Propositional logic

Solution 1

(a)

false (as (true $\Rightarrow$ false) $\Leftrightarrow$ false)

(b)

true (as (false $\Rightarrow$ false) $\Leftrightarrow$ true)

(c)

true (as (false $\Rightarrow$ true) $\Leftrightarrow$ true)

(d)

true (as (false $\Rightarrow$ false) $\Leftrightarrow$ true)

(Assuming that pigs can't fly . . .)

Solution 2

(a)

p	q	$p \wedge q$	$(p \wedge q) \Rightarrow p$
t	t	t	t
t	f	f	t
f	t	f	t
f	f	f	t

(b)

p	q	$p \wedge q$	$\neg\, p$	$\neg\, p \Rightarrow (p \wedge q)$	$(\neg\, p \Rightarrow (p \wedge q)) \Leftrightarrow p$
t	t	t	f	t	t
t	f	f	f	t	t
f	t	f	t	f	t
f	f	f	t	f	t

(c)

p	q	$p \Rightarrow q$	$p \wedge (p \Rightarrow q)$	$(p \wedge (p \Rightarrow q)) \Rightarrow q$
t	t	t	t	t
t	f	f	f	t
f	t	t	f	t
f	f	t	f	t

Solution 3

(a)

$$
\begin{aligned}
& p \Rightarrow \neg p \\
&\qquad \Leftrightarrow \neg p \vee \neg p && [\Rightarrow] \\
&\qquad \Leftrightarrow \neg p && [\text{idempotence}]
\end{aligned}
$$

(b)

$$
\begin{aligned}
& \neg p \Rightarrow p \\
&\qquad \Leftrightarrow \neg \neg p \vee p && [\Rightarrow] \\
&\qquad \Leftrightarrow p \vee p && [\neg\ \neg] \\
&\qquad \Leftrightarrow p && [\text{idempotence}]
\end{aligned}
$$

(c)

$$
\begin{aligned}
& p \Rightarrow (q \Rightarrow r) \\
&\qquad \Leftrightarrow \neg p \vee (q \Rightarrow r) && [\Rightarrow] \\
&\qquad \Leftrightarrow \neg p \vee \neg q \vee r && [\Rightarrow] \\
&\qquad \Leftrightarrow \neg p \vee \neg q \vee r && [\text{associativity}] \\
&\qquad \Leftrightarrow \neg (p \wedge q) \vee r && [\text{De Morgan}] \\
&\qquad \Leftrightarrow p \wedge q \Rightarrow r && [\Rightarrow]
\end{aligned}
$$

(d)

$$
\begin{aligned}
& q \Rightarrow (p \Rightarrow r) \\
&\qquad \Leftrightarrow \neg q \vee (p \Rightarrow r) && [\Rightarrow] \\
&\qquad \Leftrightarrow \neg q \vee \neg p \vee r && [\Rightarrow] \\
&\qquad \Leftrightarrow \neg p \vee \neg q \vee r && [\text{associativity} \wedge \text{commutativity}] \\
&\qquad \Leftrightarrow \neg p \vee (q \Rightarrow r) && [\Rightarrow] \\
&\qquad \Leftrightarrow p \Rightarrow (q \Rightarrow r) && [\Rightarrow]
\end{aligned}
$$

(e)

