Propositional logic

Solution 1

(a) $false(as(true \Rightarrow false) \Leftrightarrow false)$

(b) $true(as(false \Rightarrow false) \Leftrightarrow true)$

(c) $true(as(false \Rightarrow true) \Leftrightarrow true)$

(d) $true(as(false \Rightarrow false) \Leftrightarrow true)$

(Assuming that pigs can't fly . . .)

Solution 2

(a)

p	q	$p \wedge q$	$(\mathbf{p} \wedge \mathbf{q}) \Rightarrow \mathbf{p}$
t	t	t	**t**
t	f	f	**t**
f	t	f	**t**
f	f	f	**t**

(b)

p	q	$p \wedge q$	$\neg\, p$	$\neg\, p \Rightarrow (p \wedge q)$	$(\neg\, \mathbf{p} \Rightarrow (\mathbf{p} \wedge \mathbf{q})) \Leftrightarrow \mathbf{p}$
t	t	t	f	t	**t**
t	f	f	f	t	**t**
f	t	f	t	f	**t**
f	f	f	t	f	**t**

(c)

p	q	$p \Rightarrow q$	$p \wedge (p \Rightarrow q)$	$(\mathbf{p} \wedge (\mathbf{p} \Rightarrow \mathbf{q})) \Rightarrow \mathbf{q}$
t	t	t	t	**t**
t	f	f	f	**t**
f	t	t	f	**t**
f	f	t	f	**t**

Solution 3

(a)

$$\begin{aligned} & p \Rightarrow \neg\, p \\ \Leftrightarrow\ & \neg\, p \vee \neg\, p && [\Rightarrow] \\ \Leftrightarrow\ & \neg\, p && [\text{idempotence}] \end{aligned}$$

(b)

$$\begin{aligned} & \neg\, p \Rightarrow p \\ \Leftrightarrow\ & \neg\, \neg\, p \vee p && [\Rightarrow] \\ \Leftrightarrow\ & p \vee p && [\neg\ \neg] \\ \Leftrightarrow\ & p && [\text{idempotence}] \end{aligned}$$

(c)

$$
\begin{aligned}
& p \Rightarrow (q \Rightarrow r) \\
& \Leftrightarrow \neg\, p \vee (q \Rightarrow r) && [\Rightarrow] \\
& \Leftrightarrow \neg\, p \vee \neg\, q \vee r && [\Rightarrow] \\
& \Leftrightarrow \neg\, p \vee \neg\, q \vee r && [\text{associativity}] \\
& \Leftrightarrow \neg\, (p \wedge q) \vee r && [\text{De Morgan}] \\
& \Leftrightarrow p \wedge q \Rightarrow r && [\Rightarrow]
\end{aligned}
$$

(d)

$$
\begin{aligned}
& q \Rightarrow (p \Rightarrow r) \\
& \Leftrightarrow \neg\, q \vee (p \Rightarrow r) && [\Rightarrow] \\
& \Leftrightarrow \neg\, q \vee \neg\, p \vee r && [\Rightarrow] \\
& \Leftrightarrow \neg\, p \vee \neg\, q \vee r && [\text{associativity} \wedge \text{commutativity}] \\
& \Leftrightarrow \neg\, p \vee (q \Rightarrow r) && [\Rightarrow] \\
& \Leftrightarrow p \Rightarrow (q \Rightarrow r) && [\Rightarrow]
\end{aligned}
$$

(e)

$$
\begin{aligned}
& p \wedge q \Leftrightarrow p \\
& \Leftrightarrow (p \wedge q \Rightarrow p) \wedge (p \Rightarrow p \wedge q) && [\Leftrightarrow] \\
& \Leftrightarrow (\neg\, (p \wedge q) \vee p) \wedge (\neg\, p \vee p \wedge q) && [\Rightarrow] \\
& \Leftrightarrow (\neg\, p \vee \neg\, q \vee p) \wedge (\neg\, p \vee p \wedge q) && [\text{De Morgan}] \\
& \Leftrightarrow (\neg\, q \vee \neg\, p \vee p) \wedge (\neg\, p \vee p \wedge q) && [\text{associativity} \wedge \text{comm .}] \\
& \Leftrightarrow (\neg\, q \vee \mathit{true}) \wedge (\neg\, p \vee p \wedge q) && [\text{excluded middle}] \\
& \Leftrightarrow \mathit{true} \wedge (\neg\, p \vee p \wedge q) && [\vee \wedge \text{true}] \\
& \Leftrightarrow \neg\, p \vee p \wedge q && [\wedge \wedge \text{true}] \\
& \Leftrightarrow (\neg\, p \vee p) \wedge (\neg\, p \vee q) && [\text{distribution}] \\
& \Leftrightarrow \mathit{true} \wedge (\neg\, p \vee q) && [\text{excluded middle}] \\
& \Leftrightarrow \neg\, p \vee q && [\wedge \wedge \text{true}] \\
& \Leftrightarrow p \Rightarrow q && [\Rightarrow]
\end{aligned}
$$

(f)

$p \lor q \Leftrightarrow p$

$\Leftrightarrow (p \lor q \Rightarrow p) \land (p \Rightarrow p \lor q)$	[$\Leftrightarrow$]
$\Leftrightarrow (\neg\, (p \lor q) \lor p) \land (\neg\, p \lor p \lor q)$	[$\Rightarrow$]
$\Leftrightarrow (\neg\, p \land \neg\, q \lor p) \land (\neg\, p \lor p \lor q)$	[De Morgan]
$\Leftrightarrow (\neg\, p \lor p) \land (\neg\, q \lor p) \land (\neg\, p \lor p \lor q)$	[distribution]
$\Leftrightarrow \mathit{true} \land (\neg\, q \lor p) \land (\neg\, p \lor p \lor q)$	[excluded middle]
$\Leftrightarrow (\neg\, q \lor p) \land (\neg\, p \lor p \lor q)$	[$\land$ $\land$ true]
$\Leftrightarrow (\neg\, q \lor p) \land (\neg\, p \lor p \lor q)$	[associativity]
$\Leftrightarrow (\neg\, q \lor p) \land (\mathit{true} \lor q)$	[excluded middle]
$\Leftrightarrow (\neg\, q \lor p) \land \mathit{true}$	[$\lor$ $\land$ true]
$\Leftrightarrow \neg\, q \lor p$	[$\land$ $\land$ true]
$\Leftrightarrow q \Rightarrow p$	[$\Rightarrow$]

Solution 4

(a) (p or q) $\Leftrightarrow$ ((not p or not q) and q) is not a tautology. You might illustrate this via a truth table or via a chain of equivalences, showing that the proposition is not equivalent to true. Alternatively, you might try and find a combination of values for which the proposition is false. (In this case, the proposition is false when p and q are both true.)

