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Constructing graphs with given spectrum and the spectral radius at most 2

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ABSTRACT

Two graphs are cospectral if their spectra coincide. The set of all graphs that are cospectral to a given graph, including the graph by itself, is the cospectral equivalence class of the graph. We say that a graph is determined by its spectrum, or that it is a DS-graph, if it is a unique graph having that spectrum. Given n reals belonging to the interval $[-2, 2]$, we want to find all graphs on n vertices having these reals as the eigenvalues of the adjacency matrix. Such graphs are called Smith graphs. Our search is based on solving a system of linear Diophantine equations. We present several results on spectral characterizations of Smith graphs.

1. Introduction

In this paper, we consider only finite undirected simple graphs, i.e. graphs without loops or multiple edges. Let G be a simple graph on n vertices (or a graph of order n), and with the adjacency matrix A. The *characteristic polynomial* $P_G(x) = \det(xI - A)$

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of G is the characteristic polynomial of its adjacency matrix A. The eigenvalues of A, in non-increasing order, are denoted by $\lambda_1(G), \ldots, \lambda_n(G)$. They are called the *eigenvalues* of G and they form the *spectrum* of G. The multiplicity k of the eigenvalue λ_i in the spectrum of G will be denoted by $[\lambda_i]^k$. Since A is real and symmetric, the spectrum of G consists of reals. In particular, $\lambda_1(G)$, as the largest eigenvalue of G, is called the *spectral radius* (or *index*) of G.

The spectrum of G (as a multi-set or family of reals) will be denoted by $\widehat{G}$. The *disjoint union* of graphs G_1 and G_2 will be denoted by $G_1 + G_2$, while for the union of their spectra (i.e. the spectrum of $G_1 + G_2$) we will use the following mark $\widehat{G}_1 + \widehat{G}_2$. In the similar manner, kG ($k\widehat{G}$) stands for the union of k copies of G (the spectrum of kG). In this sense, we shall also use linear combinations of graphs and of their spectra.

The problem of determining graphs by spectral means is one of the oldest problems in the spectral graph theory. This problem is studied in the literature for various kinds of graph spectra (i.e. based on different types of graph matrices). Here we have in mind the adjacency matrix.

Two graphs G_1 and G_2 are *cospectral*, denoted by $G_1 \sim G_2$, if their spectra coincide. It is obvious that $\sim$ is an equivalence relation on the set of all graphs. So, by $[G]$ we can denote the *cospectral equivalence class* determined by G under $\sim$, i.e. the set of all graphs cospectral to G. Such a set includes G due to the reflexivity of the relation $\sim$. A graph G is said to be *determined by its spectrum* if whenever there is a graph H which is cospectral to a graph G, then H is isomorphic to G. With other words, a graph is determined by its spectrum if it is a unique graph having that spectrum. As in [6] or [13], we use mark *DS-graph* (or *not DS-graph*) to indicate that some graph is determined (or is not determined) by its spectrum. We can notice that a graph G is a DS-graph if and only if its cospectral equivalence class $[G]$ contains only one element – graph G itself, i.e. $[G] = \{G\}$. Many results on spectral characterizations of graphs can be found in [6]. For early results see [2].

In this paper, we consider the class of graphs whose spectral radius is at most 2. This class includes, for example, the graphs whose each component is either a path or a cycle. All graphs with the spectral radius at most 2 have been constructed by J.H. Smith [12]. Therefore these graphs are usually called the *Smith graphs*. Eigenvalues of these graphs have been determined in [1] and [3]. In [3], it has been proved that all eigenvalues of these graphs are of the form $2\cos\frac{p}{q}\pi$, where p, q are integers and $q \neq 0$.

A path (cycle) on n vertices will be denoted by P_n (resp. C_n). A connected graph with index ≤ 2 is either a cycle C_n ($n = 3, 4, \ldots$), or a path P_n ($n = 1, 2, \ldots$), or one of the graphs depicted in Fig. 1 (see [12]). Note that W_1 coincide with the star $K_{1,4}$, while Z_1 with P_3. In addition (see [12]), the graphs C_n ($n \geq 3$), W_n ($n \geq 2$), $K_{1,4}$, T_4, T_5, and T_6 are connected graphs with index equal to 2. Graphs P_n ($n \geq 1$), Z_n ($n \geq 2$), T_1, T_2 and T_3 are connected graphs with index less than 2. The graph Z_n is called a *snake*, while W_n is a *double snake*. We denote the set of all these graphs by $\mathfrak{S}^*$. The set of those that are bipartite (so odd cycles are excluded), will be denoted by $\mathfrak{S}$. The spectrum of each

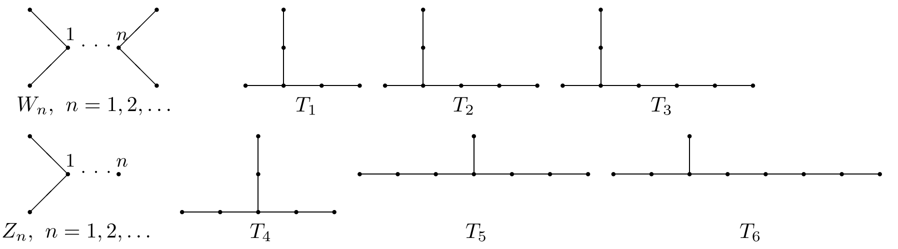

Fig. 1. Some of the Smith graphs.

graph from $\mathfrak{S}^*$ can be found (in an explicit form) in [3]. In this paper, we will consider graphs from the set $\mathfrak{S}$.

Two important problems arise in the field of spectral graph theory. Identification of DS-graphs as well as determination of the cospectral equivalence class of a given graph are both the challenging tasks for the researchers. Nevertheless, according to [13], it seems more difficult to determine the cospectral equivalence class of a given graph than prove it to be a DS-graph. In this paper, we will try to give responses to the both questions for Smith graphs by solving a system of linear Diophantine equations. In that sense, this paper can be considered as a continuation of the research initiated in [3], and further extended in [5]. Some related results on spectral determination of Smith graphs are obtained also in the papers [9,11,13].

The rest of the paper is organized as follows. Section 2 contains some preliminary results related to a system of linear Diophantine equations previously exposed in [3]. By solving such a system one can obtain all graphs from the set $\mathfrak{S}$ (and similarly $\mathfrak{S}^*$), whose spectra coincide with the given symmetric system of real numbers. On that way, just by analysing the corresponding system, we consider spectral characterizations of graphs $Z_n + P_1$ and $C_{2n} + P_1$ in Section 3. We determine the cospectral equivalence class of graph $T_5 + T_6$, as well as the cospectral equivalence classes of graphs $W_1 + T_4$ and $W_1 + T_5$. Finally, in Appendix we list the full form of the systems of linear Diophantine equations associated to some of the considered graphs.

2. Basic equations

Let G be an arbitrary graph whose each component is a path or a cycle. Then we can write:

$$G = \sum_{i \geq 1} m_i P_i + \sum_{j \geq 3} n_j C_j, \tag{1}$$

where $m_i, n_j \geq 0$ are integers. More generally, if G is any graph whose each component belongs to $\mathfrak{S}^*$, we can write

$$G = \sum_{S \in \mathfrak{S}^*} r(S) S, \tag{2}$$

where $r(S) \geq 0$ is a repetition factor, that tells us how many times S is appearing as a component in G. In the sequel, a graph G will be frequently represented in the form (1) or (2).

