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Minimizing Angular Error in Labeled Graphs

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Summary

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Abstract

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Thanksgiving

I want to thank my parents for having me as a child and not leave me hanging despite many moments of unyielding stubbornness, which made me grow up in a safe and supporting environment.

I want to thank my siblings for getting on my nerve all the time as kids, which made me become a human able to withstand hardships in any phases of my young life as well as looking at life from the bright side. As a young adult i might appreciate them ever so slightly more as we are growing older each day for that.

I want to thank all the people i have met through my life, mostly the people who took up with me for quite some time or at an intense level of love over maybe not so long a time that would have been wished for.

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Chapter 1

Introduction

As far as the field of robotics goes back in time, a prime goal has been not only to model and build robots and machines that can accomplish useful tasks that might be of too much work for a mere human to do, but to build and reconstruct the human person itself. From a biological or chemical perspective, there is no clear-cut distinction of external stimuli being received, the forwarding of this information as electrical impulses and its effect on an abundant number of neurons in the brain. Every 'step' in this information processing is in one form or the other done on a purely electrical basis. While perception and processing go wholly intertwined in the human brain, to model what is going on, we need to break down the vastly complex actions our body and brain are taking into small chunks ready to be analyzed and made sense of. This is true as well for the understanding of spatial representations of our surrounding world in our brain. How can we orientate our own person in surroundings familiar to us, if desired without help of our visual sense? Do we have integrated a map of named familiar surroundings embedded in another, bigger map of our whole ever perceived world? If so, how is the information of this map stored?

One probably has wandered around in the dark in his or her own house, myself for certain on multiple occasions. In most of these times I had no trouble finding the first step up or down the stairs or avoiding walking into the next wall head first, maybe while holding some beverage for an late hour working session. While pleasant, most of the time it's not worth mentioning or boasting of. Or, at least, so I tend to think. Admittedly, boasting should not be necessary, but mentioning and thinking about what underlies this capability of orientation on the other hand is quite interesting.

While one does not see its surroundings in the pitch black (let's assume with closed eyes, as otherwise the next light source out the window might shed enough light to grasp some visual information) this does not mean that same person is without sensory input about it. A scratch in a wooden floorboard, the ticking of some well-placed clock or even possibly the scent of previously brewed coffee on the kitchenette might give cues about the whereabouts of ones self. And cues as sensory input reach us all the time.

In this thesis, what is of interest here, is how we as humans do use cues of our surroundings to update our (spatial) representation of precisely these surroundings...

Chapter 2

Theory

In this chapter we aspire to find a reliable way of calculating good embeddings of angle-labeled graphs. The term "good embedding" is open to definition and will therefore be investigated in the first section sec:embedding.

The visualization and calculation of aspired embeddings presupposes a mathematical background, that will be established in section 2.1. With this fundamentals, the process of calculating embeddings can be stated and differentiated in sections 2.3 and 2.4.

At the end of this chapter (section 2.5) we test our theoretical procedure of finding "good embeddings" for error-prone view-graphs. As this thesis takes up on the PHD-thesis of Baumann (2019), its final view-graph will be amongst procedurally generated graphs subject to this test (see figure 2.1).

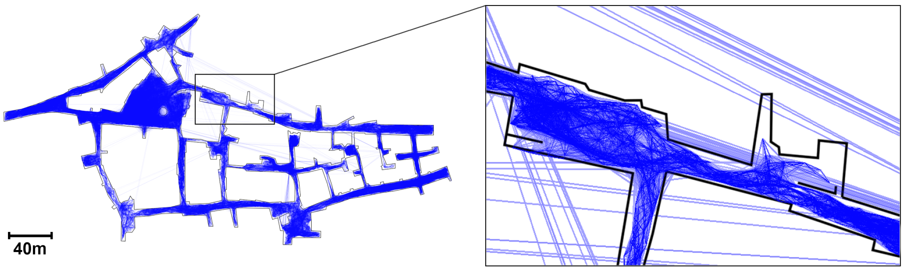

Figure 2.1: Caption and picture by courtesy of Tristan Baumann: Map of the testing environment "Virtual Tuebingen" with view graph. The view graph can be embedded into a map by placing each feature at the agent's position from where it was first detected, and drawing the graph's edges between them (blue lines). The shown graph completely maps the virtual environment and consists of 222,433 nodes and 3,492,096 edges. Some of the edges connect very distant features (long blue lines crossing the empty white space). These are wrong connections resulting from aliasing.