(b) (p or q) $\Leftrightarrow$ ((not p and not q) or q) is not a tautology. In this case, the proposition is false when p is true and q is false.

Solution 5

(a) $\exists\, d \colon \mathit{Dog} \bullet \mathit{gentle}(d) \land \mathit{well\_trained}(d)$

(b) $\forall\, d \colon \mathit{Dog} \bullet \mathit{neat}(d) \land \mathit{well\_trained}(d) \Rightarrow \mathit{attractive}(d)$

(c) $\exists\, d \colon \mathit{Dog} \bullet \mathit{gentle}(d) \Rightarrow \forall\, t \colon \mathit{Trainer} \bullet \mathit{groomed}(d, t)$

Solution 6

(a) This is a true proposition: whatever the value of x, the expression x^2 - x + 1 denotes a natural number. If we choose y to be this natural number, we will find that p is true.

(b) This is a false proposition. We cannot choose a large enough value for y such that p will hold for any value of x.

(c) This is a false proposition. It is an implication whose antecedent part is true and whose consequent part is false.

(d) This is a true proposition. It is an implication whose antecedent part is false and whose consequent part is true.

Solution 7

(a) We must define a predicate p that is false for at least one value of x, and is true for at least one other value. A suitable solution would be $p \Leftrightarrow x > 1$.

(b) With the above choice of p, we require only that q is sometimes false when p is true (for else the universal quantification would hold). A suitable solution would be $q \Leftrightarrow x > 3$.

Solution 8

(a) $\forall x : \mathbb{N} \bullet x \geq z$

(b) $\forall z : \mathbb{N} \bullet z \geq x + y$

(c) $x + 3 > 0 \land \forall z : \mathbb{N} \bullet z \geq x + 3$

Equality

Solution 9

(a)

$$\begin{aligned}
&\exists y : \mathbb{N} \bullet y \in \{0, 1\} \land y \neq 1 \land x \neq y \\
&\Leftrightarrow \exists y : \mathbb{N} \bullet y = 0 \land x \neq y && \text{[arithmetic]} \\
&\Leftrightarrow 0 \in \mathbb{N} \land x \neq 0 && \text{[one - point rule]} \\
&\Leftrightarrow x \neq 0
\end{aligned}$$

(b)

$$\begin{aligned}
&\exists x, y : \mathbb{N} \bullet x + y = 4 \land x < y \\
&\Leftrightarrow \exists x, y : \mathbb{N} \bullet y = 4 - x \land x < y \\
&\Leftrightarrow \exists x : \mathbb{N} \bullet 4 - x \in \mathbb{N} \land x < 4 - x \\
&\Leftrightarrow \mathit{true}
\end{aligned}$$

The final equivalence holds because $0 \in N$, $4 - 0 \in N$, and $0 < 4$.

(c)

$$\begin{aligned}
&\forall x : \mathbb{N} \bullet \exists y : \mathbb{N} \bullet x = y + 1 \\
&\Leftrightarrow \forall x : \mathbb{N} \bullet \exists y : \mathbb{N} \bullet y = x - 1 \\
&\Leftrightarrow \forall x : \mathbb{N} \bullet x - 1 \in \mathbb{N}
\end{aligned}$$

The final equivalence holds because $0 \in N$ and yet $0 - 1 \notin N$. We may assume that the subtraction operator is defined for all integers.

(d)

$\exists x : \mathbb{N} \bullet x = 1 \land x > y \lor x = 2 \land x > z$
$\Leftrightarrow \exists x : \mathbb{N} \bullet x = 1 \land x > y \lor \exists x : \mathbb{N} \bullet x = 2 \land x > z$
$\Leftrightarrow 1 \in \mathbb{N} \land 1 > y \lor \exists x : \mathbb{N} \bullet x = 2 \land x > z$
$\Leftrightarrow 1 \in \mathbb{N} \land 1 > y \lor 2 \in \mathbb{N} \land 2 > z$
$\Leftrightarrow 1 > y \lor 2 > z$

Solution 10

As discussed, the quantifier $\exists_1$ can help give rise to a 'test' or 'precondition' to ensure that an application of mu will work.

So, as a simple example, as the proposition

$\exists_1 n : \mathbb{N} \bullet \forall m : \mathbb{N} \bullet n \leq m$

is equivalent to true, we can be certain that the statement

$\mu n : \mathbb{N} \bullet \forall m : \mathbb{N} \bullet n \leq m$

will return a result (which happens to be 0).

Solution 11

(a) $\mu a : \mathbb{N} \bullet a = a = 0$

is a provable statement, since 0 is the only natural number with the specified property.

(b) $\mu b : \mathbb{N} \bullet b = b = 1$

is not provable. The specified property is true of both 0 and 1, and thus the value of the mu-expression is undefined.

(c) $\mu c : \mathbb{N} \bullet c > c = \mu c : \mathbb{N} \bullet c > c$

is a provable statement. Neither expression is properly defined, but we may conclude that they are equal; there is little else that we can prove about them.

