

Propositional logic

Solution 1

(a)

$$\text{false}(\text{as}(\text{true} \Rightarrow \text{false}) \Leftrightarrow \text{false})$$

(b)

$$\text{true}(\text{as}(\text{false} \Rightarrow \text{false}) \Leftrightarrow \text{true})$$

(c)

$$\text{true}(\text{as}(\text{false} \Rightarrow \text{true}) \Leftrightarrow \text{true})$$

(d)

$$\text{true}(\text{as}(\text{false} \Rightarrow \text{false}) \Leftrightarrow \text{true})$$

(Assuming that pigs can't fly . . .)

Solution 2

(a)

p	q	$p \wedge q$	$(p \wedge q) \Rightarrow p$
t	t	t	t
t	f	f	t
f	t	f	t
f	f	f	t

(b)

p	q	$p \wedge q$	$\neg p$	$\neg p \Rightarrow (p \wedge q)$	$(\neg p \Rightarrow (p \wedge q)) \Leftrightarrow p$
t	t	t	f	t	t
t	f	f	f	t	t
f	t	f	t	f	t
f	f	f	t	f	t

(c)

p	q	$p \Rightarrow q$	$p \wedge (p \Rightarrow q)$	$(p \wedge (p \Rightarrow q)) \Rightarrow q$
t	t	t	t	t
t	f	f	f	t
f	t	t	f	t
f	f	t	f	t

Solution 3

(a)

$$\begin{aligned} p \Rightarrow \neg p & \\ \Leftrightarrow \neg p \vee \neg p & \quad [\Rightarrow] \\ \Leftrightarrow \neg p & \quad [\text{idempotence}] \end{aligned}$$

(b)

$$\begin{aligned} \neg p \Rightarrow p & \\ \Leftrightarrow \neg \neg p \vee p & \quad [\Rightarrow] \\ \Leftrightarrow p \vee p & \quad [\neg \neg] \\ \Leftrightarrow p & \quad [\text{idempotence}] \end{aligned}$$

(c)

$$\begin{aligned} p \Rightarrow (q \Rightarrow r) & \\ \Leftrightarrow \neg p \vee (q \Rightarrow r) & \quad [\Rightarrow] \\ \Leftrightarrow \neg p \vee \neg q \vee r & \quad [\Rightarrow] \\ \Leftrightarrow \neg p \vee \neg q \vee r & \quad [\text{associativity}] \\ \Leftrightarrow \neg (p \wedge q) \vee r & \quad [\text{De Morgan}] \\ \Leftrightarrow p \wedge q \Rightarrow r & \quad [\Rightarrow] \end{aligned}$$

(d)

$$\begin{aligned} q \Rightarrow (p \Rightarrow r) & \\ \Leftrightarrow \neg q \vee (p \Rightarrow r) & \quad [\Rightarrow] \\ \Leftrightarrow \neg q \vee \neg p \vee r & \quad [\Rightarrow] \\ \Leftrightarrow \neg p \vee \neg q \vee r & \quad [\text{associativity} \wedge \text{commutativity}] \\ \Leftrightarrow \neg p \vee (q \Rightarrow r) & \quad [\Rightarrow] \\ \Leftrightarrow p \Rightarrow (q \Rightarrow r) & \quad [\Rightarrow] \end{aligned}$$

(e)

$$\begin{aligned}
p \wedge q &\Leftrightarrow p && [\Leftrightarrow] \\
&\Leftrightarrow (p \wedge q \Rightarrow p) \wedge (p \Rightarrow p \wedge q) && [\Rightarrow] \\
&\Leftrightarrow (\neg(p \wedge q) \vee p) \wedge (\neg p \vee p \wedge q) && [\text{De Morgan}] \\
&\Leftrightarrow (\neg p \vee \neg q \vee p) \wedge (\neg p \vee p \wedge q) && [\text{associativity } \wedge \text{ comm.}] \\
&\Leftrightarrow (\neg q \vee \neg p \vee p) \wedge (\neg p \vee p \wedge q) && [\text{excluded middle}] \\
&\Leftrightarrow (\neg q \vee \text{true}) \wedge (\neg p \vee p \wedge q) && [\vee \wedge \text{true}] \\
&\Leftrightarrow \text{true} \wedge (\neg p \vee p \wedge q) && [\wedge \wedge \text{true}] \\
&\Leftrightarrow \neg p \vee p \wedge q && [\text{distribution}] \\
&\Leftrightarrow (\neg p \vee p) \wedge (\neg p \vee q) && [\text{excluded middle}] \\
&\Leftrightarrow \text{true} \wedge (\neg p \vee q) && [\wedge \wedge \text{true}] \\
&\Leftrightarrow \neg p \vee q && [\Rightarrow] \\
&\Leftrightarrow p \Rightarrow q
\end{aligned}$$