The repetition factor $r(S_i)$ of some of the graphs $S_i \in \mathfrak{S}^*$ for any relevant index i will be denoted by s_i. So, according to the introduced marks for graphs from the set $\mathfrak{S}$, we have the following non-negative integers:

$$p_1, p_2, p_3, \ldots, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, t_1, t_2, t_3, t_4, t_5, t_6.$$

We have omitted z_1 since Z_1 and P_3 represent the same graph, and therefore the variable p_3 is relevant.

We shall use $c_2, c_3, \ldots,$ for repetition factors of the even cycles $C_4, C_6, \ldots$. For non-bipartite graphs from $\mathfrak{S}^*$ one can introduce variables $o_3, o_5, o_7, \ldots$ counting the numbers of odd cycles $C_3, C_5, C_7, \ldots$.

For a given graph $G \in \mathfrak{S}^*$ the above variables which do not vanish, together with their values, are called *parameters* of G. Parameters of a graph indicate the actual number of components of particular types present in G.

Let us consider only bipartite graphs whose spectral radius is at most 2, i.e. graphs from the set $\mathfrak{S}$. It is well known (see, for example, Theorem 3.2.3 from [4]) that bipartite graphs have a symmetric spectrum with respect to the zero point. As we pointed out in the introductory part, it has been proved in [3] that all eigenvalues of any graph from the set $\mathfrak{S}$ are of the form $2\cos\frac{p}{q}\pi$, for some integers p and q, such that $q \neq 0$. Knowing the spectra of graphs from the set $\mathfrak{S}^*$ allowed construction of the following pairs of cospectral non-isomorphic graphs in [3]:

$$\begin{aligned}
\widehat{W}_n &= \widehat{C}_4 + \widehat{P}_n, \\
\widehat{Z}_n + \widehat{P}_n &= \widehat{P}_{2n+1} + \widehat{P}_1, \\
\widehat{C}_{2n} + 2\widehat{P}_1 &= \widehat{C}_4 + 2\widehat{P}_{n-1}, \\
\widehat{T}_1 + \widehat{P}_5 + \widehat{P}_3 &= \widehat{P}_{11} + \widehat{P}_2 + \widehat{P}_1, \\
\widehat{T}_2 + \widehat{P}_8 + \widehat{P}_5 &= \widehat{P}_{17} + \widehat{P}_2 + \widehat{P}_1, \\
\widehat{T}_3 + \widehat{P}_{14} + \widehat{P}_9 + \widehat{P}_5 &= \widehat{P}_{29} + \widehat{P}_4 + \widehat{P}_2 + \widehat{P}_1, \\
\widehat{T}_4 + \widehat{P}_1 &= \widehat{C}_4 + 2\widehat{P}_2, \\
\widehat{T}_5 + \widehat{P}_1 &= \widehat{C}_4 + \widehat{P}_3 + \widehat{P}_2, \\
\widehat{T}_6 + \widehat{P}_1 &= \widehat{C}_4 + \widehat{P}_4 + \widehat{P}_2.
\end{aligned} \tag{3}$$

Given relations enable construction of all pairs of cospectral nonisomorphic graphs in the set $\mathfrak{S}$.

If $\widehat{S}_1, \widehat{S}_2, \ldots \widehat{S}_m$ are some systems of numbers and $\sigma_1, \sigma_2, \ldots \sigma_m$ integers such that the expression

$$\sigma_1\widehat{S}_1 + \sigma_2\widehat{S}_2 + \ldots + \sigma_m\widehat{S}_m \tag{4}$$

can be calculated in at least one way by successive performing the quoted operations, then (4) defines a systems $\widehat{S}$ and we shall say that $\widehat{S}$ is a linear combination of $\widehat{S}_1, \widehat{S}_2, \ldots \widehat{S}_m$.

The following theorem has been also proved in [3]:

Theorem 2.1. *Let $G \in \mathfrak{S}$. Then its spectrum can be represented in a unique way as a linear combination of the form:*

$$\sigma_0\widehat{C}_4 + \sum_{i=1}^{m} \sigma_i\widehat{P}_i,$$

where the number m is bounded by a function of the number of vertices, while σ_0 is always non-negative and the non-vanishing coefficient σ_i with the greatest i is positive.

In the sequel, the representation of the spectrum of $G \in \mathfrak{S}$ given by Theorem 2.1 will be called *canonical*. The integers σ_0 and σ_i, for $1 \leq i \leq m$, represent the *coefficients* of such representation. This representation for all bipartite graphs from $\mathfrak{S}$ can be obtained by using the equalities (3).

In [3] an effective procedure which enables the determination of all graphs having the spectrum equal to a given system of numbers of the form $2\cos\frac{p}{q}\pi$ is exposed. These graphs can be obtained by solving a system of linear Diophantine equations as follows. Recall that we will consider only bipartite graphs.

Given a symmetric system $\widehat{S}$ of numbers of the form 2 cos $\frac{p}{q}\pi$, we try to represent it as a linear combination of $\widehat{C}_4, \widehat{P}_1, \widehat{P}_2, \ldots$. If this is not possible, $\widehat{S}$ is not a spectrum of any graph. In the case such a representation is possible, the mentioned linear combination is unique. Principles of finding the corresponding coefficients are clear since among $\widehat{C}_4, \widehat{P}_1, \widehat{P}_2, \ldots$ no two systems have the same greatest element.

Let now $\widehat{S}$ be represented as:

$$\widehat{S} = \sigma_0\widehat{C}_4 + \sigma_1\widehat{P}_1 + \sigma_2\widehat{P}_2 + \cdots + \sigma_m\widehat{P}_m. \tag{5}$$

Suppose that $\widehat{S}$ is the spectrum of a graph G. Presenting $\widehat{S}$ as a linear combination of spectra of the components we get:

$$\begin{aligned}\widehat{S} = {} & p_1\widehat{P}_1 + p_2\widehat{P}_2 + p_3\widehat{P}_3 + \cdots + z_2\widehat{Z}_2 + z_3\widehat{Z}_3 + \cdots + w_1\widehat{W}_1 + w_2\widehat{W}_2 + w_3\widehat{W}_3 + \cdots \\ & c_2\widehat{C}_4 + c_3\widehat{C}_6 + \cdots + t_1\widehat{T}_1 + t_2\widehat{T}_2 + t_3\widehat{T}_3 + t_4\widehat{T}_4 + t_5\widehat{T}_5 + t_6\widehat{T}_6,\end{aligned} \tag{6}$$

for some non-negative integers (i.e. parameters of G):

$$p_1, p_2, p_3, \ldots, z_2, z_3, \ldots, w_1, w_2, w_3, \ldots, c_2, c_3, \ldots, t_1, t_2, t_3, t_4, t_5, t_6. \tag{7}$$

The number of terms in (6) is finite, and is bounded by a function of the number of vertices of a given graph G.