The overarching theoretical substance developed in this chapter will finally be applied for the analysis of a small survey in chapter 3.

2.1 Mathematical Background

TODO: (cite wikipedia? https://en.wikipedia.org/wiki/Graph_embedding) For this thesis only a small portion of graph theory is needed and all relevant subjects for this thesis will be established in this section. For more detail upon this subject see the preliminaries in Pich (2009).

A graph is a mathematical construct of the form of an ordered pair

$$G = (V, E) \tag{2.1}$$

which consists of a set of vertices V and a set of edges E connecting vertices in a pairwise fashion. As we want to represent our graphs as directed, we define E as a subset of the cartesian product over the set of vertices V.

$$E \subseteq \{(v_1, v_2) | (v_1, v_2) \in V^2 \text{ and } v_1 \neq v_2\} \tag{2.2}$$

Let us denote here the number of vertices and edges in the graph as

$$\mathbf{n} \coloneqq |V| \tag{2.3}$$

$$\mathbf{e} \coloneqq |E| \tag{2.4}$$

An *embedding* of a graph does represent the mathematical structure of G in space Σ, e.g. it associates vertices of G with points in Σ and edges of G with arcs or straight lines connecting these points. As the view-graphs we want to analyse are basically top-down seen maps of the environment, we limit the surface for embedding our graphs to the 2-dimensional Euclidean space $\mathbb{R}^2$. Mark here, that one graph can have an infinite number of different embeddings. The art of finding an aesthetic embedding for a given graph, from which one can easily deduce some of the graphs properties, is named the field of study of *graph drawing*.

Taking up the work of Baumann (2019), we define a global reference direction ν, which can be seen as the objective *"north"* direction of a compass. We extend the vertices v_i of a given graph G by a local reference direction ϕ_i. This local reference direction can be thought of as the subjective estimate of ν at the given point. Deviations of ϕ from ν in a mapping process are due to accumulating errors as occurring in most of the literature on *simultaneous learning and mapping (SLAM)*. In Baumann (2019) the accumulating deviation from the global reference direction is simulated through ϕ, which is updated while the agent explores the environment and is stored as ϕ_i with a vertex v_i upon its finding in the mapping process.

2.2 Finding embeddings

2.2.1 Problem

Given a graph $G = (V, E)$ with adjacency matrix $A = \{a_{ij}\}$, a set of connection angles ϕ_{ij} for all pairs $(v_i, v_j) \in E$ (or $a_{ij} \neq 0$) and the local reference directions ν_i, find an embedding $\mathbf{X} = (\mathbf{x}_1\mathbf{x}_2 \dots \mathbf{x_n})^T \in \mathbb{R}^{\mathbf{n}\times 2}, \mathbf{x}_i \in \mathbb{R}^2$ for the vertices satisfying

$$\begin{aligned} \vec{\mathbf{x}}_{ij} &\approx \lambda_{ij} \cdot \hat{\mathbf{p}}_{ij} \\ \iff \hat{\mathbf{x}}_{ij} &\approx \hat{\mathbf{p}}_{ij} \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \vec{\mathbf{x}}_{ij} &\coloneqq \mathbf{x}_j - \mathbf{x}_i \\ \lambda_{ij} &\coloneqq \left\| \mathbf{x}_j - \mathbf{x}_i \right\| \\ \hat{\mathbf{x}}_{ij} &\coloneqq \frac{\vec{\mathbf{x}}_{ij}}{\lambda_{ij}} \end{aligned} \tag{2.6}$$

as well as the unit vector in direction ϕ_{ij} being denoted as

$$\hat{\mathbf{p}}_{ij} \coloneqq \begin{pmatrix} \cos \phi_{ij} \\ \sin \phi_{ij} \end{pmatrix} \tag{2.7}$$