(f)

$$\begin{aligned}
p \vee q &\Leftrightarrow p && [\Leftrightarrow] \\
&\Leftrightarrow (p \vee q \Rightarrow p) \wedge (p \Rightarrow p \vee q) && [\Rightarrow] \\
&\Leftrightarrow (\neg(p \vee q) \vee p) \wedge (\neg p \vee p \vee q) && [\text{De Morgan}] \\
&\Leftrightarrow (\neg p \wedge \neg q \vee p) \wedge (\neg p \vee p \vee q) && [\text{distribution}] \\
&\Leftrightarrow (\neg p \vee p) \wedge (\neg q \vee p) \wedge (\neg p \vee p \vee q) && [\text{excluded middle}] \\
&\Leftrightarrow \text{true} \wedge (\neg q \vee p) \wedge (\neg p \vee p \vee q) && [\wedge \wedge \text{true}] \\
&\Leftrightarrow (\neg q \vee p) \wedge (\neg p \vee p \vee q) && [\text{associativity}] \\
&\Leftrightarrow (\neg q \vee p) \wedge (\neg p \vee p \vee q) && [\text{excluded middle}] \\
&\Leftrightarrow (\neg q \vee p) \wedge (\text{true} \vee q) && [\vee \wedge \text{true}] \\
&\Leftrightarrow (\neg q \vee p) \wedge \text{true} && [\wedge \wedge \text{true}] \\
&\Leftrightarrow \neg q \vee p && [\Rightarrow] \\
&\Leftrightarrow q \Rightarrow p
\end{aligned}$$

Solution 4

(a) $(p \text{ or } q) \Leftrightarrow ((\text{not } p \text{ or not } q) \text{ and } q)$ is not a tautology. You might illustrate this via a truth table or via a chain of equivalences, showing that the proposition is not equivalent to true. Alternatively, you might try and find a combination of values for which the proposition is false. (In this case, the proposition is false when p and q are both true.)

(b) $(p \text{ or } q) \Leftrightarrow ((\text{not } p \text{ and not } q) \text{ or } q)$ is not a tautology. In this case, the proposition is false when p is true and q is false.

Solution 5

(a)

$$\exists d : \text{Dog} \bullet \text{gentle}(d) \wedge \text{well}_t \text{rained}(d)$$

(b)

$$\forall d : Dog \bullet neat(d) \wedge well_trained(d) \Rightarrow attractive(d)$$

(c) (Requires nested quantifier in implication - parser limitation)

Solution 6

(a) This is a true *proposition* : *whatever the value of x, the expression* $x^2 - x + 1$ *denotes a natural number. If we choose y to be this natural number, we will find that p is true.*

(b) This is a false proposition. We cannot choose a large enough value for y such that p will hold for any value of x.

(c) This is a false proposition. It is an implication whose antecedent part is true and whose consequent part is false.

(d) This is a true proposition. It is an implication whose antecedent part is false and whose consequent part is true.

Solution 7

(a) We must define a predicate p that is false for at least one value of x, and is true for at least one other value. A suitable solution would be $p \Leftrightarrow x > 1$.

(b) With the above choice of p, we require only that q is sometimes false when p is true (for else the universal quantification would hold). A suitable solution would be $q \Leftrightarrow x > 3$.

Solution 8

(a)

$$\forall x : N \bullet x \geq z$$

Equality

Solution 9

(d)

$$\exists x : N \bullet x = 1 \wedge x > y \vee x = 2 \wedge x > z$$

$$\Leftrightarrow \exists x : N \bullet x = 1 \wedge x > y \vee \exists x : N \bullet x = 2 \wedge x > z$$

$$\Leftrightarrow 1 \in N \wedge 1 > y \vee \exists x : N \bullet x = 2 \wedge x > z$$

$$\Leftrightarrow 1 \in N \wedge 1 > y \vee 2 \in N \wedge 2 > z$$

$$\Leftrightarrow 1 > y \vee 2 > z$$

Solution 10

As discussed, the quantifier \exists_1 can help give rise to a 'test' or 'precondition' to ensure that an application of mu will work.

So, as a simple example, as the proposition

$$\exists_1 n : N \bullet \forall m : N \bullet n \leq m$$

is equivalent to true, we can be certain that the statement

$$\mu n : N \bullet \forall m : N \bullet n \leq m$$

will return a result (which happens to be 0).

Solution 11

(a) $(\mu a : N \mid a = a) = 0$ is a provable statement, since 0 is the only natural number with the specified property.

(b) $(\mu b : N \bullet b = b) = 1$ is not provable. The specified property is true of both 0 and 1, and thus the value of the mu-expression is undefined.

(c) $(\mu c : N \bullet c > c) = (\mu c : N \bullet c < c)$ is a provable statement. Neither expression is properly defined, but we may conclude that they are equal; there is little else that we can prove about them.

(d) $(\mu d : N \bullet d = d) = 1$ is not a provable statement. We cannot confirm that 1 is the only natural number with the specified property; we do not know what value is taken by undefined operations.

Solution 12

(Requires mu-operator with expression part - not yet implemented)

(a)

$$\mu m : Mountain \mid \forall n : Mountain \bullet height(n) \leq height(m) \bullet height(m)$$

(b)

$$\mu c : Chapter \mid \exists_1 d : Chapter \bullet length(d) > length(c) \bullet length(c)$$

(c) Assuming the existence of a suitable function, max: $(\mu n : N \bullet n = \max(\{m : N \mid 8 * m < 100.8 * m\}) \cdot 100 - n)$

Deductive proofs

Solution 13

$$\begin{array}{c} [\\ \Rightarrow\text{-intro}^{[1]}]p \wedge (p \Rightarrow q) \Rightarrow p \wedge qp \wedge (p \Rightarrow q)^{[1]}[\wedge \text{ intro}]p \wedge q[\wedge\text{-elim}^{[1]}]pp \wedge (p \Rightarrow q)[\Rightarrow \text{ elim}]q[\wedge\text{-elim}^{[1]}]p \Rightarrow qp \wedge (p \end{array}$$

