

# Propositional logic

## Solution 1

(a)

$$\text{false}(\text{as}(\text{true} \Rightarrow \text{false}) \Leftrightarrow \text{false})$$

(b)

$$\text{true}(\text{as}(\text{false} \Rightarrow \text{false}) \Leftrightarrow \text{true})$$

(c)

$$\text{true}(\text{as}(\text{false} \Rightarrow \text{true}) \Leftrightarrow \text{true})$$

(d)

$$\text{true}(\text{as}(\text{false} \Rightarrow \text{false}) \Leftrightarrow \text{true})$$

(Assuming that pigs can't fly . . . )

## Solution 2

(a)

$p$	$q$	$p \wedge q$	$(p \wedge q) \Rightarrow p$
t	t	t	t
t	f	f	t
f	t	f	t
f	f	f	t

(b)

$p$	$q$	$p \wedge q$	$\neg p$	$\neg p \Rightarrow (p \wedge q)$	$(\neg p \Rightarrow (p \wedge q)) \Leftrightarrow p$
t	t	t	f	t	t
t	f	f	f	t	t
f	t	f	t	f	t
f	f	f	t	f	t

(c)

$p$	$q$	$p \Rightarrow q$	$p \wedge (p \Rightarrow q)$	$(p \wedge (p \Rightarrow q)) \Rightarrow q$
t	t	t	t	t
t	f	f	f	t
f	t	t	f	t
f	f	t	f	t

### Solution 3

(a)

$$\begin{aligned}
 p \Rightarrow \neg p & \\
 &\Leftrightarrow \neg p \vee \neg p && [\Rightarrow] \\
 &\Leftrightarrow \neg p && [\text{idempotence}]
 \end{aligned}$$

(b)

$$\begin{aligned}
 \neg p \Rightarrow p & \\
 &\Leftrightarrow \neg \neg p \vee p && [\Rightarrow] \\
 &\Leftrightarrow p \vee p && [\neg \neg] \\
 &\Leftrightarrow p && [\text{idempotence}]
 \end{aligned}$$

(c)

$$\begin{aligned}
 p \Rightarrow (q \Rightarrow r) & \\
 &\Leftrightarrow \neg p \vee (q \Rightarrow r) && [\Rightarrow] \\
 &\Leftrightarrow \neg p \vee \neg q \vee r && [\Rightarrow] \\
 &\Leftrightarrow \neg p \vee \neg q \vee r && [\text{associativity}] \\
 &\Leftrightarrow \neg (p \wedge q) \vee r && [\text{De Morgan}] \\
 &\Leftrightarrow p \wedge q \Rightarrow r && [\Rightarrow]
 \end{aligned}$$

(d)

$$\begin{aligned}
 q \Rightarrow (p \Rightarrow r) & \\
 &\Leftrightarrow \neg q \vee (p \Rightarrow r) && [\Rightarrow] \\
 &\Leftrightarrow \neg q \vee \neg p \vee r && [\Rightarrow] \\
 &\Leftrightarrow \neg p \vee \neg q \vee r && [\text{associativity} \wedge \text{commutativity}] \\
 &\Leftrightarrow \neg p \vee (q \Rightarrow r) && [\Rightarrow] \\
 &\Leftrightarrow p \Rightarrow (q \Rightarrow r) && [\Rightarrow]
 \end{aligned}$$

(e)

$$\begin{aligned}
 p \wedge q &\Leftrightarrow p && [\Leftrightarrow] \\
 &\Leftrightarrow (p \wedge q \Rightarrow p) \wedge (p \Rightarrow p \wedge q) && [\Rightarrow] \\
 &\Leftrightarrow (\neg(p \wedge q) \vee p) \wedge (\neg p \vee p \wedge q) && [\text{De Morgan}] \\
 &\Leftrightarrow (\neg p \vee \neg q \vee p) \wedge (\neg p \vee p \wedge q) && [\text{associativity } \wedge \text{ comm.}] \\
 &\Leftrightarrow (\neg q \vee \neg p \vee p) \wedge (\neg p \vee p \wedge q) && [\text{excluded middle}] \\
 &\Leftrightarrow (\neg q \vee \text{true}) \wedge (\neg p \vee p \wedge q) && [\vee \wedge \text{true}] \\
 &\Leftrightarrow \text{true} \wedge (\neg p \vee p \wedge q) && [\wedge \wedge \text{true}] \\
 &\Leftrightarrow \neg p \vee p \wedge q && [\text{distribution}] \\
 &\Leftrightarrow (\neg p \vee p) \wedge (\neg p \vee q) && [\text{excluded middle}] \\
 &\Leftrightarrow \text{true} \wedge (\neg p \vee q) && [\wedge \wedge \text{true}] \\
 &\Leftrightarrow \neg p \vee q && [\Rightarrow] \\
 &\Leftrightarrow p \Rightarrow q && [\Rightarrow]
 \end{aligned}$$

