{1}{2}, \frac{1}{2}\rangle \, |1, 0\rangle \end{aligned}$$

where the coefficients of the states are called Clebsch-Gordan coefficients. Continuing analogous to before we decompose the direct product representation to the sum of the representation $\mathbf{4}$ and $\mathbf{2}$, i.e. $\mathbf{2} \otimes \mathbf{3} = \mathbf{4} \oplus \mathbf{2}$.

4.2 Groups

Let T^A, $A \in \{1, 2, 3\}$ denote the hermitian generators of the Lie algebra of $SU(2)$. From (3.3) we know that the exponential parametrisation

$$U(\boldsymbol{\theta}) = e^{i\theta_A T^A} \ , \qquad \boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} \in \mathbb{R}^3 \ , \tag{4.4}$$

satisfy the group axioms where we showed closure under multiplication up to leading order.

Fundamental representation: The group generated by (4.4) in the fundamental representation is $SU(2)$. The group elements read

$$U(\boldsymbol{\theta}) = \exp\left(i\theta_A \frac{\sigma^A}{2}\right) = \cos\left(\frac{\theta}{2}\right)\mathbb{1}_2 + i\hat{\theta}_A\sigma^A \sin\left(\frac{\theta}{2}\right) \tag{4.5}$$

where

$$\theta = \sqrt{\theta_A\theta_A} \ , \qquad \hat{\theta}_A = \frac{\theta_A}{\theta} \ ,$$

and σ^A are the Pauli matrices. From (4.5) it is easy to see that the group elements transform under $\theta \mapsto \theta + 2\pi$ like

$$U(\boldsymbol{\theta}) \mapsto -U(\boldsymbol{\theta}) \ .$$

Because the group elements are unitary, and have unit determinant their general form is given by

$$\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha* \end{pmatrix} \ ,$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. Writing

$$\alpha = \alpha_1 + i\alpha_2 \ ,$$
$$\beta = \beta_1 + i\beta_2 \ ,$$

such that $\alpha_i, \beta_i \in \mathbb{R}$ we can represent the group elements as points on the surface of the three-sphere S^3,

$$\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1 \ .$$

$SU(2)$ is isomorphic to S^3 and the group manifold of $SU(2)$ is simply connected.

Adjoint representation: In the adjoint representation the group $SO(3)$ is generated by (4.4) (remember as mentioned before that the Lie algebra of $SU(2)$ is isomorphic to the Lie algebra of $SO(3)$). In the exponential parametrisation the group elements read

$$\big(R(\boldsymbol{\theta})\big)_{bc} = \exp(\theta^A \epsilon_{Abc}) \ , \tag{4.6}$$

generating rotations in three-dimensional Euclidean space such that for $\mathbf{v} \in \mathbb{R}^3$ transforms under the group action as

$$\mathbf{v} \mapsto \mathbf{v}' = R(\boldsymbol{\theta})\mathbf{v} \ .$$

Because the scalar product is invariant under these transformations,

$$\mathbf{v}\cdot\mathbf{v} \mapsto \mathbf{v}'\cdot\mathbf{v}' = \mathbf{v}\cdot\mathbf{v} \ ,$$

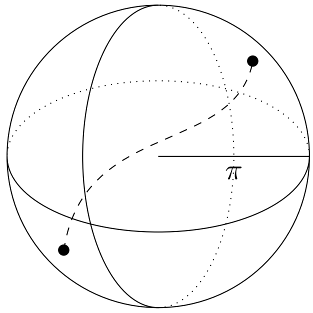
we have

$$R(\boldsymbol{\theta})^T R(\boldsymbol{\theta}) = \mathbb{1}_3 \ , \qquad \det(R(\boldsymbol{\theta})) = 1 \ .$$

The group elements (4.6) can be written

$$\big(R(\boldsymbol{\theta})\big)_{bc} = \delta_{bc}\cos(\theta) + \epsilon_{bcA}\hat{\theta}_A \sin(\theta) + \hat{\theta}_b\hat{\theta}_c(1 - \cos(\theta)) \ ,$$

which is symmetric under $\theta \mapsto \theta + 2\pi$. Limiting the range to $\theta \in (-\pi, \pi)$ we can represent the group elements as points in the closed three-ball where the antipodal points of the surface are to be identified because $\theta = \pi$ and $\theta = -\pi$ represent the same group element. From this consideration we can already infer that the group manifold of $SO(3)$ is not simply connected. A loop in the group manifold extending to the boundary and closing by running through the antipodal point can not be contracted to a single point. Indeed $SO(3)$ is double connected and $SU(2)$ is its double universal cover.



4.3 Three-dimensional harmonic oscillator

The quantum harmonic oscillator is described by the Hamiltonian

$$\begin{aligned} H &= \frac{\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}}{2m} + \frac{1}{2} m\omega^2 \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} \\ &= \hbar\omega \left(\mathbf{A}^\dagger \cdot \mathbf{A} + \frac{3}{2} \right) , \end{aligned}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{p}}$ are position and momentum operators respectively, m is the particles mass and ω the angular frequency of the oscillator. The Hamiltonian is then rewritten in terms of the creation and annihilation operators $\mathbf{A}^\dagger$ and $\mathbf{A}$ respectively which obey the commutation relations

$$[A_i, A_j^\dagger] = \delta_{ij} \ , \quad [A_i^\dagger, A_j^\dagger] = [A_i, A_j] = 0 \, .$$

Then the eigenstates are

$$H \left| n_1, n_2, n_3 \right\rangle = \hbar\omega \left(n_1 + n_2 + n_3 + \frac{3}{2} \right) \left| n_1, n_2, n_3 \right\rangle \ ,$$

for $n_i \in \mathbb{N}$, $i \in \{1, 2, 3\}$ where

$$\left| n_1, n_2, n_3 \right\rangle = \prod_{i=1}^{3} \frac{(A_i^\dagger)^{n_i}}{\sqrt{n_i!}} \left| 0 \right\rangle \ .$$

Define transition operators for the first exited states such that

$$\begin{aligned} P_{12} \left| 1, 0, 0 \right\rangle &= \left| 0, 1, 0 \right\rangle \ , \\ P_{31} \left| 0, 0, 1 \right\rangle &= \left| 1, 0, 0 \right\rangle \ , \\ P_{23} \left| 0, 1, 0 \right\rangle &= \left| 0, 0, 1 \right\rangle \ . \end{aligned}$$

Since these states have the same energy the operators P_{ij} commute with the Hamiltonian

$$[H, P_{ij}] = 0 \ , \tag{4.7}$$

and can be written in terms of creation and annihilation operators,

$$P_{ij} = i(A_i^\dagger A_j - A_j^\dagger A_i) \ .$$

Moreover, the operators P_{ij} satisfy the Lie algebra of $SU(2)$. In fact, they can be identified with the angular momentum operators

$$\begin{aligned} L_1 &= P_{23} = x_2 p_3 - x_3 p_2 \ , \\ L_2 &= P_{31} = x_3 p_1 - x_1 p_3 \ , \\ L_3 &= P_{12} = x_1 p_2 - x_2 p_1 \ , \end{aligned}$$

which have the commutation relation

$$[L_i, L_j] = i\epsilon_{ijk} L_k \ .$$

Moreover, from angular momentum conservation, or directly from equation (4.7) we obtain

$$[H, L_i] = 0 \ ,$$

for all $i \in \{1, 2, 3\}$ and, as for any vector operator,

$$[L_i, A_j^\dagger] = -i\epsilon_{ijk} A_k^\dagger \ .$$

The operators $A_j^\dagger$ thus span the $\mathbf{3}$ representation of the Lie algebra of $SU(2)$. The Nth excited state transforms as the N-fold symmetric direct product

$$(\underbrace{\mathbf{3} \otimes \ldots \otimes \mathbf{3}}_{N \text{ times}})_{\text{sym}} \ .$$

Therefore we can decompose any exited level, e.g. the second excited state,

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{5} \oplus \mathbf{1} \ ,$$

where the quadrupole is a symmetric traceless tensor,

$$\left[A_i^\dagger A_j^\dagger + A_j^\dagger A_i^\dagger - \frac{2}{3}\, \delta_{ij}\, \mathbf{A}^\dagger \cdot \mathbf{A}^\dagger \right] |0\rangle \ ,$$

and the singlet

$$(\mathbf{A}^\dagger \cdot \mathbf{A}^\dagger)\, |0\rangle \ .$$

In general we can decompose a tensor of rank two into three independent representations, a symmetric and trace-less tensor with five independent components, an anti-symmetric tensor with three independent components, and a trace corresponding to one component.

$$\begin{aligned} &\mathbf{3} \otimes \mathbf{3} = \mathbf{5} \oplus \mathbf{3} \oplus \mathbf{1} \\ &\quad T_{ij} = T_{ij}^{\text{sym, trace-less}} + \underbrace{T_{ij}^{\text{anti-sym}}}_{\epsilon_{ijk} t_k} + \delta_{ij} t \ . \end{aligned}$$

4.4 Bohr atom

As another application we can consider Bohr's Hamiltonian for the relative motion of the electron and the proton in a simple hydrogen atom,

$$H = \frac{\mathbf{p}\cdot\mathbf{p}}{2m} - \frac{e^2}{r} \ .$$

The spectrum of bound states is given by

$$E_n = -\frac{\mathrm{Ry}}{n^2}$$

where n is the principal quantum number, with some degeneracy in the angular momentum quantum number $l = 0, 1, ..., n-1$ and azimuthal quantum number $m = -l, ..., 0, ..., l$. The states for $n = 2$ are in the representations

$$\begin{cases} l = 0 : \text{singlet } \mathbf{0} \\ l = 1 : \text{triplet } \mathbf{3} \end{cases}$$

and have the same energy. The fact that these two states have the same energy poses the question whether there is a (hidden) symmetry that connects them. We would need a transition operator from **0** to **3** that commutes with the Hamiltonian. To be in agreement with rotation symmetry, this transition operator must be vector-like. In classical mechanics there is the Laplace-Runge-Lenz vector which precisely meets there requirements,

$$A_i^{\text{classical}} = \epsilon_{ijk} p_j L_k - me^2 \frac{x_i}{r}; \ .$$

For quantum theory we define the hermitian form

$$A_i = \frac{1}{2}\epsilon_{ijk}(p_j L_k + L_k p_j) - me^2 \frac{x_i}{r} \ ,$$

which obeys

$$\mathbf{L}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{L} = 0 \ ,$$

$$[L_i, A_j] = i\varepsilon_{ijk} A_k \ ,$$

as well as

$$[H, A_j] = 0 \ .$$

This invariance is a hidden symmetry for $1/r$-potentials. We will now use this to find the spectrum. We start with the commutation relation

$$[A_i, A_j] = i\epsilon_{ijk} L_k (-2mH) \ ,$$

and define the rescaled operator

$$\hat{A}_i = \frac{A_i}{\sqrt{-2mH}} \ ,$$

to give the simple commutation relation

$$[\hat{A}_i, \hat{A}_j] = i\epsilon_{ijk} L_k \ ,$$

as well as

$$[L_i, \hat{A}_j] = i\epsilon_{ijk}\hat{A}_k \ .$$

Additionally we know

$$[L_i, L_j] = i\varepsilon_{ijk}L_k \ .$$

We define now the linear combinations

$$X_j^{(+)} \equiv \frac{1}{2}(L_j + \hat{A}_j) \ ,$$
$$X_j^{(-)} \equiv \frac{1}{2}(L_j - \hat{A}_j) \ ,$$

which commute

$$[X_i^{(+)}, X_j^{(-)}] = 0 \ .$$

We are left with two copies of the Lie algebra of $SU(2)$,

$$[X_i^{(+)}, X_j^{(+)}] = i\epsilon_{ijk}X_k^{(+)} \ ,$$
$$[X_i^{(-)}, X_j^{(-)}] = i\epsilon_{ijk}X_k^{(-)} \ .$$

Now the Casimir operators are the same

$$\begin{aligned} C_2^{(+)} &= \frac{1}{4}(L_i + \hat{A}_i)(L_i + \hat{A}_i) \\ &= \frac{1}{4}(L_i - \hat{A}_i)(L_i - \hat{A}_i) = C_2^{(-)} \ , \end{aligned}$$

because of $\mathbf{A} \cdot \mathbf{L} = \mathbf{L} \cdot \mathbf{A} = 0$ and therefore

$$C_2^{(+)} = C_2^{(-)} = j(j+1) \ ,$$

where $j_1 = j_2 = j$. To express the Hamiltonian in terms of the Casimir operator we calculate (somewhat lengthy)

$$\begin{aligned} A_i A_i &= (-2mH)\hat{A}_i\hat{A}_i \\ &= \underbrace{\left(\mathbf{p} \cdot \mathbf{p} - \frac{2me^2}{r}\right)}_{-2mH}(L_iL_i + 1) + m^2e^4 \ , \end{aligned}$$

and arrive at

$$\begin{aligned} H &= -\frac{\frac{1}{2}me^4}{\mathbf{L} \cdot \mathbf{L} + \hat{\mathbf{A}} \cdot \hat{\mathbf{A}} + 1} \\ &= -\frac{\frac{1}{2}me^4}{4C_2^{(+)} + 1} \\ &= -\frac{me^4}{2(2j+1)^2} \ . \end{aligned}$$

Therefore we get the well-known result forthe spectrum! The principle quantum number is related to the Casimir by

$$n = 2j + 1 \; .$$

To find the degeneracy of these states we note that the angular momentum operator is given by

$$L_j = X_j^{(+)} + X_j^{(-)} \; .$$

For given value of j we have therefore states with different values of angular momentum as in the direct product representation $(\mathbf{2j+1}) \times (\mathbf{2j+1})$. The possible values for l are according to the rules of angular momentum addition

$$\begin{aligned} l &= 2j = n - 1 \; , \\ l &= 2j - 1 = n - 2 \; , \\ \ldots & \\ l &= 0 \; , \end{aligned}$$

with the usual degeneracy in the quantum number $m = -l, ..., l$.

4.5 Isospin

Fermi-Yang model: Even though protons (mass $m_p = 938$ MeV) carry electromagnetic charge and neutrons (mass $m_n = 939$ MeV) do not, the small mass difference of the two particles led to the assumption that they share a symmetry which leaves the strong interaction invariant. Assume the nucleons and antinucleons are fermionic states

$$\begin{aligned} |p\rangle &= b_1^\dagger \, |0\rangle \; , \qquad & |n\rangle &= b_2^\dagger \, |0\rangle \; , \\ |\bar{p}\rangle &= \bar{b}_1^\dagger \, |0\rangle \; , \qquad & |\bar{n}\rangle &= \bar{b}_2^\dagger \, |0\rangle \; , \end{aligned}$$

where $|0\rangle$ denotes the vacuum state and the creation and annihilation operators satisfy the anticommutation relations

$$\{b_i, b_j^\dagger\} = \{\bar{b}_i, \bar{b}_j^\dagger\} = \delta_{ij} \; ,$$

with all other anticommutators vanishing. From these operators one can construct the generator of the Lie algebra of $SU(2)$

$$I_j = \frac{1}{2} b_\alpha^\dagger (\sigma_j)_{\alpha\beta} b_\beta - \frac{1}{2} \bar{b}_\alpha^\dagger (\sigma_j^*)_{\alpha\beta} \bar{b}_\beta \; ,$$

as well as the generator of the Lie algebra of $U(1)$

$$I_0 = \frac{1}{2} (b_\alpha^\dagger b_\alpha - \bar{b}_\alpha^\dagger \bar{b}_\alpha) \; .$$

The direct product of the isospin representations $\mathbf{2}$

$$\begin{pmatrix} p \\ n \end{pmatrix} \; , \quad \begin{pmatrix} \bar{p} \\ \bar{n} \end{pmatrix}$$

then decomposes such that the **3** representation furnishes three particles

$$
\begin{aligned}
|\pi^+\rangle &= b_1^\dagger \bar{b}_2^\dagger \,|0\rangle \\
|\pi^0\rangle &= (b_1^\dagger \bar{b}_1^\dagger - b_2^\dagger \bar{b}_2^\dagger)\,|0\rangle \\
|\pi^-\rangle &= b_2^\dagger \bar{b}_1^\dagger \,|0\rangle
\end{aligned}
$$

of masses: $m_{\pi^\pm} = 139$ MeV, $m_{\pi^0} = 135$ MeV. The pions have here the same quantum numbers as nucleon-anti-nucleon bound states. As for the proton and neutron the mass difference is explained by the symmetry breaking when weak and electromagnetic forces are taken into account. The electro-magnetic charge is given by the combination

$$Q = I_3 + I_0 \ .$$

Wigner supermultiplet model: Extending the isospin symmetry by combining it with spin,

$$SU(4) \supset SU(2)_I \otimes SU(2)_{\text{spin}} \ ,$$

leads to nucleons N and antinucleons $\bar{N}$ transforming as isospin and spin spinors

$$
\begin{aligned}
|N\rangle &\sim \mathbf{4} = (\mathbf{2}_I, \mathbf{2}_{\text{spin}}) \ , \\
|\bar{N}\rangle &\sim \bar{\mathbf{4}} = (\mathbf{2}_I, \mathbf{2}_{\text{spin}}) \ .
\end{aligned}
$$