Solution 14

In one direction:

$$\begin{array}{c} [\\ \Rightarrow\text{-intro}^{[1]}](p \wedge q \Leftrightarrow p) \Rightarrow (p \Rightarrow q)p \wedge q \Leftrightarrow p^{[1]}[\Rightarrow\text{-intro}^{[2]}]p \Rightarrow qp^{[2]}[\wedge\text{-elim}^{[3]}]q[\Rightarrow \text{ elim from } 1 \wedge 2]p \wedge q[\textit{derived}]p \wedge q \end{array}$$

and the other:

$$\begin{array}{c} [\\ \Rightarrow\text{-intro}^{[1]}](p \Rightarrow q) \Rightarrow (p \wedge q \Leftrightarrow p)p \Rightarrow q^{[1]}[\Leftrightarrow \text{ intro}]p \wedge q \Leftrightarrow p[\Rightarrow\text{-intro}^{[2]}]p \wedge q \Rightarrow pp \wedge q^{[2]}p^{[2]}[\Rightarrow\text{-intro}^{[3]}]p \Rightarrow p \wedge q \end{array}$$

We can then combine these two proofs *with* \Leftrightarrow *intro*.

Solution 15

$$\begin{array}{c} [\\ \Rightarrow\text{-intro}^{[1]}](p \Rightarrow q) \wedge \neg q \Rightarrow \neg p(p \Rightarrow q) \wedge \neg q^{[1]}[\textit{false-elim}^{[2]}]\neg pp^{[2]}[\textit{false intro}]\textit{false}[\Rightarrow \text{ elim}]qp \Rightarrow q^{[1]}p^{[2]}\neg q^{[1]} \end{array}$$

Solution 16

In one direction:

$$\begin{array}{c} [\\ \Rightarrow\text{-intro}^{[1]}]p \wedge (q \vee r) \Rightarrow p \wedge q \vee p \wedge rp \wedge (q \vee r)^{[1]}[\vee\text{-elim}^{[2]}]p \wedge q \vee p \wedge rq \vee r^{[1]}10ex[\wedge \text{ intro}]p \wedge qp^{[1]}[\textit{case assum} \end{array}$$

In the other:

$$\begin{array}{c} [\\ \Rightarrow\text{-intro}^{[3]}]p \wedge q \vee p \wedge r \Rightarrow p \wedge (q \vee r)p \wedge q \vee p \wedge r^{[3]}[\vee\text{-elim}^{[4]}]p \wedge (q \vee r)\textit{case1} \vee \textit{case2}^{[3]}8ex[\wedge \text{ elim}]p[\vee \text{ intro}]q \vee \end{array}$$

Solution 17

In one direction:

$$\begin{array}{c} [\\ \Rightarrow\text{-intro}^{[3]}]p \vee q \wedge r \Rightarrow (p \vee q) \wedge (p \vee r)p \vee q \wedge r^{[3]}[\vee \text{ elim } \wedge \wedge \text{ intro}](p \vee q) \wedge (p \vee r) \end{array}$$

and the other:

$$\begin{array}{c} [\\ \Rightarrow\text{-intro}^{[1]}](p \vee q) \wedge (p \vee r) \Rightarrow p \vee q \wedge r(p \vee q) \wedge (p \vee r)^{[1]}p \vee q \wedge r^{[2]} \end{array}$$

Solution 18

In one direction:

$$\begin{array}{c} [\\ \Rightarrow\text{-intro}^{[1]}](p \Rightarrow q) \Rightarrow \neg p \vee qp \Rightarrow q^{[1]} \neg p \vee q \end{array}$$

and the other:

$$\begin{array}{c} [\\ \Rightarrow\text{-intro}^{[3]}] \neg p \vee q \Rightarrow (p \Rightarrow q) \neg p \vee q^{[3]} [\Rightarrow\text{-intro}^{[4]}] p \Rightarrow qp^{[4]} q^{[3]} \end{array}$$

Sets and types

Solution 19

- (a) 1 in $\{4, 3, 2, 1\}$ is true.
- (b) $\{1\}$ in $\{1, 2, 3, 4\}$ is undefined.
- (c) $\{1\}$ in $\{\{1\}, \{2\}, \{3\}, \{4\}\}$ is true.
- (d) The empty set in $\{1, 2, 3, 4\}$ is undefined.

Solution 20

(a)

$$\{1\} \times \{2, 3\}$$

is the set $\{(1, 2), (1, 3)\}$

(b) The empty set cross $\{2, 3\}$ is the empty set

(c)

$$\times \{1\}$$

is the set $\{(\cdot, 1)\}$

(d) $\{(1, 2)\}$ cross $\{3, 4\}$ is the set $\{((1, 2), 3), ((1, 2), 4)\}$

Solution 21

There are various ways of describing these sets via set comprehensions. Examples are given below.