$$
\begin{aligned}
p \wedge q \Leftrightarrow p & \\
& \Leftrightarrow (p \wedge q \Rightarrow p) \wedge (p \Rightarrow p \wedge q) && [\Leftrightarrow] \\
& \Leftrightarrow (\neg\, (p \wedge q) \vee p) \wedge (\neg\, p \vee p \wedge q) && [\Rightarrow] \\
& \Leftrightarrow (\neg\, p \vee \neg\, q \vee p) \wedge (\neg\, p \vee p \wedge q) && [\text{De Morgan}] \\
& \Leftrightarrow (\neg\, q \vee \neg\, p \vee p) \wedge (\neg\, p \vee p \wedge q) && [\text{associativity} \wedge \text{comm .}] \\
& \Leftrightarrow (\neg\, q \vee \mathit{true}) \wedge (\neg\, p \vee p \wedge q) && [\text{excluded middle}] \\
& \Leftrightarrow \mathit{true} \wedge (\neg\, p \vee p \wedge q) && [\vee \wedge \text{true}] \\
& \Leftrightarrow \neg\, p \vee p \wedge q && [\wedge \wedge \text{true}] \\
& \Leftrightarrow (\neg\, p \vee p) \wedge (\neg\, p \vee q) && [\text{distribution}] \\
& \Leftrightarrow \mathit{true} \wedge (\neg\, p \vee q) && [\text{excluded middle}] \\
& \Leftrightarrow \neg\, p \vee q && [\wedge \wedge \text{true}] \\
& \Leftrightarrow p \Rightarrow q && [\Rightarrow]
\end{aligned}
$$

(f)

$$
\begin{aligned}
p \vee q \Leftrightarrow p & \\
& \Leftrightarrow (p \vee q \Rightarrow p) \wedge (p \Rightarrow p \vee q) && [\Leftrightarrow] \\
& \Leftrightarrow (\neg\, (p \vee q) \vee p) \wedge (\neg\, p \vee p \vee q) && [\Rightarrow] \\
& \Leftrightarrow (\neg\, p \wedge \neg\, q \vee p) \wedge (\neg\, p \vee p \vee q) && [\text{De Morgan}] \\
& \Leftrightarrow (\neg\, p \vee p) \wedge (\neg\, q \vee p) \wedge (\neg\, p \vee p \vee q) && [\text{distribution}] \\
& \Leftrightarrow \mathit{true} \wedge (\neg\, q \vee p) \wedge (\neg\, p \vee p \vee q) && [\text{excluded middle}] \\
& \Leftrightarrow (\neg\, q \vee p) \wedge (\neg\, p \vee p \vee q) && [\wedge \wedge \text{true}] \\
& \Leftrightarrow (\neg\, q \vee p) \wedge (\neg\, p \vee p \vee q) && [\text{associativity}] \\
& \Leftrightarrow (\neg\, q \vee p) \wedge (\mathit{true} \vee q) && [\text{excluded middle}] \\
& \Leftrightarrow (\neg\, q \vee p) \wedge \mathit{true} && [\vee \wedge \text{true}] \\
& \Leftrightarrow \neg\, q \vee p && [\wedge \wedge \text{true}] \\
& \Leftrightarrow q \Rightarrow p && [\Rightarrow]
\end{aligned}
$$

Solution 4

(a) (p or q) $\Leftrightarrow$ ((not p or not q) and q) is not a tautology. You might illustrate this via a truth table or via a chain of equivalences, showing that the proposition is not equivalent to true. Alternatively, you might try and find a combination of values for which the proposition is false. (In this case, the proposition is false when p and q are both true.)

(b) (p or q) ⇔ ((not p and not q) or q) is not a tautology. In this case, the proposition is false when p is true and q is false.

Solution 5

Solution 6

(a)

This is a true proposition: whatever the value of x, the expression x$^2 - x + 1$ *denotesanaturalnumber.Ifwechooseytobethisnaturalnumber*, *wewillfindthatpistrue*.

(b)

This is a false proposition. We cannot choose a large enough value for y such that p will hold for any value of x.

(c)

This is a false proposition. It is an implication whose antecedent part is true and whose consequent part is false.

(d)

This is a true proposition. It is an implication whose antecedent part is false and whose consequent part is true.

Solution 7

(a)

We must define a predicate p that is false for at least one value of x, and is true for at least one other value. A suitable solution would be p ⇔ x ¿ 1.

(b)

With the above choice of p, we require only that q is sometimes false when p is true (for else the universal quantification would hold). A suitable solution would be q $\Leftrightarrow$ x ¿ 3.