We can now decompose

$$\mathbf{4} \otimes \bar{\mathbf{4}} = \mathbf{15} \oplus \mathbf{1}$$

and the pions belong to the 15-dimensional representation which now includes also other mesons

$$\mathbf{15} = \underbrace{(\mathbf{3}_I, \mathbf{3}_{\text{spin}})}_{\substack{\rho^+,\rho^0,\rho^- \\ \rho\text{-meson (vector)} \\ 1^-,149 \text{ MeV}}} \oplus \underbrace{(\mathbf{1}_I, \mathbf{3}_{\text{spin}})}_{\substack{\omega \\ \omega\text{-meson (vector)} \\ 1^-,783 \text{ MeV}}} \oplus \underbrace{(\mathbf{3}_I, \mathbf{1}_{\text{spin}})}_{\substack{\pi^+,\pi^0,\pi^- \\ \text{pion (scalar)} \\ 0^-}}$$

and the singlet state corresponds to the scalar η-meson (0^-, 538 MeV). The modern understanding of mesons is not as nucleon-anti-nucleon bound states but as quark-anti-quark bound states, for example the π^+

$$u\ u\ d\ \bar{d}\ \bar{u}\ \bar{d} \sim u\bar{d} \sim \pi^+ \ .$$

4.6 $SU(1,1)$ and the harmonic oscillator

We create a new Lie algebra with three elements

$$
\begin{aligned}
L^1 &= iT^1 \ , \\
L^2 &= iT^2 \ , \\
L^3 &= T^3 \ ,
\end{aligned}
$$

where T^1, T^2, T^3 are generators of the Lie algebra of $SU(2)$. These are no longer hermitian and the new algebra is

$$[L^1, L^2] = -iL^3 \ ,$$
$$[L^2, L^3] = iL^1 \ ,$$
$$[L^3, L^1] = iL^2 \ .$$

The Casimir operator

$$Q = (L^1)^2 + (L^2)^2 - (L^3)^2 \ ,$$

obeys

$$[Q, L^j] = 0 \ ,$$

but is no longer bounded. In the adjoint representation the group generated is $SU(1,1)$ whos elements are of the form

$$e^{i\theta_A L^A} = \exp\left(\frac{1}{2}\begin{pmatrix} i\theta_3 & -\theta_1 + i\theta_2 \\ -\theta_1 - i\theta_2 & -i\theta_3 \end{pmatrix}\right) .$$

They are not unitary and of the general form

$$\begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix} ,$$

with unit determinant

$$uu^* - vv^* = 1 \ .$$

Consider the commutation relation for a bosonic harmonic oscillator with creation and annihilation operators $a^\dagger$ and a respectively satisfying

$$[a, a^\dagger] = 1 \ .$$

Consider the canonical Bogoliubov transformation between two sets of creation and annihilation operators

$$\begin{pmatrix} a \\ a^\dagger \end{pmatrix} \mapsto \begin{pmatrix} b \\ b^\dagger \end{pmatrix} = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}$$

where $u, v \in \mathbb{C}$ and the operators $b, b^\dagger$ satisfy

$$[b, b^\dagger] = 1 \ .$$

Then

$$[b, b^\dagger] = [ua + va^\dagger, v^*a + u^*a^\dagger] = (|u|^2 - |v|^2)[a, a^\dagger]$$

and thus

$$1 = |u|^2 - |v|^2 \ .$$

Therefore the set of canonical Bogoliubov transformations forms the group $SU(1,1)$. Further, consider the canonical commutation relation of quantum mechanics

$$[x, p] = i \ ,$$

for position and momentum operators x and p respectively. With

$$q = \begin{pmatrix} x \\ p \end{pmatrix} ,$$

these can be rewritten as

$$[q_i, q_j] = i\Omega_{ij} ,$$

with the symplectic matrix

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

The set of all transformations S such that

$$S \cdot \Omega \cdot S^T = \Omega$$

form the symplectic group $Sp(2, \mathbb{R})$. One can verify that there is a one-to-one correspondence

$$SU(1,1) \leftrightarrow Sp(2, \mathbb{R}) ,$$

by a change of basis from $(a, a^\dagger)$ to (x, p).

We now use the $SU(1,1)$ algebra to determine the harmonic oscillator states and the corresponding spectrum. Consider the operators constructed from a bosonic harmonic oscillator with creation and annihilation operators $a^\dagger$ and a respectively,

$$L^+ = L^1 + iL^2 = \frac{1}{2} a^\dagger a^\dagger , \qquad L^- = L^1 - iL^2 = \frac{1}{2} aa .$$

The commutator defines another operator L^3,

$$[L^+, L^-] = -\left(\frac{1}{2} + a^\dagger a\right) = -2L^3 .$$

Note that L^3 is proportional to the Hamiltonian of the harmonic oscillator. Then we have

$$[L_3, L^\pm] = \pm L^\pm .$$

These relations correspond to the $SU(1,1)$ algebra. In contrast to $SU(2)$ there is no highest weight state since the Casimir operator is not bounded. Starting from the vacuum state $|0\rangle$ we get

$$L^- |0\rangle = 0 ,$$

and

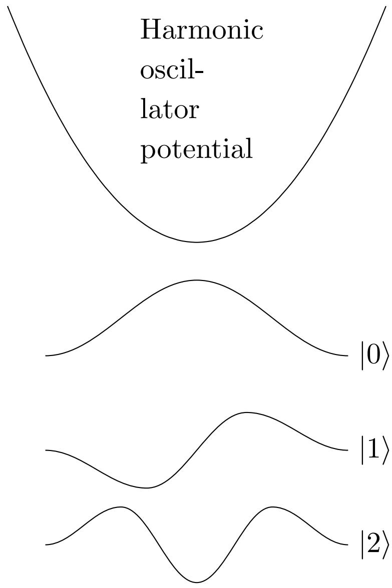
$$(L^+)^n |0\rangle \propto |2n\rangle .$$

In this way we can generate all states of even occupation number

$$|0\rangle , |2\rangle , |4\rangle , \ldots .$$

This is an infinite representation because there is no upper bound. This representation corresponds to the parity even states of a harmonic oscillator. There also is another (partiy odd) representation of $SU(1,1)$ generating all states of odd occupation number

$$|1\rangle , |3\rangle , |5\rangle , \ldots .$$



It is quite remarkable that the spectrum of the harmonic oscillator can be constructed from group theoretic arguments only. In particular because this simple problem serves also as basis for many other theoretical developments, in particular in quantum field theory.

5 SO(N) and SU(N)

5.1 $SO(N)$

The group of rotations in N-dimensional Euclidean space is $SO(N)$ where the group elements are $N \times N$ matrices R (fundamental representation) satisfying

$$R^T \cdot R = \mathbb{1} \tag{5.1}$$

and

$$\det(R) = 1 \tag{5.2}$$

(demanding only $\det(R))^2 = 1$ generates the group $O(N)$). A vector is defined by the transformation under rotation

$$v^i \mapsto R^{ij} v^j$$

where $i, j = 1, ..., N$. Analogously we define tensors be to objects transforming like

$$\begin{aligned} T^{ij} &\mapsto R^{im} R^{jn} T^{mn} \\ W^{ijn} &\mapsto R^{is} R^{jt} R^{nu} W^{stu} \end{aligned}$$

exemplary for second- and third-rank tensors, which can be generalised to arbitrary indexed tensors. To be complete a scalar does not transform at all,

$$s \mapsto s \ .$$

The anti-symmetric combination

$$A^{ij} = \frac{1}{2}(T^{ij} - T^{ji}) = -A^{ji}$$

again is a tensor and transforms to an anti-symmetric one

$$A^{ij} \mapsto R^{im} R^{jn} A^{mn} \ .$$

There are $\frac{N(N-1)}{2}$ independent components for anti-symmetric second-rank tensor. Analogously we can do for a symmetric second-rank tensor

$$S^{ij} = \frac{1}{2}(T^{ij} + T^{ji})$$

with

$$S^{ij} \mapsto R^{im} R^{jn} S^{mn}$$

and $\frac{1}{2}N(N+1)$ independent components because of the additional trace. Now the trace is a scalar since

$$S^{ii} \mapsto \underbrace{R^{ik} R^{il}}_{(R^T)^{ki} R^{il} = \delta^{kl}} S^{kl} = S^{ii}$$

Constructing the symmetric trace-less second-rank tensor

$$\tilde{S}^{ij} = S^{ij} - \delta^{ij} \frac{S^{kk}}{N}$$

we are left with $\frac{N(N+1)}{2} - 1$ components. Therefore we decomposed the representation

$$\mathbf{N} \otimes \mathbf{N} = \left[\frac{\mathbf{N}(\mathbf{N}-\mathbf{1})}{\mathbf{2}}\right] \oplus \left[\frac{\mathbf{N}(\mathbf{N}+\mathbf{1})}{\mathbf{2}} - \mathbf{1}\right] \oplus \mathbf{1} \ .$$

For example for $SO(3)$: $\mathbf{3} \otimes \mathbf{3} = \mathbf{5} \oplus \mathbf{3} \oplus \mathbf{1}$. For a general treatment of $T^{ijk...m}$ Young tableaux was invented to keep take of patterns. For rotations we can use (5.1) to write

$$R^{ik} R^{il} \delta kl = R^{ik} (R^T)^{kj} = \delta^{ij}$$

and thus δ^{ij} is an invariant symbol. Using the N-dimensional Levi-Civita symbol $\varepsilon^{ijk...n}$ which is totally antisymmetric

$$\varepsilon^{...k...m...} = -\varepsilon^{...m...k...}$$

and

$$\varepsilon^{123...N} = 1$$

we are able to rewrite the determinant of (5.2) to give

$$R^{ip} R^{jq} ... R^{ns} \varepsilon^{pq...s} = \underbrace{(\det(R))}_{=1} \varepsilon^{ij...n}$$

and therefore $\varepsilon^{ijk...n}$ is also an invariant symbol.

Dual tensors Let A^{ij} be an anti-symmetric second-rank tensor. We define its dual to be the totally anti-symmetric $(N-2)$-rank tensor

$$B^{k...n} = \varepsilon^{ijk...n} A^{ij} \ .$$

For different values of N we get

$$\begin{aligned} N = 3 &: B^k = \varepsilon^{ijk} A^{ij} \ \text{(vector)} \\ N = 2 &: B = \varepsilon^{ij} A^{ij} \ \text{(scalar)} \ . \end{aligned}$$

In the case of $N = 4$ we again get a anti-symmetric second-rank tensor which is used in electromagnetics, namely the fields strength tensor and its dual in 4-dimensional Euclidean space (for the treatment in Minkowski space we are dealing with $SO(3,1)$ not $SO(4)$ anymore),

$$\tilde{F}^{kl} = \varepsilon^{ijkl} A^{ij} \ .$$

Self-dual / Anti-self-dual For $SO(2n)$ we can construct two irreducible representations with an additional feature. Consider an totally anti-symmetric n-rank tensor $A^{i_1 i_2 ... i_n}$ which has $\frac{(2n)(2n-1)...(n+1)}{n!} = \frac{(2n)!}{(n!)^2}$ components. Constructing the dual tensor

$$B^{i_1 i_2 ... i_n} = \frac{1}{n!} \varepsilon^{i_1 i_2 ... i_n j_1 j_2 ... j_n} A^{j_1 j_2 ... j_n}$$

and its dual

$$A^{i_1 i_2 ... i_n} = \frac{1}{n!} \varepsilon^{i_1 i_2 ... i_n j_1 j_2 ... j_n} B^{j_1 j_2 ... j_n}$$

we see that the are dual to each other. Therefore the tensors

$$T_{\pm}^{i_1\dots i_n} = \frac{1}{2}(A^{i_1\dots i_n} \pm B^{i_1\dots i_n})$$

are self-dual and anti-self-dual with $\frac{(2n)!}{2(n!)^2}$ independent components. which can be schematically seen by considering

$$\varepsilon T_{\pm} = \frac{1}{2}(\varepsilon A \pm \varepsilon B) = \pm\frac{1}{2}(A \pm B) = \pm T_{\pm} \ .$$

Lie algebra of $SO(N)$ The expansion around the identity element for $SO(2)$ is

$$R = \mathbb{1} + d\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

whereas for $SO(3)$

$$R = \mathbb{1} + id\theta^A J^A$$

with the already introduces matrices

$$J^1 = -i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, J^2 = -i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, J^3 = -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \ .$$

In general we can infer for $SO(N)$

$$\begin{aligned} R &= \mathbb{1} + A \\ R^T \cdot R &= \mathbb{1} \\ \Rightarrow A &= -A^T \ . \end{aligned}$$

The generators then are

$$(J_{(mn)})^{(ij)} = -i(\delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni})$$

which obey the algebra

$$[J_{(mn)}, J_{(pq)}] = i(\delta_{mp}J_{(nq)} + \delta_{np}J_{(mp)} - \delta_{np}J_{(mp)} - \delta_{mq}J_{(np)}) \ .$$

Finally let us count the entries for the above considered tensors. Assume we have $T^{ijk\dots} = -T^{jik\dots} = \varepsilon^{ijs}\tilde{T}^{sk\dots}$ antisymmetric in the first index, then we need not take anti-symmetric pair into account. For a symmetric traceless tensor, $S^{ij} \Rightarrow S^{ii} = 0, S^{ijk} \Rightarrow S^{iik} = 0$. In general $S^{i_1 i_2 \dots i_n}$ a symmetric traceless tensor we will have

$$\varepsilon^{ij} S^{ij i_3 \dots i_n} = 0$$

and counting the components by the routine

$$S^{11\dots1}, S^{11\dots2}, S^{11\dots22}, \dots, S^{22\dots2}$$

will give $(n+1)$ entries, an continuing by

$$S^{33\ldots 3x}, S^{33\ldots xx}, \ldots, S^{xx\ldots xx}$$

where the x denotes 1 or 2 will yield

$$\sum_{k=0}^{n}(k+1) = \frac{1}{2}(n+1)(n+2)$$

components without the traceless constraint. Therefore we need to take $\delta^{ij}S^{ijm_1\ldots M_{n-2}} = 0$ into account and get

$$\frac{1}{2}(n-2+1)(n-2+2) = \frac{1}{2}(n-1)n$$

traceless constraints. Finally we have

$$d = 2n+1 = 2j+1, j \in \mathbb{N}$$

dimension of representation as $S^{i_1\ldots i_n}$.

5.2 $SU(N)$

Now that we have discussed $SO(N)$ we are interested in how to get to $SU(N)$. We have already seen that $SU(2)$ is the universal cover of $SO(3)$ for any $\mathbf{2j+1}, j \in \mathbb{N}$. To start off we have

$$\begin{aligned} &O(N) : O^T \cdot O = \mathbb{1} \Rightarrow (\det(O))^2 = 1 \Rightarrow \det(O) = \pm 1 \\ &U(N) : U^\dagger U = \mathbb{1} \Rightarrow \det(U)^* \det(U) = 1 \Rightarrow \det(U) = e^{i\varphi} \ . \end{aligned}$$

For $U(N)$ we know

- elements $e^{i\varphi}\mathbb{1}$ form $U(1)$
- elements with $\det(U) = 1 \Rightarrow SU(N)$.