(a)

$$\{z : Z \mid 0 \leq z \wedge z \leq 100\}$$

(b)

$$\{z : Z \mid z = 10\}$$

(c)

$$\{z : Z \mid z \bmod 2 = 0 \vee z \bmod 3 = 0 \vee z \bmod 5 = 0\}$$

Solution 22

(a)

$$\{n : N \mid n \leq 4 \bullet n^2\}$$

(b)

$$\{n : N \mid n \leq 4 \bullet (n, n^2)\}$$

(c) $n : P0, 1$

(d) $\{n : \{0, 1\} \mid true \bullet (n, \#n)\}$

Solution 23

(a)

$$\begin{aligned} x \in a \cap a \\ \Leftrightarrow x \in a \wedge x \in a \\ \Leftrightarrow x \in a \end{aligned}$$

(b)

$$\begin{aligned} x \in a \cup a \\ \Leftrightarrow x \in a \vee x \in a \\ \Leftrightarrow x \in a \end{aligned}$$

Solution 24

(a)

The set of all pairs of integers is \mathbb{Z} cross \mathbb{Z} . To give it a name, we could write:

$$\text{Pairs} == \mathbb{Z} \times \mathbb{Z}$$

(b)

The set of all integer pairs in which each element is strictly greater than zero could be defined by:

$$\text{StrictlyPositivePairs} == \{ m, n : \mathbb{Z} \mid m > 0 \wedge n > 0 \bullet (m, n) \}$$

(c)

It is intuitive to use a singular noun for the name of a basic type; we define the set of all people by writing:

$$[Person]$$

(d)

The set of all couples could be defined by:

$$\text{Couples} == \{ s : Person \mid \#s = 2 \}$$

Solution 25

(Requires generic set notation and Cartesian product)

Solution 26

(Requires generic parameters and relation type notation)

Relations

Solution 27

(a)

The power set of $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ is:

$$\{, \{(0, 0)\}, \{(0, 1)\}, \{(1, 0)\}, \{(1, 1)\}, \{(1, 0), (1, 1)\}, \{(0, 0), (0, 1)\}, \{(0, 1), (1, 1)\}, \{(0, 1), (1, 0)\}, \{(0, 0), (1, 1)\}, \{(0, 0),$$

(b)

$$\{\{\{0, 0\}\}, \{\{0, 1\}\}, \{\{0, 0\}, \{0, 1\}\}\}$$

(c)

$$\{\}$$

(d)

$$\{\}$$

Solution 28

(a)

$$\text{dom } R = \{0, 1, 2\}$$

(b)

$$\text{ran } R = \{1, 2, 3\}$$

(c)

$$\{1, 2\} \quad R = \{1 \mapsto 2, 1 \mapsto 3, 2 \mapsto 3\}$$

Solution 29

(a)

$$\{2 \mapsto 4, 3 \mapsto 3, 3 \mapsto 4, 4 \mapsto 2\}$$

(b)

$$\{1 \mapsto 3, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1\}$$

(c)

$$\{1 \mapsto 1, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 2, 3 \mapsto 3, 4 \mapsto 4\}$$

(d)

$$\{1 \mapsto 4, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 2, 3 \mapsto 3, 4 \mapsto 1\}$$

Solution 30

$$\mid \quad \textit{childOf} : \textit{Person} \textit{Person}$$

(a)

$$\textit{parentOf} == \textit{childOf}^{-1}$$

This is a good example of how there are many different ways of writing the same thing. An alternative abbreviation is:

$$\textit{parentOf} == \{ x, y : \textit{Person} \mid x \mapsto y \in \textit{childOf} \bullet y \mapsto x \}$$

Or, via an axiomatic definition:

$$\frac{\mid \quad \textit{parentOf} : \textit{Person} \textit{Person}}{\mid \quad \textit{parentOf} = \textit{childOf}^{-1}}$$

(b)

$$\textit{siblingOf} == (\textit{childOf} \circ \textit{parentOf}) \setminus \textit{id}$$

(c)

$$\textit{cousinOf} == \textit{childOf} \circ \textit{siblingOf} \circ \textit{parentOf}$$

(d)

$$\textit{ancestorOf} == \textit{parentOf}^+$$

Solution 31

(Requires compound identifiers with operators - \mathbb{R}^+ , \mathbb{R}^*)

(a)

$$\mathbb{R} == \{ a, b : \mathbb{N} \mid b = a \vee b = a \}$$

(b)

$$\mathbb{S} == \{ a, b : \mathbb{N} \mid b = a \vee b = a \}$$

$$(c) \mathbb{R}^+ == \{ a, b : \mathbb{N} \mid b > a \}$$

$$(d) \mathbb{R}^* == \{ a, b : \mathbb{N} \mid b \geq a \}$$

Solution 32

(a)

$$x \mapsto y \in A \cap B \iff R$$

$$\Leftrightarrow x \in A \wedge x \mapsto y \in (B \cap R)$$

$$\Leftrightarrow x \in A \wedge x \in B \wedge x \mapsto y \in R$$

$$\Leftrightarrow x \in A \cap B \wedge x \mapsto y \in R$$

$$\Leftrightarrow x \mapsto y \in A \cap B \iff R$$

(b)

$$x \mapsto y \in R \cup S \iff C$$

$$\Leftrightarrow x \mapsto y \in R \cup S \wedge y \in C$$

$$\Leftrightarrow (x \mapsto y \in R \vee x \mapsto y \in S) \wedge y \in C$$

$$\Leftrightarrow x \mapsto y \in R \wedge y \in C \vee x \mapsto y \in S \wedge y \in C$$

$$\Leftrightarrow x \mapsto y \in R \iff C \vee x \mapsto y \in S \iff C$$

$$\Leftrightarrow x \mapsto y \in (R \iff C) \cup (S \iff C)$$

Functions

Solution 33

The set of 9 functions:

$$\{, \{(0, 0)\}, \{(0, 1)\}, \{(1, 1)\}, \{(1, 0)\}, \{(0, 0), (1, 1)\}, \{(0, 1), (1, 1)\}, \{(1, 0), (0, 0)\}, \{(0, 1), (1, 0)\}\}$$