(f)

$$\begin{aligned}
 p \vee q &\Leftrightarrow p && [\Leftrightarrow] \\
 &\Leftrightarrow (p \vee q \Rightarrow p) \wedge (p \Rightarrow p \vee q) && [\Rightarrow] \\
 &\Leftrightarrow (\neg(p \vee q) \vee p) \wedge (\neg p \vee p \vee q) && [\text{De Morgan}] \\
 &\Leftrightarrow (\neg p \wedge \neg q \vee p) \wedge (\neg p \vee p \vee q) && [\text{distribution}] \\
 &\Leftrightarrow (\neg p \vee p) \wedge (\neg q \vee p) \wedge (\neg p \vee p \vee q) && [\text{excluded middle}] \\
 &\Leftrightarrow \text{true} \wedge (\neg q \vee p) \wedge (\neg p \vee p \vee q) && [\wedge \wedge \text{true}] \\
 &\Leftrightarrow (\neg q \vee p) \wedge (\neg p \vee p \vee q) && [\text{associativity}] \\
 &\Leftrightarrow (\neg q \vee p) \wedge (\neg p \vee p \vee q) && [\text{excluded middle}] \\
 &\Leftrightarrow (\neg q \vee p) \wedge (\text{true} \vee q) && [\vee \wedge \text{true}] \\
 &\Leftrightarrow (\neg q \vee p) \wedge \text{true} && [\wedge \wedge \text{true}] \\
 &\Leftrightarrow \neg q \vee p && [\Rightarrow] \\
 &\Leftrightarrow q \Rightarrow p && [\Rightarrow]
 \end{aligned}$$

#### Solution 4

(a)  $(p \text{ or } q) \Leftrightarrow ((\text{not } p \text{ or not } q) \text{ and } q)$  is not a tautology. You might illustrate this via a truth table or via a chain of equivalences, showing that the proposition is not equivalent to true. Alternatively, you might try and find a combination of values for which the proposition is false. (In this case, the proposition is false when  $p$  and  $q$  are both true.)

(b)  $(p \text{ or } q) \Leftrightarrow ((\text{not } p \text{ and not } q) \text{ or } q)$  is not a tautology. In this case, the proposition is false when p is true and q is false.

### Solution 5

(a)

$$\exists d: Dog \bullet gentle(d) \wedge well\_trained(d)$$

(b)

$$\forall d: Dog \bullet neat(d) \wedge well\_trained(d) \Rightarrow attractive(d)$$

(c)

(Requires nested quantifier in implication - parser limitation)

### Solution 6

(a)

This is a true *proposition* : *whatever the value of x, the expression*  $x^2 - x + 1$  *denotes a natural number. If we choose y to be this natural number, we will find that p is true.*

(b)

This is a false proposition. We cannot choose a large enough value for y such that p will hold for any value of x.

(c)

This is a false proposition. It is an implication whose antecedent part is true and whose consequent part is false.

(d)

This is a true proposition. It is an implication whose antecedent part is false and whose consequent part is true.

#### Solution 7

(a)

We must define a predicate  $p$  that is false for at least one value of  $x$ , and is true for at least one other value. A suitable solution would be  $p \Leftrightarrow x > 1$ .

(b)

With the above choice of  $p$ , we require only that  $q$  is sometimes false when  $p$  is true (for else the universal quantification would hold). A suitable solution would be  $q \Leftrightarrow x > 3$ .

#### Solution 8

(a)

$$\forall x: N \bullet x \geq z$$

## Equality

#### Solution 9

(d)

$$\begin{aligned} \exists x: N \bullet x = 1 \wedge x > y \vee x = 2 \wedge x > z \\ \Leftrightarrow \exists x: N \bullet x = 1 \wedge x > y \vee \exists x: N \bullet x = 2 \wedge x > z \\ \Leftrightarrow 1 \in N \wedge 1 > y \vee \exists x: N \bullet x = 2 \wedge x > z \\ \Leftrightarrow 1 \in N \wedge 1 > y \vee 2 \in N \wedge 2 > z \\ \Leftrightarrow 1 > y \vee 2 > z \end{aligned}$$

#### Solution 10

As discussed, the quantifier  $\exists_1$  can help give rise to a 'test' or 'precondition' to ensure that an application of mu will work.

So, as a simple example, as the proposition

$$\exists_1 n: N \bullet \forall m: N \bullet n \leq m$$

is equivalent to true, we can be certain that the statement

$$\mu n: N \bullet \forall m: N \bullet n \leq m$$

will return a result (which happens to be 0).

### Solution 11

(a)

$(\mu a: N \bullet a = a) = 0$  is a provable statement, since 0 is the only natural number with the specified property.

(b)

$(\mu b: N \bullet b = b) = 1$  is not provable. The specified property is true of both 0 and 1, and thus the value of the mu-expression is undefined.

(c)

$(\mu c: N \bullet c > c) = (\mu c: N \bullet c > c)$  is a provable statement. Neither expression is properly defined, but we may conclude that they are equal; there is little else that we can prove about them.

(d)

$(\mu d: N \bullet d = d) = 1$  is not a provable statement. We cannot confirm that 1 is the only natural number with the specified property; we do not know what value is taken by undefined operations.

### Solution 12

(Requires mu-operator with expression part - not yet implemented)

(a)

$$\mu m: Mountain \mid \forall n: Mountain \bullet height(n) \leq height(m) \bullet height(m)$$

(b)

$$\mu c: Chapter \mid \exists_1 d: Chapter \bullet length(d) > length(c) \bullet length(c)$$

(c)

Assuming the existence of a suitable function, max:  $(\mu n: N \bullet n = \max(\{m: N \mid 8 * m < 100.8 * m\}) \cdot 100 - n)$

## Deductive proofs

### Solution 13

$$\frac{\frac{\frac{\frac{\frac{p \wedge (p \Rightarrow q)}{p} [\wedge\text{-elim}^{[1]}]}{p \wedge (p \Rightarrow q) \Rightarrow p \wedge q} [\Rightarrow\text{-intro}^{[1]}]}{\frac{\frac{\frac{p \wedge (p \Rightarrow q)}{p \Rightarrow q} [\wedge\text{-elim}^{[1]}]}{p} [\Rightarrow\text{elim}]}{p \wedge (p \Rightarrow q) \Rightarrow p \wedge q} [\wedge\text{intro}]}}{\frac{\frac{p \wedge (p \Rightarrow q)}{p \wedge (p \Rightarrow q) \Rightarrow p \wedge q} [\wedge\text{-elim}^{[1]}]}{p \wedge (p \Rightarrow q) \Rightarrow p \wedge q} [\Rightarrow\text{-intro}^{[1]}]}$$