Using relations (3) one can express the equation (6) in the form:

$$\widehat{S} = F_0\widehat{C}_4 + F_1\widehat{P}_1 + F_2\widehat{P}_2 + \cdots, \tag{8}$$

where the coefficients F_i, $i = 0, 1, \ldots$ are functions of variables (7). Hence,

$$F_0 = (w_1 + w_2 + w_3 + \cdots) + (c_2 + c_3 + \cdots) + t_4 + t_5 + t_6, \tag{9}$$

$$\begin{aligned} F_1 = {} & p_1 + w_1 + (z_2 + z_3 + \cdots) - 2\,(c_3 + c_4 + \cdots) + \\ & t_1 + t_2 + t_3 - t_4 - t_5 - t_6, \end{aligned} \tag{10}$$

and for $i > 1$ and $i \neq 2, 3, 4, 5, 8, 9, 11, 14, 17, 29$ we have

$$F_i = \tilde{F}_i, \tag{11}$$

where

$$\tilde{F}_i = \begin{cases} p_i - z_i + w_i + 2\,c_{i+1}, & \text{if } i \text{ even} \\ p_i + z_{\frac{i-1}{2}} - z_i + w_i + 2\,c_{i+1}, & \text{if } i \text{ odd} \end{cases} \tag{12}$$

For the excluded values of i we have

$$F_i = \begin{cases} p_3 - z_3 + w_3 + 2\,c_4 + h_3, & \text{if } i = 3 \\ \tilde{F}_i + h_i, & \text{in all other cases,} \end{cases} \tag{13}$$

where

$$\begin{aligned} & h_2 = t_1 + t_2 + t_3 + 2\,t_4 + t_5 + t_6; \\ & h_3 = -t_1 + t_5; \quad h_4 = t_3 + t_6; \\ & h_5 = -t_1 - t_2 - t_3; \quad h_8 = -t_2; \quad h_9 = -t_3; \\ & h_{11} = t_1; \quad h_{14} = -t_3; \quad h_{17} = t_2; \quad h_{29} = t_3. \end{aligned} \tag{14}$$

Comparing (5) and (8) we get the following system of linear algebraic equations in unknowns (7):

$$F_i = \sigma_i, \qquad i = 0, 1, 2, \ldots, m. \tag{15}$$

Equation $F_i = \sigma_i$ will be denoted by E_i, for any non-negative integer i.

According to this, the following theorem is given in [3]:

Theorem 2.2. *Let $\widehat{S}$ be a symmetric system of numbers of the form $2\cos\frac{p}{q}\pi$, where p, q are integers and $q \neq 0$. A necessary condition for $\widehat{S}$ to be a graph spectrum is that $\widehat{S}$ can be represented in the form (5). In this case, to every solution of the system of equations (15) in unknowns (7), these quantities being non-negative integers, a graph corresponds, the spectrum of which is $\widehat{S}$. All graphs having the spectrum equal to $\widehat{S}$ can be obtained in this way.*

Remark 2.1. Equality (5) can be formulated as $\widehat{S} = \sigma_0\widehat{C}_4 + \sum_{i=1}^{+\infty}\sigma_i\widehat{P}_i$, with $\sigma_i = 0$ for $i > m$. Together with equalities (15) we can consider equalities $F_i = 0$ for $i > m$ and they are also fulfilled.

Example 2.1. Let us find all graphs with the spectrum $\widehat{S}$: $2, [0]^3, -2$. We have $\widehat{S} = \widehat{C}_4 + \widehat{P}_1$. The corresponding graph has 5 vertices, so the parameters of the graph i.e. the variables of the system of linear equations are: $p_1, \ldots, p_5, z_2, z_3, w_1$, and c_2. Exploring the system (15), we find:

$$F_0 = w_1 + c_2 = 1$$

$$F_1 = p_1 + w_1 + z_2 + z_3 = 1$$

$$F_2 = p_2 - z_2 = 0$$

$$F_3 = p_3 - z_3 = 0$$

$$F_4 = p_4 = 0$$

$$F_5 = p_5 + z_2 = 0$$

$$F_7 = z_3 = 0,$$

wherefrom we easily find that $p_2 = p_3 = p_4 = p_5 = 0$ and $z_2 = z_3 = 0$. Therefore, the equations E_0 and E_1 reduce to the following:

$$w_1 + c_2 = 1$$

$$p_1 + w_1 = 1,$$

and their solution is $p_1 = 1$, $w_1 = 0$, $c_2 = 1$ or $p_1 = 0$, $w_1 = 1$, $c_2 = 0$. Together with the previous, graphs $C_4 + P_1$ and W_1 are cospectral.

By considering the system (15) one can distinguish between the following two outcomes:

1. system (15) has a unique solution of non-negative integers, which implies that a considered graph G is DS-graph, or
2. system (15) has a non-unique solution over the set of non-negative integers, which means that a considered graph G is not DS-graph.

In the subsequent section we will consider the cospectrality of Smith graphs from the set $\mathfrak{S}$ by solving the corresponding system of linear equations (15). An efficient general theory of systems of linear Diophantine equations does not exist (see, for example, [8]) and therefore we have to use specific features of the system (15) when we are looking for solutions and their properties.

3. Cospectrality of Smith graphs

Given a bipartite graph G, we can represent it in the canonical form, defined by Theorem 2.1, and find the corresponding canonical coefficients $\sigma_0, \sigma_1, \ldots, \sigma_m$. The corresponding system of equations (15) will be called the system *associated* to the graph G. We shall assume in this section that the system we are considering is associated to a graph.

Proposition 3.1. *If $\sigma_0, \sigma_1, \ldots, \sigma_m$ are the coefficients of the canonical representation of the spectrum of a bipartite graph G from $\mathfrak{S}$, then the number n of vertices of G is given by*

$$n = 4\sigma_0 + \Sigma_{i=1}^{m} i\sigma_i.$$

Proof. The number of vertices of G is equal to the number of eigenvalues in $\widehat{G}$. We have

$$\widehat{G} = \sigma_0\widehat{C}_4 + \sigma_1\widehat{P}_1 + \sigma_2\widehat{P}_2 + \cdots + \sigma_m\widehat{P}_m.$$

If all σ_i's are non-negative, the conclusion is clear. If $\sigma_i < 0$ for some $i > 0$, the eigenvalues of $\widehat{P}_i$ have negative multiplicities. Hence we have for each i for which $\sigma_i < 0$ to subtract $i|\sigma_i|$ from the sum of positive summands. $\square$

Example 3.1. Based on equations (3) we have the following canonical forms for the spectra of graphs Z_n and T_3 respectively:

$$\widehat{Z}_n = \widehat{P}_1 - \widehat{P}_n + \widehat{P}_{2n+1},$$
$$\widehat{T}_3 = \widehat{P}_1 + \widehat{P}_2 + \widehat{P}_4 - \widehat{P}_5 - \widehat{P}_9 - \widehat{P}_{14} + \widehat{P}_{29}.$$

By Proposition 3.1 we have for the number of vertices $1 - n + 2n + 1 = n + 2$ for Z_n and $1 + 2 + 4 - 5 - 9 - 14 + 29 = 8$ for T_3.

The number n determines the set of variables in the system (15). To make and solve the system (15) one should include variables indicating the number of components whose number of vertices is at most n.