(a)

The set of total functions:

$$\{\{(0, 0), (1, 1)\}, \{(0, 1), (1, 1)\}, \{(1, 0), (0, 0)\}, \{(0, 1), (1, 0)\}\}$$

(b)

The set of functions which are neither injective nor surjective:

$$\{\{(0, 1), (1, 1)\}, \{(0, 0), (1, 0)\}\}$$

(c)

The set of functions which are injective but not surjective:

$$\{\{\{0,0\}\}, \{\{0,1\}\}, \{\{1,0\}\}, \{\{1,1\}\}\}$$

(d) There are no functions (of this type) which are surjective but not injective.

(e)

The set of bijective functions:

$$\{\{(0,0), (1,1)\}, \{(0,1), (1,0)\}\}$$

Solution 34

(a)

$$\{1 \mapsto a, 2 \mapsto b, 3 \mapsto c, 4 \mapsto b\}$$

(b)

$$\{1 \mapsto c, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d\}$$

(c)

$$\{1 \mapsto c, 2 \mapsto b, 3 \mapsto c, 4 \mapsto b\}$$

(d)

$$\{1 \mapsto c, 2 \mapsto b, 3 \mapsto c, 4 \mapsto b\}$$

Solution 35

(Requires power set notation P and relational image)

(a)

$$\frac{\text{children} : \text{Person} \rightarrow \text{Person}}{\text{children} = \{p : \text{Person} \bullet p \mapsto \text{parentOf}(\{p\})\}}$$

(b)

$$\frac{\text{number_of_randchildren} : \text{Person} \rightarrow \mathbb{N}}{\text{number_of_randchildren} = \{p : \text{Person} \bullet p \mapsto \# \text{parentOf} \circ \text{parentOf}(\{p\})\}}$$

Solution 36

(Note: This solution demonstrates relation types in quantifier domains)

$$\frac{\text{number_of_drivers} : \text{Drivers} \rightarrow \text{Cars} \rightarrow (\text{Cars} \rightarrow \mathbb{N})}{\text{number_of_drivers} = \lambda r : \text{Drivers} \rightarrow \text{Cars} \bullet \{c : \text{ran } r \bullet c \mapsto \#\{d : \text{Drivers} \mid d \mapsto c \in r\}\}}$$

Sequences

Solution 37

(a)

$$\langle a \rangle$$

(b)

$$\{1 \mapsto a, 2 \mapsto b, 2 \mapsto a, 3 \mapsto c, 3 \mapsto b, 4 \mapsto d\}$$

(c)

$$\{2 \mapsto b, 3 \mapsto c, 4 \mapsto d\}$$

(d)

$$\{1, 2, 3, 4\}$$

(e)

$$\{a, b\}$$

(f)

$$\{a \mapsto 1, b \mapsto 2, c \mapsto 3, d \mapsto 4\}$$

(g)

$$\langle a, b \rangle$$

(h)

$$\{3 \mapsto b\}$$

(i)

$$\{a\}$$

(j)

$$c$$

Solution 38

(a)

$$\frac{f : Place \rightarrow Place}{\forall p : Place \bullet f(p) = \{q : Place \mid p \mapsto q \in \text{ran } \text{trains}\}}$$

(b)

$$\{p : Place \mid \exists_1 x : \text{dom } \text{trains} \bullet \text{trains}(x).2 = p\}$$

(c)

$$\mu p : Place \bullet \forall q : Place \bullet p \neq q \wedge \#\{x : \text{dom } \text{trains} \mid \text{trains}(x).2 = p\} > \#\{x : \text{dom } \text{trains} \mid \text{trains}(x).2 = q\}$$

Solution 39

(a)

$$\text{large}_c \text{oins} : \text{Collection} \rightarrow \mathcal{N}$$

$$\forall c : \text{Collection} \bullet \text{large}_c \text{oins}(c) = c(\text{large})$$

(Blocked by : *underscoreinidentifierforfuzzcompatibility*)

(b)

$$\text{add}_c \text{oin} : \text{Collection} * \text{Coin} \rightarrow \text{Collection}$$

$$\forall c : \text{Collection} \bullet \forall d : \text{Coin} \bullet \text{add}_c \text{oin}(c, d) = c \cup d$$

(Blocked by : *underscoreinidentifierandbagunion*)

Modelling

Solutions 40-52 are work in progress - many require features not yet implemented

Solution 40

(Work in progress - requires semicolon-separated bindings in set comprehensions)

(a)

$hd : seq(Title * Length * Viewed)$

$cumulative_total(hd) \leq 12000$

$\forall p : \text{ran } hd \bullet p.2 \leq 360$

Note that $cumulative_total$ is defined in part (d).