As a side remark: $e^{i\frac{k}{N}2n}\mathbb{1}_N$ has $\det = 1, k \in \mathbb{N}$ and forms Z_N group and therefore

$$U(N) = U(1) \times (SU(N)/Z_N) \ .$$

Now for $U(N)$ the quadratic form

$$\left.\begin{aligned} \psi^i &\mapsto U^{ij}\psi^j \\ \xi^i &\mapsto U^{ij}\xi^j \end{aligned}\right\} \xi^\dagger\psi = \xi^{*T}\psi = \sum_{j=1}^{N}\xi^{*j}\psi^j$$

is invariant. For the conjugate spinors we have

$$\xi^{i^*} \mapsto U^{*ij}\xi^{j^*} = \xi^{*j}(U^\dagger)^{ji}$$

We will now use the notation $\psi_i = \psi^{*i}$ for conjugate spinors. The transformation laws then take the form

$$\begin{aligned} \psi^i &\mapsto \psi'^i = U^i{}_j\psi^j \\ \psi_i &\mapsto \psi'_i = \psi_j(U^\dagger)^j{}_i \end{aligned}$$

and therefore

$$\xi_i\psi^i \mapsto \xi_j \underbrace{(U^\dagger)^j{}_i U^i{}_k}_{\delta^j{}_k} \psi^k = \xi_i\psi^i \ .$$

From this we can infer that $\delta^i{}_j$ exists as an invariant symbol, whereas δ^{ij}, δ_{ij} do not. We can now write the tensors in $SU(N)$ to have two indices $\varphi^{i_1...i_m}_{j_1...j_n}$. The transformation law is then

$$\varphi^{ij}_k \mapsto \varphi'^{ij}_k = U^i{}_l U^j{}_m (U^\dagger)^n{}_k \varphi^{lm}_n$$

and contractions are only allowed between upper and lower indices,

$$\varphi^{ij}_j = \delta^k{}_j \varphi^{ij}_k \overset{SU(N)}{\rightarrow} U^i{}_l \varphi^{lj}_j \ .$$

To lower or raise indices we use the antisymmetric symbol

$$\varepsilon_{i_1 i_2 ... i_n} (U^\dagger)^{i_1}_{\ j_1} ... (U^\dagger)^{i_n}_{\ j_n} = \underbrace{(\det(U))^*}_{=1} \varepsilon_{j_1...j_n}$$
$$U^{i_1}_{\ j_1} ... U^{i_n}_{\ j_n} \varepsilon^{j_1...j_n} = \underbrace{\det(U)}_{=1} \varepsilon^{i_1...i_n}$$

making $\varepsilon_{i_1...i_n}$ and $\varepsilon^{i_1...i_n}$ invariant symbols. For example for $SU(4)$ we can lower indices: $\varphi_{kpq} = \varepsilon_{ijpq}\varphi^{ij}_k$. In case of $SU(2)$ the speciality is that it needs only upper indices with the invariant symbols $\varepsilon_{ij}, \varepsilon^{ij}, \delta^i{}_j$. We can upper all indices

$$\varphi^{ij...k}_{mn...s} = \varepsilon_{mn}\varepsilon_{nb}...\varepsilon_{st}\varphi^{ab...tij...k}$$

and moreover only need symmetric tensors. In general this does not work,

$$\psi_{mnp} = \varepsilon_{mnpj}\psi^j$$

but for $SU(2)$

$$\psi_m = \varepsilon_{mj}\psi^j \ .$$

In general an element of $SU(2)$

$$U = \exp(\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma})$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices, is not real. With

$$\sigma_2\sigma_a^*\sigma_2 = -\sigma_a$$

we get

$$\sigma_2[e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}]^*\sigma_2 = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}$$

thus there exists a similarity transformation

$$S[e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}]^* S^{-1} = e^{\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}}$$

where

$$S = -i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, S^{-1} = i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Now for general $SU(N)$ we write the elements

$$U = e^{iM} = \sum_{k=0}^{\infty} \frac{1}{k!}(iM)^k$$

where $M = M^\dagger$ is hermitian and therefore $U^\dagger U = \mathbb{1}$. Due to $\ln(\det(M)) = \mathrm{tr}(\ln(M))$ we get

$$\det(U) = e^{i\mathrm{tr}(M)} = 1 \Rightarrow \mathrm{tr}(M) = 0 .$$

As seen for $SU(2)$ we therefore have

$$M = \boldsymbol{\theta} \cdot \frac{\boldsymbol{\sigma}}{2} = \theta^A T^A$$

6 SU(3)

For $SU(3)$ the analogous to the Pauli matrices of $SU(2)$ as generators are the Gell Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} ,$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} ,$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

which are normalised to $\mathrm{tr}(\lambda_a \lambda_b) = 2\delta_{ab}$. The first three Gell-Mann matrices contain the Pauli matrices $(\lambda_1, \lambda_2, \lambda_3) \sim (\sigma_1, \sigma_2, \sigma_3)$ acting on a subspace. We can now look at commutators and since λ_3 and λ_8 are diagonal,

$$[\lambda_3, \lambda_8] = 0 \ .$$

Moreover

$$[\lambda_4, \lambda_5] = 2i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} =: 2i\lambda_{[4,5]} = i(\lambda_3 + \sqrt{3}\lambda_8)$$

and therefore $\lambda_4, \lambda_5, \lambda_{[4,5]}$ form the Lie algebra of $SU(2)$. The same we can do for

$$[\lambda_6, \lambda_7] = 2i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} =: 2i\lambda_{[6,7]} = i(-\lambda_3 + \sqrt{3}\lambda_8)$$

and are then left with three copies of the Lie algebra of $SU(2)$ which overlap. The Lie algebra of $SU(3)$ is generated in the fundamental representation by

$$T^A = \frac{\lambda_A}{2} \ .$$

In the light of the above we define $I_\pm = T^1 \pm iT^2, U_\pm = T^6 \pm iT^7, V_\pm = T^4 \pm iT^5$ which form a basis together with T^3 and T^8 for which

$$[T^3, T^8] = 0 \ .$$

This generates a subalgebra,

$$
\begin{aligned}
[T^3, I_\pm] &= \pm I_\pm \\
[T^3, U_\pm] &= \mp\frac{1}{2}U_\pm \\
[T3, V_\pm] &= \pm\frac{1}{2}V_\pm \\
[T^8, I_\pm] &= 0 \\
[T^8, U_\pm] &= \pm\frac{\sqrt{3}}{2}U_\pm \\
[T^8, V_\pm] &= \pm\frac{\sqrt{3}}{2}V_\pm \\
[I_+, I_-] &= 2T^3 \\
[U_+, U_-] &= \sqrt{3}T^8 - T^3 \\
[V_+, V_-] &= \sqrt{3}T^8 + T^3 \\
[I_+, V_-] &= -U_- \\
[I_+, U_+] &= V_+ \\
[U_+, V_-] &= I_- \\
[I_+, V_+] &= 0 \\
[I_+, U_-] &= 0 \\
[U_+, V_+] &= 0
\end{aligned}
$$

with all the hermitian conjugate relations. Since there are two Casimir operators we can label the states in every irreducible representation by eigenstates of T^3 and T^8,

$$|i_3, i_8\rangle$$

such that

$$
\begin{aligned}
T^3 |i_3, i_8\rangle &= i_3 |i_3, i_8\rangle \\
T^8 |i_3, i_8\rangle &= i_8 |i_3, i_8\rangle \ .
\end{aligned}
$$

One can then verify that

$$
\begin{aligned}
T^3 I_\pm |i_3, i_8\rangle &= (I_\pm T^3 \pm I_\pm) |i_3, i_8\rangle = (i_3 \pm 1) I_\pm |i_3, i_8\rangle \\
T^8 I_\pm |i_3, i_8\rangle &= I_\pm T^8 |i_3, i_8\rangle = i_8 I_\pm |i_3, i_8\rangle
\end{aligned}
$$

and thus

$$I_\pm |i_3, i_8\rangle \propto |i_3 \pm 1, i_8\rangle \ .$$

Analogous

$$
\begin{aligned}
T^3 U_\pm |i_3, i_8\rangle &= (i_3 \mp \frac{1}{2}) U_\pm |i_3, i_8\rangle \\
T^3 U_\pm |i_3, i_8\rangle &= (i_8 \pm \frac{\sqrt{3}}{2}) U_\pm |i_3, i_8\rangle
\end{aligned}
$$

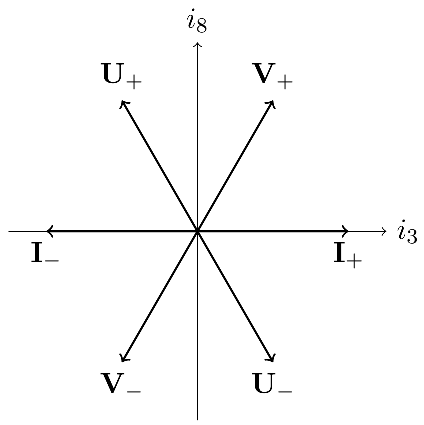
giving

$$U_\pm \left|i_3, i_8\right\rangle \propto \left|i_3 \mp \frac{1}{2}, i_8 \pm \frac{\sqrt{3}}{2}\right\rangle$$

and finally in the same manner

$$V_\pm \left|i_3, i_8\right\rangle \propto \left|i_3 \pm \frac{1}{2}, i_8 \pm \frac{\sqrt{3}}{2}\right\rangle \ .$$

Therefore considering the lattice spanned by i_3 and i_8 the operators $U_\pm$, $V_\pm$ and $I_\pm$ can be represented by



and are called root vectors. Correctly normalized they read

$$\begin{aligned}
\mathbf{I}_\pm &= (\pm 1, 0) \\
\mathbf{U}_\pm &= (\mp\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) \\
\mathbf{V}_\pm &= (\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})
\end{aligned}$$

Associating root vectors to T^3 and T^8 we would get $(0,0)$ for both. We can now make a few observations:

- Root vectors have equal lengths or vanish
- Angles between them are $\frac{\pi}{3}, \frac{2\pi}{3}$ and so on
- sometimes the sum of two roots give another root sometimes not
- the sum of a root vector with itself is not a root vector
- positive roots and simple roots

Positive roots are characterised by $i_3 > 0$. Therefore V_+, I_+, U_- are positive and V_-, I_-, U_+ are negative. Moreover

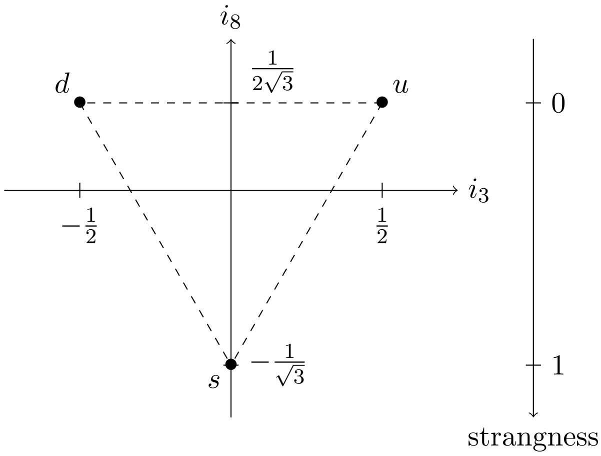
$$\mathbf{I}_+ = \mathbf{V}_+ + \mathbf{U}_-$$

and $\mathbf{V}_+, \mathbf{U}_-$ are the simple roots.

Weight diagrams and representations Let $\mathcal{R}$ be some (irreducible or reducible) representation of $SU(3)$ and plot the states $|i_3, i_8\rangle$ in the i_3, i_8 plane. For the fundamental representation **3** we identify the states with quark states

$$|u\rangle := |\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle$$
$$|d\rangle := |-\frac{1}{2}, \frac{1}{2\sqrt{3}}\rangle$$
$$|s\rangle := |0, -\frac{1}{\sqrt{3}}\rangle$$

such that the weight diagram is



Using the root vectors we see that

$$V_- |u\rangle \propto |s\rangle$$
$$I_- |u\rangle \propto |d\rangle$$
$$U_+ |s\rangle \propto |d\rangle$$
$$V_+ |s\rangle \propto |u\rangle$$

as well as

$$V_+ |u\rangle = I_+ |u\rangle = U_+ |U\rangle = U_- |u\rangle = 0 \ .$$

We can then consider the representation under charge conjugation such that the generators

$$T_C^A = -(T^A)^\dagger = -(T^A)^T$$

with

$$[T^A, T^B] = if^{ABC} T^C$$

and therefore

$$[T_C^A, T_C^B] = if^{ABC} T_C^C \ .$$

We are then left with

$$T_C^A = T^A$$

for $A = 2, 5, 7$ and

$$T_C^A = -T^A$$

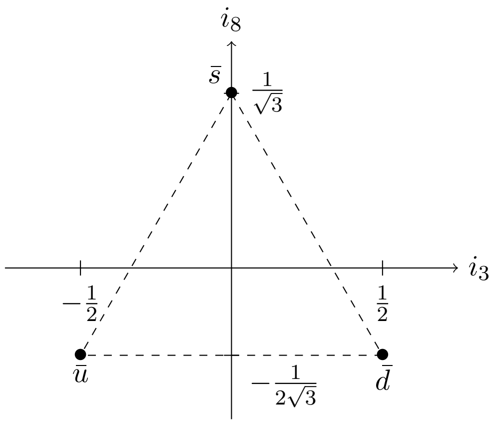
for $A = 1, 3, 4, 6, 8$. Thus we conclude

$$C\,|i_3, i_8\rangle \sim |-i_3, -i_8\rangle \ .$$

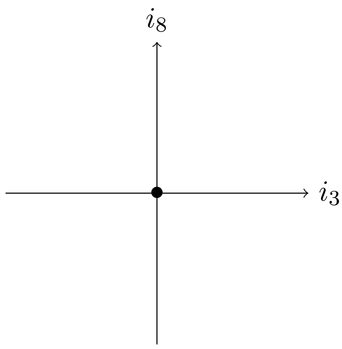
We are now ready to look at the charge conjugate representation $\mathbf{3}^*$ where we identify the states as the antiparticles of the $\mathbf{3}$ states, namely

$$|\bar{u}\rangle = |-\frac{1}{2}, -\frac{1}{2\sqrt{3}}\rangle$$
$$|\bar{d}\rangle = |\frac{1}{2}, -\frac{1}{2\sqrt{3}}\rangle$$
$$|\bar{s}\rangle = |0, \frac{1}{\sqrt{3}}\rangle$$

and the weight diagram reads



For the singlet representation $\mathbf{1}$ the weight diagram is almost trivial with only one state $(i_3, i_8) = (0, 0)$



In the adjoint representation we have

$$\mathrm{adj}(T^A)X = [T^A, X]$$

with $X = X^C T^C$. This satisfies the algebra since

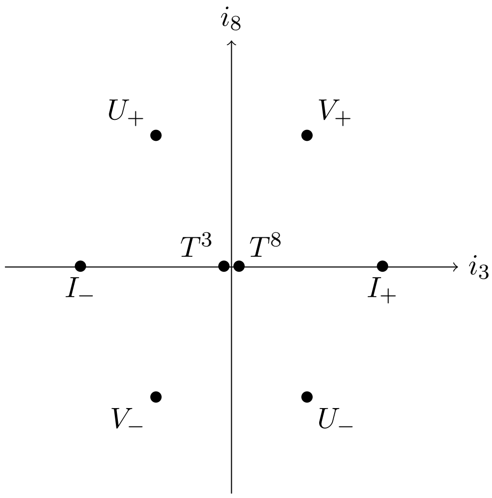
$$[\mathrm{adj}(T^A), \mathrm{adj}(T^B)]X \stackrel{?}{=} \mathrm{adj}([T^A, T^B])X \Leftrightarrow [T^A, [T^B, X]] - [T^B, [T^A, X]] \stackrel{?}{=} [[T^A, T^B], X]$$

$$\Leftrightarrow [T^A, [T^B, T^C]] + [T^B, [T^C, T^A]] + [T^C, [T^A, T^B]] \stackrel{?}{=} 0$$

is the Jacobi identity. The weight table for **8** can then be computed using the commutation relations,

state	$[T^3, V]$	$[T^8, V]$	weight (i_3, i_8)
T^3	0	0	$(0, 0)$
T^8	0	0	$(0, 0)$
I_+	$+1$	0	$(+1, 0)$
I_-	-1	0	$(-1, 0)$
U_+	$-\frac{1}{2}$	$+\frac{\sqrt{3}}{2}$	$(-\frac{1}{2}, +\frac{\sqrt{3}}{2})$
U_-	$+\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$(+\frac{1}{2}, -\frac{\sqrt{3}}{2})$
V_+	$+\frac{1}{2}$	$+\frac{\sqrt{3}}{2}$	$(+\frac{1}{2}, +\frac{\sqrt{3}}{2})$
V_-	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

and has the weight diagram



Analogous to the $SU(2)$ case we can look at the tensor product representation with two representations $T^A_{(1)}$ and $T^A_{(2)}$ with states $|i_3, i_8\rangle_{(1)}$ and $|j_3, j_8\rangle_{(2)}$ respectively. Define

$$T^A := T^A_{(1)} + T^A_{(2)}$$

and as before we have

$$T^3 |i_3, i_8\rangle_{(1)} |j_3, j_8\rangle_{(2)} = (T^A_{(1)} + T^A_{(2)}) |i_3, i_8\rangle_{(1)} |j_3, j_8\rangle_{(2)} = (i_3 + j_3) |i_3, i_8\rangle_{(1)} |j_3, j_8\rangle_{(2)}$$

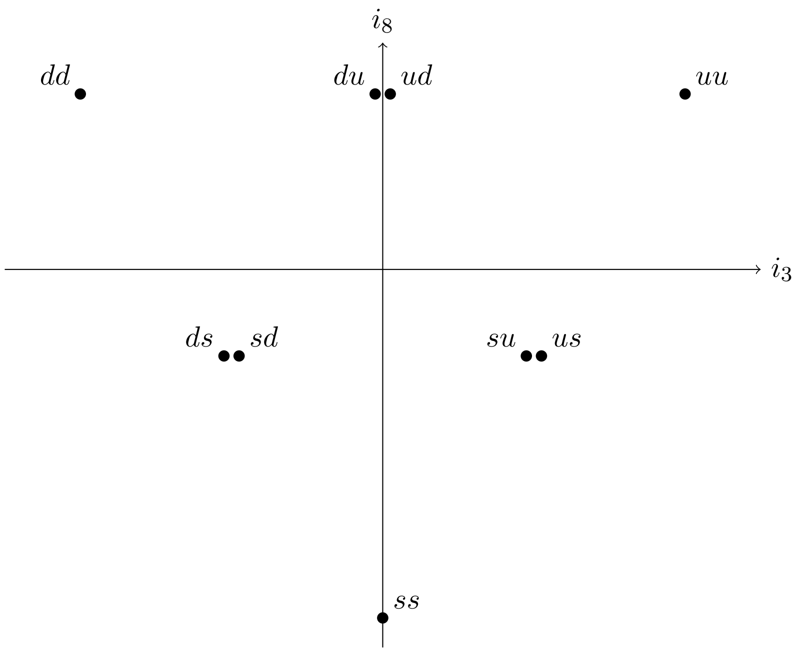
where the difference to $SU(2)$ is that we have two labels for $SU(3)$ since its Lie algebra is of rank three. Let us denote

$$\{I_+, U_+, V_+\} \ni E_+^m$$
$$\{I_-, U_-, V_-\} \ni E_-^m$$

$m = 1, 2, 3$ and study $\mathbf{3} \otimes \mathbf{3}$. The weight table is then

state	weight (i_3, i_8)
$u \otimes u$	$(1, \frac{1}{\sqrt{3}})$
$d \otimes d$	$(-1, \frac{1}{\sqrt{3}})$
$s \otimes s$	$(0, -\frac{2}{\sqrt{3}})$
$u \otimes d, d \otimes u$	$(0, \frac{1}{\sqrt{3}})$
$u \otimes s, s \otimes u$	$(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$
$d \otimes s, s \otimes d$	$(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$

