Propositional logic

Solution 1

(a)

false (as (true $\Rightarrow$ false) $\Leftrightarrow$ false)

(b)

true (as (false $\Rightarrow$ false) $\Leftrightarrow$ true)

(c)

true (as (false $\Rightarrow$ true) $\Leftrightarrow$ true)

(d)

true (as (false $\Rightarrow$ false) $\Leftrightarrow$ true)

(Assuming that pigs can't fly . . .)

Solution 2

(a)

p	q	$p \wedge q$	$(p \wedge q) \Rightarrow p$
t	t	t	t
t	f	f	t
f	t	f	t
f	f	f	t

(b)

p	q	$p \wedge q$	$\neg\, p$	$\neg\, p \Rightarrow (p \wedge q)$	$(\neg\, p \Rightarrow (p \wedge q)) \Leftrightarrow p$
t	t	t	f	t	t
t	f	f	f	t	t
f	t	f	t	f	t
f	f	f	t	f	t

(c)

p	q	$p \Rightarrow q$	$p \wedge (p \Rightarrow q)$	$(p \wedge (p \Rightarrow q)) \Rightarrow q$
t	t	t	t	t
t	f	f	f	t
f	t	t	f	t
f	f	t	f	t

Solution 3

(a)

$$
\begin{aligned}
& p \Rightarrow \neg\, p \\
& \qquad \Leftrightarrow \neg\, p \vee \neg\, p && [\Rightarrow] \\
& \qquad \Leftrightarrow \neg\, p && [\text{idempotence}]
\end{aligned}
$$

(b)

$$
\begin{aligned}
& \neg\, p \Rightarrow p \\
& \qquad \Leftrightarrow \neg\, \neg\, p \vee p && [\Rightarrow] \\
& \qquad \Leftrightarrow p \vee p && [\neg\ \neg\,] \\
& \qquad \Leftrightarrow p && [\text{idempotence}]
\end{aligned}
$$

(c)

$$
\begin{aligned}
& p \Rightarrow (q \Rightarrow r) \\
& \qquad \Leftrightarrow \neg\, p \vee (q \Rightarrow r) && [\Rightarrow] \\
& \qquad \Leftrightarrow \neg\, p \vee \neg\, q \vee r && [\Rightarrow] \\
& \qquad \Leftrightarrow \neg\, p \vee \neg\, q \vee r && [\text{associativity}] \\
& \qquad \Leftrightarrow \neg\, (p \wedge q) \vee r && [\text{De Morgan}] \\
& \qquad \Leftrightarrow p \wedge q \Rightarrow r && [\Rightarrow]
\end{aligned}
$$

(d)

$$
\begin{aligned}
& q \Rightarrow (p \Rightarrow r) \\
& \qquad \Leftrightarrow \neg\, q \vee (p \Rightarrow r) && [\Rightarrow] \\
& \qquad \Leftrightarrow \neg\, q \vee \neg\, p \vee r && [\Rightarrow] \\
& \qquad \Leftrightarrow \neg\, p \vee \neg\, q \vee r && [\text{associativity} \wedge \text{commutativity}] \\
& \qquad \Leftrightarrow \neg\, p \vee (q \Rightarrow r) && [\Rightarrow] \\
& \qquad \Leftrightarrow p \Rightarrow (q \Rightarrow r) && [\Rightarrow]
\end{aligned}
$$

(e)