Example 3.2. For considering graphs on 6 vertices the following variables are relevant: $p_1, p_2, p_3, p_4, p_5, p_6, z_2, z_3, z_4, w_1, w_2, c_2, c_3$ and t_1. If we take 6 equations, the matrix of the system (15) reads:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Remark 3.1. System (15) always has a solution $c_2 = \sigma_0, p_1 = \sigma_1, \ldots, p_m = \sigma_m$ with other variables being equal to 0, giving rise to a hypothetical graph $\sigma_0 C_4 + \sigma_1 P_1 + \sigma_2 P_2 + \cdots + \sigma_m P_m$. However, this formal linear combination does not correspond to a graph if among coefficients σ_i are some which are negative. In this case, we know that still a solution exists since we assume that the system is associated to a graph G. This solution is expressed through parameters of G. Such a solution is called standard solution of system (15). Obviously, a graph G is a DS-graph if and only if the system (15), associated to G, has a unique solution (i.e. only standard solution). In the contrary, in order to determine the cospectral equivalence class of some not DS-graph, we are interested in non-standard solutions of the associated system.

Theorem 3.1. *The graph Z_n is DS-graph.*

Proof. Since $\widehat{Z}_n = \widehat{P}_1 - \widehat{P}_n + \widehat{P}_{2n+1}$, we have:

$$\sigma_0 = 0, \sigma_1 = 1, \sigma_2 = 0, \ldots, \sigma_{n-1} = 0, \sigma_n = -1, \sigma_{n+1} = 0, \ldots, \sigma_{2n} = 0, \sigma_{2n+1} = 1,$$

while by Proposition 3.1 the number of vertices is $n + 2$. We will carry out the proof assuming that $n \geq 9$, which means that we can consider all of the following variables as relevant:

$$p_1, p_2, \ldots, p_{n+2}, z_2, z_3, \ldots, z_n, w_1, w_2, \ldots, w_{n-2}, c_2, c_3, \ldots, c_{\lfloor \frac{n+2}{2} \rfloor}, t_1, t_2, t_3, t_4, t_5, t_6.$$

The cases when $n < 9$ can be considered on the similar fashion.

The equation E_0 of the system of linear equations (15) associated to Z_n reads:

$$F_0 = (w_1 + w_2 + \ldots + w_{n-2}) + (c_2 + c_3 + \ldots + c_{\lfloor \frac{n+2}{2} \rfloor}) + t_4 + t_5 + t_6 = 0,$$

wherefrom we conclude that $w_1 = w_2 = \ldots = w_{n-2} = 0$, $c_2 = c_3 = \ldots = c_{\lfloor \frac{n+2}{2} \rfloor} = 0$ and $t_4 = t_5 = t_6 = 0$.

Therefore, the equation E_1 becomes:

$$F_1 = p_1 + (z_2 + z_3 + \ldots + z_n) + t_1 + t_2 + t_3 = 1, \tag{16}$$

which means that exactly one of the involved variables should be equal to one.

If $p_1 = 1$, then the equation E_n is of the following form: $F_n = p_n = -1$, that is the contradiction with the assumption that the variables should be non-negative integers. So, $p_1 = 0$.

If $t_i = 1$, for exactly one $i \in \{1, 2, 3\}$, then the equation E_2 becomes $F_2 = p_2 + 1 = 0$, and therefore $t_1 = t_2 = t_3 = 0$.

From the considered cases it follows from (16) that exactly one of the z_i's is equal to 1. From the equation E_{2n+1} we have $F_{2n+1} = z_n = 1$, since the other possible variables are not relevant.

According to the determined values, the equations E_i, for $i \in \{2, 3, \dots, n+2\}$ and $i \neq n$ are of the following form $F_i = p_i = 0$, while for $i = n$ we have $F_n = p_n - 1 = -1$. Therefore, $p_2 = p_3 = \ldots = p_{n+2} = 0$.

Since the system (15) has unique solution, Z_n is DS-graph. □

Theorem 3.1 has already been proved in [11] by a different technique. Here we have given an alternative proof using system of linear equations (15).

It is well known that all connected Smith graphs, except for the double snake W_n and the tree T_4 (see [7]), are DS-graphs, and this can be proved in the same spirit as in Theorem 3.1. Graph that is cospectral with T_4 is $C_6 + P_1$. Furthermore, together with Theorem 3.1, it is proved in [11] that $Z_{n_1} + Z_{n_2} + \cdots + Z_{n_k}$ is DS-graph whenever $n_1, n_2, \dots, n_k$ are positive integers greater than 1. This result is generalized in [9], where it is proved that a graph of type $Z_{j_1} + Z_{j_2} + \cdots + Z_{j_k} + t_1T_1 + t_2T_2 + t_3T_3$, for some natural numbers $j_1, j_2, \dots, j_k, t_1, t_2, t_3$ is determined by its adjacency spectrum under some circumstances.

Cospectral equivalence classes of $P_n + P_1$ and of $W_n + P_1$ have been determined in [13], while bellow we consider graphs $Z_n + P_1$ and $C_{2n} + P_1$. In principal, all results on the spectral characterization and cospectrality of Smith graphs from the papers [11,9,13] can be reproduced using our new technique. Also, note that the main result of the paper [10] can be proved by our technique, as well.

Theorem 3.2. *Graph $Z_n + P_1$, for $n \geq 9$ is DS-graph.*

Proof. Graph $Z_n + P_1$ has $n + 3$ vertices, and according to (3) we have $\widehat{Z}_n + \widehat{P}_1 = 2\widehat{P}_1 - \widehat{P}_n + \widehat{P}_{2n+1}$. The relevant variables are:

$$p_1, p_2, \dots, p_{n+3}, z_2, z_3, \dots, z_{n+1}, w_1, w_2, \dots, w_{n-1}, c_2, c_3, \dots, c_{\lfloor \frac{n+3}{2} \rfloor}, t_1, t_2, t_3, t_4, t_5, t_6.$$

Equation E_0 of the system of linear equations (15) that is associated to $Z_n + P_1$ reads:

$$F_0 = w_1 + w_2 + \ldots + w_{n-1} + c_2 + c_3 + \ldots + c_{\lfloor \frac{n+3}{2} \rfloor} + t_4 + t_5 + t_6 = 0,$$

wherefrom we get $w_1 = w_2 = \ldots = w_{n-1} = 0$, $c_2 = c_3 = \ldots = c_{\lfloor \frac{n+3}{2} \rfloor} = 0$ and $t_4 = t_5 = t_6 = 0$. Therefore the equation E_1 becomes:

$$F_1 = p_1 + z_2 + z_3 + \ldots + z_{n+1} + t_1 + t_2 + t_3 = 2. \tag{17}$$

Let us find the non-negative solutions of the equation (17). First, let us consider the possible values of the variables t_1, t_2 and t_3.