(d) $\mu d : \mathbb{N} \bullet d = d = 1$

is not a provable statement. We cannot confirm that 1 is the only natural number with the specified property; we do not know what value is taken by undefined operations.

Solution 12

(Requires mu-operator with expression part - not yet implemented)

(a) $\mu m : \mathit{Mountain} \mid \forall n : \mathit{Mountain} \bullet \mathit{height}(n) \leq \mathit{height}(m) \bullet \mathit{height}(m)$

(b) $\mu c : \mathit{Chapter} \mid \exists_1 d : \mathit{Chapter} \bullet \mathit{length}(d) > \mathit{length}(c) \bullet \mathit{length}(c)$

(c) Assuming the existence of a suitable function, max: ($\mu n : \mathbb{N} \bullet n = \mathit{max}(\{m : \mathbb{N} \mid 8 * m < 100.8 * m\})$. 100 - n)

Deductive proofs

Solution 13

$$\dfrac{\ulcorner p \wedge (p \Rightarrow q)\urcorner^{[1]} \qquad \dfrac{\dfrac{p \wedge (p \Rightarrow q)}{p}\;[\wedge\text{-elim}^{[1]}] \qquad \dfrac{\dfrac{p \wedge (p \Rightarrow q)}{p \Rightarrow q}\;[\wedge\text{-elim}^{[1]}] \qquad \dfrac{p \wedge (p \Rightarrow q)}{p}\;[\wedge\text{-elim}^{[1]}]}{q}\;[\Rightarrow \text{elim}]}{p \wedge q}\;[\wedge \text{ intro}]}{p \wedge (p \Rightarrow q) \Rightarrow p \wedge q}\;[\Rightarrow\text{-intro}^{[1]}]$$

Solution 14

In one direction:

$$\dfrac{\ulcorner p \wedge q \Leftrightarrow p\urcorner^{[1]} \qquad \dfrac{\ulcorner p\urcorner^{[2]} \qquad \dfrac{\dfrac{\dfrac{}{p \wedge q}\;[\text{derived}]}{p \wedge q}\;[\Rightarrow \text{elim from } 1 \wedge 2]}{q}\;[\wedge\text{-elim}^{[3]}]}{p \Rightarrow q}\;[\Rightarrow\text{-intro}^{[2]}]}{(p \wedge q \Leftrightarrow p) \Rightarrow (p \Rightarrow q)}\;[\Rightarrow\text{-intro}^{[1]}]$$

and the other:

$$\dfrac{\ulcorner p \Rightarrow q\urcorner^{[1]} \qquad \dfrac{\dfrac{\ulcorner p \wedge q\urcorner^{[2]} \quad \ulcorner p\urcorner^{[2]}}{p \wedge q \Rightarrow p}\;[\Rightarrow\text{-intro}^{[2]}] \qquad \dfrac{\ulcorner p\urcorner^{[3]} \quad \ulcorner p \wedge q\urcorner^{[1]}}{p \Rightarrow p \wedge q}\;[\Rightarrow\text{-intro}^{[3]}]}{p \wedge q \Leftrightarrow p}\;[\Leftrightarrow \text{ intro}]}{(p \Rightarrow q) \Rightarrow (p \wedge q \Leftrightarrow p)}\;[\Rightarrow\text{-intro}^{[1]}]$$

We can then combine these two proofs *with* $\Leftrightarrow$ *intro*.

Solution 15

$$\dfrac{\ulcorner (p \Rightarrow q) \wedge \neg\, q\urcorner^{[1]} \qquad \dfrac{\ulcorner p\urcorner^{[2]} \qquad \dfrac{\dfrac{\ulcorner p \Rightarrow q\urcorner^{[1]} \quad \ulcorner p\urcorner^{[2]}}{q}\;[\Rightarrow \text{elim}] \qquad \ulcorner \neg\, q\urcorner^{[1]}}{\mathit{false}}\;[\text{false intro}]}{\neg\, p}\;[\text{false-elim}^{[2]}]}{(p \Rightarrow q) \wedge \neg\, q \Rightarrow \neg\, p}\;[\Rightarrow\text{-intro}^{[1]}]$$

Solution 16

In one direction:

$$\dfrac{\ulcorner p \wedge (q \vee r) \urcorner^{[1]} \qquad \dfrac{\ulcorner q \vee r \urcorner^{[1]} \qquad \dfrac{\dfrac{\ulcorner p \urcorner^{[1]} \quad \overline{q}\ \text{[case assumption]}}{p \wedge q}\ \text{[}\wedge\text{ intro]}}{p \wedge q \vee p \wedge r}\ \text{[}\vee\text{ intro]} \qquad \dfrac{\dfrac{\ulcorner p \urcorner^{[1]} \quad \overline{r}\ \text{[case assumption]}}{p \wedge r}\ \text{[}\wedge\text{ intro]}}{p \wedge q \vee p \wedge r}\ \text{[}\vee\text{ intro]}}{p \wedge q \vee p \wedge r}\ \text{[}\vee\text{-elim}^{[2]}\text{]}}{p \wedge (q \vee r) \Rightarrow p \wedge q \vee p \wedge r}\ \text{[}\Rightarrow\text{-intro}^{[1]}\text{]}$$

In the other:

$$\dfrac{\ulcorner p \wedge q \vee p \wedge r \urcorner^{[3]} \qquad \dfrac{\ulcorner case1 \vee case2 \urcorner^{[3]} \qquad \dfrac{\dfrac{\overline{p}\ \text{[}\wedge\text{ elim]}}{q \vee r}\ \text{[}\vee\text{ intro]}}{p \wedge (q \vee r)}\ \text{[}\wedge\text{ intro]} \qquad \dfrac{\dfrac{\overline{p}\ \text{[}\wedge\text{ elim]}}{q \vee r}\ \text{[}\vee\text{ intro]}}{p \wedge (q \vee r)}\ \text{[}\wedge\text{ intro]}}{p \wedge (q \vee r)}\ \text{[}\vee\text{-elim}^{[4]}\text{]}}{p \wedge q \vee p \wedge r \Rightarrow p \wedge (q \vee r)}\ \text{[}\Rightarrow\text{-intro}^{[3]}\text{]}$$