$$
\begin{aligned}
& p \wedge q \Leftrightarrow p \\
& \qquad \Leftrightarrow (p \wedge q \Rightarrow p) \wedge (p \Rightarrow p \wedge q) && [\Leftrightarrow] \\
& \qquad \Leftrightarrow (\neg\, (p \wedge q) \vee p) \wedge (\neg\, p \vee p \wedge q) && [\Rightarrow] \\
& \qquad \Leftrightarrow (\neg\, p \vee \neg\, q \vee p) \wedge (\neg\, p \vee p \wedge q) && [\text{De Morgan}] \\
& \qquad \Leftrightarrow (\neg\, q \vee \neg\, p \vee p) \wedge (\neg\, p \vee p \wedge q) && [\text{associativity} \wedge \text{comm .}] \\
& \qquad \Leftrightarrow (\neg\, q \vee \mathit{true}) \wedge (\neg\, p \vee p \wedge q) && [\text{excluded middle}] \\
& \qquad \Leftrightarrow \mathit{true} \wedge (\neg\, p \vee p \wedge q) && [\vee \wedge \text{true}] \\
& \qquad \Leftrightarrow \neg\, p \vee p \wedge q && [\wedge \wedge \text{true}] \\
& \qquad \Leftrightarrow (\neg\, p \vee p) \wedge (\neg\, p \vee q) && [\text{distribution}] \\
& \qquad \Leftrightarrow \mathit{true} \wedge (\neg\, p \vee q) && [\text{excluded middle}] \\
& \qquad \Leftrightarrow \neg\, p \vee q && [\wedge \wedge \text{true}] \\
& \qquad \Leftrightarrow p \Rightarrow q && [\Rightarrow]
\end{aligned}
$$

(f)

$$
\begin{aligned}
& p \vee q \Leftrightarrow p \\
& \qquad \Leftrightarrow (p \vee q \Rightarrow p) \wedge (p \Rightarrow p \vee q) && [\Leftrightarrow] \\
& \qquad \Leftrightarrow (\neg\, (p \vee q) \vee p) \wedge (\neg\, p \vee p \vee q) && [\Rightarrow] \\
& \qquad \Leftrightarrow (\neg\, p \wedge \neg\, q \vee p) \wedge (\neg\, p \vee p \vee q) && [\text{De Morgan}] \\
& \qquad \Leftrightarrow (\neg\, p \vee p) \wedge (\neg\, q \vee p) \wedge (\neg\, p \vee p \vee q) && [\text{distribution}] \\
& \qquad \Leftrightarrow \mathit{true} \wedge (\neg\, q \vee p) \wedge (\neg\, p \vee p \vee q) && [\text{excluded middle}] \\
& \qquad \Leftrightarrow (\neg\, q \vee p) \wedge (\neg\, p \vee p \vee q) && [\wedge \wedge \text{true}] \\
& \qquad \Leftrightarrow (\neg\, q \vee p) \wedge (\neg\, p \vee p \vee q) && [\text{associativity}] \\
& \qquad \Leftrightarrow (\neg\, q \vee p) \wedge (\mathit{true} \vee q) && [\text{excluded middle}] \\
& \qquad \Leftrightarrow (\neg\, q \vee p) \wedge \mathit{true} && [\vee \wedge \text{true}] \\
& \qquad \Leftrightarrow \neg\, q \vee p && [\wedge \wedge \text{true}] \\
& \qquad \Leftrightarrow q \Rightarrow p && [\Rightarrow]
\end{aligned}
$$

Solution 4

(a) (p or q) $\Leftrightarrow$ ((not p or not q) and q) is not a tautology. You might illustrate this via a truth table or via a chain of equivalences, showing that the proposition is not equivalent to true. Alternatively, you might try and find a combination of values for which the proposition is false. (In this case, the proposition is false when p and q are both true.)

(b) (p or q) ⇔ ((not p and not q) or q) is not a tautology. In this case, the proposition is false when p is true and q is false.

Solution 5

Solution 6

(a)

This is a true proposition: whatever the value of x, the expression x$^2 - x + 1 denotesanaturalnumber. Ifwechooseytobethisnaturalnumber, wewillfindthatpistrue.$

(b)

This is a false proposition. We cannot choose a large enough value for y such that p will hold for any value of x.

(c)

This is a false proposition. It is an implication whose antecedent part is true and whose consequent part is false.

(d)

This is a true proposition. It is an implication whose antecedent part is false and whose consequent part is true.

Solution 7

(a)

We must define a predicate p that is false for at least one value of x, and is true for at least one other value. A suitable solution would be p ⇔ x ¿ 1.

(b)

With the above choice of p, we require only that q is sometimes false when p is true (for else the universal quantification would hold). A suitable solution would be q ⇔ x ¿ 3.