Notably, equation 2.5 only exists for such $i, j \in \{1, ..., n\}$ for which $a_{ij} \neq 0$ and, therefore, ϕ_{ij} is given. Whenever $\hat{\mathbf{p}}_{ij}$ is used in the following, this exact notion would to have be implicit, which makes of poor mathematical syntax. Therefore, we establish

$$(v_i, v_j) \notin E \implies \phi_{ij} = 0 \tag{2.8}$$

2.2.2 Objective Functions

TODO: Mallot hat hier x_2 genommen, statt x_1, um direkt anzudeuten, dass x_1 auf 0 liegt...
TODO: Das Quadrat wird im folgenden als die squared magnitude beschrieben, vielleicht sollte ich das nochmal mit Tristan und Mallot absprechen, ob das so gemeint war, außerdem vielleicht dafür ein Symbol einführen?
From the constraint equation 2.5 we can derive objective functions

$$\omega = \Omega(G, \mathbf{X}) \tag{2.9}$$

such as

$$\Omega_{init}(G, \mathbf{X}) = \sum_{ij} a_{ij} \left(\vec{\mathbf{x}}_{ij} - \hat{\mathbf{p}}_{ij} \right)^2 \tag{2.10}$$

$$\Omega_{norm}(G, \mathbf{X}) = \sum_{ij} a_{ij} \left(\hat{\mathbf{x}}_{ij} - \hat{\mathbf{p}}_{ij}\right)^2 \tag{2.11}$$

that assign a single numerical value, the *objective value* ω, to one specific graph embedding $\mathbf{X}$ of a given graph G. ω gives (regardless of the specific Ω) a measure of how well the given embedding does in minimizing angular errors of the angle-labeled graph. Which objective value is implying a good embedding (see subsection 2.2.4) is dependent on the Ω it is calculated upon.

To get a straightforward assessment of the objective value in the normalized case of equation 2.11, we can fit this function to a range of values $W = [0, 1]$, basically resulting in an additional objective function. As $\hat{\mathbf{x}}_{ij}$ and $\hat{\mathbf{p}}_{ij}$ are both unit vectors with length one, equation 2.11 can be simplified through equation 2.12, which states as follows:

Be s,t unit vectors $\in \mathbb{R}^2$ and the scalar product given as $\langle s, t\rangle \in [-1, 1]$, then

$$\begin{aligned} \|s-t\|^2 &= (s_x - t_x)^2 + (s_y - t_y)^2 \\ &= s_x^2 - 2s_xt_x + t_x^2 + s_y^2 - 2s_yt_y + t_y^2 \\ &= \|s\|^2 + \|t\|^2 - 2\cdot(s_xt_x + s_yt_y) \\ &= 2\cdot(1 - \langle s, t\rangle) \end{aligned} \tag{2.12}$$

It follows that

$$\Omega_{norm}(G, \mathbf{X}) = \sum_{ij} a_{ij} \left\|\hat{\mathbf{x}}_{ij} - \hat{\mathbf{p}}_{ij}\right\|^2 \tag{2.13}$$

$$\iff \quad \Omega_{norm}(G, \mathbf{X}) = \sum_{ij} 2a_{ij} \left(1 - \langle \hat{\mathbf{x}}_{ij}, \hat{\mathbf{p}}_{ij}\rangle\right) \in [0, 4\mathbf{e}] \tag{2.14}$$

$$\iff \quad \frac{\Omega_{norm}(G, \mathbf{X})}{2} = \mathbf{e} - \sum_{ij} a_{ij} \langle \hat{\mathbf{x}}_{ij}, \hat{\mathbf{p}}_{ij}\rangle \quad \in [0, 2\mathbf{e}] \tag{2.15}$$

$$\iff \quad 1 - \frac{1}{2\mathbf{e}} \cdot \Omega_{norm}(G, \mathbf{X}) = \frac{\sum_{ij} a_{ij} \langle \hat{\mathbf{x}}_{ij}, \hat{\mathbf{p}}_{ij}\rangle}{\mathbf{e}} \quad \in [-1, 1] \tag{2.16}$$

$$\iff \quad \left|1 - \frac{1}{2\mathbf{e}} \cdot \Omega_{norm}(G, \mathbf{X})\right| = \left|\frac{\sum_{ij} a_{ij} \langle \hat{\mathbf{x}}_{ij}, \hat{\mathbf{p}}_{ij}\rangle}{\mathbf{e}}\right| \quad \in [0, 1] \tag{2.17}$$