### Solution 14

In one direction:

$$\frac{\frac{\frac{\frac{\frac{\frac{p \wedge q}{p \wedge q} [\text{derived}]}{p \wedge q} [\Rightarrow\text{elim from } 1 \wedge 2]}{p \wedge q} [\wedge\text{-elim}^{[3]}]}{p \wedge q} [\Rightarrow\text{-intro}^{[2]}]}{\frac{\frac{p \wedge q}{p \wedge q} [\Rightarrow\text{-intro}^{[1]}]}{p \wedge q} [\Rightarrow\text{-intro}^{[1]}]}$$

and the other:

$$\frac{\frac{\frac{\frac{\Gamma p \wedge q^{\neg[2]} \quad \Gamma p^{\neg[2]}}{p \wedge q \Rightarrow p} [\Rightarrow\text{-intro}^{[2]}] \quad \frac{\frac{\Gamma p^{\neg[3]} \quad \Gamma p \wedge q^{\neg[1]}}{p \Rightarrow p \wedge q} [\Rightarrow\text{-intro}^{[3]}]}{p \Rightarrow p \wedge q} [\Leftrightarrow \text{intro}]}{p \wedge q \Leftrightarrow p} [\Rightarrow\text{-intro}^{[1]}]}{\Gamma p \Rightarrow q^{\neg[1]}} [\Rightarrow\text{-intro}^{[1]}]$$

We can then combine these two proofs *with*  $\Leftrightarrow$  *intro*.

### Solution 15

$$\frac{\frac{\frac{\frac{\Gamma p \Rightarrow q^{\neg[1]} \quad \Gamma p^{\neg[2]}}{q} [\Rightarrow \text{elim}] \quad \frac{\Gamma \neg q^{\neg[1]}}{\text{false}} [\text{false intro}]}{\text{false}} [\text{false-elim}^{[2]}]}{\frac{\Gamma p^{\neg[2]}}{\neg p} [\Rightarrow\text{-intro}^{[1]}]} [\Rightarrow\text{-intro}^{[1]}]$$

### Solution 16

In one direction:

$$\frac{\frac{\frac{\frac{\frac{\Gamma p^{\neg[1]} \quad \bar{r} [\text{case assumption}]}{p \wedge r} [\wedge \text{intro}]}{p \wedge q \vee p \wedge r} [\vee \text{intro}]}{\frac{\frac{\frac{\Gamma p^{\neg[1]} \quad \bar{q} [\text{case assumption}]}{p \wedge q} [\wedge \text{intro}]}{p \wedge q \vee p \wedge r} [\vee \text{intro}]} [\vee\text{-elim}^{[2]}]}{\frac{\Gamma p \wedge (q \vee r)^{\neg[1]}}{p \wedge (q \vee r) \Rightarrow p \wedge q \vee p \wedge r} [\Rightarrow\text{-intro}^{[1]}]} [\Rightarrow\text{-intro}^{[1]}]$$

In the other:



$$\begin{array}{c}
\frac{}{p} [\wedge \text{ elim}] \\
\frac{}{q \vee r} [\vee \text{ intro}] \\
\frac{}{p} [\wedge \text{ elim}] \quad \frac{}{p \wedge (q \vee r)} [\wedge \text{ intro}] \\
\frac{}{q \vee r} [\vee \text{ intro}] \\
\frac{}{p \wedge (q \vee r)} [\wedge \text{ intro}] \\
\frac{\ulcorner p \wedge q \vee p \wedge r \urcorner^{[3]} \quad \ulcorner \text{case1} \vee \text{case2} \urcorner^{[3]}}{p \wedge (q \vee r)} [\vee\text{-elim}^{[4]}] \\
\frac{\ulcorner p \wedge q \vee p \wedge r \urcorner^{[3]} \quad p \wedge (q \vee r)}{p \wedge q \vee p \wedge r \Rightarrow p \wedge (q \vee r)} [\Rightarrow\text{-intro}^{[3]}]
\end{array}$$

### Solution 17

In one direction:

$$\frac{\frac{\ulcorner p \vee q \wedge r \urcorner^{[3]} \quad \overline{(p \vee q) \wedge (p \vee r)}}{p \vee q \wedge r \Rightarrow (p \vee q) \wedge (p \vee r)} [\Rightarrow\text{-intro}^{[3]}] \quad [\vee \text{ elim } \wedge \wedge \text{ intro}]$$

and the other:

$$\frac{\ulcorner (p \vee q) \wedge (p \vee r) \urcorner^{[1]} \quad \ulcorner p \vee q \wedge r \urcorner^{[2]}}{(p \vee q) \wedge (p \vee r) \Rightarrow p \vee q \wedge r} [\Rightarrow\text{-intro}^{[1]}]$$

### Solution 18

In one direction:

$$\frac{\ulcorner p \Rightarrow q \urcorner^{[1]} \quad \neg p \vee q}{(p \Rightarrow q) \Rightarrow \neg p \vee q} [\Rightarrow\text{-intro}^{[1]}]$$

and the other:

$$\frac{\frac{\frac{\neg p \vee q}{\neg p \vee q} \quad \frac{\frac{p \Rightarrow q}{p \Rightarrow q}}{\neg p \vee q \Rightarrow (p \Rightarrow q)} [\Rightarrow\text{-intro}^{[4]}] \quad \frac{\neg p \vee q \Rightarrow (p \Rightarrow q)}{\neg p \vee q \Rightarrow (p \Rightarrow q)} [\Rightarrow\text{-intro}^{[3]}]}{\neg p \vee q \Rightarrow (p \Rightarrow q)}$$

## Sets and types

### Solution 19

(a)

1 in  $\{4, 3, 2, 1\}$  is true.