Solution 17

In one direction:

$$\dfrac{\ulcorner p \vee q \wedge r \urcorner^{[3]} \qquad \overline{(p \vee q) \wedge (p \vee r)}\ \text{[}\vee\text{ elim }\wedge\ \wedge\text{ intro]}}{p \vee q \wedge r \Rightarrow (p \vee q) \wedge (p \vee r)}\ \text{[}\Rightarrow\text{-intro}^{[3]}\text{]}$$

and the other:

$$\dfrac{\ulcorner (p \vee q) \wedge (p \vee r) \urcorner^{[1]} \qquad \ulcorner p \vee q \wedge r \urcorner^{[2]}}{(p \vee q) \wedge (p \vee r) \Rightarrow p \vee q \wedge r}\ \text{[}\Rightarrow\text{-intro}^{[1]}\text{]}$$

Solution 18

In one direction:

$$\dfrac{\ulcorner p \Rightarrow q \urcorner^{[1]} \quad \neg\, p \vee q}{(p \Rightarrow q) \Rightarrow \neg\, p \vee q} \; [\Rightarrow\text{-intro}^{[1]}]$$

and the other:

$$\dfrac{\ulcorner \neg\, p \vee q \urcorner^{[3]} \quad \dfrac{\ulcorner p \urcorner^{[4]} \quad \ulcorner q \urcorner^{[3]}}{p \Rightarrow q} \; [\Rightarrow\text{-intro}^{[4]}]}{\neg\, p \vee q \Rightarrow (p \Rightarrow q)} \; [\Rightarrow\text{-intro}^{[3]}]$$

Sets and types

Solution 19

(a) 1 in $\{4, 3, 2, 1\}$ is true.

(b) $\{1\}$ in $\{1, 2, 3, 4\}$ is undefined.

(c) $\{1\}$ in $\{\{1\}, \{2\}, \{3\}, \{4\}\}$ is true.

(d) The empty set in $\{1, 2, 3, 4\}$ is undefined.

Solution 20

(a) $\{1\} \times \{2, 3\}$

is the set $\{(1, 2), (1, 3)\}$

(b) The empty set cross $\{2, 3\}$ is the empty set

(c) $\mathbb{P}\,\emptyset \times \{1\}$

is the set $\{(\emptyset, 1)\}$

(d) $\{(1, 2)\}$ cross $\{3, 4\}$ is the set $\{((1, 2), 3), ((1, 2), 4)\}$

Solution 21

There are various ways of describing these sets via set comprehensions. Examples are given below.

(a) $\{z : \mathbb{Z} \mid 0 \leq z \wedge z \leq 100\}$

(b) $\{z : \mathbb{Z} \mid z = 10\}$

(c) $\{z : \mathbb{Z} \mid z \bmod 2 = 0 \vee z \bmod 3 = 0 \vee z \bmod 5 = 0\}$

Solution 22

(a) $\{n : \mathbb{N} \mid n \leq 4 \bullet n^2\}$

(b) $\{n\colon \mathbb{N} \mid n \leq 4 \bullet (n, n^2)\}$

(c) $\{n\colon \mathbb{P}\{0,1\}\}$

(d) $\{n\colon \mathbb{P}\{0,1\} \mid \mathit{true} \bullet (n, \#n)\}$

Solution 23

(a)

$$
\begin{aligned}
& x \in a \cap a \\
& \Leftrightarrow x \in a \land x \in a \\
& \Leftrightarrow x \in a
\end{aligned}
$$

(b)

$$
\begin{aligned}
& x \in a \cup a \\
& \Leftrightarrow x \in a \lor x \in a \\
& \Leftrightarrow x \in a
\end{aligned}
$$

Solution 24

(a)

The set of all pairs of integers is Z cross Z. To give it a name, we could write:

Pairs == Z $\times\mathbb{Z}$

(b)

The set of all integer pairs in which each element is strictly greater than zero could be defined by:

StrictlyPositivePairs == { m, n : $\mathbb{Z} \mid m > 0 \land n > 0 \bullet (m, n)\}$

(c)

It is intuitive to use a singular noun for the name of a basic type; we define the set of all people by writing:

$$[\mathit{Person}]$$

(d)

The set of all couples could be defined by:

Couples == { s : $\mathbb{P}$ $\mathit{Person} \mid \#s = 2\}$

Solution 25

(Requires generic set notation and Cartesian product)

Solution 26

(Requires generic parameters and relation type notation)

Relations

Solution 27

(a)

The power set of $\{(0,0),(0,1),(1,0),(1,1)\}$ is:

$\{\emptyset, \{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\}, \{(1,0),(1,1)\}, \{(0,0),(0,1)\}, \{(0,1),(1,1)\}, \{(0,1),(1,0)\}, \{(0,0),(1,1)\}, \{(0,0)$

(b) $\{\emptyset, \{(0,0)\}, \{(0,1)\}, \{(0,0),(0,1)\}\}$

(c) $\{\emptyset\}$

(d) $\{\emptyset\}$

Solution 28

(a) $\operatorname{dom} R = \{0,1,2\}$

(b) $\operatorname{ran} R = \{1,2,3\}$

(c) $\{1,2\} \lhd R = \{1 \mapsto 2, 1 \mapsto 3, 2 \mapsto 3\}$

Solution 29

(a) $\{2 \mapsto 4, 3 \mapsto 3, 3 \mapsto 4, 4 \mapsto 2\}$

(b) $\{1 \mapsto 3, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1\}$

(c) $\{1 \mapsto 1, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 2, 3 \mapsto 3, 4 \mapsto 4\}$

(d) $\{1 \mapsto 4, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 2, 3 \mapsto 3, 4 \mapsto 1\}$