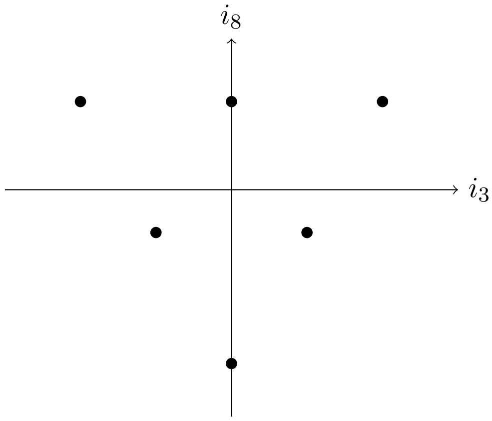
and the corresponding weight diagram



For the highest weight state we need $E_+^m |s\rangle = 0$ and therefore it is $u \otimes u$. Applying $E_-^m |u \otimes u\rangle$ generates the representation $\mathbf{6}$ with weight table

state	weight (i_3, i_8)
$u \otimes u$	$(1, \frac{1}{\sqrt{3}})$
$d \otimes d$	$(-1, \frac{1}{\sqrt{3}})$
$s \otimes s$	$(0, -\frac{2}{\sqrt{3}})$
$\frac{1}{\sqrt{2}}(u \otimes d + d \otimes u)$	$(0, \frac{1}{\sqrt{3}})$
$\frac{1}{\sqrt{2}}(u \otimes s + s \otimes u)$	$(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$
$\frac{1}{\sqrt{2}}(d \otimes s + s \otimes d)$	$(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$

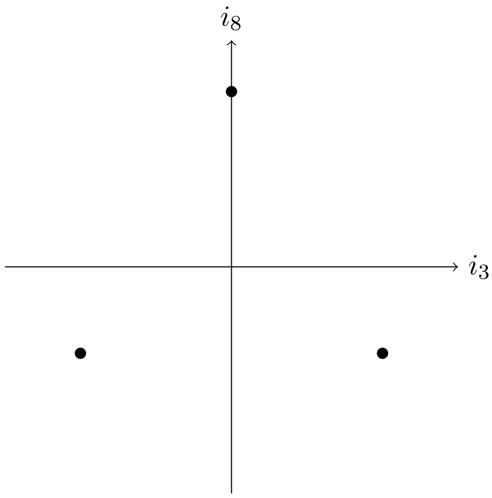
and weight diagram



The remainder of the decomposition is $\mathbf{3}^*$ with the weight table

state	weight (i_3, i_8)
$\frac{1}{\sqrt{2}}(d \otimes u - u \otimes d)$	$(0, \frac{1}{\sqrt{3}})$
$\frac{1}{\sqrt{2}}(d \otimes s - s \otimes d)$	$(-\frac{1}{2}, -\frac{1}{2\sqrt{3}})$
$\frac{1}{\sqrt{2}}(s \otimes u - u \otimes s)$	$(\frac{1}{2}, -\frac{1}{2\sqrt{3}})$

and weight diagram



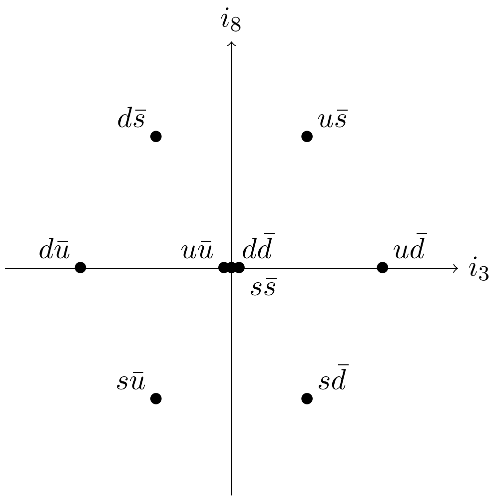
We have thus decomposed $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \mathbf{3}^*$ where for $\mathbf{3}^*$

$$|i_3, i_8\rangle^* = -|i_3, i_8\rangle$$

denotes the charge conjugation. Similar one can derive the weight tables and diagrams for $\mathbf{3}^* \otimes \mathbf{3}^* = \mathbf{6}^* \oplus \mathbf{3}$. Now for $\mathbf{3} \otimes \mathbf{3}^*$. The weight table is

state	weight (i_3, i_8)
$u\otimes\bar{u}, d\otimes\bar{d}, s\otimes\bar{s}$	$(0,0)$
$u\otimes\bar{s}$	$(\frac{1}{2},\frac{\sqrt{3}}{2})$
$u\otimes\bar{d}$	$(1,0)$
$d\otimes\bar{s}$	$(-\frac{1}{2},\frac{\sqrt{3}}{2})$
$d\otimes\bar{u}$	$(-1,0)$
$s\otimes\bar{u}$	$(-\frac{1}{2},-\frac{\sqrt{3}}{2})$
$s\otimes\bar{d}$	$(\frac{1}{2},-\frac{\sqrt{3}}{2})$

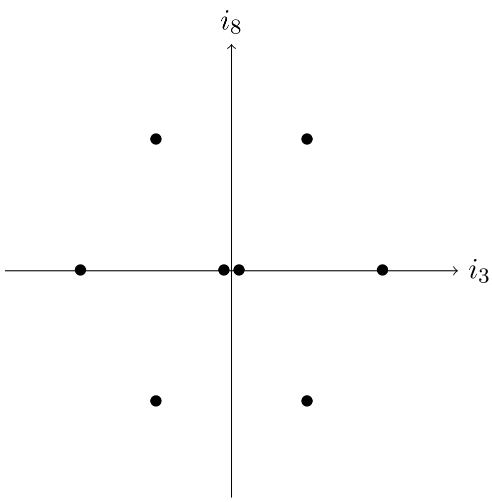
and the weight diagram



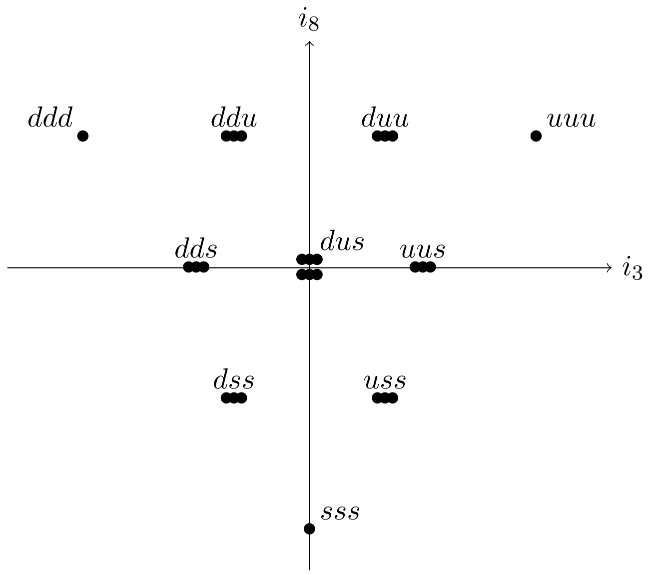
Applying lowering operators to the highest weight state $u\otimes\bar{s}$ give the states

$$u\otimes\bar{s}\,,\quad u\otimes\bar{d}\,,\quad d\otimes\bar{s}\,,\quad d\otimes\bar{u}\,,\quad s\otimes\bar{u}\,,\quad s\otimes\bar{d}\,,\quad \frac{1}{\sqrt{2}}(d\otimes\bar{d}-u\otimes\bar{u})\,,\quad \frac{1}{2}(d\otimes\bar{d}+u\otimes\bar{u}-2s\otimes\bar{s})$$

which furnish the **8** representation with weight diagram



There is a singlet representation left with the state $\frac{1}{\sqrt{3}}(u \otimes \bar{u} + d \otimes \bar{d} + s \otimes \bar{s})$ which is evident since it is annihilated by E_+^m. Thus we have decomposed $\mathbf{3} \otimes \mathbf{3}^* = \mathbf{8} \oplus \mathbf{1}$. Finally considering $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ the weight table can be derived analogously and the weight diagram is



In the exact same way as before applying lowering operators to the highest weight state (uuu) we can decompose $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$. The first irreducible representation are the symmetric combinations

$$u \otimes u \otimes u \ , \quad d \otimes d \otimes d \ , \quad s \otimes s \otimes s \ , \quad \frac{1}{\sqrt{3}}(u \otimes u \otimes s + u \otimes s \otimes u + s \otimes u \otimes u) \ , \quad \ldots$$

forming $\mathbf{10}$ whereas the singlet is the totally anti-symmetric combination

$$\frac{1}{\sqrt{6}}(s \otimes d \otimes u - s \otimes u \otimes d + d \otimes u \otimes s - d \otimes s \otimes u + u \otimes s \otimes d - u \otimes d \otimes s)$$

forming $\mathbf{1}$. The octet states have mixed symmetry. One can now assign particle states to the derived decomposed representations, namely mesons to the decomposition

$$\mathbf{3} \otimes \mathbf{3}^* = \mathbf{8} \oplus \mathbf{1}$$

and baryons to

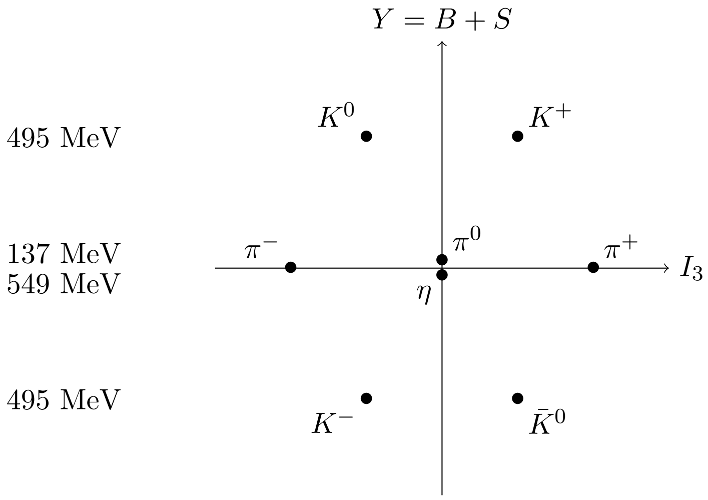
$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1} \ .$$

We denote

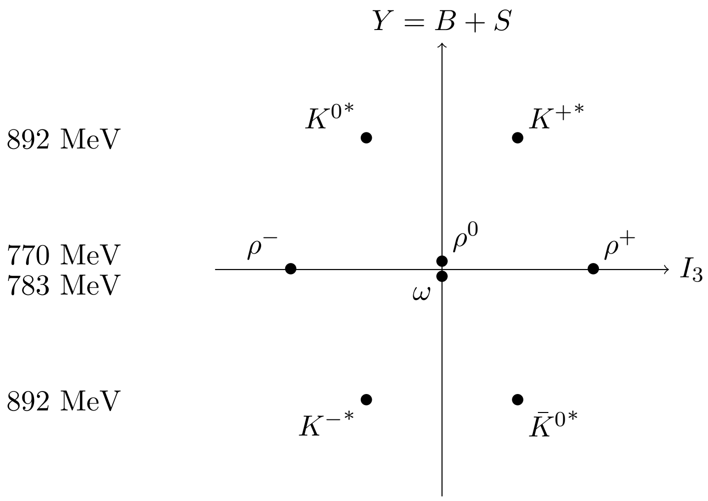
- B: baryon number
- S: strangeness (number of $\bar{s}$ quarks $-$ number of s quarks)
- J: spin
- I_3: isospin (third component)

- $Y = B + S$: hypercharge .

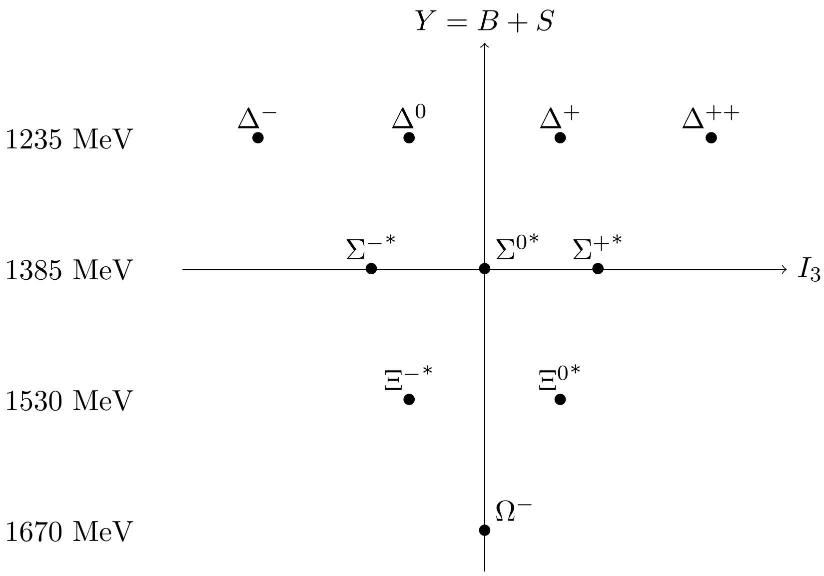
We then identify the pseudoscalar mesons $(B = 0, J = 0)$ with the weight diagram **8**



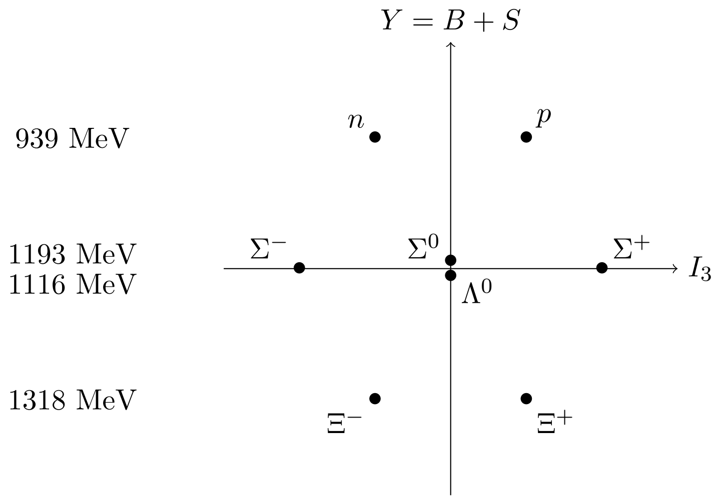
and the additional singlet with the η'-particle. Analogous for the vector mesons $(B = 0, J = 1)$ with weight diagram



and the additional singlet is the ϕ-particle. Now the baryons $(B = 1, J = \frac{3}{2})$ are in the **10** representation



and the baryons ($B = 1, J = \frac{1}{2}$) in $\mathbf{8}$



Now the question might arise why there are no $\mathbf{3} \otimes \mathbf{3}$ or $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}^*$ state. The reason is that one assumes that all asymptotic states are singlets with respect to a $SU(3)_C$ colour symmetry and since

$$\mathbf{3}_C \otimes \mathbf{3}^*{}_C = \mathbf{8}_C \oplus \mathbf{1}_C$$

as well as

$$\mathbf{3}_C \otimes \mathbf{3}_C \otimes \mathbf{3}_C = \mathbf{10}_C \oplus \mathbf{8}_C \oplus \mathbf{8}_C \oplus \mathbf{1}_C$$

decompose into a singlet representation and e.g.

$$\mathbf{3}_C \otimes \mathbf{3}_C = \mathbf{6}_C \oplus \mathbf{3}^*{}_C$$

does not there are no $\mathbf{3}_C \otimes \mathbf{3}_C$ states.

Let us turn again to the Isospin model of the Lie algebra of $SU(2)$ in the fundamental representation we already encountered and denote the nucleons and antinucleons

$$N^i = \begin{pmatrix} p \\ n \end{pmatrix} \ , \quad \bar{N}_i = \begin{pmatrix} \bar{p} \\ \bar{n} \end{pmatrix} \ .$$

Their transformation in the notation established for general $SU(N)$ then is

$$N^i \mapsto U^i{}_j N^j \ , \quad \bar{N}_i \mapsto \bar{N}_j (U^\dagger)^j{}_i$$

Turning to the adjoint representation we can write

$$\begin{aligned}
\phi = \boldsymbol{\pi} \cdot \boldsymbol{\sigma} &= \pi_1\sigma_1 + \pi_2\sigma_2 + \pi_3\sigma_3 \\
&= \begin{pmatrix} \pi_3 & \pi_1 - i\pi_2 \\ \pi_1 + i\pi_2 & -\pi_3 \end{pmatrix} \\
&= \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}
\end{aligned}$$

which transforms like

$$\phi^i{}_j \mapsto \phi'^i{}_j = U^i{}_l \phi^l{}_m (U^\dagger)^m_j \ .$$

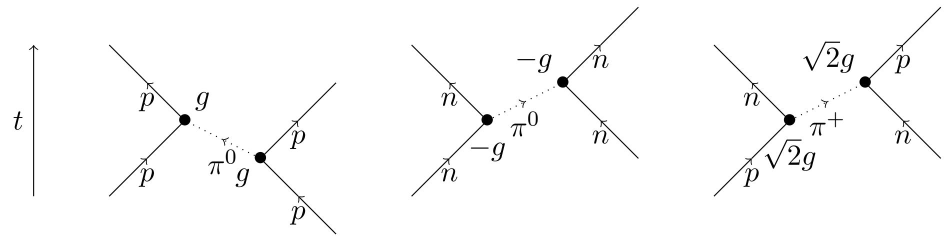
For Lagrangian density which has a term of the form

$$\mathcal{L} = ... + g\bar{N}_i \phi^i{}_j N^j$$

is then $SU(2)$ invariant. Explicitly

$$g \begin{pmatrix} \bar{p} & \bar{n} \end{pmatrix} \cdot \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} \cdot \begin{pmatrix} p \\ n \end{pmatrix} = g\Big(\bar{p}\pi^0 p - \bar{n}\pi^0 n + \sqrt{2}(\bar{p}\pi^+ n + \bar{n}\pi^- p)\Big)$$

with Feynman diagrams, e.g.