Solution 8

(a)

$\forall x\colon N \bullet x \geq z$

Equality

Solution 9

(d)

$$\begin{aligned}
&\exists x\colon N \bullet x = 1 \land x > y \lor x = 2 \land x > z \\
&\qquad \Leftrightarrow \exists x\colon N \bullet x = 1 \land x > y \lor \exists x\colon N \bullet x = 2 \land x > z \\
&\qquad \Leftrightarrow 1 \in N \land 1 > y \lor \exists x\colon N \bullet x = 2 \land x > z \\
&\qquad \Leftrightarrow 1 \in N \land 1 > y \lor 2 \in N \land 2 > z \\
&\qquad \Leftrightarrow 1 > y \lor 2 > z
\end{aligned}$$

Solution 10

Solution 11

Solution 12

Deductive proofs

Solution 13

$$\dfrac{\ulcorner p \land (p \Rightarrow q)\urcorner^{[1]} \qquad \dfrac{\dfrac{p \land (p \Rightarrow q)}{p}\,[\land\text{-elim}^{[1]}] \qquad \dfrac{\dfrac{p \land (p \Rightarrow q)}{p \Rightarrow q}\,[\land\text{-elim}^{[1]}] \qquad \dfrac{p \land (p \Rightarrow q)}{p}\,[\land\text{-elim}^{[1]}]}{q}\,[\Rightarrow \text{elim}]}{p \land q}\,[\land \text{ intro}]}{p \land (p \Rightarrow q) \Rightarrow p \land q}\,[\Rightarrow\text{-intro}^{[1]}]$$

Solution 14

In one direction:

$$\dfrac{\ulcorner p \wedge q \Leftrightarrow p \urcorner^{[1]} \qquad \dfrac{\ulcorner p \urcorner^{[2]} \qquad \dfrac{\dfrac{\dfrac{}{p \wedge q}\ [\text{derived}]}{p \wedge q}\ [\Rightarrow \text{ elim from } 1 \wedge 2]}{q}\ [\wedge\text{-elim}^{[3]}]}{p \Rightarrow q}\ [\Rightarrow\text{-intro}^{[2]}]}{(p \wedge q \Leftrightarrow p) \Rightarrow (p \Rightarrow q)}\ [\Rightarrow\text{-intro}^{[1]}]$$

and the other:

$$\dfrac{\ulcorner p \Rightarrow q \urcorner^{[1]} \qquad \dfrac{\dfrac{\ulcorner p \wedge q \urcorner^{[2]} \quad \ulcorner p \urcorner^{[2]}}{p \wedge q \Rightarrow p}\ [\Rightarrow\text{-intro}^{[2]}] \qquad \dfrac{\ulcorner p \urcorner^{[3]} \quad \ulcorner p \wedge q \urcorner^{[1]}}{p \Rightarrow p \wedge q}\ [\Rightarrow\text{-intro}^{[3]}]}{p \wedge q \Leftrightarrow p}\ [\Leftrightarrow \text{ intro}]}{(p \Rightarrow q) \Rightarrow (p \wedge q \Leftrightarrow p)}\ [\Rightarrow\text{-intro}^{[1]}]$$

We can then combine these two proofs with $\Leftrightarrow$ intro.

Solution 15

$$\dfrac{\ulcorner (p \Rightarrow q) \wedge \neg\, q \urcorner^{[1]} \qquad \dfrac{\ulcorner p \urcorner^{[2]} \qquad \dfrac{\dfrac{\ulcorner p \Rightarrow q \urcorner^{[1]} \quad \ulcorner p \urcorner^{[2]}}{q}\ [\Rightarrow \text{ elim}] \qquad \ulcorner \neg\, q \urcorner^{[1]}}{\mathit{false}}\ [\text{false intro}]}{\neg\, p}\ [\text{false-elim}^{[2]}]}{(p \Rightarrow q) \wedge \neg\, q \Rightarrow \neg\, p}\ [\Rightarrow\text{-intro}^{[1]}]$$