Solution 30

| $\mathit{childOf} : \mathit{Person} \leftrightarrow \mathit{Person}$

(a)

parentOf == childOf^{-1}

This is a good example of how there are many different ways of writing the same thing. An alternative abbreviation is:

parentOf == { x, y : $\mathit{Person} \mid x \mapsto y \in \mathit{childOf} \bullet y \mapsto x\}$

Or, via an axiomatic definition:

$$\begin{array}{|l} parentOf : Person \leftrightarrow Person \\ \hline parentOf = childOf^{-1} \end{array}$$

(b)

siblingOf == (childOf $\circ parentOf) \setminus id$

(c)

cousinOf == childOf $\circ siblingOf \circ parentOf$

(d)

ancestorOf == parentOf$^+$

Solution 31

(Requires compound identifiers with operators - R+, R*)

(a)

R == { a, b : $\mathbb{N} \mid b = a \vee b = a\}$

(b)

S == { a, b : $\mathbb{N} \mid b = a \vee b = a\}$

(c) R+ == $\{a, b \colon \mathbb{N} \mid b > a\}$

(d) R* == $\{a, b \colon \mathbb{N} \mid b \geq a\}$

Solution 32

(a)

$$\begin{array}{l} x \mapsto y \in A \lhd B \lhd R \\ \Leftrightarrow x \in A \wedge x \mapsto y \in (B \lhd R) \\ \Leftrightarrow x \in A \wedge x \in B \wedge x \mapsto y \in R \\ \Leftrightarrow x \in A \cap B \wedge x \mapsto y \in R \\ \Leftrightarrow x \mapsto y \in A \cap B \lhd R \end{array}$$

(b)

$$\begin{array}{l} x \mapsto y \in R \cup S \rhd C \\ \Leftrightarrow x \mapsto y \in R \cup S \wedge y \in C \\ \Leftrightarrow (x \mapsto y \in R \vee x \mapsto y \in S) \wedge y \in C \\ \Leftrightarrow x \mapsto y \in R \wedge y \in C \vee x \mapsto y \in S \wedge y \in C \\ \Leftrightarrow x \mapsto y \in R \rhd C \vee x \mapsto y \in S \rhd C \\ \Leftrightarrow x \mapsto y \in (R \rhd C) \cup (S \rhd C) \end{array}$$

Functions

Solution 33

The set of 9 functions:

$\{\emptyset, \{(0,0)\}, \{(0,1)\}, \{(1,1)\}, \{(1,0)\}, \{(0,0),(1,1)\}, \{(0,1),(1,1)\}, \{(1,0),(0,0)\}, \{(0,1),(1,0)\}\}$
(a)

The set of total functions:

$\{\{(0,0),(1,1)\}, \{(0,1),(1,1)\}, \{(1,0),(0,0)\}, \{(0,1),(1,0)\}\}$

(b)

The set of functions which are neither injective nor surjective:

$\{\{(0,1),(1,1)\}, \{(0,0),(1,0)\}\}$

(c)

The set of functions which are injective but not surjective:

$\{\emptyset, \{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\}\}$

(d) There are no functions (of this type) which are surjective but not injective.

(e)

The set of bijective functions:

$\{\{(0,0),(1,1)\}, \{(0,1),(1,0)\}\}$

Solution 34

(a) $\{1 \mapsto a, 2 \mapsto b, 3 \mapsto c, 4 \mapsto b\}$

(b) $\{1 \mapsto c, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d\}$

(c) $\{1 \mapsto c, 2 \mapsto b, 3 \mapsto c, 4 \mapsto b\}$

(d) $\{1 \mapsto c, 2 \mapsto b, 3 \mapsto c, 4 \mapsto b\}$

Solution 35

(Requires power set notation P and relational image)

(a)

$$\begin{array}{|l}
children : Person \rightarrow \mathbb{P}\ Person \\
\hline
children = \{p : Person \bullet p \mapsto parentOf(\!|\ \{p\}\ |\!)\}
\end{array}$$

(b)

$$\begin{array}{|l}
number_of_grandchildren : Person \rightarrow \mathbb{N} \\
\hline
number_of_grandchildren = \{p \colon Person \bullet p \mapsto \# parentOf \circ parentOf (\llparenthesis \{p\} \rrparenthesis)\}
\end{array}$$

Solution 36

(Note: This solution demonstrates relation types in quantifier domains)

$$\begin{array}{|l}
number_of_drivers : Drivers \leftrightarrow Cars \rightarrow (Cars \rightarrow \mathbb{N}) \\
\hline
number_of_drivers = \lambda\, r : Drivers \leftrightarrow Cars \bullet \{c \colon \operatorname{ran} r \bullet c \mapsto \#\{d \colon Drivers \mid d \mapsto c \in r\}\}
\end{array}$$

Sequences

Solution 37

(a) $\langle a \rangle$

(b) $\{1 \mapsto a, 2 \mapsto b, 2 \mapsto a, 3 \mapsto c, 3 \mapsto b, 4 \mapsto d\}$

(c) $\{2 \mapsto b, 3 \mapsto c, 4 \mapsto d\}$

(d) $\{1, 2, 3, 4\}$

(e) $\{a, b\}$

(f) $\{a \mapsto 1, b \mapsto 2, c \mapsto 3, d \mapsto 4\}$

(g) $\langle a, b \rangle$

(h) $\{3 \mapsto b\}$

(i) $\{a\}$

(j) c

Solution 38

(a)

$$\begin{array}{|l}
f : Place \rightarrow \mathbb{P}\ Place \\
\hline
\forall\, p \colon Place \bullet f(p) = \{q \colon Place \mid p \mapsto q \in \operatorname{ran} trains\}
\end{array}$$

(b) $\{p \colon Place \mid \exists_1\, x \colon \operatorname{dom} trains \bullet trains(x).2 = p\}$