(b)

$\{1\}$  in  $\{1, 2, 3, 4\}$  is undefined.

(c)

$\{1\}$  in  $\{\{1\}, \{2\}, \{3\}, \{4\}\}$  is true.

(d)

The empty set in  $\{1, 2, 3, 4\}$  is undefined.

### Solution 20

(a)

$$\{1\} \times \{2, 3\}$$

is the set  $\{(1, 2), (1, 3)\}$

(b)

The empty set cross  $\{2, 3\}$  is the empty set

(c)

$$\mathbb{P} \emptyset \times \{1\}$$

is the set  $\{(\emptyset, 1)\}$

(d)

$\{(1, 2)\}$  cross  $\{3, 4\}$  is the set  $\{((1, 2), 3), ((1, 2), 4)\}$

### Solution 21

There are various ways of describing these sets via set comprehensions. Examples are given below.

(a)

$$\{z : Z \mid 0 \leq z \wedge z \leq 100\}$$

(b)

$$\{z : Z \mid z = 10\}$$

(c)

$$\{z : Z \mid z \bmod 2 = 0 \vee z \bmod 3 = 0 \vee z \bmod 5 = 0\}$$

### Solution 22

(a)

$$\{n : N \mid n \leq 4 \bullet n^2\}$$

(b)

$$\{n : N \mid n \leq 4 \bullet (n, n^2)\}$$

(c)

$$n : P0, 1$$

(d)

$$\{n : \mathbb{P}\{0, 1\} \mid true \bullet (n, \#n)\}$$

### Solution 23

(a)

$$\begin{aligned} x \in a \cap a \\ \Leftrightarrow x \in a \wedge x \in a \\ \Leftrightarrow x \in a \end{aligned}$$

(b)

$$\begin{aligned} x \in a \cup a \\ \Leftrightarrow x \in a \vee x \in a \\ \Leftrightarrow x \in a \end{aligned}$$

### Solution 24

(a)

The set of all pairs of integers is  $Z$  cross  $Z$ . To give it a name, we could write:

$$\text{Pairs} == Z \times Z$$

(b)

The set of all integer pairs in which each element is strictly greater than zero could be defined by:

$$\text{StrictlyPositivePairs} == \{ m, n : Z \mid m > 0 \wedge n > 0 \bullet (m, n) \}$$

(c)

It is intuitive to use a singular noun for the name of a basic type; we define the set of all people by writing:

$$[Person]$$

(d)

The set of all couples could be defined by:

$$\text{Couples} == \{ s : \mathbb{P} \text{ Person} \mid \#s = 2 \}$$

### Solution 25

(Requires generic set notation and Cartesian product)

### Solution 26

(Requires generic parameters and relation type notation)

## Relations

### Solution 27

(a)

The power set of  $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$  is:

$$\{\emptyset, \{(0, 0)\}, \{(0, 1)\}, \{(1, 0)\}, \{(1, 1)\}, \{(1, 0), (1, 1)\}, \{(0, 0), (0, 1)\}, \{(0, 1), (1, 1)\}, \{(0, 1), (1, 0)\}, \{(0, 0), (1, 1)\}, \{(0, 0), (0, 1), (1, 0)\}, \{(0, 0), (0, 1), (1, 1)\}, \{(0, 1), (1, 0), (1, 1)\}, \{(0, 0), (1, 0), (1, 1)\}, \{(0, 0), (0, 1), (1, 1)\}\}$$

(b)

$$\{\emptyset, \{(0, 0)\}, \{(0, 1)\}, \{(0, 0), (0, 1)\}\}$$

(c)

$$\{\emptyset\}$$

(d)

$$\{\emptyset\}$$

**Solution 28**

(a)

$$\text{dom } R = \{0, 1, 2\}$$

(b)

$$\text{ran } R = \{1, 2, 3\}$$

(c)

$$\{1, 2\} \triangleleft R = \{1 \mapsto 2, 1 \mapsto 3, 2 \mapsto 3\}$$

**Solution 29**

(a)

$$\{2 \mapsto 4, 3 \mapsto 3, 3 \mapsto 4, 4 \mapsto 2\}$$

(b)

$$\{1 \mapsto 3, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1\}$$

(c)

$$\{1 \mapsto 1, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 2, 3 \mapsto 3, 4 \mapsto 4\}$$

(d)

$$\{1 \mapsto 4, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 2, 3 \mapsto 3, 4 \mapsto 1\}$$

**Solution 30**

$$\mid \quad \text{childOf} : \text{Person} \leftrightarrow \text{Person}$$

(a)

$$\text{parentOf} == \text{childOf}^{-1}$$

This is a good example of how there are many different ways of writing the same thing. An alternative abbreviation is:

$$\text{parentOf} == \{ x, y : \text{Person} \mid x \mapsto y \in \text{childOf} \bullet y \mapsto x \}$$

Or, via an axiomatic definition:

$$\left| \begin{array}{l} \text{parentOf} : \text{Person} \leftrightarrow \text{Person} \\ \hline \text{parentOf} = \text{childOf}^{-1} \end{array} \right|$$

(b)

$$\text{siblingOf} == (\text{childOf} \circ \text{parentOf}) \setminus \text{id}$$

(c)

$$\text{cousinOf} == \text{childOf} \circ \text{siblingOf} \circ \text{parentOf}$$