Now in the same manner for Lie algebra of $SU(3)$ in the adjoint representation,

$$\begin{aligned}
\phi &= \frac{1}{\sqrt{2}} \sum_{a=1}^{8} \phi_a \lambda_a \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K^0} & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}
\end{aligned}$$

Start by assuming a Lagrangian with $SU(3)$ symmetry

$$\mathcal{L}_0 = \frac{1}{2} g_{\mu\nu} \mathrm{tr}\Big((\partial^\mu \phi)(\partial^\nu \phi) \Big) - \frac{1}{2} m_0^2 \mathrm{tr}(\phi^2) \ .$$

Here all mesons masses would be equal, which clearly is not the case and therefore we have to break the symmetry. Suppose we have a symmetric Hamiltonian H_0 and introduce a H_1 which breaks the symmetry,

$$H = H_0 + \alpha H_1 \ .$$

As long as α stays small we are in the perturbative regime and know that we just shift the eigenvalue of H_0 by a small amount

$$\begin{aligned} \langle s|H_0|s\rangle &= E_s \\ \langle s|\alpha H_1|s\rangle &= \Delta E \end{aligned}$$

and thus write

$$E = E_s + \Delta E + \mathcal{O}(\alpha^2) \ .$$

Thus we are interested in introducing a symmetry breaking term in the Lagrangian. Since ϕ lives in the $\mathbf{8}$ representation and $\mathcal{L}_0 \propto \phi^2$ we look at the decomposition

$$\mathbf{8} \otimes \mathbf{8} = \mathbf{27} \oplus \mathbf{10} \oplus \mathbf{10}^* \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$$

where the $\mathbf{1}$ is the mass term in $\mathcal{L}_0$. Since ϕ^2 is symmetric if view as direct product of representations only the symmetric part is interesting and since

$$\begin{aligned} (\mathbf{8} \otimes \mathbf{8})_s &= \mathbf{27} \oplus \mathbf{8} \oplus \mathbf{1} \\ (\mathbf{8} \otimes \mathbf{8})_a &= \mathbf{10} \oplus \mathbf{10}^* \oplus \mathbf{8} \end{aligned}$$

one is able to choose $\mathbf{27}$ or $\mathbf{8}$ for the breaking term. Since it is always natural to go the simpler way first we pick $\mathbf{8}$ and add $\mathrm{tr}(\phi \cdot \phi \cdot \lambda_8)$ as the breaking term. Note that we need to preserve $SU(2)$ in the upper part of ϕ to keep the pion structure and do not break isospin. Now write

$$\lambda_8 = c \mathbb{1}_3 + d \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

such that the first term adds to the mass term in the Lagrangian and the second is new to break the $SU(3)$ symmetry in the right way, namely

$$\mathrm{tr}\left(\phi \cdot \phi \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = K^- K^+ + K^0 \overline{K^0} + \frac{2}{3} \eta^2$$

and therefore

$$\mathcal{L} = \mathcal{L}_0 - \frac{1}{2} \kappa \ \mathrm{tr}\left(\phi \cdot \phi \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)$$

corrects the mass term of the K's and η but not the pion, comparing to

$$\mathrm{tr}(\phi \cdot \phi) = (\pi^0)^2 + \eta^2 + 2K^- K^+ + ... \;.$$

To first order in perturbation theory around the symmetric situation one gets

$$\begin{aligned} m_\pi^2 &= m_0^2 \\ m_\eta^2 &= m_0^2 + \frac{2}{3}\kappa \\ m_K^2 &= m_0^2 + \frac{1}{2}\kappa \end{aligned}$$

which yields the Gell-Mann Okubo mass formula

$$\Rightarrow 4m_K^2 = 3m_\eta^2 + m_\pi^2 \;.$$

7 Lorentz and Pioncaré group

7.1 Real rotations and Lorentz transformations

We use here conventions where the metric in four dimensional Minkowski space is given by

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \ .$$

Infinitesimal Lorentz transformations and rotations in Minkowski space are of the form

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu} \tag{7.1}$$

with $\Lambda^{\mu}{}_{\nu} \in \mathbb{R}$ such that the metric $\eta_{\mu\nu}$ is invariant.

Exercise: *Show that this implies*

$$\delta\omega_{\mu\nu} = -\delta\omega_{\nu\mu} \ .$$

The spatial-spatial components describe rotations the three dimensional subspace and the spatial-temporal components Lorentz boost in Minkowski space or rotations around a particular three-dimensional direction in Euclidian space. Representations of the Lorentz group with

$$U(\Lambda'\Lambda) = U(\Lambda')U(\Lambda)$$

can be written in infinitesimal form as

$$U(\Lambda) = \mathbb{1}_4 + \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu} \ ,$$

where $M^{\mu\nu} = -M^{\nu\mu}$ are the generators of the Lorentz algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho}\right). \tag{7.2}$$

The fundamental representation (7.1) has the generators

$$(M_F^{\mu\nu})^{\alpha}{}_{\beta} = -i(\eta^{\mu\alpha}\delta^{\nu}_{\beta} - \eta^{\nu\alpha}\delta^{\mu}_{\beta}) \ .$$

It acts on the space of four-dimensional vectors x^{α} and the infinitesimal transformation in (7.1) induces the infinitesimal change

$$\delta x^{\alpha} = \frac{i}{2}\delta\omega_{\mu\nu}(M_F^{\mu\nu})^{\alpha}{}_{\beta}\, x^{\beta} \ .$$

One can decompose the generators into the spatial-spatial part

$$J_i = \frac{1}{2}\varepsilon_{ijk}M^{jk}, \tag{7.3}$$

and a spatial-temporal part,

$$K_j = M^{j0}. \tag{7.4}$$

Equation (7.2) implies the commutation relations

$$
\begin{aligned}
[J_i, J_j] &= i\ \varepsilon_{ijk} J_k, \\
[J_i, K_j] &= i\ \varepsilon_{ijk} K_k, \\
[K_i, K_j] &= -i\ \varepsilon_{ijk} J_k.
\end{aligned}
$$

In the fundamental representation one has

$$
(J_i^F)^j{}_k = -i\varepsilon_{ijk}
$$

where j, k are spatial indices. All other components vanish, $(J_i^F)^0{}_0 = (J_i^F)^0{}_j = (J_i^F)^j{}_0 = 0$. Note that J_i^F is hermitian, $(J_i^F)^\dagger = J_i^F$. The generator K_j has the fundamental representation

$$
(K_j^F)^0{}_m = -i\delta_{jm}, \qquad (K_j^F)^m{}_0 = -i\delta_{jm}
$$

and all other components vanish, $(K_j^F)^0{}_0 = (K_j^F)^m{}_n = 0$. From these expression one finds that the conjugate of the fundamental representation of the Lorentz algebra has the generators

$$
J_j^C = (J_j^F)^\dagger = J_j^F, \qquad K_j^C = (K_j^F)^\dagger = -K_j^F. \tag{7.5}
$$

This implies that K_j^F is anti-hermitian,

$$
(K_j^F)^\dagger = -K_j^F\ .
$$

One can define the linear combinations of generators

$$
N_j = \frac{1}{2}(J_j - iK_j), \qquad \tilde{N}_j = \frac{1}{2}(J_j + iK_j),
$$

for which the commutation relations become

$$
\begin{aligned}
[N_i, N_j] &= i\varepsilon_{ijk} N_k, \\
[\tilde{N}_i, \tilde{N}_j] &= i\varepsilon_{ijk} \tilde{N}_k, \\
[N_i, \tilde{N}_j] &= 0.
\end{aligned}
$$

This shows that the representations of the Lorentz algebra can be decomposed into two representations of SU(2) with generators N_j and $\tilde{N}_j$, respectively. Note that N_j and $\tilde{N}_j$ are hermitian and linearly independent. Nevertheless, there is an interesting relation between the two: Consider the hermitian conjugate representation of the Lorentz group as related to the fundamental one by eq. (7.5). The representation of the generators $N_j, \tilde{N}_j$ is

$$
\begin{aligned}
N_j^C &= \frac{1}{2}(J_j^C - iK_j^C) = \frac{1}{2}(J_j^F + iK_j^F) = \tilde{N}_j^F, \\
\tilde{N}_j^C &= \frac{1}{2}(J_j^C + iK_j^C) = \frac{1}{2}(J_j^F - iK_j^F) = N_j^F.
\end{aligned}
$$

This implies that the role of N_j and $\tilde{N}_j$ is interchanged in the hermitian conjugate representation. Representations of SU(2) are characterized by spin n of half integer or integer value.

Accordingly, the representations of the Lorentz group can be classified as $(2n+1, 2\tilde{n}+1)$. For example

$$\begin{aligned}
(1,1) &= \text{scalar or singlet},\\
(2,1) &= \text{left-handed spinor},\\
(1,2) &= \text{right-handed spinor},\\
(2,2) &= \text{vector}.
\end{aligned}$$

7.2 Pauli formalism

In the non-relativistic description of spin-1/2 particles due to Pauli the generators of rotation are given by

$$J_i = \frac{1}{2}\sigma_i \ ,$$

where the hermitian Pauli matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ ,$$

and fulfill the algebraic relation

$$\sigma_i \cdot \sigma_j = \delta_{ij}\mathbb{1}_2 + i\varepsilon_{ijk}\sigma_k.$$

In other words, the Pauli matrices provide a mapping between the space of rotations SO(3) and the space of unitary matrices SU(2). More concrete, a rotation

$$\Lambda^i{}_j = \delta^i{}_j + \delta\omega^i{}_j$$

corresponds to

$$L(\Lambda) = \mathbb{1}_2 + \frac{i}{4}\delta\omega_{ij}\,\varepsilon_{ijk}\,\sigma_k.$$

By exponentiating this one obtains the mapping. Note, however, that the group SU(2) covers SO(3) twice in the sense that a rotation by 360 degree corresponds to $L(\Lambda) = -\mathbb{1}_2$.

7.3 Dirac formalism

Left and right handed spinor representation We will construct the left and right handed spinor representations of the Lorentz group by using that they agree with the Pauli representation for normal (spatial) rotations. When acting on the left-handed representation (2,1), the generator $\tilde{N}_j$ vanishes. Since $J_j = N_j + \tilde{N}_j$ and $K_j = i(N_j - \tilde{N}_j)$ one has

$$N_j = J_j = -iK_j = \frac{1}{2}\sigma_j, \qquad \tilde{N}_j = 0.$$

Using (7.3) and (7.4) this yields for the left handed spinor representation

$$\begin{aligned}
(M_L^{jk}) &= \varepsilon_{jkl}N_l = \frac{1}{2}\varepsilon_{jkl}\,\sigma_l\\
(M_L^{j0}) &= iN_j = i\frac{1}{2}\sigma_j.
\end{aligned} \tag{7.6}$$

Note that (M_L^{j0}) receives a factor $1/v$ in Wick space so that it becomes $-\frac{1}{2}\sigma_j$ in Euclidean space. As the name suggests, this representation acts in the space of left-handed spinors which are two-components entities, for example

$$\psi_L = \begin{pmatrix} (\psi_L)_1 \\ (\psi_L)_2 \end{pmatrix}.$$

We also use a notation with explicit indices $(\psi_L)_a$ with $a = 1, 2$. The infinitesimal transformation in (7.1) reads with the matrices (7.6)

$$\delta(\psi_L)_a = \frac{i}{2}\delta\omega_{\mu\nu}(M_L^{\mu\nu})_a{}^b\,(\psi_L)_b. \tag{7.7}$$

Similarly one finds for the right-handed spinor representation (1,2) using

$$N_j = 0, \qquad \tilde{N}_j = J_j = iK_j = \frac{1}{2}\sigma_j,$$

the relations

$$\begin{aligned} (M_R^{jk}) &= \varepsilon_{jkl}\tilde{N}_l = \frac{1}{2}\varepsilon_{jkl}\,\sigma_l \\ (M_R^{j0}) &= -i\tilde{N}_j = -i\frac{1}{2}\sigma_j. \end{aligned} \tag{7.8}$$

In Wick space (M_R^{j0}) receives an additional factor $1/v$ and becomes $\frac{1}{2}\sigma_j$ in the Euclidean limit. The representation (7.8) acts in the space of right handed spinors, for example

$$\psi_R = \begin{pmatrix} (\psi_R)^1 \\ (\psi_R)^2 \end{pmatrix}.$$

For right handed spinors we will also use a notation with an explicit index that has a dot in order to distinguish it from a left-handed index, $(\psi_R)^{\dot{a}}$ with $\dot{a} = 1, 2$. The infinitesimal transformation in (7.1) reads with the matrices in (7.8)

$$\delta(\psi_R)^{\dot{a}} = \frac{i}{2}\delta\omega_{\mu\nu}(M_R^{\mu\nu})^{\dot{a}}{}_{\dot{b}}\,(\psi_R)^{\dot{b}}.$$

Invariant symbols From the group-theoretic relation

$$(2, 1) \times (2, 1) = (1, 1)_A + (3, 1)_S,$$

it follows that there must be a Lorentz-singlet with two left-handed spinor indices and that it has to be anti-symmetric. The corresponding invariant symbol can be taken as ε_{ab} with components $\varepsilon_{21} = 1$, $\varepsilon_{12} = -1$ and $\varepsilon_{11} = \varepsilon_{22} = 0$. Indeed one finds that

$$(M_L^{\mu\nu})_a{}^c\varepsilon_{cb} + (M_L^{\mu\nu})_b{}^c\varepsilon_{ac} = 0. \tag{7.9}$$

(This is essentialy due to $\sigma_j\sigma_2 + \sigma_2\sigma_j^T = 0$ for $j = 1, 2, 3$.) It is natural to use ε_{ab} and its inverse ε^{ab} to pull the indices a, b, c up and down. For clarity the non-vanishing components are

$$\varepsilon^{12} = -\varepsilon^{21} = \varepsilon_{21} = -\varepsilon_{12} = 1. \tag{7.10}$$

The symbol δ^a_b is also invariant when spinors with upper left-handed indices have the Lorentz-transformation behavior

$$\delta(\psi_L)^a = -\frac{i}{2}\delta\omega_{\mu\nu}(\psi_L)^b(M_L^{\mu\nu})_b{}^a.$$

From eq. (7.9) it follows also that

$$(M_L^{\mu\nu})_{ab} = (M_L^{\mu\nu})_{ba},$$

so that

$$\varepsilon^{ab}(M_L^{\mu\nu})_{ab} = (M_L^{\mu\nu})_a{}^a = 0.$$

In a completely analogous way the relation

$$(1,2)\times(1,2) = (1,1)_A + (1,3)_S$$

implies that there is a Lorentz singlet with two right-handed spinor indices. The corresponding symbol can be taken as $\varepsilon^{\dot{a}\dot{b}}$, with inverse $\varepsilon_{\dot{a}\dot{b}}$, with components as in (7.10). This symbol is used to lower and raise right-handed indices. Spinors with lower right handed index transform under Lorentz-transformations as

$$\delta(\psi_R)_{\dot{a}} = -\frac{i}{2}\delta\omega_{\mu\nu}(\psi_R)_{\dot{b}}(M_R^{\mu\nu})^{\dot{b}}{}_{\dot{a}}. \tag{7.11}$$

Consider now an object with a left-handed and a right-handed index. It is in the representation $(2,2)$ which should also contain the vector. There is therefore an invariant symbol which can be chosen as

$$(\sigma^\mu)_{a\dot{a}} = (\mathbb{1}_2, \vec{\sigma}),$$

and similarly

$$(\bar{\sigma}^\mu)^{\dot{a}a} = (\mathbb{1}_2, -\vec{\sigma}).$$

It turns out that the matrices for infinitesimal Lorentz transformations can be written as

$$\begin{aligned}(M_L^{\mu\nu})_a{}^b &= \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_a{}^b,\\ (M_R^{\mu\nu})^{\dot{a}}{}_{\dot{b}} &= \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{a}}{}_{\dot{b}}.\end{aligned}$$

Some useful identities are

$$\begin{aligned}(\sigma^\mu)_{a\dot{a}}(\sigma_\mu)_{b\dot{b}} &= -2\,\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}}\,,\\ (\bar{\sigma}^\mu)^{\dot{a}a}(\sigma_\mu)_{\dot{b}b} &= -2\varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}},\\ \varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}(\sigma^\mu)_{a\dot{a}}(\sigma^\nu)_{b\dot{b}} &= -2\,\eta^{\mu\nu},\\ (\bar{\sigma}^\mu)^{\dot{a}a} &= \varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}(\sigma^\mu)_{b\dot{b}},\\ (\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu)_a{}^b &= -2\,\eta^{\mu\nu}\delta_a^b,\\ \mathrm{Tr}(\sigma^\mu\bar{\sigma}^\nu) &= \mathrm{Tr}(\bar{\sigma}^\mu\sigma^\nu) = -2\,\eta^{\mu\nu},\\ \bar{\sigma}^\mu\sigma^\nu\bar{\sigma}_\mu &= 2\,\bar{\sigma}^\nu,\\ \sigma^\mu\bar{\sigma}^\nu\sigma_\mu &= 2\,\sigma^\nu.\end{aligned}$$