(b)

$\{p : \text{ran } hd \mid p.2 > 120 \bullet p.1\}$

(c)

These can be defined recursively:

$$\frac{\text{viewed} : seq \text{ Programme } seq \text{ Programme}}{\text{viewed}(\langle \rangle) = \langle \rangle \wedge \forall x : \text{Programme} \bullet \forall s : seq \text{ Programme} \bullet \text{viewed}(\langle x \rangle s) = (\text{if } x.3 = yes \text{ then } \langle x \rangle \text{ viewed}(s))}$$

or otherwise (omitted - requires semicolon-separated bindings in set comprehension)

(d)

$$\frac{\text{cumulative_total} : seq \text{ Title } * \text{Length } * \text{Viewed } N}{\text{cumulative_total}(\langle \rangle) = 0 \wedge \forall x : \text{Title } * \text{Length } * \text{Viewed} \bullet \forall s : seq \text{ Title } * \text{Length } * \text{Viewed} \bullet \text{cumulative_total}(\langle x \rangle s) = \text{if } x.3 = yes \text{ then } x.2 + \text{cumulative_total}(s) \text{ else } \text{cumulative_total}(s)}$$

(e)

$(\mu p : \text{ran } hd \mid \forall q : \text{ran } hd \bullet p \neq q \wedge p.2 > q.2 \mid p.1)$

(This, of course, assumes that there is a unique element with this property.)

(f)

(f) Omitted - requires semicolon-separated bindings in nested set comprehension

(g)

axdef

$g : \text{seq}(\text{Title} * \text{Length} * \text{Viewed}) \rightarrow \text{seq}(\text{Title} * \text{Length} * \text{Viewed})$

where

$\forall s : \text{seq} \text{ Title} * \text{Length} * \text{Viewed} \bullet g(s) = s \setminus \{x : \text{ran } s \mid x \neq \text{longest}_{\text{viewed}}(s)\}$

end

Where $\text{longest}_{\text{viewed}}$ is defined as

axdef

$\text{longest}_{\text{viewed}} : \text{seq}(\text{Title} * \text{Length} * \text{Viewed}) \rightarrow \text{Title} * \text{Length} * \text{Viewed}$

where

$\forall s : \text{seq} \text{ Title} * \text{Length} * \text{Viewed} \bullet \text{longest}_{\text{viewed}}(s) = (\mu p : \text{ran } s \bullet p.3 = \text{yes} \text{ and } \forall q : \text{ran } s \bullet p \neq q \wedge q.3 = \text{yes} \wedge p.2 > q.2)$

end

This, of course, assumes that there is at least one viewed programme (and one of a unique maximum length).

(h)

$s : \text{seq} \text{ Title} * \text{Length} * \text{Viewed}$	$\text{seq} \text{ Title} * \text{Length} * \text{Viewed}$
$\forall x : \text{seq} \text{ Title} * \text{Length} * \text{Viewed} \bullet \text{items}(s(x)) = \text{items}(x) \wedge \forall i, j : \text{dom } s(x) \bullet i < j \Rightarrow s(x)(i).2 \geq s(x)(j).2$	$\text{seq} \text{ Title} * \text{Length} * \text{Viewed}$

Solution 41

(a)

axdef

$\text{records} : \text{Year} \rightarrow \text{Table}$

where

$\text{dom}(\text{records}) = 1993..\text{current}$

$\forall y : \text{dom } \text{records} \bullet \# \text{records}(y) \leq 50$

$\forall y : \text{dom}(\text{records}) \mid \forall e : \text{ran } \text{records}(y) \bullet \text{year}(e.1) = y$

$\forall r : \text{ran}(\text{records}) \mid \forall i1, i2 : \text{dom } r \bullet i1 \neq i2 \wedge r(i1).1 = r(i2).1 \Rightarrow r(i1).3 \neq r(i2).3$

end

(b)

(i)

$\{e : \text{Entry} \mid \exists r : \text{ran } \text{records} \bullet e \in \text{ran } r \wedge e.3 = 479\}$

ii

$\{e : \text{Entry} \mid \exists r : \text{ran } \text{records} \bullet e \in \text{ran } r \wedge e.6 > e.5\}$

iii

$\{e : \text{Entry} \mid \exists r : \text{ran } \text{records} \bullet e \in \text{ran } r \wedge e.7 \geq 70\}$

iv

$\{c : \text{Course} \mid \forall r : \text{ran } \text{records} \bullet \forall e : \text{ran } r \bullet e.2 = c \Rightarrow e.7 \geq 70\}$

v

$\{y : \text{Year} \mid y \in \text{dom } \text{records} \bullet y \mapsto \{l : \text{Lecturer} \mid \#\{c : \text{ran } \text{records}(y) \mid c.4 = l\} > 6\}\}$

(c)

axdef

where

$\forall x : \text{Entry} \bullet \forall s : \text{seq } \text{Entry} \bullet 479_{\text{courses}}(\langle \rangle) = \langle \rangle \text{ and } 479_{\text{courses}}(\langle x \rangle^s) = \text{if } x.3 = 479 \text{ then } \langle x \rangle$
 $479_{\text{courses}}(s) \text{ else } 479_{\text{courses}}(s)$

end

(Blocked by : *underscoreinidentifier – usecamelCaseforfuzzcompatibility*)

(d)

$$\overline{\quad} \quad \forall x : \text{Entry} \bullet \forall s : \text{seq Entry} \bullet \text{total}(\langle \rangle) = 0 \wedge \text{total}(\langle x \rangle s) = x.5 + \text{total}(s)$$

Solution 42

[*Person*]

axdef

State : $P(\text{seq}(\text{iseq}(\text{Person})))$

where

$$\forall s : \text{State} \mid \forall i, j : \text{dom } s \bullet i \neq j \wedge \text{ran } s(i) \cap \text{ran } s(j) = \{\}$$

end

(b)

axdef

add : $N * \text{Person} * \text{State} \rightarrow \text{State}$

where

$$\forall n : N \bullet \forall p : \text{Person} \bullet \forall s : \text{State} \bullet n \in \text{dom } s \wedge p \notin \bigcup \text{ran } s \mid$$

$$\text{add}(n, p, s) = s ++ n \mapsto s(n) \langle p \rangle$$

end

(Blocked by: operator not implemented)