Solution 16

In one direction:

$$\dfrac{\dfrac{\ulcorner p \urcorner^{[1]} \quad \overline{r}\ \text{[case assumption]}}{p \wedge r}\ [\wedge\ \text{intro}]}{p \wedge q \vee p \wedge r}\ [\vee\ \text{intro}]$$

$$\dfrac{\dfrac{\ulcorner p \urcorner^{[1]} \quad \overline{q}\ \text{[case assumption]}}{p \wedge q}\ [\wedge\ \text{intro}]}{p \wedge q \vee p \wedge r}\ [\vee\ \text{intro}]$$

$$\dfrac{\ulcorner p \wedge (q \vee r) \urcorner^{[1]} \qquad \dfrac{\ulcorner q \vee r \urcorner^{[1]} \qquad \vdots \qquad \vdots}{p \wedge q \vee p \wedge r}\ [\vee\text{-elim}^{[2]}]}{p \wedge (q \vee r) \Rightarrow p \wedge q \vee p \wedge r}\ [\Rightarrow\text{-intro}^{[1]}]$$

In the other:

$$\dfrac{\dfrac{\overline{p}\ [\wedge\ \text{elim}]}{q \vee r}\ [\vee\ \text{intro}]}{p \wedge (q \vee r)}\ [\wedge\ \text{intro}]$$

$$\dfrac{\dfrac{\overline{p}\ [\wedge\ \text{elim}]}{q \vee r}\ [\vee\ \text{intro}]}{p \wedge (q \vee r)}\ [\wedge\ \text{intro}]$$

$$\dfrac{\ulcorner p \wedge q \vee p \wedge r \urcorner^{[3]} \qquad \dfrac{\ulcorner case1 \vee case2 \urcorner^{[3]} \qquad \vdots \qquad \vdots}{p \wedge (q \vee r)}\ [\vee\text{-elim}^{[4]}]}{p \wedge q \vee p \wedge r \Rightarrow p \wedge (q \vee r)}\ [\Rightarrow\text{-intro}^{[3]}]$$

Solution 17

In one direction:

$$\dfrac{\ulcorner p \vee q \wedge r \urcorner^{[3]} \qquad \overline{(p \vee q) \wedge (p \vee r)}\ [\vee\ \text{elim} \wedge \wedge\ \text{intro}]}{p \vee q \wedge r \Rightarrow (p \vee q) \wedge (p \vee r)}\ [\Rightarrow\text{-intro}^{[3]}]$$

and the other:

$$\dfrac{\ulcorner (p \vee q) \wedge (p \vee r) \urcorner^{[1]} \quad \ulcorner p \vee q \wedge r \urcorner^{[2]}}{(p \vee q) \wedge (p \vee r) \Rightarrow p \vee q \wedge r} \; [\Rightarrow\text{-intro}^{[1]}]$$

Solution 18

In one direction:

$$\dfrac{\ulcorner p \Rightarrow q \urcorner^{[1]} \quad \neg\, p \vee q}{(p \Rightarrow q) \Rightarrow \neg\, p \vee q} \; [\Rightarrow\text{-intro}^{[1]}]$$

and the other:

$$\dfrac{\ulcorner \neg\, p \vee q \urcorner^{[3]} \quad \dfrac{\ulcorner p \urcorner^{[4]} \quad \ulcorner q \urcorner^{[3]}}{p \Rightarrow q} \; [\Rightarrow\text{-intro}^{[4]}]}{\neg\, p \vee q \Rightarrow (p \Rightarrow q)} \; [\Rightarrow\text{-intro}^{[3]}]$$

Sets and types

Solution 19

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Solution 21

Solution 22

Solution 23

Solution 24

Solution 25

Solution 26

Relations

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Solution 28

Solution 29

Solution 30

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