TODO: Can the norm of the last equation be taken so the range goes from [0,1] with 1 being good and 0 bad embeddings? 0 would be 90° across all scalar products, whereas 1 would mean either overlapping (0°) or mirrored (180°) bad embeddings. The latter might be swapped along the north direction to get the "more positive" partner embedding...?
which we will denote as

$$\Omega_{fitted}(G, \mathbf{X}) = \left|\frac{\sum_{ij} a_{ij} \langle \hat{\mathbf{x}}_{ij}, \hat{\mathbf{p}}_{ij}\rangle}{\mathbf{e}}\right|, W = [0, 1] \tag{2.18}$$

2.2.3 Visualization

To get a straightforward understanding of what these objective functions actually compute, let us visualize a simple graph embedding and the objective value in figure 2.2.

2.2.4 "Good" embeddings

As briefly stated in subsection 2.2.2, the meaning of a given objective value ω depends on the objective function Ω its calculated from. For example, an objective value of $\omega_{fitted} = 0$ would describe a *worst possible embedding* of perpendicularity between each pair of $(\hat{\mathbf{x}}_{ij}, \hat{\mathbf{p}}_{ij})$ of a given graph G, whereas $\omega_{fitted} = 1$ would announce its perfect embedding. On the other hand, the perfect embedding for Ω_{norm} would be described by the value $\omega_{norm} = 0$. And for Ω_{init} there is no value for the *worst possible embedding.*

As can be seen, the notion of ω is not unified and has to be interpreted for each objective function differently. In addition, the construction of different objective functions can be founded on any measurable criterion of a graph embedding (not for the graph itself, as we look for the best embedding of one fixed graph). In our objective functions thus far we used as concept the *spatial* distances between "should be" and "in fact" positions of the neighbouring vertices (see figure 2.2). Additional concepts could be *angular*, which evaluates the angular error rather than the distance, or *areal*, which spans areas by means of the vector(cross?)-product and thus combines the spatial and angular concepts.

2.2.5 Analytical solution for the non-normalized case

For the non-normalized version of the objective function in equation 2.10, the ideal solution should satisfy

$$\hat{\mathbf{p}}_{ij} = \vec{\mathbf{x}}_{ij} \quad \text{only when } a_{ij} = 1 \tag{2.19}$$

For the construction of the respective matrix form we will need the indexing functions

$$i(k) = \left\lceil \frac{k}{\mathbf{n} - 1} \right\rceil \tag{2.20}$$

$$j(k) = \begin{cases} \mathrm{mod}(k, \mathbf{n} - 1) & \text{for} \quad \mathrm{mod}(k, \mathbf{n} - 1) < i(k) \\ 1 + \mathrm{mod}(k, \mathbf{n} - 1) & \text{for} \quad \mathrm{mod}(k, \mathbf{n} - 1) \geq i(k) \end{cases} \tag{2.21}$$