Complex conjugation Note that the matrices (7.6) and (7.8) are hermitian conjugate of each other, i. e.

$$(M_L^{\mu\nu})^\dagger = M_R^{\mu\nu}, \qquad (M_R^{\mu\nu})^\dagger = M_L^{\mu\nu}.$$

The hermitian conjugate of the Lorentz transformation (7.7) is given by

$$[\delta(\psi_L)_a]^\dagger = -\frac{i}{2}\delta\omega^*_{\mu\nu}\,[(\psi_L)_b]^\dagger \underbrace{\left[(M_L^{\mu\nu})_a{}^b\right]^\dagger}_{=(M_R^{\mu\nu})^{\dot b}{}_{\dot a}}. \tag{7.12}$$

For $\delta\omega_{\mu\nu} \in \mathbb{R}$ this is of the same form as eq. (7.11). In Minkowski space it is therefore consistent to take $\psi_L^\dagger$ to be a right-handed spinor with lower dotted index, we write

$$[(\psi_L)_a]^\dagger = (\psi_L^\dagger)_{\dot a},$$

and in an analogous way one finds that it is consistent to write

$$\left[(\psi_R)^{\dot a}\right]^\dagger = (\psi_R^\dagger)^a.$$

So far we have considered Minkowski space only. In Euclidean space or for more general complex $\delta\omega_{\mu\nu}$ the hermitian conjugation is more complicated. For complex $\delta\omega_{\mu\nu}$ eq. (7.12) constitutes a transformation behavior that is not of any already discussed type. For a consistent analytic continuation it is actually necessary to have all fields transforming such that the infinitesimal transformation law involves only $\delta\omega_{\mu\nu}$ (and not $\delta\omega^*_{\mu\nu}$). We define therefore new fields

$$(\bar\psi_L)_{\dot a}, \qquad (\psi_R)^a,$$

with transformation laws

$$\delta(\bar\psi_L)_{\dot a} = -\frac{i}{2}\delta\omega_{\mu\nu}(\bar\psi_L)_{\dot b}(M_R^{\mu\nu})^{\dot b}{}_{\dot a},$$
$$\delta(\bar\psi_R)^a = -\frac{i}{2}\delta\omega_{\mu\nu}(\bar\psi_R)^b(M_L^{\mu\nu})_b{}^a.$$

Only in Minkowski space one may identify $(\bar\psi_L)_{\dot a} = (\psi_L^\dagger)_{\dot a}$ and $(\bar\psi_R)^a = (\psi_R^\dagger)^a$.

Connection between Lorentz group and $SL(2,\mathbb{C})$ Consider the Lorentz transformation of a left-handed spinor

$$(\delta\psi_L)_a = \frac{i}{2}\delta\omega_{\mu\nu}(M_L^{\mu\nu})_a{}^b(\psi_L)_b\ .$$

Decompose the infinitesimal Lorentz transformation into a boost

$$\delta\omega_{0j} = -\delta\omega_{j0} = \delta\varphi_j$$

and rotation

$$\delta\omega_{jk} = \varepsilon_{jkm}\delta\theta_m$$

and use

$$M^{j0} = -M^{0j} = \frac{i}{2}\sigma_j$$
$$M^{jk} = \frac{1}{2}\varepsilon_{jkl}\sigma_l$$

so that

$$(\delta\psi_L)_a = \frac{i}{2}\Big(- i\delta\varphi_j + \delta\theta_j\Big)(\sigma_j)_a{}^b(\psi_L)_b \ .$$

In other words, the representation is of the form

$$U(\Lambda) = \exp\left(\frac{1}{2}(\varphi_j + i\theta_j)\sigma_j\right) \ .$$

Because there are real and imaginary parts, this is a general 2×2 matrix, although with vanishing determinant. We have established a map between Lorentz transformations $SO(3,1)$ and the group $SL(2,\mathbb{C})$ of linear transformations by complex 2×2 matrices with uni determinant. The Lorentz transformations of left-handed spinors can be represented as elements of $SL(2,\mathbb{C})$.

Exercise: *Show that for a given matrix B in $SL(2,\mathbb{C})$ the associated Lorentz transformation matrix $\Lambda^\mu{}_\nu$ is*

$$\Lambda^\mu{}_\nu = \frac{1}{2}\mathrm{tr}(\sigma^\mu \cdot B \cdot \sigma_\nu \cdot B^\dagger) \ .$$

Weyl fermions We now have everything to understand massless relativistic fermions. The Lagrange density for Weyl fermions is

$$\mathcal{L} = i(\bar\psi_L)_{\dot a}(\bar\sigma^\mu)^{\dot a b}\partial_\mu(\psi_L)_b \ .$$

Exercise: *Check invariance under rotations and Lorentz boosts.*

To obtain the equation of motion we vary with respect to $\bar\psi_L$

$$\begin{aligned}\frac{\delta}{\delta(\bar\psi_L)_{\dot a}(x)}S &= \frac{\delta}{\delta(\bar\psi_L)_{\dot a}(x)}\int d^4x\mathcal{L}\\ &= i(\bar\sigma^\mu)^{\dot a b}\partial_\mu(\psi_L)_b(x) \stackrel{!}{=} 0 \ .\end{aligned}$$

This is the Weyl equation for a left-handed fermion. In Fourier representation

$$\psi_L(x) \sim e^{iEt + i\boldsymbol{p}\cdot\boldsymbol{x}}$$

one has

$$(E\ \mathbb{1}_2 + \boldsymbol{p}\cdot\boldsymbol{\sigma})\psi_L = 0 \ .$$

Multiplying with $E\ \mathbb{1}_2 - \boldsymbol{p}\cdot\boldsymbol{\sigma}$ from the left gives

$$\Big(E^2\mathbb{1}_2 - p_ip_j\{\sigma_i,\sigma_j\}\Big)\psi_L = 0 \ .$$

With $\{\sigma_i,\sigma_j\} = \delta_{ij}\ \mathbb{1}_2$ the dispersion relation is therefore

$$E^2 - \boldsymbol{p}^2 = 0$$

which describes massless particles, indeed.

Helicity Particle spin was defined for massive particles in the rest frame and we could choose one axis, say the z-axis for the label of states. For massless particles this does not work, they have no rest frame. One defines spin with respect to the momentum axis and defines helicity to be determined by the operator

$$h = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{E} = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{|\boldsymbol{p}|} \ .$$

For the left-handed fermions

$$h\psi_L = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{p}}{|\boldsymbol{p}|}\psi_L = (-1)\psi_L$$

so they have helicity -1. Analogously for right-handed fermions one sees they are massless particles with helicity 1. These are right-handed Weyl fermions.

Exercise: *Consider the Lagrangian*

$$\mathcal{L} = i(\bar{\psi}_R)^a (\sigma^\mu)_{a\dot{b}} \partial_\mu (\psi_R)^{\dot{b}}$$

and show it describes massless particles with helicity 1*.*

Transformations of fields So far we have discussed how the "internal" indices of a field transform under Lorentz transformations. However, a field depends on a space-time position x^μ which also transforms. This is already the case for a scalar field

$$\phi(x) \mapsto \phi'(x) = \phi(\Lambda^{-1}x) \ .$$

(A maximum at x^μ is moved to a maximum at $\Lambda^\mu{}_\nu x^\nu$.) In infinitesimal form

$$(\Lambda^{-1})^\mu{}_\nu = \delta^\mu{}_\nu - \delta\omega^\mu{}_\nu$$

and thus

$$\phi(x) \mapsto \phi'(x) = \phi(x) - x^\nu \delta\omega^\mu{}_\nu \partial_\mu \phi(x) \ .$$

This can be written as

$$\phi'(x) = (1 + \frac{i}{2}\delta\omega_{\mu\nu}\mathcal{M}^{\mu\nu})$$

with generator

$$\mathcal{M}^{\mu\nu} = -i(x^\mu \partial^\nu - x^\nu \partial^\mu) \ .$$

Indeed, these generators form a representation of the Lorentz algebra

$$[\mathcal{M}^{\mu\nu}, \mathcal{M}^{\rho\sigma}] = i(\eta^{\mu\rho}\mathcal{M}^{\nu\sigma} - \eta^{\mu\sigma}\mathcal{M}^{\nu\rho} - \eta^{\nu\rho}\mathcal{M}^{\mu\sigma} + \eta^{\nu\sigma}\mathcal{M}^{\mu\rho}) \ .$$

For higher representation fields, the complete generator contains $\mathcal{M}^{\mu\nu}$ and the generator of "internal" transformations, for example

$$(\psi_L)_a(x) \mapsto (\psi'_L)_a(x) = (\delta_a{}^b + \frac{i}{2}\delta\omega_{\mu\nu}(M^{\mu\nu})_a{}^b)(\psi_L)_b(x)$$

with

$$(M^{\mu\nu})_a{}^b := (M_L^{\mu\nu})_a{}^b + \mathcal{M}^{\mu\nu}\delta_a{}^b \ .$$

7.4 Pioncaré group

Poincaré transformations consist of Lorentz transformations plus translations

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu - b^\mu \ .$$

It is clear that these transformations form a group.

Exercise: *Show the Poincaré group indeed is a group with the composition law*

$$(\Lambda_2, b_2) \circ (\Lambda_1, b_1) = (\Lambda_2\Lambda_1, \Lambda_2 b_1 + b_2) \ .$$

As transformations of fields, translations are generated by the momentum operator

$$P_\mu := -i\partial_\mu \ .$$

For example

$$\begin{aligned} \phi(x) \mapsto \phi'(x) &= \phi(\Lambda^{-1}(x+b)) \\ &\approx \phi(x^\mu - \delta\omega^\mu{}_\nu x^\nu + b^\mu) \\ &= (1 + \frac{i}{2}\delta\omega_{\mu\nu}\mathcal{M}^{\mu\nu} + ib^\mu P_\mu)\phi(x) \ . \end{aligned}$$

One finds easily

$$[P_\mu, P_\nu] = 0 \tag{7.13}$$

and

$$\begin{aligned} [\mathcal{M}^{\mu\nu}, P_\rho] &= i(\delta_\rho{}^\mu P^\nu - \delta^\nu{}_\rho P^\mu) \\ [\mathcal{M}^{\mu\nu}, P^\rho] &= i(\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu) \end{aligned} \tag{7.14}$$

which together with

$$[\mathcal{M}^{\mu\nu}, \mathcal{M}^{\rho\sigma}] = i(\eta^{\mu\rho}\mathcal{M}^{\nu\sigma} - \eta^{\mu\sigma}\mathcal{M}^{\nu\rho} - \eta^{\nu\rho}\mathcal{M}^{\mu\sigma} + \eta^{\nu\sigma}\mathcal{M}^{\mu\rho})$$

forms the Poincaré algebra. The commutator (7.13) tells that the different components of the energy-momentum operator can be diagonalized simultaneously, while (7.14) says that P^ρ transforms as a vector under Lorentz transformations.

Particles as representations One can understand particles as representations of the Poincaré algebra. Energy and momentum are the eigenvalues of

$$P_\mu := -i\partial_\mu$$

and the spin tells information about $\mathcal{M}^{\mu\nu}$. One Casimir operator is

$$P^2 := P_\mu P^\mu$$

which obviously commutes with $\mathcal{M}^{\mu\nu}$ and P_μ.

$$P^2 \left|p\right\rangle = -m^2 \left|p\right\rangle$$

gives the particle mass. The other Casimir follows from the Pauli-Lubanski vector

$$W_\sigma = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\mathcal{M}^{\mu\nu}P^\rho \ .$$

In the particles rest frame P^ρ gives $p_*^\rho = (m, 0, 0, 0)$ and

$$\begin{aligned} W_0 &= 0 \\ W_j &= \frac{1}{2}\varepsilon_{jmn}\mathcal{M}^{mn} = mJ_j \end{aligned}$$

with spin operator J_j. The second Casimir of the Poincaré algebra is $W_\mu W^\mu$ and

$$\frac{1}{m^2}W_\mu W^\mu \left|p, j\right\rangle = \boldsymbol{J}^2 \left|p, j\right\rangle = j(j+1) \left|p, j\right\rangle$$

so the states are characterized by the labels p and j. The commutation relations

$$[W_\mu, W_\nu] = i\varepsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma$$

reduce to the $SO(3)$ algebra for the rest frame case: $P^\sigma \rightsquigarrow p_*^\sigma = (m, 0, 0, 0)$.

The little group For massive particles, spin can be characterized in terms of the rotation group $SO(3)$ in the particles rest frame. For massless particles this is more complicated and one must pick another frame, for example where $p^\mu = (p, 0, 0, p)$. The little group consists of transformations that leave this invariant. This leads to a characterization in terms of helicity.

8 Tetrad formalism and local Lorentz transformations

The tetrad can be defined through a local choice of an orthogonal frame or formally as a Lorentz vector valued one-form $V_\mu{}^A(x)dx^\mu$. The latin index A is here a Lorentz index (in a sense to be made more precise below), while the Greek index μ is a standard coordinate index. With Minkowski metric $\eta_{AB} = \mathrm{diag}(-1, +1, +1, +1)$ one can write the coordinate metric $g_{\mu\nu}(x)$ as

$$g_{\mu\nu}(x) = \eta_{AB} V_\mu{}^A(x) V_\nu{}^B(x). \tag{8.1}$$

We also introduce the *inverse* tetrad $V^\mu{}_A(x)$ such that

$$V_\mu{}^A(x) V^\nu{}_A(x) = \delta_\mu{}^\nu, \qquad V_\mu{}^A(x) V^\mu{}_B(x) = \delta^A{}_B. \tag{8.2}$$

Under a coordinate transformation or diffeomorphism $x^\mu \to x'^\mu(x)$, the tetrad transforms like a one-form

$$V_\mu{}^A(x) \to V'^A_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu{}^A(x) \tag{8.3}$$

Changing afterwards the label or integration variable from x'^μ back to x^μ gives the transformation rule

$$V_\mu{}^A(x) \to V'^A_\mu(x) = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu{}^A(x) - \left[V'^A_\mu(x') - V'^A_\mu(x) \right]. \tag{8.4}$$

For an infinitesimal transformation $x'^\mu = x'^\mu - \varepsilon^\mu(x)$ this reads

$$V_\mu{}^A(x) \to V_\mu{}^A(x) + \varepsilon^\nu(x) \partial_\nu V_\mu{}^A(x) + \left(\partial_\mu \varepsilon^\rho(x)\right) V_\rho{}^A(x) = V_\mu{}^A(x) + \mathcal{L}_\varepsilon V_\mu{}^A(x). \tag{8.5}$$

We are using here the *Lie derivative* $\mathcal{L}_\varepsilon$ in the direction $\varepsilon^\mu(x)$. More general, any coordinate tensor transforms under such infinitesimal coordinate transformations with the corresponding Lie derivative $\mathcal{L}_\varepsilon$.

In addition to coordinate transformations one may also consider *local* Lorentz transformations acting on the tetrad according to

$$V_\mu{}^A(x) \to V'^A_\mu(x) = \Lambda^A{}_B(x)\, V_\mu{}^B(x), \tag{8.6}$$

where $\Lambda^A{}_B(x)$ is at every point x a Lorentz transformation matrix such that

$$\Lambda^A{}_B(x) \Lambda^C{}_D(x) \eta_{AC} = \eta_{BD}. \tag{8.7}$$

Note that these local Lorentz transformations are *internal*, i. e. they do not act on the space-time argument x of a field as a conventional Lorentz transformation would do.

In infinitesimal form, the local Lorentz transformation (8.6) reads

$$V_\mu{}^A(x) \to V'^A_\mu(x) = V_\mu{}^A(x) + \delta\omega^A{}_B(x) V_\mu{}^B(x), \tag{8.8}$$

where $\omega_{AB}(x) = -\omega_{BA}(x)$ is anti-symmetric.

Coordinate vectors and tensors can be transformed using the tetrad and its inverse to become scalars under general coordinate transformations according to

$$A^B(x) = V_\mu{}^B(x) A^\mu(x), \qquad T^{AB}(x) = V_\mu{}^A(x) V_\nu{}^B(x) T^{\mu\nu}(x). \tag{8.9}$$

The results are then Lorentz vectors and tensors, respectively.

More generally, a field Ψ might transform in some representation $\mathcal{R}$ with respect to the local, internal Lorentz transformations

$$\Psi(x) \to \Psi'(x) = L_{\mathcal{R}}(\Lambda(x))\Psi(x), \tag{8.10}$$

or infinitesimally

$$\Psi(x) \to \Psi'(x) = \Psi(x) + \frac{i}{2}\omega_{AB}(x)M_{\mathcal{R}}^{AB}\Psi(x). \tag{8.11}$$

In addition to various fields, we also need to consider their derivatives. Of course, the standard coordinate covariant derivative ∇_μ creates coordinate tensors of higher rank when acting on coordinate scalar, vector or tensor fields. Moreover, one could contract with the inverse tetrad $V^\mu{}_A(x)$ to change the additional index from a coordinate index into a Lorentz index.

It is less clear at this point how to deal with fields that are already in some non-trivial representation with respect to local Lorentz transformations. For Lorentz vectors or tensors one could use the strategy to first transform them to coordinate vectors and tensors with help of the tetrad, use then the standard coordinate covariant derivative, and then transform the result back using again the tetrad. This shows that a real difficulty arises only for spinor fields in some general representation $\mathcal{R}$ with respect to the Lorentz group. The problem is solved by the Lorentz covariant derivative in terms of the *spin connection*.