Solution 43

(a)

$$(i) \forall i : dombookings \mid \forall x, y : bookings(i) \bullet x \neq y \wedge x.2 \dots x.3 \cap y.2 \dots y.3 = \{\}$$

$$(ii) \forall i : dombookings \mid \forall x : bookings(i) \mid \{x.2, x.3\} \text{ subseq } 1..max(i.1)$$

$$(iii) \forall i : dombookings \mid \forall b : bookings(i) \bullet b.2 \leq b.3$$

(iv) This is enforced by the constraint for part (i).

(b)

$$(i) \{i : dom bookings \mid i.1 = Banbury \bullet i.2\}$$

$$(ii) \{i : dom bookings \mid i.1 = Banbury \wedge \exists b : bookings(i) \bullet 50 \in b.2 \dots b.3\}$$

$$(iii) r : Room; s : N \mid \exists i : dom bookings \bullet i.1 = r \wedge i.2 = s. \quad (r, s)$$

$$(iv) r : Room \mid \exists i : dom bookings \bullet i.1 = r \wedge \#bookings(i) \geq 10$$

Free types and induction

$[N]$

$Tree ::= stalk \mid leaf\ N \mid branch\ Tree \times Tree$

Solution 44

The two cases of the proof are established by equational *reasoning* : *thefirstby*

$$\text{reverse } (\langle \rangle^t) = \text{reverset}[cat.1a] = (\text{reverset})\langle \rangle \text{ [cat.1b]}$$

where cat.1a is $\langle \rangle s = sandcat.1biss\langle \rangle = s$

and the second by

$$\text{reverse } ((\langle x \rangle^u)^t) = \text{reverse}(\langle x \rangle^{\langle u^t \rangle})[cat.2]$$

$$= \text{reverse } (u^t)\langle x \rangle \text{ [reverse.2]}$$

$$= (\text{reverse } t^r \text{everseu})\langle x \rangle \text{ [anti-distributive]}$$

$$= \text{reverse } t^r (\text{reverseu}\langle x \rangle) \text{ [cat.2]}$$

$$= \text{reverse } t^r \text{everse}(\langle x \rangle^u) \text{ [reverse.2]}$$

Solution 45

The base case:

$$\text{reverse } (\text{reverse } \langle \rangle) = \text{reverse } \langle \rangle \text{ [reverse.1]} = \langle \rangle \text{ [reverse.1]}$$

The inductive step:

$$\begin{aligned}
& \text{reverse } (\text{reverse } (\langle x \rangle^t)) \\
&= \text{reverse } ((\text{reverse } t) \langle x \rangle) [\text{reverse.2}] \\
&= \text{reverse } (\langle x \rangle)^{\text{reverse}(\text{reverse } t)} [\text{anti - distributive}] \\
&= \text{reverse } (\langle x \rangle \langle \rangle)^{\text{reverse}(\text{reverse } t)} [\text{cat.1}] \\
&= ((\text{reverse } \langle \rangle) \langle x \rangle)^{\text{reverse}(\text{reverse } t)} [\text{reverse.2}] \\
&= (\langle \rangle \langle x \rangle)^{\text{reverse}(\text{reverse } t)} [\text{reverse.1}] \\
&= \langle x \rangle^{\text{reverse}(\text{reverse } t)} [\text{cat.1}] \\
&= \langle x \rangle^t [\text{reverse}(\text{reverse } t) = t]
\end{aligned}$$

Solution 46

(a)

$\text{count} : \text{Tree } N$

$\text{count stalk} = 0$

$\forall n : N \bullet \text{count}(\text{leaf}(n)) = 1$

$\forall t1, t2 : \text{Tree} \bullet \text{count}(\text{branch}(t1, t2)) = \text{count}(t1) + \text{count}(t2)$

(Blocked by : *recursive freetypes and pattern matching*)

(b)

$\text{flatten} : \text{Tree } \text{seq } N$

$\text{flatten stalk} = \langle \rangle$

$$\forall n : N \bullet \text{flatten}(\text{leaf}(n)) = \langle n \rangle$$

$$\forall t1, t2 : \text{Tree} \bullet \text{flatten}(\text{branch}(t1, t2)) = \text{flatten}(t1^{\text{flatten}})(t2)$$

(Blocked by : *recursive freetypes and pattern matching*)

Solution 47

First, exhibit the induction principle for the free type:

$$\text{P stalk and } (\forall n : N \bullet \text{leaf}(n)) \text{ and } (\forall t1, t2 : \text{Tree} \bullet t1 \wedge t2 \Rightarrow \text{branch}(t1, t2))$$

implies $\forall t : \text{Tree} \bullet t$

This gives three cases for the proof:

$$(\text{flatten stalk}) = \langle \rangle [\text{flatten}] = 0 [] = \text{count stalk} [\text{count}]$$

(Remaining cases omitted - require equational reasoning with recursive functions)