(d)

$$\text{ancestorOf} == \text{parentOf}^+$$

### Solution 31

(Requires compound identifiers with operators -  $\mathbb{R}^+$ ,  $\mathbb{R}^*$ )

(a)

$$\mathbb{R} == \{ a, b : \mathbb{N} \mid b = a \vee b = a \}$$

(b)

$$\mathbb{S} == \{ a, b : \mathbb{N} \mid b = a \vee b = a \}$$

(c)

$$\mathbb{R}^+ == \{ a, b : \mathbb{N} \mid b > a \}$$

(d)

$$\mathbb{R}^* == \{ a, b : \mathbb{N} \mid b \geq a \}$$

### Solution 32

(a)

$$\begin{aligned}
x \mapsto y \in A \triangleleft B \triangleleft R \\
&\Leftrightarrow x \in A \wedge x \mapsto y \in (B \triangleleft R) \\
&\Leftrightarrow x \in A \wedge x \in B \wedge x \mapsto y \in R \\
&\Leftrightarrow x \in A \cap B \wedge x \mapsto y \in R \\
&\Leftrightarrow x \mapsto y \in A \cap B \triangleleft R
\end{aligned}$$

(b)

$$\begin{aligned}
x \mapsto y \in R \cup S \triangleright C \\
&\Leftrightarrow x \mapsto y \in R \cup S \wedge y \in C \\
&\Leftrightarrow (x \mapsto y \in R \vee x \mapsto y \in S) \wedge y \in C \\
&\Leftrightarrow x \mapsto y \in R \wedge y \in C \vee x \mapsto y \in S \wedge y \in C \\
&\Leftrightarrow x \mapsto y \in R \triangleright C \vee x \mapsto y \in S \triangleright C \\
&\Leftrightarrow x \mapsto y \in (R \triangleright C) \cup (S \triangleright C)
\end{aligned}$$

## Functions

### Solution 33

The set of 9 functions:

$$\{\emptyset, \{(0, 0)\}, \{(0, 1)\}, \{(1, 1)\}, \{(1, 0)\}, \{(0, 0), (1, 1)\}, \{(0, 1), (1, 1)\}, \{(1, 0), (0, 0)\}, \{(0, 1), (1, 0)\}\}$$

(a)

The set of total functions:

$$\{\{(0, 0), (1, 1)\}, \{(0, 1), (1, 1)\}, \{(1, 0), (0, 0)\}, \{(0, 1), (1, 0)\}\}$$

(b)

The set of functions which are neither injective nor surjective:



$$\{\{(0, 1), (1, 1)\}, \{(0, 0), (1, 0)\}\}$$

(c)

The set of functions which are injective but not surjective:

$$\{\emptyset, \{(0, 0)\}, \{(0, 1)\}, \{(1, 0)\}, \{(1, 1)\}\}$$

(d)

There are no functions (of this type) which are surjective but not injective.

(e)

The set of bijective functions:

$$\{\{(0, 0), (1, 1)\}, \{(0, 1), (1, 0)\}\}$$

### Solution 34

(a)

$$\{1 \mapsto a, 2 \mapsto b, 3 \mapsto c, 4 \mapsto b\}$$

(b)

$$\{1 \mapsto c, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d\}$$

(c)

$$\{1 \mapsto c, 2 \mapsto b, 3 \mapsto c, 4 \mapsto b\}$$

(d)

$$\{1 \mapsto c, 2 \mapsto b, 3 \mapsto c, 4 \mapsto b\}$$

### Solution 35

(Requires power set notation  $\mathbb{P}$  and relational image)

(a)

$$\frac{\text{children} : \text{Person} \rightarrow \mathbb{P} \text{ Person}}{\text{children} = \{p : \text{Person} \bullet p \mapsto \text{parentOf}(\llbracket \{p\} \rrbracket)\}}$$

(b)

$$\frac{\text{number\_of\_grandchildren} : \text{Person} \rightarrow N}{\text{number\_of\_grandchildren} = \{p : \text{Person} \bullet p \mapsto \# \text{parentOf} \circ \text{parentOf}(\llbracket \{p\} \rrbracket)\}}$$

### Solution 36

(Note : This solution demonstrates relation types in quantifier domains)

$$\frac{\text{number\_of\_drivers} : \text{Drivers} \leftrightarrow \text{Cars} \rightarrow (\text{Cars} \rightarrow N)}{\text{number\_of\_drivers} = \lambda r : \text{Drivers} \leftrightarrow \text{Cars} \bullet \{c : \text{ran } r \bullet c \mapsto \#\{d : \text{Drivers} \mid d \mapsto c \in r\}\}}$$

## Sequences

### Solution 37

(a)

$$\langle a \rangle$$

(b)

$$\{1 \mapsto a, 2 \mapsto b, 2 \mapsto a, 3 \mapsto c, 3 \mapsto b, 4 \mapsto d\}$$

(c)

$$\{2 \mapsto b, 3 \mapsto c, 4 \mapsto d\}$$

(d)

$$\{1, 2, 3, 4\}$$

(e)

$$\{a, b\}$$

(f)

$$\{a \mapsto 1, b \mapsto 2, c \mapsto 3, d \mapsto 4\}$$

(g)

$$\langle a, b \rangle$$

(h)

$$\{3 \mapsto b\}$$

(i)

$$\{a\}$$

(j)

$$c$$

### Solution 38

(a)

$$\frac{f : Place \rightarrow \mathbb{P} Place}{\forall p : Place \bullet f(p) = \{q : Place \mid p \mapsto q \in \text{ran } \text{trains}\}}$$