With these we formulate equation 2.19 as

$$P = M\mathbf{X} \tag{2.22}$$

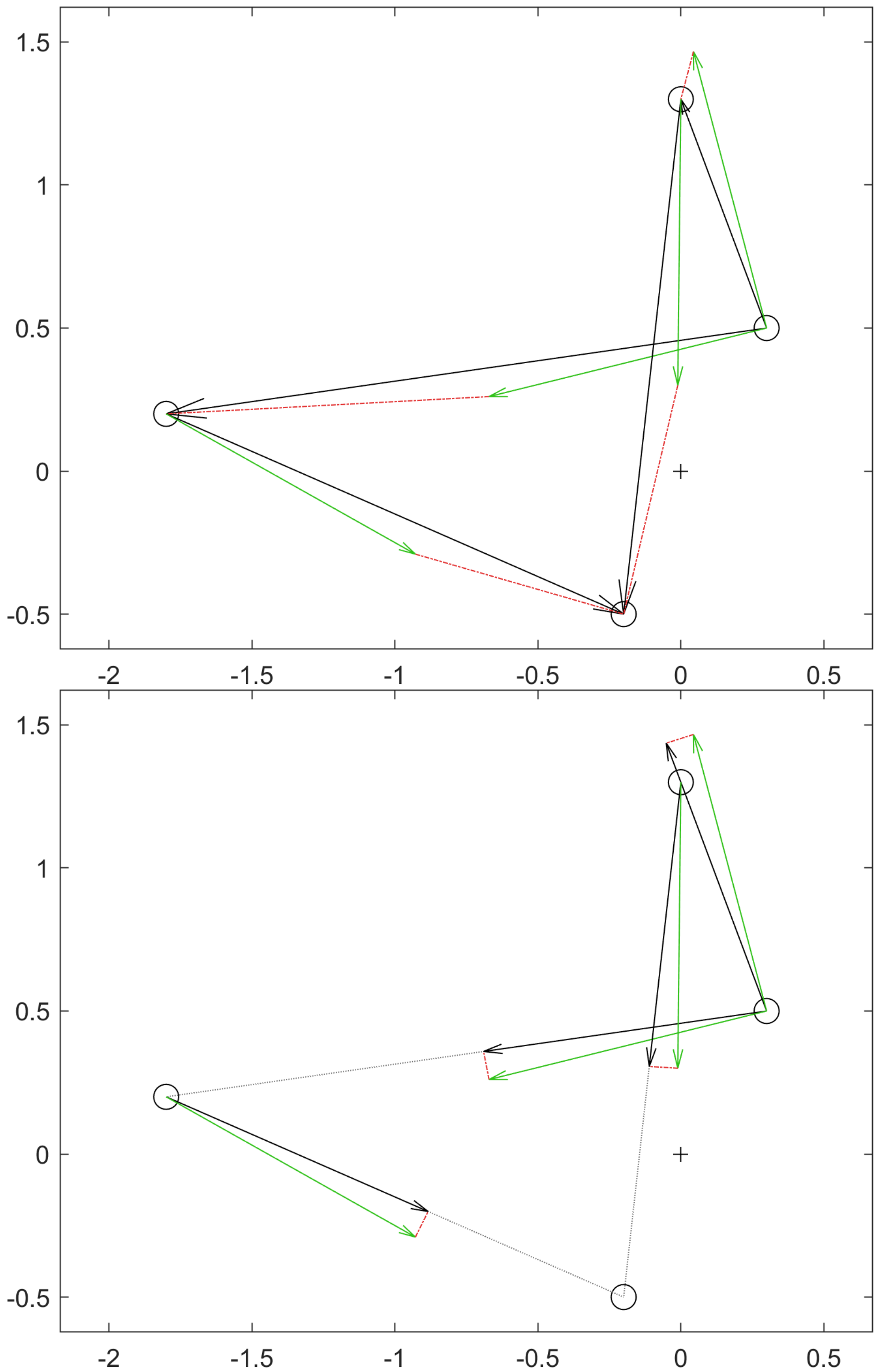

Figure 2.2: A random embedding of a directed graph with 4 vertices, its edges (black arrows) and the respective $\hat{\mathbf{p}}_{ij}$ unit vectors (green arrows) are drawn. These unit vectors are derived from the angular labels ϕ_{ij} of each vertex. **(top)** The differences between $\hat{\mathbf{p}}_{ij}$ and the non-normalized vector $\vec{\mathbf{x}}_{ij}$ are drawn as dotted red lines. This can be seen as the visualization of Ω_{init}. **(bottom)** Here the differences as dotted red lines are of the respective unit vectors $\hat{\mathbf{x}}_{ij} - \hat{\mathbf{p}}_{ij}$ instead. Ω_{norm} is visualized here.

where $\forall\ k \in \{1, 2, ..., \mathbf{n}(\mathbf{n}-1)\}$:

$$a_{i(k),j(k)} \neq 0 \implies P_k = \hat{\mathbf{p}}^T_{i(k),j(k)} \quad \text{and} \quad M_{kl} = \begin{cases} -1 & \text{for } i(k) = l \\ 1 & \text{for } j(k) = l \\ 0 & \text{otherwise} \end{cases} \tag{2.23}$$

The dimensions of the three matrices are $P \in \mathbb{R}^{\mathbf{e}\times 2}$, $M \in \mathbb{R}^{\mathbf{e}\times\mathbf{n}}$, $\mathbf{X} \in \mathbb{R}^{\mathbf{n}\times 2}$.

