

Unwrapping Graphs with Local Metrics

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BSc Project for Jakob Meyer, September 2020

1 Problem

View-graphs with local metric are generated as representations of space in the PhD project of Tristan Baumann. The nodes of these graphs are small image features (“micro-snapshots”) numbered $i = 1, \dots, n$ with adjacencies $a_{ij} \in [0, 1]$. Each link is labeled with an angle ϕ_{ij} specifying the direction of node j as seen from node i , relative to a reference direction ν_i . During exploration of the environment, this agent continuously updates the reference direction and stores it with the node as a label. The updating process is error-prone.

2 Approach

The unwrapping problem can be stated as follows: Given an adjacency matrix $A = \{a_{ij}\}$, a set of connection angles ϕ_{ij} for all pairs with $a_{ij} \neq 0$, and (maybe) the local assumed reference directions ν_i , find an embedding $(\mathbf{x}_1, \dots, \mathbf{x}_n)$, $\mathbf{x}_i \in \mathbb{R}^2$ of the nodes satisfying

$$\mathbf{x}_j - \mathbf{x}_i \approx \lambda \begin{pmatrix} \cos \phi_{ij} \\ \sin \phi_{ij} \end{pmatrix} \quad (1)$$

where λ is the distance between the two nodes’ embeddings.

In this constraint, ν_i does not show up. This is to say that it is implicitly set to zero. Once an embedding is found, we could say that ν_i has been rotated by the angle

$$\nu_i^* = \frac{\sum_j a_{ij} (\arctan(\mathbf{x}_j - \mathbf{x}_i) - \phi_{ij})}{\sum_j a_{ij}} \quad (2)$$

where the denominator is the out-degree of node i .

We now denote the unit vector in direction ϕ_{ij} as \mathbf{p}_{ij} , i.e.,

$$\mathbf{p}_{ij} := \begin{pmatrix} \cos \phi_{ij} \\ \sin \phi_{ij} \end{pmatrix}. \quad (3)$$

From the constraint equation (1), we can derive an objective function as

$$f(\mathbf{x}_2, \dots, \mathbf{x}_n) = \sum_{ij} a_{ij} \left(\frac{\mathbf{x}_j - \mathbf{x}_i}{\|\mathbf{x}_j - \mathbf{x}_i\|} - \mathbf{p}_{ij} \right)^2 \quad (4)$$

or

$$f(\mathbf{x}_2, \dots, \mathbf{x}_n) = \sum_{ij} a_{ij} (\mathbf{x}_j - \mathbf{x}_i - \mathbf{p}_{ij})^2 \quad (5)$$

which would also punish node distances differing from 1. Other objective functions might be possible.

Since we now have implicitly assumed $\nu = 0$ for all nodes, solutions cannot be rotated. The only degree of freedom remaining is shift, which is dealt with by setting $\mathbf{x}_1 = (0, 0)'$.

3 Analytical solution for the non-normalized case

For the non-normalized case, Eq. 5, an analytical solution might be possible. The ideal solution should satisfy the equations

$$a_{ij}\mathbf{p}_{ij} = \mathbf{x}_j - \mathbf{x}_i \quad \text{for all } i \neq j. \quad (6)$$

This system of equation can be expressed in matrix form as

$$\begin{pmatrix} a_{12}\mathbf{p}_{12}^\top \\ a_{13}\mathbf{p}_{13}^\top \\ a_{14}\mathbf{p}_{14}^\top \\ \vdots \\ a_{1n}\mathbf{p}_{1n}^\top \\ \hline \vdots \\ \hline a_{n1}\mathbf{p}_{n1}^\top \\ a_{n2}\mathbf{p}_{n2}^\top \\ a_{n3}\mathbf{p}_{n3}^\top \\ \vdots \\ a_{n,n-1}\mathbf{p}_{n,n-1}^\top \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ -1 & 0 & 0 & 0 & \dots & 1 \\ \hline \vdots & & & & & \vdots \\ \hline 1 & 0 & 0 & \dots & 0 & -1 \\ 0 & 1 & 0 & \dots & 0 & -1 \\ 0 & 0 & 1 & \dots & 0 & -1 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{x}_1^\top \\ \vdots \\ \mathbf{x}_n^\top \end{pmatrix} \quad (7)$$

for which we write in short $P = GX$. P is $n(n-1) \times 2$, M is $n(n-1) \times n$ and X is $n \times 2$.

The least square solution for this exists if the $n \times n$ matrix $M^\top M$ is invertible and will then be given as

$$(M^\top M)^{-1}P = X \quad (8)$$

If $M^\top M$ is singular, we might try to delete the first row of M and reduce X to $(\mathbf{x}_2, \dots, \mathbf{x}_n)$, as discussed above.

4 Next Steps

- Try out the objective function with simple graphs (chains, triangles, small loops, 4-cliques etc)
- Think about analytical approaches to minimizing the objective function
- Check MDS literature on related problems. (One is Hübner & Mallot, Autonomous Robots 2007, but there the angles are local, not with respect to a global reference direction. Still. the idea of an objective function working on triples rather than pairs may be useful.)