(b)

$$\{p : Place \mid \exists_1 x : \text{dom } trains \bullet trains(x).2 = p\}$$

(c)

$$\mu p : Place \bullet \forall q : Place \bullet p \neq q \wedge \#\{x : \text{dom } trains \mid trains(x).2 = p\} > \#\{x : \text{dom } trains \mid trains(x).2 = q\}$$

### Solution 39

(a)

$$large\_coins : Collection \rightarrow N$$

$$\forall c : Collection \bullet large\_coins(c) = c(large)$$

(Blocked by : *underscoreinidentifierforfuzzcompatibility*)

(b)

$$add\_coin : Collection * Coin \rightarrow Collection$$

$$\forall c : Collection \bullet \forall d : Coin \bullet add\_coin(c, d) = c \cup \llbracket d \rrbracket$$

(Blocked by : *underscoreinidentifierandbagunion*)

## Modelling

Solutions 40-52 are work in progress - many require features not yet implemented

### Solution 40

(Work in progress - requires semicolon-separated bindings in set comprehensions)

(a)

$hd : seq(Title * Length * Viewed)$

$cumulative\_total(hd) \leq 12000$

$\forall p : \text{ran } hd \bullet p.2 \leq 360$

Note that  $cumulative\_total$  is defined in part (d).

(b)

$\{p : \text{ran } hd \mid p.2 > 120 \bullet p.1\}$

(c)

These can be defined recursively:

$$\left| \begin{array}{l} viewed : seq \ Programme \rightarrow seq \ Programme \\ viewed(\langle \rangle) = \langle \rangle \wedge \forall x : Programme \bullet \forall s : seq \ Programme \bullet viewed(\langle x \rangle \frown s) = (\text{if } x.3 = yes \text{ then } \langle x \rangle \frown s) \end{array} \right|$$

or otherwise (omitted - requires semicolon-separated bindings in set comprehension)

(d)

$$\left| \begin{array}{l} cumulative\_total : seq \ Title * Length * Viewed \rightarrow N \\ cumulative\_total(\langle \rangle) = 0 \wedge \forall x : Title * Length * Viewed \bullet \forall s : seq \ Title * Length * Viewed \bullet cumulative\_total(\langle x \rangle \frown s) = 1 + cumulative\_total(s) \end{array} \right|$$

(e)

$(\mu p : \text{ran } hd \mid \forall q : \text{ran } hd \bullet p \neq q \wedge p.2 > q.2 \mid p.1)$

(This, of course, assumes that there is a unique element with this property.)

(f)

(f) Omitted - requires semicolon-separated bindings in nested set comprehension

(g)

axdef

$g : seq(Title * Length * Viewed) \rightarrow seq(Title * Length * Viewed)$

where

$\forall s : seq\ Title * Length * Viewed \bullet g(s) = s \triangleright \{x : \text{ran } s \mid x \neq longest\_viewed(s)\}$

end

Where  $longest\_viewed$  is defined as

axdef

$longest\_viewed : seq(Title * Length * Viewed)^+ \rightarrow Title * Length * Viewed$

where

$\forall s : seq\ Title * Length * Viewed \bullet longest\_viewed(s) = (\mu p : \text{ran } s \bullet p.3 = yes \wedge \forall q : \text{ran } s \bullet p \neq q \wedge q.3 = yes \wedge p.2 > q.2)$

end

This, of course, assumes that there is at least one viewed programme (and one of a unique maximum length).

(h)

$$\left| \begin{array}{l} s : \text{seq } Title * Length * Viewed \rightarrow \text{seq } Title * Length * Viewed \\ \hline \forall x : \text{seq } Title * Length * Viewed \bullet \text{items}(s(x)) = \text{items}(x) \wedge \forall i, j : \text{dom } s(x) \bullet i < j \Rightarrow s(x)(i).2 \geq s(x)(j).2 \end{array} \right|$$

**Solution 41**

(a)

axdef

*records* : Year  $\leftrightarrow$  Table

where

$\text{dom}(\text{records}) = 1993..current$

$\forall y : \text{dom } records \bullet \#records(y) \leq 50$

$\forall y : \text{dom}(records) \mid \forall e : \text{ran } records(y) \bullet \text{year}(e.1) = y$

$\forall r : \text{ran}(records) \mid \forall i1, i2 : \text{dom } r \bullet i1 \neq i2 \wedge r(i1).1 = r(i2).1 \Rightarrow r(i1).3 \neq r(i2).3$

end

(b)

(i)

$\{e : Entry \mid \exists r : \text{ran } records \bullet e \in \text{ran } r \wedge e.3 = 479\}$

ii

$$\{e: Entry \mid \exists r: \text{ran } records \bullet e \in \text{ran } r \wedge e.6 > e.5\}$$

iii

$$\{e: Entry \mid \exists r: \text{ran } records \bullet e \in \text{ran } r \wedge e.7 \geq 70\}$$

iv

$$\{c: Course \mid \forall r: \text{ran } records \bullet \forall e: \text{ran } r \bullet e.2 = c \Rightarrow e.7 \geq 70\}$$

v

$$y: Year \mid y \text{indom} records. \ y \mapsto l: Lecturer \mid c: \text{ran}(recordsy) \mid c.4 = 1 > 6$$

(c)

axdef

where

$$\forall x: Entry \bullet \forall s: \text{seq } Entry \bullet 479_{courses}(\langle \rangle) = \langle \rangle \text{ and } 479_{courses}(\langle x \rangle^s) = \text{if } x.3 = 479 \text{ then } \langle x \rangle^{479_{courses}(s)} \text{ else } 479_{courses}(s)$$

end

(Blocked by : *underscore in identifier – use camelCase for fuzz compatibility*)