(c) $\mu\, p \colon Place \bullet \forall\, q \colon Place \bullet p \neq q \land \#\{x \colon \operatorname{dom} trains \mid trains(x).2 = p\} > \#\{x \colon \operatorname{dom} trains \mid trains(x).2 = q\}$

Solution 39

(a)

$large\_coins : Collection \to N$

$\forall c\colon Collection \bullet large\_coins(c) = c(large)$

(Blocked by: underscore in identifier for fuzz compatibility)

(b)

$add\_coin : Collection * Coin \to Collection$

$\forall c\colon Collection \bullet \forall d\colon Coin \bullet add\_coin(c, d) = c \cup [\![d]\!]$

(Blocked by: underscore in identifier and bag union)

Modelling

Solutions 40-52 are work in progress - many require features not yet implemented

Solution 40

(Work in progress - requires semicolon-separated bindings in set comprehensions)

(a)

$hd : seq(Title * Length * Viewed)$

$cumulative\_total(hd) \leq 12000$

$\forall p\colon \operatorname{ran} hd \bullet p.2 \leq 360$

Note that cumulative$_t otalisdefinedinpart(d)$.

(b) $\{p\colon \operatorname{ran} hd \mid p.2 > 120 \bullet p.1\}$

(c)

These can be defined recursively:

$viewed : \text{seq } Programme \rightarrow \text{seq } Programme$

$viewed(\langle\rangle) = \langle\rangle \wedge \forall x : Programme \bullet \forall s : \text{ seq } Programme \bullet viewed(\langle x\rangle \frown s) = (\text{if } x.3 = yes \text{ then } \langle x\rangle \frown viewe$

or otherwise (omitted - requires semicolon-separated bindings in set comprehension)

(d)

$cumulative\_total : \text{seq } Title * Length * Viewed \rightarrow \mathbb{N}$

$cumulative\_total(\langle\rangle) = 0 \, \forall x : Title * Length * Viewed \bullet \forall s : \text{ seq } Title * Length * Viewed \bullet cumulative\_total$

(e)

(mu $p : ranhd \; | \forall q : \text{ ran } hd \bullet p \neq q \wedge p.2 > q.2|$ p.1)

(This, of course, assumes that there is a unique element with this property.)

(f)

(f) Omitted - requires semicolon-separated bindings in nested set comprehension

(g)

axdef

$g : seq(Title * Length * Viewed) \rightarrow seq(Title * Length * Viewed)$

where

$\forall s : \text{ seq } Title * Length * Viewed \bullet g(s) = s \rhd \{x : \text{ ran } s \mid x \neq longest\_viewed(s)\}$

end

Where longest$_v iewedisdefinedas$

axdef

$longest\_viewed : seq(Title * Length * Viewed) + \rightarrow Title * Length * Viewed$

where

$\forall s\colon \operatorname{seq} \mathit{Title} * \mathit{Length} * \mathit{Viewed} \bullet \mathit{longest_viewed}(s) = (\mu\, p\colon \operatorname{ran} s \bullet p.3 = \mathit{yes}$and $\forall q\colon \operatorname{ran} s \bullet p \neq q \land q.3 = \mathit{yes} \land p.2 > q.2)$

end

This, of course, assumes that there is at least one viewed programme (and one of a unique maximum length).

(h)

$$\begin{array}{|l}
s : \operatorname{seq} \mathit{Title} * \mathit{Length} * \mathit{Viewed} \rightarrow \operatorname{seq} \mathit{Title} * \mathit{Length} * \mathit{Viewed} \\
\hline
\forall x\colon \operatorname{seq} \mathit{Title} * \mathit{Length} * \mathit{Viewed} \bullet \mathit{items}(s(x)) = \mathit{items}(x) \land \forall i, j\colon \operatorname{dom} s(x) \bullet i < j \Rightarrow s(x)(i).2 \geq s(x)(j).2
\end{array}$$

Solution 41

(a)

axdef

$\mathit{records} : \mathit{Year} \rightarrowtail\!\!\!\rightarrow$ Table

where

dom(records) = 1993..current

$\forall y\colon \operatorname{dom} \mathit{records} \bullet \#\mathit{records}(y) \leq 50$

$\forall\ y : \mathit{dom}(\mathit{records})\ |\forall e\colon \operatorname{ran} \mathit{records}(y) \bullet \mathit{year}(e.1) = y$

$\forall\ r : \mathit{ran}(\mathit{records})\ |\forall i1, i2\colon \operatorname{dom} r \bullet i1 \neq i2 \land r(i1).1 = r(i2).1 \Rightarrow r(i1).3 \neq r(i2).3$

end

(b)

(i) $\{e\colon \mathit{Entry} \mid \exists r\colon \operatorname{ran} \mathit{records} \bullet e \in \operatorname{ran} r \land e.3 = 479\}$
ii
$\{e\colon \mathit{Entry} \mid \exists r\colon \operatorname{ran} \mathit{records} \bullet e \in \operatorname{ran} r \land e.6 > e.5\}$
iii

$\{e\colon Entry \mid \exists\, r\colon \operatorname{ran} records \bullet e \in \operatorname{ran} r \land e.7 \geq 70\}$
iv
$\{c\colon Course \mid \forall\, r\colon \operatorname{ran} records \bullet \forall\, e\colon \operatorname{ran} r \bullet e.2 = c \Rightarrow e.7 \geq 70\}$
v

$\{y\colon Year \mid y \in \operatorname{dom} records \bullet y \mapsto \{l\colon Lecturer \mid \#\{c\colon \operatorname{ran} records(y) \mid c.4 = l\} > 6\}\}$

(c)

axdef

where

$\forall\, x\colon Entry \bullet \forall\, s\colon \operatorname{seq} Entry \bullet 479\_courses(\langle\rangle) = \langle\rangle$ and $479_courses(\langle x\rangle\ ^s) = ifx.3 = 479then\langle x\rangle\ ^479_courses(s)else479_courses(s)$

end

(Blocked by: underscore in identifier - use camelCase for fuzz compatibility)