We have:

$$\begin{cases} F_2 = p_2 + 2 = 0, & \text{if } t_i = 2 \text{ for exactly one } i \in \{1,2,3\}; \\ F_2 = p_2 + 2 = 0, & \text{if } t_i = 1 \text{ and } t_j = 1 \text{ for exactly one } i \neq j \in \{1,2,3\}; \\ F_2 = p_2 + 1 = 0, & \text{if } p_1 = 1 \text{ and } t_i = 1 \text{ for exactly one } i \in \{1,2,3\}; \\ F_2 = p_2 + 1 = 0, & \text{if } t_i = 1 \text{ for exactly one } i \in \{1,2,3\} \text{ and} \\ & z_j = 1, \text{ for exactly one } j \in \{3,4,\ldots,n+1\}. \end{cases}$$

Let us now assume that $t_i = 1$ for exactly one $i \in \{1,2,3\}$ and $z_2 = 1$. Then we have:

$$\begin{cases} F_{11} = p_{11} + 1 = \begin{cases} 0, & \text{if } n \neq 11; \\ -1, & \text{if } n = 11. \end{cases}, & \text{if } t_1 = 1; \\ F_{17} = p_{17} + 1 = \begin{cases} 0, & \text{if } n \neq 17; \\ -1, & \text{if } n = 17. \end{cases}, & \text{if } t_2 = 1; \\ F_4 = p_4 + 1 = 0, & \text{if } t_3 = 1. \end{cases}$$

From the considered cases we conclude $t_1 = t_2 = t_3 = 0$.

From the equation E_{2n+1} we find $F_{2n+1} = z_n = 1$, which together with (17) means that exactly one of the variables $p_1, z_2, z_3, \ldots, z_{n-1}, z_{n+1}$ is equal to 1. Let us suppose that $z_i = 1$, for exactly one $i \in \{2,3,\ldots,n-1,n+1\}$. Then the equation E_{2i+1} is of the form: $F_{2i+1} = p_{2i+1} = -1$. Since this gives the contradiction, we have $z_2 = z_3 = \ldots = z_{n-1} = z_{n+1} = 0$. Therefore, from (17) we have $p_1 = 1$.

Now, equations E_i, for $i \in \{2,3,\ldots,n+3\}$ are of the form $F_i = p_i = 0$, so we find $p_2 = p_3 = \ldots = p_{n+3} = 0$.

Since the associated system (15) has unique solution, graph $Z_n + P_1$ is DS-graph. □

Theorem 3.3. *Graph $C_{2n} + P_1$, for $n \geq 4$ is DS-graph.*

Proof. Graph $C_{2n} + P_1$ has $2n+1$ vertices, and according to (3) one can find $\widehat{C}_{2n} + \widehat{P}_1 = \widehat{C}_4 - \widehat{P}_1 + 2\widehat{P}_{n-1}$. The relevant variables are:

$$p_1, p_2, \ldots, p_{2n+1}, z_2, \ldots, z_{2n-1}, w_1, w_2, \ldots, w_{2n-3}, c_2, c_3, \ldots, c_n, t_1, t_2, t_3, t_4, t_5, t_6.$$

Let us determine the possible non-negative values of these variables.

The equation E_0 of the system of linear equations (15) that is associated to $C_{2n} + P_1$ reads:

$$F_0 = w_1 + w_2 + \ldots + w_{2n-3} + c_2 + c_3 + \ldots + c_n + t_4 + t_5 + t_6 = 1,$$

wherefrom we conclude that exactly one of the variables $w_1, w_2, \ldots, w_{2n-3}, c_2, c_3, \ldots, c_n, t_4, t_5, t_6$ is equal to 1.

If $w_i = 1$ for exactly one $i \in \{1, 2, \ldots, 2n-3\}$, then we have:

$$F_1 = p_1 + z_2 + z_3 + \ldots + z_{2n-1} + t_1 + t_2 + t_3 = \begin{cases} -2, & \text{if } w_1 = 1; \\ -1, & \text{if } w_i = 1, \text{ for } i \neq 1, \end{cases}$$

which means that $w_1 = w_2 = \ldots = w_{2n-3} = 0$.

Let us now suppose that $t_i = 1$, for exactly one $i \in \{4, 5, 6\}$. Then we have:

$$F_1 = p_1 + z_2 + z_3 + \ldots + z_{2n-1} + t_1 + t_2 + t_3 = 0,$$

which implies $p_1 = z_2 = z_3 = \ldots = z_{2n-1} = t_1 = t_2 = t_3 = 0$. In that case the equation E_2 becomes:

$$\begin{cases} F_2 = p_2 + 2 = 0, & \text{if } t_4 = 1; \\ F_2 = p_2 + 1 = 0, & \text{if } t_5 = 1, \text{ or } t_6 = 1, \end{cases}$$

that is a contradiction, so $t_4 = t_5 = t_6 = 0$.

It holds that $c_2 = 0$, since if $c_2 = 1$, we find that $F_1 = p_1 + z_2 + z_3 + \ldots + z_{2n-1} + t_1 + t_2 + t_3 = -1$. Therefore, let us suppose that $c_i = 1$, for exactly one $i \in \{3, 4, \ldots, n\}$. Then equation E_1 reads:

$$F_1 = p_1 + z_2 + z_3 + \ldots + z_{2n-1} + t_1 + t_2 + t_3 = 1, \tag{18}$$

which means that exactly one of the variables $p_1, z_2, z_3, \ldots, z_{2n-1}, t_1, t_2, t_3$ is equal to 1.

If $t_j = 1$ for exactly one $j \in \{1, 2, 3\}$, we find $F_2 = p_2 + 2c_3 = -1$. Therefore, $t_1 = t_2 = t_3 = 0$.

Let us now suppose that $z_j = 1$, for exactly one $j \in \{2, 3, \ldots, 2n-1\}$. Then the equation E_{2j+1} reads:

$$F_{2j+1} = p_{2j+1} + 2c_{2j+2} = \begin{cases} 1, & \text{if } j = \frac{n-2}{2} \text{ and } n \text{ is even;} \\ -1, & \text{in all remaining cases.} \end{cases}$$

In the second case we have a contradiction, while in the first one we find $p_{2j+1} = p_{n-1} = 1$ and $c_{2j+2} = c_n = 0$. Therefore, the equations E_k, for $k \in \{2, 3, \ldots, 2n+1\}$ and $k \neq (n-1)$ of the associated system are of the following form:

$$F_k = \begin{cases} p_{\frac{n-2}{2}} + 2c_{\frac{n}{2}} = 1, & \text{if } k = \frac{n-2}{2} \text{ and } n \text{ is even;} \\ p_k + 2c_{k+1} = 0, & \text{in all remaining cases.} \end{cases}$$

So, we find that $c_i = 0$, for each i, that is the contradiction with the assumption that $c_i = 1$, for exactly one $i \in \{3, 4, \ldots, n\}$. Therefore, $z_2 = z_3 = \ldots = z_{2n-1} = 0$, and according to (18) we have $p_1 = 1$.

So, from the previous conclusions we have that the equations E_i, for $i \in \{2, 3, \ldots, 2n+1\}$ have the following form:

$$F_i = \begin{cases} p_i + 2c_{i+1} = 0, & \text{if } i \neq n-1; \\ p_{n-1} + 2c_n = 2, & \text{if } i = n-1. \end{cases}$$

From the first equality we find that $p_i = 0$ and $c_{i+1} = 0$, for each $i \neq n-1$, while the second equality gives two possible solutions: $(p_{n-1}, c_n) = (2, 0)$ or $(p_{n-1}, c_n) = (0, 1)$. The first one is not valid since then we have the contradiction with the assumption that $c_i = 1$, for exactly one $i \in \{3, 4, \ldots, n\}$. Therefore, the resulting graph is $C_{2n} + P_1$, and the proof follows. □

Let us now consider some not DS-graphs from the set $\mathfrak{S}$.