Solution 8

(a)

$\forall x : N \bullet x \geq z$

Equality

Solution 9

(d)

$$\begin{aligned}
&\exists x : N \bullet x = 1 \wedge x > y \vee x = 2 \wedge x > z \\
&\qquad \Leftrightarrow \exists x : N \bullet x = 1 \wedge x > y \vee \exists x : N \bullet x = 2 \wedge x > z \\
&\qquad \Leftrightarrow 1 \in N \wedge 1 > y \vee \exists x : N \bullet x = 2 \wedge x > z \\
&\qquad \Leftrightarrow 1 \in N \wedge 1 > y \vee 2 \in N \wedge 2 > z \\
&\qquad \Leftrightarrow 1 > y \vee 2 > z
\end{aligned}$$

Solution 10

Solution 11

Solution 12

Deductive proofs

Solution 13

$$\dfrac{\ulcorner p \wedge (p \Rightarrow q) \urcorner^{[1]} \qquad \dfrac{\dfrac{p \wedge (p \Rightarrow q)}{p}\,[\wedge\text{-elim}^{[1]}] \qquad \dfrac{\dfrac{p \wedge (p \Rightarrow q)}{p \Rightarrow q}\,[\wedge\text{-elim}^{[1]}] \qquad \dfrac{p \wedge (p \Rightarrow q)}{p}\,[\wedge\text{-elim}^{[1]}]}{q}\,[\Rightarrow \text{elim}]}{p \wedge q}\,[\wedge \text{ intro}]}{p \wedge (p \Rightarrow q) \Rightarrow p \wedge q}\,[\Rightarrow\text{-intro}^{[1]}]$$

Solution 14

In one direction:

$$\dfrac{\ulcorner p \wedge q \Leftrightarrow p \urcorner^{[1]} \qquad \dfrac{\ulcorner p \urcorner^{[2]} \qquad \dfrac{\dfrac{\dfrac{}{p \wedge q}\;[\text{derived}]}{p \wedge q}\;[\Rightarrow \text{ elim from } 1 \wedge 2]}{q}\;[\wedge\text{-elim}^{[3]}]}{p \Rightarrow q}\;[\Rightarrow\text{-intro}^{[2]}]}{(p \wedge q \Leftrightarrow p) \Rightarrow (p \Rightarrow q)}\;[\Rightarrow\text{-intro}^{[1]}]$$

and the other:

$$\dfrac{\ulcorner p \Rightarrow q \urcorner^{[1]} \qquad \dfrac{\dfrac{\ulcorner p \wedge q \urcorner^{[2]} \quad \ulcorner p \urcorner^{[2]}}{p \wedge q \Rightarrow p}\;[\Rightarrow\text{-intro}^{[2]}] \qquad \dfrac{\ulcorner p \urcorner^{[3]} \quad \ulcorner p \wedge q \urcorner^{[1]}}{p \Rightarrow p \wedge q}\;[\Rightarrow\text{-intro}^{[3]}]}{p \wedge q \Leftrightarrow p}\;[\Leftrightarrow \text{ intro}]}{(p \Rightarrow q) \Rightarrow (p \wedge q \Leftrightarrow p)}\;[\Rightarrow\text{-intro}^{[1]}]$$

We can then combine these two proofs with $\Leftrightarrow$ intro.

Solution 15

$$\dfrac{\ulcorner (p \Rightarrow q) \wedge \neg\, q \urcorner^{[1]} \qquad \dfrac{\ulcorner p \urcorner^{[2]} \qquad \dfrac{\dfrac{\ulcorner p \Rightarrow q \urcorner^{[1]} \quad \ulcorner p \urcorner^{[2]}}{q}\;[\Rightarrow \text{ elim}] \qquad \ulcorner \neg\, q \urcorner^{[1]}}{\mathit{false}}\;[\text{false intro}]}{\neg\, p}\;[\text{false-elim}^{[2]}]}{(p \Rightarrow q) \wedge \neg\, q \Rightarrow \neg\, p}\;[\Rightarrow\text{-intro}^{[1]}]$$