(d)

$\forall\, x\colon Entry \bullet \forall\, s\colon \operatorname{seq} Entry \bullet total(\langle\rangle) = 0 \land total(\langle x\rangle \frown s) = x.5 + total(s)$

Solution 42

$[Person]$

axdef

$State : P(seq(iseq(Person)))$

where

$\forall\ s : State \mid \forall\, i, j\colon \operatorname{dom} s \bullet i \neq j \land \operatorname{ran} s(i) \cap \operatorname{ran} s(j) = \{\}$

end

(b)

axdef

$add : N * Person * State \rightarrowtail\!\!\!\!\!\!\twoheadrightarrow$ State

where

$\forall\, n \colon \mathbb{N} \bullet \forall\, p \colon Person \bullet \forall\, s \colon State \bullet n \in \operatorname{dom} s \wedge p \notin \bigcup \operatorname{ran} \operatorname{ran} s |$

add(n, p, s) = s ++ n $\mapsto$ s(n) $\langle p \rangle$

end

(Blocked by: $\rightarrowtail\!\!\!\!\!\!\twoheadrightarrow$ operator not implemented)

Solution 43

(a)

(i) $\forall\ i : dombookings\ |\forall\, x, y \colon bookings(i) \bullet x \neq y \wedge x.2 \mathrel{..} x.3 \cap y.2 \mathrel{..} y.3 = \{\}$

(ii) $\forall\ i : dombookings\ |\forall$ x : $bookings(i)|$ $\{x.2, x.3\}$ subseteq 1..max(i.1)

(iii) $\forall\ i : dombookings\ |\forall\, b \colon bookings(i) \bullet b.2 \leq b.3$

(iv) This is enforced by the constraint for part (i).

(b)

(i) $\{i \colon \operatorname{dom} bookings \mid i.1 = Banbury \bullet i.2\}$

(ii) $\{i \colon \operatorname{dom} bookings \mid i.1 = Banbury \wedge \exists\, b \colon bookings(i) \bullet 50 \in b.2 \mathrel{..} b.3\}$

(iii) $r : Room;\ s : N\ |\exists\, i \colon \operatorname{dom} bookings \bullet i.1 = r \wedge i.2 = s.$ (r, s)

(iv) $r : Room\ |\exists\, i \colon \operatorname{dom} bookings \bullet i.1 = r \wedge \#bookings(i) \geq 10$

Free types and induction

$[N]$

$Tree ::= stalk \mid leaf\langle\langle \mathbb{N} \rangle\rangle \mid branch\langle\langle Tree \times Tree \rangle\rangle$

Solution 44

The two cases of the proof are established by equational reasoning: the first by

reverse $(\langle\rangle\ ^{t}) = reverset[cat.1a] = (reverset)\langle\rangle$ [cat.1b]

where cat.1a is $\langle\rangle$ s = s$andcat.1biss\langle\rangle$ = s

and the second by

reverse $((\langle x\rangle\ ^{u})^{t}) = reverse(\langle x\rangle\ ^{(}u^{t}))[cat.2]$

= reverse (u $^{t})\langle x\rangle$ [reverse.2]

= (reverse t $^{r}everseu)\langle x\rangle$ [anti-distributive]

= reverse t $^{(}reverseu\langle x\rangle)$ [cat.2]

= reverse t $^{r}everse(\langle x\rangle\ ^{u})[reverse.2]$

Solution 45

The base case:

reverse (reverse $\langle\rangle$) = reverse $\langle\rangle$ [reverse.1] = $\langle\rangle$ [reverse.1]

The inductive step:

reverse (reverse ($\langle x \rangle$ ^{t}))

= reverse ((reverse t) $\langle x \rangle$) [reverse.2]

= reverse ($\langle x \rangle$) $^r everse(reverset)[anti - distributive]$

= reverse ($\langle x \rangle$ $\langle \rangle$) $^r everse(reverset)[cat.1]$

= ((reverse $\langle \rangle$) $\langle x \rangle$) $^r everse(reverset)[reverse.2]$

= ($\langle \rangle$ $\langle x \rangle$) $^r everse(reverset)[reverse.1]$

= $\langle x \rangle$ $^r everse(reverset)[cat.1]$

= $\langle x \rangle$ $^t[reverse(reverset) = t]$

Solution 46

(a)

$count : Tree \rightarrow N$

count $stalk = 0$

$\forall\, n \colon \mathbb{N} \bullet count(leaf(n)) = 1$

$\forall\, t1, t2 \colon Tree \bullet count(branch(t1, t2)) = count(t1) + count(t2)$

(Blocked $by : recursivefreetypesandpatternmatching$)

(b)

$flatten : Tree \rightarrow seqN$

flatten stalk = $\langle \rangle$

$\forall n : \mathbb{N} \bullet \mathit{flatten}(\mathit{leaf}(n)) = \langle n \rangle$

$\forall t1, t2 : \mathit{Tree} \bullet \mathit{flatten}(\mathit{branch}(t1, t2)) = \mathit{flatten}(t1^{\mathit{flatten}})(t2)$

(Blocked by : $\mathit{recursivefreetypesandpatternmatching}$)

Solution 47

First, exhibit the induction principle for the free type:

P stalk and ($\forall n : \mathbb{N} \bullet \mathbb{P}\ \mathit{leaf}(n)$) and ($\forall t1, t2 : \mathit{Tree} \bullet \mathbb{P}\ t1 \wedge \mathbb{P}\ t2 \Rightarrow \mathbb{P}\ \mathit{branch}(t1, t2)$)

implies $\forall t : \mathit{Tree} \bullet \mathbb{P}\ t$

This gives three cases for the proof:

(flatten stalk) = $\langle\rangle$ [flatten] = 0 [] = count stalk [count]

(Remaining cases omitted - require equational reasoning with recursive functions)