With the established established construction of the matrices, we are able to compute directly the optimal embedding minimizing distances between $\hat{\mathbf{p}}_{ij}$ and $\vec{\mathbf{x}}_{ij}$ by means of the *method of least sqaures* (see ()). For this method the $\mathbf{n} \times \mathbf{n}$ matrix $M^T M$ has to be invertible. If it is singular instead, we remove $\mathbf{x}_1$ from X as well as clear the matrix M of its first column, which will result in an implicit fixing of $\mathbf{x}_1$ to point $(0, 0)$.

The *least sqaure solution* then is given by

$$\mathbf{X}^* = (M^T M)^{-1} P \tag{2.24}$$

2.3 Embedding calculation and assessment

2.4 Embedding with Normalization

2.5 Applying the Theory

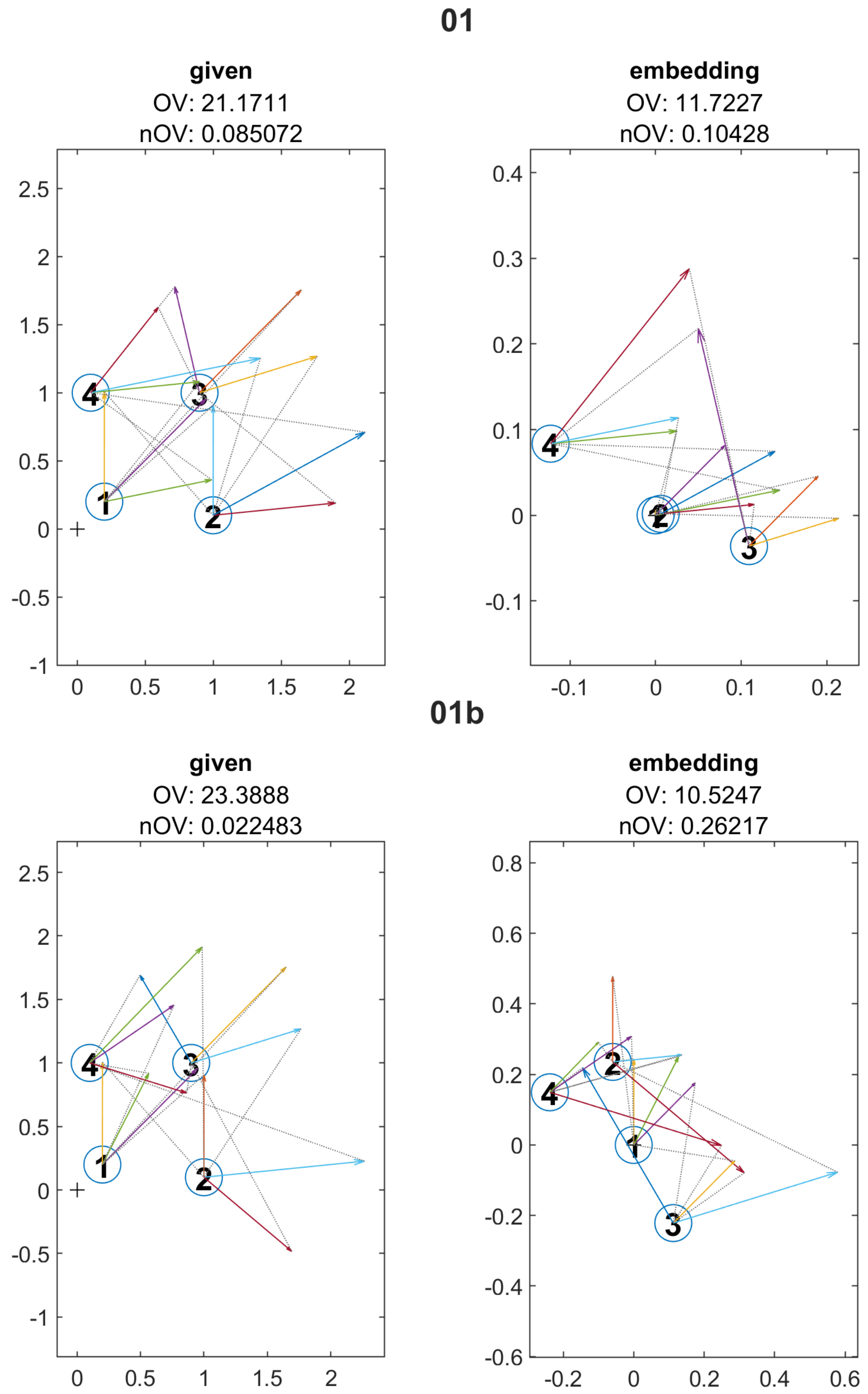

Figure 2.3: Comparisons of example graphs **01** and **01b** as given (left column) and the optimized embeddings calculated minimizing the objective function (right column). **OV** and **nOV** refer to the objective values in non-normalized and normalized fashion respectively. Between **01** and **01b** only the angular values were tweaked in a random fashion.

Chapter 3

Practice

To apply the theoretical mechanisms of the previous chapter, a survey of angular perception in two loci was conducted. In the following this survey will be described and hereinafter its data analyzed.

3.1 Survey design

The survey was constructed as a travel-and-measure procedure and did take place either in Tuebingen, Baden-Wuerttemberg or around my homeplace of Loeschenmuehle, Bavaria. Starting at a specific point X_0, the participants were given the true north direction ν as well as a list of n locations in the near surroundings by name. These locations along with the starting point will be called *landmarks* and denoted the letter X.

$$X = \{X_i | X_i \text{ for } i \in \{0, ..., n\}\} \tag{3.1}$$

The maximum distance of the landmarks from the starting point was in Tuebingen ... km and around Loeschenmuehle ...km. A list of used landmarks as well as their position on a map of the respective surroundings is given in the appendix.