Proposition 3.2. *The cospectral equivalence class of graph $W_1 + T_4$ is: $[W_1 + T_4] = \{W_1 + T_4, P_1 + C_6 + W_1, P_1 + C_4 + T_4, P_2 + C_4 + W_2, 2P_2 + 2C_4, 2W_2, C_6 + C_4 + 2P_1\}$.*

Proof. Graph $W_1 + T_4$ has 12 vertices, and according to (3) we find: $\widehat{W}_1 + \widehat{T}_4 = 2\widehat{C}_4 + 2\widehat{P}_2$. Relevant variables are:

$$p_1, p_2, \ldots, p_{12}, z_2, z_3, \ldots, z_{10}, w_1, w_2, \ldots, w_8, c_2, c_3, \ldots, c_6, t_1, t_2, t_3, t_4, t_5, t_6,$$

while the system (15) associated to this graph is given as item A1 in Appendix. By considering the equations E_i of this system, for $i \in \{11, 12, 13, 14, 15, 17, 19, 21\}$, we find that

$$p_{11} = p_{12} = 0,\ z_5 = z_6 = \ldots = z_{10} = 0,\ t_1 = t_2 = t_3 = 0. \tag{19}$$

Using equalities (19), from equations E_i, for $i \in \{5, 6, 7, 8, 9, 10\}$, we get:

$$p_5 = p_6 = \ldots = p_{10} = 0,\ z_2 = z_3 = z_4 = 0,\ w_5 = w_6 = w_7 = w_8,\ c_6 = 0. \tag{20}$$

Using (19) and (20), from equations E_3 and E_4, we have:

$$p_3 = p_4 = 0,\ w_3 = w_4 = 0,\ c_4 = c_5 = 0,\ t_5 = t_6 = 0. \tag{21}$$

So, having in mind (19), (20) and (21), equations E_0, E_1 and E_2 reduce to:

$$\left.\begin{array}{lcccccccccccc} F_0 = & & & & w_1 & + & w_2 & + & c_2 & + & c_3 & + & t_4 & = & 2 \\ F_1 = & p_1 & & + & w_1 & & & & & - & 2c_3 & - & t_4 & = & 0 \\ F_2 = & & p_2 & & & + & w_2 & & & + & 2c_3 & + & 2t_4 & = & 2 \end{array}\right\} \tag{22}$$

By considering the equation $F_2 = 2$ of system (22), one can distinguish the following five cases.

Case 1: If $c_3 = 1$ and $p_2 = w_2 = t_4 = 0$, then there are two sets of possible solutions. In the first one, we have $w_1 = 1$, $c_2 = 0$ and $p_1 = 1$, while in the second one, we have $c_2 = 1$, $w_1 = 0$ and $p_1 = 2$. Therefore, $P_1 + C_6 + W_1$ and $C_6 + C_4 + 2P_1$ are the graphs cospectral to $W_1 + T_4$.

Case 2: If $t_4 = 1$ and $p_2 = w_2 = c_3 = 0$, then there are also two sets of possible solutions. In the first one, we have $p_1 = 1$, $w_1 = 0$ and $c_2 = 1$, in the second one we have $p_1 = 0$, $w_1 = 1$ and $c_2 = 0$, so $P_1 + C_4 + T_4$ and $W_1 + T_4$ are the corresponding resulting graphs.

Case 3: For $p_2 = w_2 = 1$ and $c_3 = t_4 = 0$, one finds $p_1 = w_1 = 0$ and $c_2 = 1$, so $P_2 + C_4 + W_2$ is the graph cospectral to $W_1 + T_4$.

Case 4: If $p_2 = 2$ and $w_2 = c_3 = t_4 = 0$, then $p_1 = w_1 = 0$ and $c_2 = 2$, so $2P_2 + 2C_4$ is graph cospectral to $W_1 + T_4$.

Case 5: If $w_2 = 2$ and $p_2 = c_3 = t_4 = 0$, then $p_1 = w_1 = c_2 = 0$, and $2W_2$ is the graph cospectral to $W_1 + T_4$. $\square$

Proposition 3.3. *The cospectral equivalence class of graph* $W_1 + T_5$ *is:* $[W_1 + T_5] = \{W_1 + T_5, P_2 + P_3 + 2C_4, P_2 + C_4 + W_3, W_2 + P_3 + C_4, W_2 + W_3, T_5 + C_4 + P_1\}$.

Proof. Graph $W_1 + T_5$ has 13 vertices, and according to (3) we find $\widehat{W_1} + \widehat{T_5} = 2\widehat{C}_4 + \widehat{P}_2 + \widehat{P}_3$. Relevant variables are:

$$p_1, p_2, \ldots, p_{13}, z_2, z_3, \ldots, z_{11}, w_1, w_2, \ldots, w_9, c_2, c_3, \ldots, c_6, t_1, t_2, t_3, t_4, t_5, t_6,$$

while the system (15) associated to this graph is given as item A2 in Appendix. By considering this system, from the equations E_i, for $i \in \{12, 13, 14, 15, 17, 19, 21, 23\}$ we directly get:

$$p_{12} = p_{13} = 0,\ t_2 = t_3 = 0,\ z_6 = z_7 = \ldots = z_{11} = 0. \tag{23}$$

By using identities (23) from equations E_i, for $i \in \{6, 7, 8, 9, 10, 11\}$ we find:

$$p_6 = p_7 = \ldots = p_{11} = 0,\ z_3 = z_4 = z_5 = 0,\ t_1 = 0,\ w_6 = w_7 = w_8 = w_9 = 0. \tag{24}$$

From (23) and (24) and equations E_4 and E_5, we have:

$$p_4 = p_5 = 0,\ z_2 = 0,\ w_4 = w_5 = 0,\ c_5 = c_6 = 0,\ t_6 = 0. \tag{25}$$

Now, having in mind (23), (24) and (25), the equations E_0, E_1, E_2 and E_3 become:

$$\begin{array}{rcccccccccccccccccc} F_0 = & w_1 & + & w_2 & + & w_3 & + & c_2 & + & c_3 & + & c_4 & + & t_4 & + & t_5 & = & 2 \\ F_1 = & p_1 & + & w_1 & - & 2c_3 & - & 2c_4 & - & t_4 & - & t_5 & & & & & = & 0 \\ F_2 = & p_2 & + & w_2 & + & 2c_3 & + & 2t_4 & + & t_5 & & & & & & & = & 1 \\ F_3 = & p_3 & + & & & w_3 & + & 2c_4 & + & t_5 & & & & & & & = & 1 \end{array} \tag{26}$$

From the equations $F_2 = 1$ and $F_3 = 1$ of the system (26) we get that $c_3 = c_4 = t_4 = 0$, so the system (26) becomes:

$$\begin{array}{llllllllllll} F_0 = & w_1 & + & w_2 & + & w_3 & + & c_2 & + & t_5 & = & 2 \\ F_1 = & p_1 & + & w_1 & - & t_5 & & & & & = & 0 \\ F_2 = & p_2 & + & w_2 & + & t_5 & & & & & = & 1 \\ F_3 = & p_3 & & & + & w_3 & + & t_5 & & & = & 1 \end{array} \tag{27}$$

By analysing the equation $F_2 = 1$ of system (27) we can distinguish the following three cases:

Case 1: If $p_2 = 1$ and $w_2 = t_5 = 0$, then $p_1 = w_1 = 0$, and there are two subcases:

Subcase 1: if $p_3 = 1$ and $w_3 = 0$, then $c_2 = 2$, and it follows $W_1 + T_5 \sim P_2 + P_3 + 2C_4$;

Subcase 2: if $w_3 = 1$ and $p_3 = 0$, then $c_2 = 1$, and we have $W_1 + T_5 \sim P_2 + C_4 + W_3$.

