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Assignment 3

Sommersemester 2024

Part 2:

Proof the following identity.

$$\int \mathcal{N}(y; x, \beta^2) \cdot \mathcal{N}(x; \mu, \sigma^2)\, dx = \mathcal{N}(y; \mu, \beta^2 + \sigma^2)$$

We will transform and simplify the integrals step by step:

$$\begin{aligned}
&\int \mathcal{N}(y; x, \beta^2) \cdot \mathcal{N}(x; \mu, \sigma^2)\, dx && \text{(Equality of terms)}\\
=&\int \mathcal{N}(x; y, \beta^2) \cdot \mathcal{N}(x; \mu, \sigma^2)\, dx && \text{(Natural Parameters)}\\
=&\int G(x; \frac{y}{\beta^2}, \frac{1}{\beta^2}) \cdot G(x; \frac{\mu}{\sigma^2}, \frac{1}{\sigma^2})\, dx && \text{(Parameter Conversion)}\\
=&\int G(x; \tau_1, \rho_1) \cdot G(x; \tau_2, \rho_2)\, dx && \text{(Multiplication Theorem)}\\
=&\int G(x; \tau_1 + \tau_2, \rho_1 + \rho_2) \cdot \mathcal{N}(\frac{\tau_1}{\rho_1}; \frac{\tau_2}{\rho_2}, \frac{1}{\rho_1} + \frac{1}{\rho_2})\, dx && \text{(Constant in Integral)}\\
=&\mathcal{N}(\frac{\tau_1}{\rho_1}; \frac{\tau_2}{\rho_2}, \frac{1}{\rho_1} + \frac{1}{\rho_2}) \cdot \int G(x; \tau_1 + \tau_2, \rho_1 + \rho_2)\, dx && \text{(Gaussian integrates to 1)}\\
=&\mathcal{N}(\frac{\tau_1}{\rho_1}; \frac{\tau_2}{\rho_2}, \frac{1}{\rho_1} + \frac{1}{\rho_2}) \cdot 1 && \text{(Parameter Conversion)}\\
=&\mathcal{N}(y; \mu, \beta^2 + \sigma^2)
\end{aligned}$$