In the first measurement the participants, currently at the starting point, were to give an estimate of the angular direction for each other landmark.

$$\alpha_0 = (\alpha_{01}, ..., \alpha_{0n})^T \tag{3.2}$$

After that, the participants did wander from one landmark to the next in fashion of the shortest path, while at every landmark measuring again the estimated angular direction towards each other landmark.

$$\alpha_i = (\alpha_{ij})^T \text{ with } i \neq j \text{ and } i, j \in \{0, ..., n\} \tag{3.3}$$

In addition, after finishing these measurements at the current landmark X_i, the participant was to give an estimate of the north direction ϕ_i. Upon finishing measurements at every landmark, the participant was to do one last measurement again at the starting point. If for any reason a participant was not familiar with a given landmark, this landmark was included in measuring only after its visit on the travel route.

All angular directions for landmarks were measured by the participant itself by means of an analog compass (compass app?). The north direction estimate was signaled through an outstretched arm and compared to the true north by the experimental head, my humble self, again by means of the same compass.

3.2 Data presentation

3.3 Conclusion

References

Baumann, T. (2019). *Dual population coding for path planning in graphs with overlapping place representations* (Unpublished doctoral dissertation). University of Tuebingen, Germany, ... (to be filled out)

Pich, C. (2009). *Applications of multidimensional scaling to graph drawing* (dissertation). University of Konstanz.

Selbstständigkeitserklärung

Hiermit versichere ich, dass ich die vorliegende Bachelorarbeit selbstständig und nur mit den angegebenen Hilfsmitteln angefertigt habe und dass alle Stellen, die dem Wortlaut oder dem Sinne nach anderen Werken entnommen sind, durch Angaben von Quellen als Entlehnung kenntlich gemacht worden sind.
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