Case 2: If $w_2 = 1$ and $p_2 = t_5 = 0$, we find that $p_1 = w_1 = 0$, so we have the following:

Subcase 1: if $w_3 = 1$ and $c_2 = 0$, it follows $p_3 = 0$, and $W_1 + T_5 \sim W_2 + W_3$;

Subcase 2: if $w_3 = 0$ and $c_2 = 1$, then $p_3 = 1$, so we have $W_1 + T_5 \sim W_2 + P_3 + C_4$.

Case 3: If $t_5 = 1$ and $p_2 = w_2 = 0$, we find $p_3 = w_3 = 0$, and then:

Subcase 1: if $p_1 = 1$, $w_1 = 0$ and $c_2 = 1$, we have $W_1 + T_5 \sim T_5 + C_4 + P_1$;

Subcase 2: if $p_1 = 0$, $w_1 = 1$ and $c_2 = 0$, we get $W_1 + T_5 \sim T_5 + W_1$. □

Theorem 3.4. *Graph T_5+T_6 is not DS-graph. Its cospectral equivalence class is:* $[T_5+T_6] = \{T_5 + T_6, P_3 + P_4 + C_4 + C_6, P_4 + C_6 + W_3, P_3 + C_6 + W_4\}$.

Proof. Graph $T_5 + T_6$ has 17 vertices, and according to (3) we find $\widehat{T}_5 + \widehat{T}_6 = 2\widehat{C}_4 - 2\widehat{P}_1 + 2\widehat{P}_2 + \widehat{P}_3 + \widehat{P}_4$. The corresponding variables are:

$$p_1, p_2, \ldots, p_{17}, z_2, z_3, \ldots, z_{15}, w_1, w_2, \ldots, w_{13}, c_2, c_3, \ldots, c_8, t_1, t_2, t_3, t_4, t_5, t_6,$$

while the system of linear equations associated to this graphs is given as item A3 in Appendix.

From the equations E_i, for $i \in \{16, 17, 19, 21, 23, 25, 27, 29, 31\}$ we directly get:

$$p_{16} = p_{17} = 0,\ z_8 = z_9 = \ldots = z_{15} = 0,\ t_2 = t_3 = 0. \tag{28}$$

By using (28), from the equations E_i, for $i \in \{8, 9, 10, 11, 12, 13, 14, 15\}$, we have:

$$p_8 = p_9 = \ldots = p_{15} = 0,\ z_4 = \ldots = z_7 = 0,\ w_8 = w_9 = \ldots = w_{13} = 0,\ t_1 = 0. \tag{29}$$

Now, using (28) and (29) from equations E_5, E_6 and E_7 we find:

$$p_5 = p_6 = p_7 = 0,\ z_2 = z_3 = 0,\ w_5 = w_6 = w_7 = 0,\ c_6 = c_7 = c_8 = 0. \tag{30}$$

According to (28), (29) and (30), equations E_0, E_1, E_2, E_3 and E_4 become:

$$\begin{aligned} F_0 &= w_1 + w_2 + w_3 + w_4 + c_2 + c_3 + c_4 + c_5 + t_4 + t_5 + t_6 = 2 \\ F_1 &= p_1 + w_1 - 2c_3 - 2c_4 - 2c_5 - t_4 - t_5 - t_6 = -2 \\ F_2 &= p_2 + w_2 + 2c_3 + 2t_4 + t_5 + t_6 = 2 \\ F_3 &= p_3 + w_3 + 2c_4 + t_5 = 1 \\ F_4 &= p_4 + w_4 + 2c_5 + t_6 = 1. \end{aligned} \tag{31}$$

From the relations $F_3 = 1$ and $F_4 = 1$ we find $c_4 = c_5 = 0$, so the system (31) reduces to:

$$\begin{aligned} F_0 &= w_1 + w_2 + w_3 + w_4 + c_2 + c_3 + t_4 + t_5 + t_6 = 2 \\ F_1 &= p_1 + w_1 - 2c_3 - t_4 - t_5 - t_6 = -2 \\ F_2 &= p_2 + w_2 + 2c_3 + 2t_4 + t_5 + t_6 = 2 \\ F_3 &= p_3 + w_3 + t_5 = 1 \\ F_4 &= p_4 + w_4 + t_6 = 1. \end{aligned} \tag{32}$$

By considering the equation $F_4 = 1$ of the system (32), we can distinguish between three cases:

Case 1: If $p_4 = 1$ and $w_4 = t_6 = 0$, then by considering the equation $F_3 = 1$ we can distinguish between the following three subcases:

- **Subcase 1:** for $p_3 = 1$ and $w_3 = t_5 = 0$, we find that $c_2 = c_3 = 1$ and $p_1 = p_2 = w_1 = w_2 = t_4 = 0$, so the corresponding graph is $C_4 + C_6 + P_3 + P_4$;
- **Subcase 2:** if $w_3 = 1$ and $p_3 = t_5 = 0$, we find $c_3 = 1$ and $p_1 = p_2 = w_1 = w_2 = c_2 = t_4 = 0$, and the corresponding graph is $C_6 + W_3 + P_4$;
- **Subcase 3:** if $t_5 = 1$ and $p_3 = w_3 = 0$, then we have $F_2 = p_2 + w_2 + 2c_3 + 2t_4 = 1$, wherefrom we find $c_3 = t_4 = 0$, that implies $F_1 = p_1 + w_1 = -1$. Therefore, in this subcase we do not have non-negative solutions.

Case 2: If $w_4 = 1$ and $p_4 = t_6 = 0$, then by analysing the equation $F_3 = 1$ we can distinguish between the following three subcases:

- **Subcase 1:** if $p_3 = 1$ and $w_3 = t_5 = 0$, we find that $c_3 = 1$ and $p_1 = p_2 = w_1 = w_2 = c_2 = t_4 = 0$, so the corresponding graph is $C_6 + P_3 + W_4$;
- **Subcase 2:** if $w_3 = 1$ and $p_3 = t_5 = 0$, then we have $F_0 = w_1 + w_2 + c_2 + c_3 + t_4 = 0$, wherefrom we get $w_1 = w_2 = c_2 = c_3 = t_4 = 0$. This implies that $F_1 = p_1 = -2$, that is the contradiction.
- **Subcase 3:** if $t_5 = 1$ and $p_3 = w_3 = 0$, then we have $F_0 = w_1 + w_2 + c_2 + c_3 + t_4 = 0$, wherefrom we get $w_1 = w_2 = c_2 = c_3 = t_4 = 0$. Therefore, we have $F_1 = p_1 = -1$, that is the contradiction.