(d)

$$\left| \forall x: Entry \bullet \forall s: \text{seq } Entry \bullet total(\langle \rangle) = 0 \wedge total(\langle x \rangle \frown s) = x.5 + total(s) \right|$$



**Solution 42**

$[Person]$

axdef

$State : P(seq(iseq(Person)))$

where

$\forall s : State \mid \forall i, j : \text{dom } s \bullet i \neq j \wedge \text{ran } s(i) \cap \text{ran } s(j) = \{\}$

end

(b)

axdef

$add : N * Person * State \mapsto State$

where

$\forall n : N \bullet \forall p : Person \bullet \forall s : State \bullet n \in \text{dom } s \wedge p \notin \bigcup \text{ran } s$

$add(n, p, s) = s ++ n \mapsto s(n) \langle p \rangle$

end

(Blocked by:  $\mapsto$  operator not implemented)

**Solution 43**

(a)

(i)  $\forall i : \text{dom bookings} \mid \forall x, y : \text{bookings}(i) \bullet x \neq y \wedge x.2 \dots x.3 \cap y.2 \dots y.3 = \{\}$

(ii)  $\forall i : \text{dom bookings} \bullet \text{forall } x : \text{bookings}(i) \mid \{x.2, x.3\} \text{ subseq } 1.. \text{max}(i.1)$

(iii)  $\forall i : \text{dom bookings} \mid \forall b : \text{bookings}(i) \bullet b.2 \leq b.3$

(iv) This is enforced by the constraint for part (i).

(b)

(i)  $\{i : \text{dom bookings} \mid i.1 = \text{Banbury} \bullet i.2\}$

(ii)  $\{i : \text{dom bookings} \mid i.1 = \text{Banbury} \wedge \exists b : \text{bookings}(i) \bullet 50 \in b.2 \dots b.3\}$

(iii)  $r : \text{Room}; s : N \mid \exists i : \text{dom bookings} \bullet i.1 = r \wedge i.2 = s. \text{ (r, s)}$

(iv)  $r : \text{Room} \mid \exists i : \text{dom bookings} \bullet i.1 = r \wedge \# \text{bookings}(i) \geq 10$

## Free types and induction

$[N]$

$Tree ::= stalk \mid leaf \langle\langle N \rangle\rangle \mid branch \langle\langle Tree \times Tree \rangle\rangle$

### Solution 44

The two cases of the proof are established by equational *reasoning* : *thefirstby*

$$\text{reverse } (\langle \rangle^t) = \text{reverset}[cat.1a] = (\text{reverset})\langle \rangle \text{ [cat.1b]}$$

where cat.1a is  $\langle \rangle s = \text{sandcat.1biss} \langle \rangle = s$

and the second by

$$\text{reverse } ((\langle x \rangle^u)^t) = \text{reverse}(\langle x \rangle^{\langle u^t \rangle})[cat.2]$$

$$= \text{reverse } (u^t \langle x \rangle) [\text{reverse.2}]$$

$$= (\text{reverse } t^{\text{reverse} u}) \langle x \rangle [\text{anti-distributive}]$$

$$= \text{reverse } t^{\langle \text{reverse} u \rangle} \langle x \rangle [\text{cat.2}]$$

$$= \text{reverse } t^{\text{reverse}(\langle x \rangle^u)} [\text{reverse.2}]$$

### Solution 45

The base case:

$$\text{reverse}(\text{reverse } \langle \rangle) = \text{reverse } \langle \rangle [\text{reverse.1}] = \langle \rangle [\text{reverse.1}]$$

The inductive step:

$$\begin{aligned} & \text{reverse}(\text{reverse}(\langle x \rangle^t)) \\ &= \text{reverse}((\text{reverse } t) \langle x \rangle) [\text{reverse.2}] \\ &= \text{reverse}(\langle x \rangle)^{\text{reverse}(\text{reverse } t)} [\text{anti-distributive}] \\ &= \text{reverse}(\langle x \rangle \langle \rangle)^{\text{reverse}(\text{reverse } t)} [\text{cat.1}] \\ &= ((\text{reverse } \langle \rangle) \langle x \rangle)^{\text{reverse}(\text{reverse } t)} [\text{reverse.2}] \\ &= (\langle \rangle \langle x \rangle)^{\text{reverse}(\text{reverse } t)} [\text{reverse.1}] \\ &= \langle x \rangle^{\text{reverse}(\text{reverse } t)} [\text{cat.1}] \\ &= \langle x \rangle^t [\text{reverse}(\text{reverse } t) = t] \end{aligned}$$

#### Solution 46

(a)

$$\text{count} : \text{Tree} \rightarrow N$$

$$\text{count stalk} = 0$$

$$\forall n: N \bullet \text{count}(\text{leaf } n) = 1$$

$$\forall t1, t2: \text{Tree} \bullet \text{count}(\text{branch}(t1, t2)) = \text{count } t1 + \text{count } t2$$

(Blocked *by* : *recursive freetypes and pattern matching*)

(b)

$$\text{flatten} : \text{Tree} \rightarrow \text{seq } N$$

$$\text{flatten stalk} = \langle \rangle$$

$$\forall n: N \bullet \text{flatten}(\text{leaf } n) = \langle n \rangle$$

$$\forall t1, t2: \text{Tree} \bullet \text{flatten}(\text{branch}(t1, t2)) = \text{flatten } t1 \mathbin{^f} \text{flatten } t2$$

(Blocked *by* : *recursive freetypes and pattern matching*)