Supplementary material : assignment practice

Solution 48

$$[\text{SongId}, \text{UserId}, \text{PlaylistId}, \text{Playlist}]$$

$$\frac{\text{songs} : \text{SongId} \text{ users} : \text{UserId} \text{ playlists} : \text{PlaylistId} \text{ Playlist} \text{ playlistOwner} : \text{PlaylistId} \text{ UserId} \text{ playlistSubscribe}}{\forall i : \text{dom playlists} \bullet \text{ran playlists}(i)(\text{subseq})(\text{songs}) \text{ dom playlistOwner}(\text{subseq})(\text{dom playlists}) \text{ ran playlistSubscribe}(i)}$$

Solution 49

$$\frac{\text{hated} : \text{UserId} \text{ SongId} \text{ loved} : \text{UserId} \text{ SongId}}{\text{dom hated}(\text{subseq})(\text{users}) \forall i : \text{dom hated} \bullet \text{hated}(i)(\text{subseq})(\text{songs}) \text{ dom loved}(\text{subseq})(\text{users}) \forall i : \text{dom loved} \bullet \text{loved}(i)(\text{subseq})(\text{songs})}$$

Solution 50

(a)

$$A == \text{users} \setminus \bigcup \text{ran } \text{playlistSubscribers}$$

(b)

$$B == \{ p : \text{dom } \text{playlistSubscribers} \mid \# \text{playlistSubscribers}(p) \geq 100 \}$$

(c)

$$C == \mu u : \text{dom } \text{loved} \bullet \forall v : \text{dom } \text{loved} \bullet u \neq v \wedge \# \text{loved}(u) > \# \text{loved}(v)$$

(d)

$$D == \mu s : \text{songs} \bullet \forall t : \text{songs} \bullet s \neq t \wedge \# \{ u : \text{UserId} \mid s \in \text{loved}(u) \} > \# \{ u : \text{UserId} \mid t \in \text{loved}(u) \}$$

Solution 51

(a)

Let's first define two helper functions:

$$\text{loveHateScore} : \text{SongId} \rightarrow N$$

$$\forall i : \text{songs} \mid \{ u : \text{UserId} \mid i \in \text{loved}(u) \} \geq \{ u : \text{UserId} \mid i \in \text{hated}(u) \} \Rightarrow$$

$$\text{loveHateScore}(i) = \{ u : \text{UserId} \mid i \in \text{loved}(u) \} - \{ u : \text{UserId} \mid i \in \text{hated}(u) \}$$

and

$$\forall i : \text{songs} \mid \{ u : \text{UserId} \mid i \in \text{loved}(u) \} < \{ u : \text{UserId} \mid i \in \text{hated}(u) \} \Rightarrow$$

$$\text{loveHateScore}(i) = 0$$

$$\frac{\text{playlistCount} : \text{SongId} \rightarrow N}{\forall i : \text{songs} \bullet \text{playlistCount}(i) = \# \{ p : \text{dom } \text{playlist} \mid i \in \text{ran } \text{playlist}(p) \}}$$

We then have:

$$\frac{\text{length} : \text{SongId} \rightarrow \text{N} \quad \text{popularity} : \text{SongId} \rightarrow \text{N}}{\text{dom length}(\text{subseq})(\text{songs}) = \text{dom popularity}(\text{subseq})(\text{songs}) \quad \forall i : \text{songs} \bullet \text{popularity}(i) = \text{loveHateScore}(i)}$$

(b)

mostPopular : *SongId*

$(\exists_1 i : \text{songs} \mid \forall j : \text{songs} \bullet i \neq j \wedge \text{popularity}(i) > \text{popularity}(j)) \Rightarrow$

$\text{mostPopular} = (\mu i : \text{songs} \mid \forall j : \text{songs} \bullet i \neq j \wedge \text{popularity}(i) > \text{popularity}(j))$

and

$\neg \exists_1 i : \text{songs} \bullet \forall j : \text{songs} \bullet i \neq j \wedge \text{popularity}(i) > \text{popularity}(j) \Rightarrow \text{mostPopular} = \text{nullSong}$

(c) $\text{playlistsContainingMostPopularSong} == \{i : \text{dom playlists} \mid \text{mostPopular} \in \text{ran playlists}(i)\}$

Solution 52

(a)

premiumPlays : *seq(Play)* *seq(Play)*

$\text{premiumPlays}(\langle \rangle) = \langle \rangle$

$\forall x : \text{Play}; s : \text{seq}(\text{Play}) \mid$

$\text{premiumPlays}(\langle x \rangle^s) = \langle x \rangle^{\text{premiumPlays}(s)}$ if $\text{userStatus}(x.2) = \text{premium}$

$\text{premiumPlays}(s)$ if $\text{userStatus}(x.2) = \text{standard}$

(Note: Uses camelCase for fuzz compatibility)

(b)

standardPlays : *seq(Play)* *seq(Play)*

$\text{standardPlays}(\langle \rangle) = \langle \rangle$

$\forall x : Play; s : seq(Play) \mid$

$standardPlays(\langle x \rangle^s) = \langle x \rangle^s standardPlays(s) if userStatus(x.2) = standard$

$standardPlays(s) \text{ if } userStatus(x.2) = premium$

(Note: Uses camelCase for fuzz compatibility)

(c)

$cumulativeLength : seq(Play) \rightarrow N$

$cumulativeLength(\langle \rangle) = 0$

$\forall x : Play; s : seq(Play) \mid$

$cumulativeLength(\langle x \rangle^s) = length(x.1) + cumulativeLength(s)$

(Note: Uses camelCase for fuzz compatibility)