Unwrapping Graphs with Local Metrics

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BSc Project for Jakob Meyer, September 2020

1 Problem

View-graphs with local metric are generated as representations of space in the PhD project of Tristan Baumann. The nodes of these graphs are small image features ("micro-snapshots") numbered $i = 1, ..., n$ with adjacencies $a_{ij} \in [0, 1]$. Each link is labeled with an angle ϕ_{ij} specifying the direction of node j as seen from node i, relative to a reference direction ν_i. During exploration of the environment, this agent continuously updates the reference direction and stores it with the node as a label. The updating process in error-prone.

2 Approach

The unwrapping problem can be stated as follows: Given an adjacency matrix $A = \{a_{ij}\}$, a set of connection angles ϕ_{ij} for all pairs with $a_{ij} \neq 0$, and (maybe) the local assumed reference directions ν_i, find an embedding $(\mathbf{x}_1, ...\mathbf{x}_n), \mathbf{x}_i \in \mathrm{IR}^2$ of the nodes satisfying

$$\mathbf{x}_j - \mathbf{x}_i \approx \lambda \begin{pmatrix} \cos \phi_{ij} \\ \sin \phi_{ij} \end{pmatrix} \tag{1}$$

where λ is the distance between the two nodes' embeddings.

In this constraint, ν_i does not show up. This is to say that it is implicitly set to zero. Once an embedding is found, we could say that ν_i has been rotated by the angle

$$\nu_i^* = \frac{\sum_j a_{ij}(\arctan(\mathbf{x}_j - \mathbf{x}_i) - \phi_{ij})}{\sum_j a_{ij}} \tag{2}$$

where the denominator is the out-degree of node i.

We now denote the unit vector in direction ϕ_{ij} as $\mathbf{p}_{ij}$, i.e.,

$$\mathbf{p}_{ij} := \begin{pmatrix} \cos \phi_{ij} \\ \sin \phi_{ij} \end{pmatrix}. \tag{3}$$

From the constraint equation (1), we can derive an objective function as

$$f(\mathbf{x}_2, ..., \mathbf{x}_n) = \sum_{ij} a_{ij} \left(\frac{\mathbf{x}_j - \mathbf{x}_i}{\|\mathbf{x}_j - \mathbf{x}_i\|} - \mathbf{p}_{ij} \right)^2 \tag{4}$$

or

$$f(\mathbf{x}_2, ..., \mathbf{x}_n) = \sum_{ij} a_{ij} \left(\mathbf{x}_j - \mathbf{x}_i - \mathbf{p}_{ij} \right)^2 \tag{5}$$

which would also punish node distances differing from 1. Other objective functions might be possible.

Since we now have implicitly assumed $\nu = 0$ for all nodes, solutions cannot be rotated. The only degree of freedom remaining is shift, which is dealt with by setting $\mathbf{x}_1 = (0, 0)'$.

3 Analytical solution for the non-normalized case

For the non-normalized case, Eq. 5, an analytical solution might be possible. The ideal solution should satisfy the equations

$$a_{ij}\mathbf{p}_{ij} = \mathbf{x}_j - \mathbf{x}_i \quad \text{for all} \quad i \neq j. \tag{6}$$

This system of equation can be expressed in matrix form as

$$\left(\begin{array}{c} a_{12}\mathbf{p}_{12}^\top \\ a_{13}\mathbf{p}_{13}^\top \\ a_{14}\mathbf{p}_{14}^\top \\ \vdots \\ a_{1n}\mathbf{p}_{1n}^\top \\ \hline \vdots \\ \hline a_{n1}\mathbf{p}_{n1}^\top \\ a_{n2}\mathbf{p}_{n3}^\top \\ a_{n3}\mathbf{p}_{n3}^\top \\ \vdots \\ a_{n,n-1}\mathbf{p}_{n,n-1}^\top \end{array}\right) = \left(\begin{array}{cccccc} -1 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ -1 & 0 & 0 & 0 & \dots & 1 \\ \hline \vdots & & & & & \vdots \\ \hline 1 & 0 & 0 & \dots & 0 & -1 \\ 0 & 1 & 0 & \dots & 0 & -1 \\ 0 & 0 & 1 & \dots & 0 & -1 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{array}\right) \left(\begin{array}{c} \mathbf{x}_1^\top \\ \vdots \\ \mathbf{x}_n^\top \end{array}\right) \tag{7}$$

for which we write in short $P = MX$. P is $n(n-1)\times 2$, M is $n(n-1)\times n$ and X is $n \times 2$. For the construction of the matrix, we will need the index functions

$$i(k) = \text{ceiling}(k/(n-1)) \quad \text{and} \quad j(k) = \begin{cases} \text{mod}(k, n-1) & \text{for} \quad \text{mod}(k, n-1) < i(k) \\ 1 + \text{mod}(k, n-1) & \text{for} \quad \text{mod}(k, n-1) \geq i(k) \end{cases} \tag{8}$$

such that $P_k = a_{i(k),j(k)}\mathbf{p}_{i(k),j(k)}^\top$, $M_{kl} = -1$ for $i(k) = l$, $M_{kl} = 1$ for $j(k) = l$, and $M_{kl} = 0$ otherwise.

The least square solution for this exists if the $n \times n$ matrix $M^\top M$ is invertible and will then be given as

$$(M^\top M)^{-1} M^\top P = X \tag{9}$$

If $M^\top M$ is singular, we might try to delete the first row of M and reduce X to $(\mathbf{x}_2, ..., \mathbf{x}_n)$, as discussed above.

4 Next Steps

- Try out the objective function with simple graphs (chains, triangles, small loops, 4-cliques etc)
- Think about analytical approaches to minimizing the objective function
- Check MDS literature on related problems. (One is Hübner & Mallot, Autonomous Robots 2007, but there the angles are local, not with respect to a global reference direction. Still. the idea of an objective function working on triples rather than pairs may be useful.)