It is possible to define a covariant derivative $\mathscr{D}_\mu$ such that for the spinor field $\Psi(x)$ transforming under local Lorentz transformations according to (8.10). One has

$$V^\mu{}_A(x)\mathscr{D}_\mu\Psi(x) \to \Lambda_A{}^B(x)V^\mu{}_B(x)L_{\mathcal{R}}(\Lambda(x))\mathscr{D}_\mu\Psi(x). \tag{8.12}$$

In other words, the covariant derivative of some field transforms as before, with an additional transformation matrix for the new index, but without any extra non-homogeneous term.

The full covariant derivative is now

$$\mathscr{D}_\mu = \nabla_\mu + \mathbf{\Omega}_\mu(x), \tag{8.13}$$

where ∇_μ is the standard coordinate covariant derivative and where $\mathbf{\Omega}_\mu$ depends on the Lorentz representation of the field the derivative acts on. The spin connection $\mathbf{\Omega}_\mu(x)$ must transform like a non-abelian gauge field for local Lorentz transformations,

$$\mathbf{\Omega}_\mu(x) \to \mathbf{\Omega}'_\mu(x) = L_{\mathcal{R}}(\Lambda(x))\mathbf{\Omega}_\mu(x)L_{\mathcal{R}}^{-1}(\Lambda(x)) - \left[\partial_\mu L_{\mathcal{R}}(\Lambda(x))\right]L_{\mathcal{R}}^{-1}(\Lambda(x)). \tag{8.14}$$

We also write this for an infinitesimal Lorentz transformation $\Lambda^A{}_B(x) = \delta^A{}_B + \delta\omega^A{}_B(x)$ as

$$\mathbf{\Omega}_\mu(x) \to \mathbf{\Omega}'_\mu(x) = \mathbf{\Omega}_\mu(x) + \frac{i}{2}\delta\omega_{AB}(x)\left[M_{\mathcal{R}}^{AB}, \mathbf{\Omega}_\mu(x)\right] - \frac{i}{2}M_{\mathcal{R}}^{AB}\partial_\mu\delta\omega_{AB}(x). \tag{8.15}$$

This is the transformation rule of the adjoint representation plus the additional term required for a gauge field. Quite generally, one may write the spin connection as

$$\mathbf{\Omega}_\mu(x) = \Omega_{\mu AB}(x)\frac{i}{2}M_{\mathcal{R}}^{AB}, \tag{8.16}$$

where $\Omega_{\mu AB}(x)$ is anti-symmetric in the Lorentz indices A and B and now independent of the representation $\mathcal{R}$. Sometimes it is also called spin connection. Es examples we note here the covariant derivative of a Lorentz vector with upper index,

$$\mathscr{D}_\mu A^B(x) = \partial_\mu A^B(x) + \Omega_\mu{}^B{}_C(x) A^C(x), \tag{8.17}$$

and similarly for a lower Lorentz index,

$$\mathscr{D}_\mu A_B(x) = \partial_\mu A_B(x) - \Omega_\mu{}^C{}_B(x) A_C(x) = \partial_\mu A_B(x) + \Omega_{\mu B}{}^C(x) A_C(x). \tag{8.18}$$

We will define the spin connection $\Omega_\mu{}^A{}_B$ such that the fully covariant derivative of the tetrad vanishes,

$$\mathscr{D}_\mu V_\nu{}^A = \partial_\mu V_\nu{}^A + \Omega_\mu{}^A{}_B V_\nu{}^B - \Gamma^\rho_{\mu\nu} V_\rho{}^A = 0. \tag{8.19}$$

This implies directly the so-called metric compatibility condition $\nabla_\rho g_{\mu\nu} = 0$. One may solve the above relation for the spin connection, leading to

$$\Omega_\mu{}^A{}_B = -\left(\nabla_\mu V_\nu{}^A\right) V^\nu{}_B = -\left(\partial_\mu V_\nu{}^A\right) V^\nu{}_B + \Gamma^\rho_{\mu\nu} V_\rho{}^A V^\nu{}_B. \tag{8.20}$$

Note that the spin connection is here fully determined by the tetrad and its derivatives. As a consequence of (8.1) one can also express the Christoffel symbols through the tetrad. An expression for the spin connection that uses only the tetrad can be given as follows

$$\Omega_\mu{}^{AB} = \frac{1}{2} V^{A\nu}\left(\partial_\mu V^B_{\ \nu} - \partial_\nu V^B_{\ \mu}\right) - \frac{1}{2} V^{B\nu}\left(\partial_\mu V^A_{\ \nu} - \partial_\nu V^A_{\ \mu}\right) - \frac{1}{2} V^{A\rho} V^{B\sigma}\left(\partial_\rho V_{C\sigma} - \partial_\sigma V_{C\rho}\right) V^C_{\ \mu}. \tag{8.21}$$

Even though the spin connection is directly determined by the tetrad it is sometimes useful to keep it open and to vary it independent of the tetrad, at least at intermediate stages of a calculation.

It is useful to note also a relation known as Cartans first structure equation for torsion

$$T_{\mu\nu}{}^A = \partial_\mu V_\nu{}^A - \partial_\nu V_\mu{}^A + \Omega_\mu{}^A{}_B V_\nu{}^B - \Omega_\nu{}^A{}_B V_\mu{}^B. \tag{8.22}$$

In particular the right hand side vanishes in situations without torsion. We also give the behavior under infinitesimal Lorentz transformations directly for $\Omega_\mu{}^A{}_B(x)$,

$$\Omega_\mu{}^A{}_B(x) \to \Omega'^{\,A}_\mu{}_B(x) = \Omega_\mu{}^A{}_B(x) + \delta\omega^A{}_C(x)\Omega_\mu{}^C{}_B(x) - \Omega_\mu{}^A{}_C(x)\delta\omega^C{}_B(x) - \partial_\mu \delta\omega^A{}_B(x). \tag{8.23}$$

This relation will be useful below. It is automatically fulfilled by (8.20).

Finally we note a useful identity for the variation of the spin connection that can be easily derived from (8.20),

$$\delta\Omega_\mu{}^A{}_B(x) = -\left(\mathscr{D}_\mu \delta V_\nu{}^A\right) V^\nu{}_B + \delta\Gamma^\rho_{\mu\nu} V_\rho{}^A V^\nu{}_B. \tag{8.24}$$

We use here the fully covariant derivative $\mathscr{D}_\mu$ and the variation of the Christoffel symbol

$$\delta\Gamma_\mu{}^\rho{}_\nu = \frac{1}{2} g^{\rho\lambda}\left(\nabla_\mu \delta g_{\nu\lambda} + \nabla_\nu \delta g_{\mu\lambda} - \nabla_\lambda \delta g_{\mu\nu}\right). \tag{8.25}$$

Note that in contrast to the spin connection $\Omega_\mu{}^A{}_B(x)$ itself, which is a gauge field for local Lorentz transformations, its variation $\delta\Omega_\mu{}^A{}_B(x)$ transforms simply as a tensor with one upper and one lower index under local Lorentz transformations. Under coordinate transformations both $\Omega_\mu{}^A{}_B(x)$ and $\delta\Omega_\mu{}^A{}_B(x)$ transform as one-forms.

8.1 Response to coordinate transformations and local Lorentz transformations

Let us discuss here the response of a quantum field theory to both general coordinate transformations and local Lorentz transformations. We will perform this discussion for a quantum effective action which we take to depend on a collection of matter fields $\Psi(x)$ (actually field expectation values), as well as the tetrad field $V_\mu{}^A(x)$ and the spin connection field $\Omega_\mu{}^A{}_B(x)$,

$$\Gamma[\Psi, V, \Omega]. \tag{8.26}$$

Because of relation (8.20) or equivalently (8.21) the spin connection and the tetrad field are actually not independent of each other. Nevertheless, one may at an intermediate stage consider the spin connection as an independent field (for the moment anti-symmetric in A and B) and vary the effective action with respect to it. For stationary matter fields $\delta\Gamma/\delta\Psi = 0$, the variation of the effective action is

$$\delta\Gamma = \int d^dx\sqrt{g}\left\{\mathscr{T}^\mu{}_A(x)\delta V_\mu{}^A(x) - \frac{1}{2}\mathscr{S}^\mu{}_{AB}(x)\delta\Omega_\mu{}^{AB}(x)\right\}. \tag{8.27}$$

The field $\mathscr{T}^\mu{}_A(x)$ is defined through a variation with respect to the tetrad at fixed spin connection. It's physical significance will become more clear below. The variation with respect to the spin connection with otherwise fixed tetrad defines the current $\mathscr{S}^\mu{}_{AB}(x)$ which is in fact the spin current. Both $\mathscr{T}^\mu{}_A(x)$ and $\mathscr{S}^\mu{}_{AB}(x)$ transform as mixed coordinate and Lorentz tensors under coordinate transformations and local Lorentz transformations as indicated by their indices. The reason is that $\delta V_\mu{}^A(x)$ and $\delta\Omega_\mu{}^{AB}(x)$ are both transforming as tensors in this sense and the variation of the action itself must be a scalar.

We should also state here that a full variation of the effective action with respect to the tetrad (with the spin connection taken to obey relation (8.20) and therefore not taken as independent) leads to the energy momentum tensor as a mixed coordinate and Lorentz tensor,

$$\delta\Gamma = \int d^dx\sqrt{g}\, T^\mu{}_A(x)\delta V_\mu{}^A(x). \tag{8.28}$$

Using (8.24) we can relate the quantities in (8.27) and (8.28) and find

$$T^\mu{}_A(x) = \mathscr{T}^\mu{}_A(x) - \frac{1}{2}\mathscr{D}_\rho\left[-\mathscr{S}^{\rho\mu}{}_A + \mathscr{S}_A{}^{\rho\mu} + \mathscr{S}^{\mu\rho}{}_A\right]. \tag{8.29}$$

One can recognize this as the Belinfante-Rosenfeld form of the energy-momentum tensor with the first term $\mathscr{T}^\mu{}_A(x)$ being the canonical energy-momentum tensor and $T^\mu{}_A(x)$ its symmetric relative, written here as a mixed Lorentz and coordinate tensor. Note that the expression in square brackets in (8.29) is anti-symmetric in ρ and μ. This implies $\mathscr{D}_\mu T^\mu{}_A(x) = \mathscr{D}_\mu\mathscr{T}^\mu{}_A(x)$.

Let us now first discuss general coordinate transformations. The tetrad transforms according to eq. (8.5) and the spin connection in a fully analogous way with the Lie derivative $\mathcal{L}_\varepsilon$. It suffices at this point to consider the variation with spin connection taken as dependent, as in eq. (8.28). One finds after a bit of algebra

$$\begin{aligned}\delta\Gamma &= \int d^dx\sqrt{g}\, T^\mu{}_A(x)\left[\epsilon^\nu(x)\partial_\nu V_\mu{}^A(x) + V_\nu{}^A(x)\partial_\mu\epsilon^\nu(x)\right] \\ &= \int d^dx\sqrt{g}\,\epsilon^\nu(x)\left[-V_\nu{}^A(x)\mathscr{D}_\mu T^\mu{}_A(x) + T_{AB}(x)\Omega_\nu{}^{AB}(x)\right].\end{aligned} \tag{8.30}$$

We will see below that $T_{AB}(x) = T_{BA}(x)$ so that the second term on the right hand side of (8.30) drops out. Accordingly, because the variation $\delta\Gamma$ must vanish for any $\epsilon^\nu(x)$, the energy-momentum tensor $T^\mu{}_A(x)$ is covariantly conserved. One may easily bring this conservation law to the standard form $\nabla_\mu T^{\mu\nu}(x) = \nabla_\mu \mathscr{T}^{\mu\nu}(x) = 0$.

Besides general coordinate transformations, the action is invariant under local Lorentz transformations. We consider now such a transformation in infinitesimal form. The matter fields are still assumed to be stationary, $\delta\Gamma/\delta\Psi = 0$, so that it suffices to consider the variations of the tetrad and spin connection.

We first consider a variation where only the tetrad is being varied, and the spin connection is taken as dependent according to eq. (8.20). One finds

$$\delta\Gamma = \int d^dx\sqrt{g}\, T^{\mu A}(x)V_\mu{}^B(x)\delta\omega_{AB}(x). \tag{8.31}$$

Because this must vanish for arbitrary $\delta\omega_{AB}(x)$ one finds that the energy-momentum tensor is symmetric,

$$T^{AB}(x) = T^{BA}(x). \tag{8.32}$$

However, one may also do the calculation in an alternative way where the spin connection is first varied independent of the tetrad and we use then (8.8) and (8.23),

$$\begin{aligned}\delta\Gamma &= \int d^dx\sqrt{g}\left\{\mathscr{T}^\mu{}_A(x)\delta V_\mu{}^A(x) - \frac{1}{2}\mathscr{S}^\mu{}_{AB}(x)\delta\Omega_\mu{}^{AB}(x)\right\} \\ &= \int d^dx\sqrt{g}\left\{\mathscr{T}^\mu{}_A(x)\delta\omega^A{}_B(x)V_\mu{}^B(x) - \frac{1}{2}\mathscr{S}^\mu{}_{AB}(x)\left[\delta\omega^A{}_C(x)\Omega_\mu{}^C{}_B(x) - \Omega_\mu{}^A{}_C(x)\delta\omega^C{}_B(x) - \partial_\mu\delta\omega^A{}_B(x)\right]\right\}.\end{aligned} \tag{8.33}$$

Using partial integration one can rewrite this as

$$\delta\Gamma = \int d^dx\sqrt{g}\left[\mathscr{T}^{BA}(x) - \frac{1}{2}\mathscr{D}_\mu\mathscr{S}^{\mu AB}(x)\right]\delta\omega_{AB}(x). \tag{8.34}$$

For this to vanish for arbitrary $\delta\omega_{AB}(x)$ the expression in square brackets must be symmetric. Because $\mathscr{S}^{\mu AB} = -\mathscr{S}^{\mu BA}$ is anti-symmetric, we find for the divergence of the spin current

$$\mathscr{D}_\mu\mathscr{S}^{\mu AB}(x) = \left[\mathscr{T}^{BA}(x) - \mathscr{T}^{AB}(x)\right]. \tag{8.35}$$

We note that the spin current is in general *not* conserved. What needs to be conserved as a consequence of full Lorentz symmetry (also including a coordinate transformation) is the sum of spin current and orbital angular momentum current,

$$\mathscr{M}^{\mu AB}(x) = x^A(x)\mathscr{T}^{\mu B}(x) - x^B(x)\mathscr{T}^{\mu A}(x) + \mathscr{S}^{\mu AB}(x). \tag{8.36}$$

We assume here $\mathscr{D}_\mu x^A(x) = V_\mu{}^A(x)$ (which essentially defines what is meant by $x^A(x)$ in non-cartesian coordinates) one has indeed $\mathscr{D}_\mu\mathscr{M}^{\mu AB}(x) = 0$ as a consequence of (8.35) and the conservation law $\mathscr{D}_\mu\mathscr{T}^{\mu A}(x) = 0$.

9 General linear transformations, the Lorentz group and dilatations

9.1 Fundamental representation and Lie algebra

A general linear transformation $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ can be written in infinitesimal form as

$$x^\mu \to (\delta^\mu{}_\nu + \delta a^\mu{}_\nu)\, x^\nu, \tag{9.1}$$

and one may decompose

$$\delta a^\mu{}_\nu = \delta\omega^\mu{}_\nu + \delta\zeta\; \delta^\mu{}_\nu + \delta\zeta^\mu{}_\nu, \tag{9.2}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is anti-symmetric, $\zeta_{\mu\nu} = +\zeta_{\nu\mu}$ is symmetric and trace-less, $\delta\zeta^\mu{}_\mu = 0$. The anti-symmetric part generates Lorentz transformations, the second term ζ (the trace) introduces scale transformations and $\zeta^\mu{}_\nu$ introduces shear transformations. We raise and lower the indices here by the metric $\eta_{\mu\nu}$ (and its inverse $\eta^{\mu\nu}$) which, however, is itself not invariant under scale and shear transformations.

Let us now discuss the corresponding Lie algebra. We write in a general representation R the infinitesimal transformation as

$$\Lambda = \mathbb{1} + \frac{i}{2}\delta\omega_{\mu\nu}\, M_R^{\mu\nu} + i\delta\zeta\; D_R + \frac{i}{2}\delta\zeta_{\mu\nu}\, S_R^{\mu\nu}, \tag{9.3}$$

with the generators $M_R^{\mu\nu}$, D_R and $S_R^{\mu\nu}$. In the fundamental representation we have

$$\begin{aligned}(M_F^{\mu\nu})^\rho{}_\sigma &= -\,i\,(\eta^{\mu\rho}\delta^\nu{}_\sigma - \eta^{\nu\rho}\delta^\mu{}_\sigma)\,,\\ (D_F)^\rho{}_\sigma &= -\,i\,\delta^\rho{}_\sigma,\\ (S_F^{\mu\nu})^\rho{}_\sigma &= -\,i\left(\eta^{\mu\rho}\delta^\nu{}_\sigma + \eta^{\nu\rho}\delta^\mu{}_\sigma - \tfrac{2}{d}\eta^{\mu\nu}\delta^\rho{}_\sigma\right).\end{aligned} \tag{9.4}$$

The Lie brackets are as follows. First,

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\,(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} + \eta^{\nu\sigma}M^{\rho\mu} - \eta^{\mu\sigma}M^{\rho\nu})\,, \tag{9.5}$$

is the standard Lie algebra of the Lorentz group. One can read this as the statement that $M^{\rho\sigma}$ transforms as a tensor under Lorentz transformations. (A vector P^ρ has the commutation relation $[M^{\mu\nu}, P^\rho] = -i\,(\eta^{\nu\rho}P^\mu - \eta^{\mu\rho}P^\nu)$.)