Supplementary material : assignment practice

Solution 48

$[\mathit{SongId}, \mathit{UserId}, \mathit{PlaylistId}, \mathit{Playlist}]$

$\mathit{songs} : \mathbb{F}\ \mathit{SongIdusers} : \mathbb{F}\ \mathit{UserIdplaylists} : \mathit{PlaylistId} \nrightarrow \mathit{PlaylistplaylistOwner} : \mathit{PlaylistId} \nrightarrow \mathit{UserIdplaylistS}$

$\forall i : \operatorname{dom} \mathit{playlists} \bullet \operatorname{ran} \mathit{playlists}(i)(\mathit{subseteq})(\mathit{songs}) \operatorname{dom} \mathit{playlistOwner}(\mathit{subseteq})(\operatorname{dom} \mathit{playlists}) \operatorname{ran} \mathit{playlistO}$

Solution 49

$\mathit{hated} : \mathit{UserId} \nrightarrow \mathbb{F}\ \mathit{SongIdloved} : \mathit{UserId} \nrightarrow \mathbb{F}\ \mathit{SongId}$

$\operatorname{dom} \mathit{hated}(\mathit{subseteq})(\mathit{users}) \forall i : \operatorname{dom} \mathit{hated} \bullet \mathit{hated}(i)(\mathit{subseteq})(\mathit{songs}) \operatorname{dom} \mathit{loved}(\mathit{subseteq})(\mathit{users}) \forall i : \operatorname{dom}$

Solution 50

(a)

A == users $\setminus \bigcup \operatorname{ran} \mathit{playlistSubscribers}$

(b)

B == { p : dom $\mathit{playlistSubscribers} \mid \#\mathit{playlistSubscribers}(p) \geq 100\}$

(c)

C == $\mu\, u\colon \operatorname{dom} \mathit{loved} \bullet \forall\, v\colon \operatorname{dom} \mathit{loved} \bullet u \neq v \wedge \#\mathit{loved}(u) > \#\mathit{loved}(v)$

(d)

D == $\mu\, s\colon \mathit{songs} \bullet \forall\, t\colon \mathit{songs} \bullet s \neq t \wedge \#\{u\colon \mathit{UserId} \mid s \in \mathit{loved}(u)\} > \#\{u\colon \mathit{UserId} \mid t \in \mathit{loved}(u)\}$

Solution 51

(a)

Let's first define two helper functions:

$\mathit{loveHateScore} : \mathit{SongId}+ \rightarrow N$

$\forall\ i : \mathit{songs} \mid \{u\colon \mathit{UserId} \mid i \in \mathit{loved}(u)\} \geq\ \{u\colon \mathit{UserId} \mid i \in \mathit{hated}(u)\} \Rightarrow$

loveHateScore(i) = $\{u\colon \mathit{UserId} \mid i \in \mathit{loved}(u)\}$ - $\{u\colon \mathit{UserId} \mid i \in \mathit{hated}(u)\}$

and

$\forall\ i : \mathit{songs} \mid \{u\colon \mathit{UserId} \mid i \in \mathit{loved}(u)\} <\ \{u\colon \mathit{UserId} \mid i \in \mathit{hated}(u)\} \Rightarrow$

loveHateScore(i) = 0

$\mathit{playlistCount} : \mathit{SongId} \rightarrow\!\!\!\!\!+ \mathbb{N}$

$\forall\, i\colon \mathit{songs} \bullet \mathit{playlistCount}(i) = \#\{p\colon \operatorname{dom} \mathit{playlist} \mid i \in \operatorname{ran} \mathit{playlist}(p)\}$

We then have:

$length : SongId \pfun \mathbb{N} popularity : SongId \pfun \mathbb{N}$

$\mathrm{dom}\, length (subseteq)(songs)\, \mathrm{dom}\, popularity (subseteq)(songs) \forall\, i : songs \bullet popularity(i) = loveHateScore(i) +$

(b)

$mostPopular : SongId$

$(\exists_1\ i : songs\ | \forall\, j : songs \bullet i \neq j \land popularity(i) > popularity(j)) \Rightarrow$

mostPopular $= (\mathrm{mu}\ i : songs\ | \forall\, j : songs \bullet i \neq j \land popularity(i) > popularity(j))$

and

$\neg\ \exists_1\, i : songs \bullet \forall\, j : songs \bullet i \neq j \land popularity(i) > popularity(j) \Rightarrow mostPopular = nullSong$

(c) playlistsContainingMostPopularSong $== \{ i : \mathrm{dom}\, playlists \mid mostPopular \in \mathrm{ran}\, playlists(i) \}$

Solution 52

(a)

$premiumPlays : seq(Play) \rightarrow seq(Play)$

premiumPlays$(\langle\rangle) = \langle\rangle$

$\forall\ x : Play;\ s : seq(Play) \mid$

premiumPlays$(\langle x \rangle\ ^s) = \langle x \rangle\ ^p remiumPlays(s) if userStatus(x.2) = premium$

premiumPlays(s) if userStatus(x.2) = standard

(Note: Uses camelCase for fuzz compatibility)

(b)

$standardPlays : seq(Play) \rightarrow seq(Play)$

standardPlays$(\langle\rangle) = \langle\rangle$

$\forall\ x : Play;\ s : seq(Play) \mid$

standardPlays($\langle x\rangle$ ^{s}) =$\langle x\rangle$ $^s tandardPlays(s) if userStatus(x.2) = standard$

standardPlays(s) if userStatus(x.2) = premium

(Note: Uses camelCase for fuzz compatibility)

(c)

$cumulativeLength : seq(Play) \to N$

cumulativeLength($\langle\rangle$) = 0

$\forall\ x : Play;\ s : seq(Play)\ |$

cumulativeLength($\langle x\rangle$ ^{s}) = $length(x.1) + cumulativeLength(s)$

(Note: Uses camelCase for fuzz compatibility)