Case 3: If $t_6 = 1$ and $p_4 = w_4 = 0$, then from the equation $F_2 = p_2 + w_2 + 2c_3 + 2t_4 + t_5 = 1$, we find $c_3 = t_4 = 0$, so by considering this equation we can distinguish between three subcases:

Subcase 1: if $p_2 = 1$ and $w_2 = t_5 = 0$, we get $F_1 = p_1 + w_1 = -1$, which is the contradiction;

Subcase 2: if $w_2 = 1$ and $p_2 = t_5 = 0$, the equation $F_1 = p_1 + w_1 = -1$ gives the contradiction;

Subcase 3: if $t_5 = 1$ and $p_2 = w_2 = 0$, we find that $p_1 = p_3 = w_1 = w_3 = c_2 = 0$, so the resulting graph is $T_5 + T_6$. $\square$

4. Appendix

Here we list the full-form of the systems of linear equations (15) associated to the graphs $W_1 + T_4$, $W_1 + T_5$ and $T_5 + T_6$, respectively, that we are solving in the proofs of the corresponding statements in Section 3.

A1. System of linear equations (15) associated to $W_1 + T_4$.

$$F_0 = w_1 + w_2 + \ldots + w_8 + c_2 + c_3 + \ldots + c_6 + t_4 + t_5 + t_6 = 2,$$

$$F_1 = p_1 + w_1 + z_2 + z_3 + \ldots + z_{10} - 2c_3 - 2c_4 - \ldots - 2c_6 + t_1 + t_2 + t_3 - t_4 - t_5 - t_6 = 0,$$

$$F_2 = p_2 - z_2 + w_2 + 2c_3 + t_1 + t_2 + t_3 + 2t_4 + t_5 + t_6 = 2,$$

$$F_3 = p_3 - z_3 + w_3 + 2c_4 - t_1 + t_5 = 0,$$

$$F_4 = p_4 - z_4 + w_4 + 2c_5 + t_3 + t_6 = 0,$$

$$F_5 = p_5 + z_2 - z_5 + w_5 + 2c_6 - t_1 - t_2 - t_3 = 0,$$

$$F_6 = p_6 - z_6 + w_6 = 0,$$

$$F_7 = p_7 + z_3 - z_7 + w_7 = 0,$$

$$F_8 = p_8 - z_8 + w_8 - t_2 = 0,$$

$$F_9 = p_9 + z_4 - z_9 - t_3 = 0,$$

$$F_{10} = p_{10} - z_{10} = 0,$$

$$F_{11} = p_{11} + z_5 + t_1 = 0,$$

$$F_{12} = p_{12} = 0,$$

$$F_{13} = z_6 = 0,$$

$$F_{14} = -t_3 = 0,$$

$$F_{15} = z_7 = 0,$$

$$F_{17} = z_8 + t_2 = 0,$$

$$F_{19} = z_9 = 0,$$

$$F_{21} = z_{10} = 0.$$

A2. System of linear equations (15) associated to $W_1 + T_5$.

$$F_0 = w_1 + w_2 + \ldots + w_9 + c_2 + c_3 + \ldots + c_6 + t_4 + t_5 + t_6 = 2,$$

$$F_1 = p_1 + w_1 + z_2 + z_3 + \ldots + z_{11} - 2c_3 - 2c_4 - \ldots - 2c_6 + t_1 + t_2 + t_3 - t_4 - t_5 - t_6 = 0,$$

$$F_2 = p_2 - z_2 + w_2 + 2c_3 + t_1 + t_2 + t_3 + 2t_4 + t_5 + t_6 = 1,$$

$$F_3 = p_3 - z_3 + w_3 + 2c_4 - t_1 + t_5 = 1,$$

$$F_4 = p_4 - z_4 + w_4 + 2c_5 + t_3 + t_6 = 0,$$

$$F_5 = p_5 + z_2 - z_5 + w_5 + 2c_6 - t_1 - t_2 - t_3 = 0,$$

$$F_6 = p_6 - z_6 + w_6 = 0,$$

$$F_7 = p_7 + z_3 - z_7 + w_7 = 0,$$

$$F_8 = p_8 - z_8 + w_8 - t_2 = 0,$$

$$F_9 = p_9 + z_4 - z_9 + w_9 - t_3 = 0,$$

$$F_{10} = p_{10} - z_{10} = 0,$$

$$F_{11} = p_{11} + z_5 - z_{11} + t_1 = 0,$$

$$F_{12} = p_{12} = 0,$$

$$F_{13} = p_{13} + z_6 = 0,$$

$$F_{14} = -t_3 = 0,$$

$$F_{15} = z_7 = 0,$$

$$F_{17} = z_8 + t_2 = 0,$$

$$F_{19} = z_9 = 0,$$

$$F_{21} = z_{10} = 0,$$

$$F_{23} = z_{11} = 0.$$

A3. System of linear equations (15) associated to $T_5 + T_6$.

$$F_0 = w_1 + w_2 + \ldots + w_{13} + c_2 + c_3 + \ldots + c_8 + t_4 + t_5 + t_6 = 2,$$

$$F_1 = p_1 + w_1 + z_2 + z_3 + \ldots + z_{15} - 2c_3 - 2c_4 - \ldots - 2c_8 + t_1 + t_2 + t_3 - t_4$$

$$- t_5 - t_6 = -2,$$

$$F_2 = p_2 - z_2 + w_2 + 2c_3 + t_1 + t_2 + t_3 + 2t_4 + t_5 + t_6 = 2,$$

$$F_3 = p_3 - z_3 + w_3 + 2c_4 - t_1 + t_5 = 1,$$

$$F_4 = p_4 - z_4 + w_4 + 2c_5 + t_3 + t_6 = 1,$$

$$F_5 = p_5 + z_2 - z_5 + w_5 + 2c_6 - t_1 - t_2 - t_3 = 0,$$

$$F_6 = p_6 - z_6 + w_6 + 2c_7 = 0,$$

$$F_7 = p_7 + z_3 - z_7 + w_7 + 2c_8 = 0,$$

$$F_8 = p_8 - z_8 + w_8 - t_2 = 0,$$

$$F_9 = p_9 + z_4 - z_9 + w_9 - t_3 = 0,$$

$$F_{10} = p_{10} - z_{10} + w_{10} = 0,$$

$$F_{11} = p_{11} + z_5 - z_{11} + w_{11} + t_1 = 0,$$

$$F_{12} = p_{12} - z_{12} + w_{12} = 0,$$

$$F_{13} = p_{13} + z_6 - z_{13} + w_{13} = 0,$$

$$F_{14} = p_{14} - z_{14} - t_3 = 0,$$

$$F_{15} = p_{15} + z_7 - z_{15} = 0,$$

$$F_{16} = p_{16} = 0,$$

$$F_{17} = p_{17} + z_8 + t_2 = 0,$$

$$F_{19} = z_9 = 0,$$

$$F_{21} = z_{10} = 0,$$

$$F_{23} = z_{11} = 0,$$

$$F_{25} = z_{12} = 0,$$

$$F_{27} = z_{13} = 0,$$

$$F_{29} = z_{14} + t_3 = 0,$$

$$F_{31} = z_{15} = 0.$$

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