#### Solution 47

First, exhibit the induction principle for the free type:

$$\mathbb{P} \text{ stalk and } (\forall n: N \bullet \mathbb{P}(\text{leaf } n)) \text{ and } (\forall t1, t2: \text{Tree} \bullet \mathbb{P} t1 \wedge \mathbb{P} t2 \Rightarrow \mathbb{P} \text{branch}(t1, t2))$$

$$\text{implies } \forall t: \text{Tree} \bullet \mathbb{P} t$$

This gives three cases for the proof:

$$(\text{flatten stalk}) = \langle \rangle \text{ [flatten]} = 0 \text{ []} = \text{count stalk [count]}$$

(Remaining cases omitted - require equational reasoning with recursive functions)

## Supplementary material : assignment practice

### Solution 48

$[SongId, UserId, PlaylistId, Playlist]$

$songs : \mathbb{F} SongId$ $users : \mathbb{F} UserId$ $playlists : PlaylistId \rightarrow Playlist$ $playlistOwner : PlaylistId \rightarrow UserId$
$\forall i : \text{dom } playlists \bullet \text{ran } playlists(i) \subseteq songs$ $\text{dom } playlistOwner \subseteq \text{dom } playlists$ $\text{ran } playlistOwner \subseteq \text{dom } users$

### Solution 49

$hated : UserId \rightarrow \mathbb{F} SongId$ $loved : UserId \rightarrow \mathbb{F} SongId$
$\text{dom } hated \subseteq users$ $\forall i : \text{dom } hated \bullet hated(i) \subseteq songs$ $\text{dom } loved \subseteq users$ $\forall i : \text{dom } loved \bullet loved(i) \subseteq songs$

### Solution 50

(a)

*abbrev*

$A == users \setminus \bigcup \text{ran } playlistSubscribers$

(b)

*abbrev*

$B == \{ p : \text{dom } playlistSubscribers \mid \#playlistSubscribers(p) \geq 100 \}$

(c)

$C == \mu u : \text{dom } loved \bullet \forall v : \text{dom } loved \bullet u \neq v \wedge \#loved(u) > \#loved(v)$

(d)

$D == \mu s : songs \bullet \forall t : songs \bullet s \neq t \wedge \#\{u : UserId \mid s \in loved(u)\} > \#\{u : UserId \mid t \in loved(u)\}$

### Solution 51

(a)

Let's first define two helper functions:

$loveHateScore : SongId+ \rightarrow N$

$\forall i : songs \mid \{u : UserId \mid i \in loved(u)\} \geq \{u : UserId \mid i \in hated(u)\} \Rightarrow$

$loveHateScore(i) = \{u : UserId \mid i \in loved(u)\} - \{u : UserId \mid i \in hated(u)\}$

and

$\forall i : songs \mid \{u : UserId \mid i \in loved(u)\} < \{u : UserId \mid i \in hated(u)\} \Rightarrow$

$loveHateScore(i) = 0$

$$\frac{\text{playlistCount} : SongId \rightarrow N}{\forall i : songs \bullet \text{playlistCount}(i) = \#\{p : \text{dom playlist} \mid i \in \text{ran playlist}(p)\}}$$

We then have:

$$\frac{\text{length} : SongId \rightarrow N \text{ popularity} : SongId \rightarrow N}{\text{dom length} \subseteq \text{dom popularity} \wedge \forall i : songs \bullet \text{length}(i) = \text{popularity}(i) + \text{loveHateScore}(i)}$$

(b)

$mostPopular : SongId$

$(\exists_1 i : songs \mid \forall j : songs \bullet i \neq j \wedge \text{popularity}(i) > \text{popularity}(j)) \Rightarrow$

$mostPopular = (\mu i : songs \mid \forall j : songs \bullet i \neq j \wedge \text{popularity}(i) > \text{popularity}(j))$

and

$\neg \exists_1 i : songs \bullet \forall j : songs \bullet i \neq j \wedge \text{popularity}(i) > \text{popularity}(j) \Rightarrow mostPopular = nullSong$

(c)

$\text{playlistsContainingMostPopularSong} == \{i : \text{dom } \text{playlists} \mid \text{mostPopular} \in \text{ran } \text{playlists}(i)\}$

### Solution 52

(a)

$\text{premiumPlays} : \text{seq}(\text{Play}) \rightarrow \text{seq}(\text{Play})$

$\text{premiumPlays}(\langle \rangle) = \langle \rangle$

$\forall x : \text{Play}; s : \text{seq}(\text{Play}) \mid$

$\text{premiumPlays}(\langle x \rangle^s) = \langle x \rangle^{\text{premiumPlays}(s)} \text{ if } \text{userStatus}(x.2) = \text{premium}$

$\text{premiumPlays}(s) \text{ if } \text{userStatus}(x.2) = \text{standard}$

(Note : Uses camel Case for fuzz compatibility)

(b)

$\text{standardPlays} : \text{seq}(\text{Play}) \rightarrow \text{seq}(\text{Play})$

$\text{standardPlays}(\langle \rangle) = \langle \rangle$

$\forall x : \text{Play}; s : \text{seq}(\text{Play}) \mid$

$\text{standardPlays}(\langle x \rangle^s) = \langle x \rangle^{\text{standardPlays}(s)} \text{ if } \text{userStatus}(x.2) = \text{standard}$

$\text{standardPlays}(s) \text{ if } \text{userStatus}(x.2) = \text{premium}$



*(Note : Uses camelCase for fuzz compatibility)*

(c)

*cumulativeLength* : *seq*(*Play*) → *N*

*cumulativeLength*(⟨⟩) = 0

∀ *x* : *Play*; *s* : *seq*(*Play*) |

*cumulativeLength*(⟨*x*⟩<sup>*s*</sup>) = *length*(*x*.1) + *cumulativeLength*(*s*)

*(Note : Uses camelCase for fuzz compatibility)*