It is immediately clear that D commutes with itself as well as $M^{\mu\nu}$ and $S^{\mu\nu}$. The remaining commutators are non-trivial and one finds first

$$[M^{\mu\nu}, S^{\rho\sigma}] = -i\,(\eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\rho}S^{\nu\sigma} + \eta^{\nu\sigma}S^{\rho\mu} - \eta^{\mu\sigma}S^{\rho\nu})\,, \tag{9.6}$$

which tells that $S^{\rho\sigma}$ transforms as a tensor under Lorentz transformations. Finally, one has also

$$[S^{\mu\nu}, S^{\rho\sigma}] = -i\,(\eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\rho\mu} - \eta^{\mu\sigma}M^{\rho\nu})\,. \tag{9.7}$$

One observes that Lorentz transformations form a subgroup and so do dilatations. Shear transformations do not form a subgroup, however.

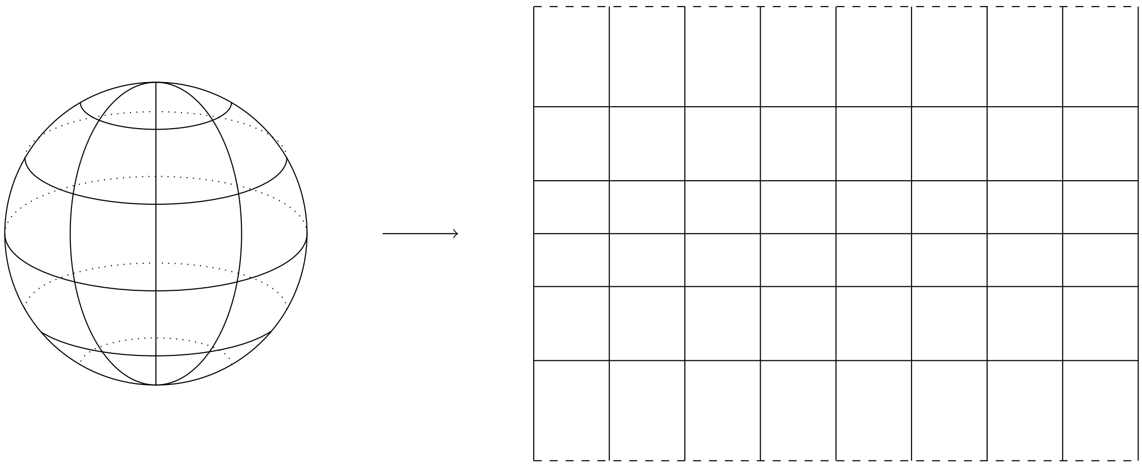
9.2 Representations

The representations of the Lorentz group in terms of scalar, vector, tensor and spinor fields are well studied. The representations of the dilatation group are not very difficult, because the latter is abelian.

The situation is more involved for the infinitesimal shear transformations, though. Of course, a representation for scalar, vector and tensor fields is easy to define from the fundamental representation, but the transformation behavior of spinor fields is a priori unclear. In particular, it may be not possible to find generators for shear transformations within the Clifford algebra.

10 Conformal group

We are interested in transformations which describe a change of scale but preserves the angle between line segments. These will lead to the conformal algebra which generates the conformal group. An example of a conformal map is the Mercator projection



Consider transformations of the form

$$x^\mu \mapsto x^\mu + \epsilon\, \xi^\mu(x) =: x'^\mu$$

for some infinitesimal ϵ. For a general metric tensor $g_{\mu\nu}$ the transformation is given by

$$g_{\mu\nu} \mapsto g_{\mu\nu} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} =: g'_{\mu\nu}$$

and any transformations such that

$$g'_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$$

is a conformal transformation. These form a group called the conformal group. Expanding

$$\Omega^2(x) = 1 + \epsilon\kappa(x) + \mathcal{O}(\epsilon^2) \tag{10.1}$$

yields the conformal Killing equation

$$g_{\mu\sigma}\partial_\rho \xi^\mu + g_{\rho\nu}\partial_\sigma \xi^\nu + \xi^\lambda \partial_\lambda g_{\rho\sigma} + \kappa g_{\rho\sigma} = 0 \tag{10.2}$$

up to first order in ϵ and ξ^μ is called the conformal Killing vector.

Exercise: *Show that the expansion* (10.1) *implies the conformal Killing equation* (10.2).

In the special case of the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$ the conformal Killing equation (10.2) reduces to

$$\partial_\sigma \xi_\rho + \partial_\rho \xi_\sigma + \kappa \eta_{\rho\sigma} = 0 \ .$$

For the case $\kappa = 0$ the solutions are of the form

$$\xi^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu$$

with an antisymmetric $\omega^\mu{}_\nu = -\omega^\nu{}_\mu$. For the more interesting case of $\kappa \neq 0$ we consider

$$x^\mu \mapsto \lambda x^\mu \tag{10.3}$$

where $\lambda = 1 + \epsilon c$ with a real number $c \in \mathbb{R}$.

Exercise: *Show that for the transformation* (10.3) *the conformal Killing vector and* κ *are given by*

$$\xi^\mu = cx^\mu \ , \quad \kappa = -2c \ .$$

Now turning to inversions

$$x^\mu \mapsto \frac{x^\mu}{x^2}$$

we have

$$dx^\mu = \frac{\delta^\mu_\lambda x^2 - 2x_\lambda x^\mu}{x^4} dx^\lambda$$

and thus

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \mapsto \frac{1}{x^4} \eta_{\mu\nu} dx^\mu dx^\nu$$

is a conformal transformation. Still we want to find conformal transformations which can be expressed in an infinitesimal form. Therefore consider

$$x^\mu \overset{\text{inversion}}{\mapsto} \frac{x^\mu}{x^2} \overset{\text{translation}}{\mapsto} \frac{x^\mu}{x^2} + a^\mu \overset{\text{inversion}}{\mapsto} \frac{\frac{x^\mu}{x^2} + a^\mu}{\eta_{\rho\sigma}\left(\frac{x^\rho}{x^2} + a^\rho\right)\left(\frac{x^\sigma}{x^2} + a^\sigma\right)}$$
$$\overset{\text{infinitesimal}}{\approx} x^\mu + a_\lambda(\eta^{\mu\lambda}x^2 - 2x^\mu x^\lambda) + \mathcal{O}(a^2)$$

and thus

$$\xi^\mu = a_\lambda(\eta^{\mu\lambda}x^2 - 2x^\mu x^\lambda)$$

is a conformal Killing vector. Taking the derivative of the conformal Killing equation yields

$$\partial^\rho\partial_\rho\xi_\sigma + \partial_\sigma\partial^\rho\xi_\rho + \partial_\sigma\kappa = 0$$

and with $\kappa = -2\frac{\partial_\mu\xi^\mu}{d}$ where $d = \eta_{\mu\nu}\eta^{\mu\nu}$ is the dimension of space-time we arrive at

$$d\partial_\mu\partial^\mu\xi_\sigma = (2-d)\partial_\sigma\partial_\mu\xi^\mu \ .$$

For $d = 2$ the equation $\partial_\mu\partial^\mu\xi_\sigma = 0$ has infinitely many solutions but for $d > 2$ ξ_σ can be at most quadratic in x^μ. The generators of the conformal algebra are

$$\begin{aligned}
D &= -ix^\mu\partial_\mu \\
P_\mu &= -i\partial_\mu \\
K^\mu &= -i(\eta^{\mu\nu}x^2 - 2x^\mu x^\nu)\partial_\nu \\
\mathcal{M}^{\mu\nu} &= -i(x^\mu\partial^\nu - x^\nu\partial^\mu)
\end{aligned}$$

where we already know the commutation relations of P_μ and $\mathcal{M}^{\mu\nu}$. We notice that

$$\begin{aligned}
[D, (x^{\alpha_1}...x^{\alpha_d})] &= d(x^{\alpha_1}...x^{\alpha_d}) \\
[D, (\partial_{\alpha_1}...\partial_{\alpha_d})] &= -(\partial_{\alpha_1}...\partial_{\alpha_d})
\end{aligned}$$

and the commutation relations read

$$\begin{aligned}
[D, P^\mu] &= iP^\mu \\
[D, \mathcal{M}^{\mu\nu}] &= 0 \\
[K^\mu, K^\nu] &= 0 \\
[\mathcal{M}^{\mu\nu}, K^\rho] &= i(\eta^{\mu\rho}K^\nu - \eta^{\nu\rho}K^\mu) \\
[K^\mu, P^\nu] &= 2i(\mathcal{M}^{\mu\nu} + \eta^{\mu\nu}D) \ .
\end{aligned} \tag{10.4}$$

Exercise: *Explicitly derive the commutation relations* (10.4).

11 Non-Abelian gauge theories

A gauge theory has a local symmetry as opposed to a global symmetry. The fields are invariant under

$$\psi_a(x) \to U_a{}^b(x)\ \psi_b(x) = \exp\left[i\omega_A(x)\ T^A(x)\right]_a{}^b\ \psi_b(x)$$

where $U_a{}^b(x)$ depends on space and time. This is possible with the help of gauge fields, like for example the photon field $A_\mu(x)$ and its non-Abelian generalisation. Let us concentrate on the group theoretic aspects. The gauge group of the Standard Model is

$$SU(3) \times SU(2) \times U(1)\ .$$

The fermion fields and the Higgs boson scalar field can be classified into representations of the corresponding algebras. With respect to the strong interaction group $SU(3)_{\text{colour}}$ we need the

singlet	**1**
triplet	**3**
anti-triplet	$\mathbf{3}^*$

representations. With respect to the weak interaction group $SU(2)$ we need

singlets	**1**
doublets	**2** .

Recall that $SU(2)$ is pseudo-real so there is no $\mathbf{2}^*$. Finally with respect to the hypercharge group $U(1)_Y$ we will classify fields by their charge as generalisations of electric charge q. The charge turn out to be

$$0\ , \quad \pm\frac{1}{6}\ , \quad \pm\frac{1}{3}\ , \quad \pm\frac{1}{2}\ , \quad \frac{2}{3}\ , \quad \pm 1\ .$$

Moreover the fermions transform as Weyl spinors under the Lorentz group, either left- or right-handed. There are the following fields

$$
\begin{array}{llll}
\begin{pmatrix} \nu_L \\ e_L \end{pmatrix} : & \begin{array}{c}\text{neutrino}\\ \text{electron}\end{array} , & \text{left-handed} , & \left(\mathbf{1},\ \mathbf{2},\ -\frac{1}{2}\right) \\
\left(\bar{\nu}_L\ \bar{e}_L\right) : & \begin{array}{c}\text{anti-neutrino}\\ \text{anti-electron}\end{array} , & \text{right-handed} , & \left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right) \\
e_R : & \text{electron} , & \text{right-handed} , & \left(\mathbf{1},\ \mathbf{1},\ -1\right) \\
\bar{e}_R : & \text{anti-electron} , & \text{left-handed} , & \left(\mathbf{1},\ \mathbf{1},\ 1\right) \\
\begin{pmatrix} u_L \\ d_L \end{pmatrix} : & \begin{array}{c}\text{up-quark}\\ \text{down-quark}\end{array} , & \text{left-handed} , & \left(\mathbf{3},\ \mathbf{2},\ \frac{1}{6}\right) \\
\left(\bar{u}_L\ \bar{d}_L\right) : & \begin{array}{c}\text{anti-up-quark}\\ \text{anti-down-quark}\end{array} , & \text{right-handed} , & \left(\mathbf{3}^*,\ \mathbf{2},\ -\frac{1}{6}\right) \\
u_R : & \text{up-quark} , & \text{right-handed} , & \left(\mathbf{3},\ \mathbf{1},\ \frac{2}{3}\right) \\
\bar{u}_R : & \text{anti-up-quark} , & \text{left-handed} , & \left(\mathbf{3}^*,\ \mathbf{1},\ -\frac{2}{3}\right) \\
d_R : & \text{down-quark} , & \text{right-handed} , & \left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right) \\
\bar{d}_R : & \text{anti-down-quark} , & \text{left-handed} , & \left(\mathbf{3}^*,\ \mathbf{1},\ \frac{1}{3}\right) \\
\phi : & \text{Higgs-doublet} , & \text{scalar} , & \left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right)
\end{array}
$$

where the last expression determines the particles local symmetries: $\left(SU(3)_{\text{colour}},\ SU(2),\ Y\right)$. The fields have several indices corresponding to the different groups, for example

$$(u_R)^{\dot{a}m}$$

where $\dot{a} \in \{1,2\}$ is the Lorentz spinor index and $m \in \{r,g,b\}$ is the $SU(3)_{\text{colour}}$ index.

12 Grand unification

We now discuss a proposed extension of the Standard Model which leads to a unification of the gauge groups into $SU(5)$. Note that the $SU(3)$ and $SU(2)$ generators naturally fit into $SU(5)$ generators and similar for the spinors

$$\begin{pmatrix} \begin{pmatrix} 3\times 3 \end{pmatrix}_{SU(3)} & \\ & \begin{pmatrix} 2\times 2 \end{pmatrix}_{SU(2)} \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \\ \psi^5 \end{pmatrix} .$$

There are $5^2 - 1 = 24$ generators of $SU(5)$ corresponding to the hermitian traceless 5×5 matrices. Out of them eight generate $SU(3)$ while three generate $SU(2)$. Moreover there is one hermitian traceless matrix

$$\frac{1}{2}Y = \begin{pmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{pmatrix} .$$

That generates a $U(1)$ subgroup which actually gives $U(1)_Y$. The remaining generators correspond to additional gauge bosons not present in the Standard Model so they are supposedly very heavy or confined. We find the embedding

$$SU(5) \to SU(3) \times SU(2) \times U(1) .$$

Now let us consider representations. Take the fundamental representation of $SU(5)$ the spinor ψ^m. It decomposes

$$\mathbf{5} = \left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right) \oplus \left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right)$$

in a natural way. The conjugate decomposes

$$\mathbf{5}^* = \left(\mathbf{3}^*,\ \mathbf{1},\ \frac{1}{3}\right) \oplus \left(\mathbf{1},\ \mathbf{2},\ -\frac{1}{2}\right) .$$

Indeed this could be the representation for

$$\begin{aligned} &d_R\ , \quad \begin{pmatrix} \bar{\nu}_L\ \bar{e}_L \end{pmatrix} \\ &\bar{d}_R\ , \quad \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} . \end{aligned}$$

So what about the other representations? The next smallest representation is the anti-symmetric tensor ψ^{mn} with dimension ten. We still need

$$\left(\mathbf{3},\ \mathbf{2},\ \frac{1}{6}\right)\ , \quad \left(\mathbf{3}^*,\ \mathbf{1},\ -\frac{2}{3}\right)\ , \quad \left(\mathbf{1},\ \mathbf{1},\ 1\right)$$

and the corresponding anti-fields. These are ten fields indeed. Now ψ^{mn} decomposes into irreducible representations

$$\begin{aligned}
\left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right) \otimes_A \left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right) \quad &= \left(\mathbf{3}^*,\ \mathbf{1},\ -\frac{2}{3}\right) && \bar{u}_R \\
\left(\mathbf{3},\ \mathbf{1},\ -\frac{1}{3}\right) \otimes_A \left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right) \quad &= \left(\mathbf{3},\ \mathbf{2},\ \frac{1}{6}\right) && \begin{pmatrix} u_L \\ d_L \end{pmatrix} \\
\left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right) \otimes_A \left(\mathbf{1},\ \mathbf{2},\ \frac{1}{2}\right) \quad &= \left(\mathbf{1},\ \mathbf{1},\ 1\right) && \bar{e}_R\ .
\end{aligned}$$

Note that we have used here non-trivial relations discussed before such as for $SU(3)$

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{3}^*{}_A \oplus \mathbf{6}_S$$

or for $SU(2)$

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1}_A \oplus \mathbf{3}_S\ .$$

The $U(1)$ charges are simply added. Indeed things work out! The fermion fields of a single generation in the Standard Model can be organised into the $SU(5)$ representations

$$\mathbf{5}^* : \quad \bar{d}_R,\ \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

and

$$\mathbf{10} : \quad \bar{u}_R,\ \bar{e}_L,\ \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

as well as corresponding anti-fields. The scalar Higgs field could be part of a $\mathbf{5}$ scalar representation but the corresponding field with quantum numbers

$$\left(\mathbf{3},\ \mathbf{1},\ -\frac{2}{3}\right)$$

is not present in the Standard Model and must be very heavy or otherwise suppressed. The hypothetical $SU(5)$ gauge bosons that are neither $SU(2)$ nor $SU(3)$ bosons could in principle induce transitions

$$\begin{aligned}
d &\to e^+ \\
u &\to \bar{u}
\end{aligned}$$

and thus $u + d \to \bar{u} + e^+$ causing

$$\begin{aligned}
uud &\to u\bar{u} + e^+ \\
p &\to \pi^0 + e^+\ .
\end{aligned}$$

The proton could therefore in principle decay. This has not been observed so the transition rate must be very small.