Solution 16

In one direction:

$$\dfrac{\ulcorner p \urcorner^{[1]} \qquad \dfrac{}{r}\,[\text{case assumption}]}{p \wedge r}\,[\wedge\ \text{intro}]$$

$$\dfrac{}{p \wedge q \vee p \wedge r}\,[\vee\ \text{intro}]$$

$$\dfrac{\ulcorner p \urcorner^{[1]} \qquad \dfrac{}{q}\,[\text{case assumption}]}{p \wedge q}\,[\wedge\ \text{intro}]$$

$$\dfrac{}{p \wedge q \vee p \wedge r}\,[\vee\ \text{intro}]$$

$$\dfrac{\ulcorner p \wedge (q \vee r) \urcorner^{[1]} \qquad \dfrac{\ulcorner q \vee r \urcorner^{[1]}}{p \wedge q \vee p \wedge r}\,[\vee\text{-elim}^{[2]}]}{p \wedge (q \vee r) \Rightarrow p \wedge q \vee p \wedge r}\,[\Rightarrow\text{-intro}^{[1]}]$$

In the other:

$$\dfrac{}{p}\,[\wedge\ \text{elim}]$$

$$\dfrac{}{q \vee r}\,[\vee\ \text{intro}]$$

$$\dfrac{}{p \wedge (q \vee r)}\,[\wedge\ \text{intro}]$$

$$\dfrac{}{p}\,[\wedge\ \text{elim}]$$

$$\dfrac{}{q \vee r}\,[\vee\ \text{intro}]$$

$$\dfrac{}{p \wedge (q \vee r)}\,[\wedge\ \text{intro}]$$

$$\dfrac{\ulcorner p \wedge q \vee p \wedge r \urcorner^{[3]} \qquad \dfrac{\ulcorner case1 \vee case2 \urcorner^{[3]}}{p \wedge (q \vee r)}\,[\vee\text{-elim}^{[4]}]}{p \wedge q \vee p \wedge r \Rightarrow p \wedge (q \vee r)}\,[\Rightarrow\text{-intro}^{[3]}]$$

Solution 17

In one direction:

$$\dfrac{\ulcorner p \vee q \wedge r \urcorner^{[3]} \qquad \dfrac{}{(p \vee q) \wedge (p \vee r)}\,[\vee\ \text{elim} \wedge \wedge\ \text{intro}]}{p \vee q \wedge r \Rightarrow (p \vee q) \wedge (p \vee r)}\,[\Rightarrow\text{-intro}^{[3]}]$$

and the other:

$$\dfrac{\ulcorner (p \vee q) \wedge (p \vee r) \urcorner^{[1]} \quad \ulcorner p \vee q \wedge r \urcorner^{[2]}}{(p \vee q) \wedge (p \vee r) \Rightarrow p \vee q \wedge r} \ [\Rightarrow\text{-intro}^{[1]}]$$

Solution 18

In one direction:

$$\dfrac{\ulcorner p \Rightarrow q \urcorner^{[1]} \quad \neg\, p \vee q}{(p \Rightarrow q) \Rightarrow \neg\, p \vee q} \ [\Rightarrow\text{-intro}^{[1]}]$$

and the other:

$$\dfrac{\ulcorner \neg\, p \vee q \urcorner^{[3]} \quad \dfrac{\ulcorner p \urcorner^{[4]} \quad \ulcorner q \urcorner^{[3]}}{p \Rightarrow q} \ [\Rightarrow\text{-intro}^{[4]}]}{\neg\, p \vee q \Rightarrow (p \Rightarrow q)} \ [\Rightarrow\text{-intro}^{[3]}]$$

Sets and types

Solution 19

Solution 20

Solution 21

Solution 22

Solution 23

(a)

$$\begin{aligned} & x \in a \cap a \\ & \qquad \Leftrightarrow x \in a \wedge x \in a \\ & \qquad \Leftrightarrow x \in a \end{aligned}$$

(b)

$$
\begin{aligned}
& x \in a \cup a \\
& \qquad \Leftrightarrow x \in a \vee x \in a \\
& \qquad \Leftrightarrow x \in a
\end{aligned}
$$

Solution 24

Solution 25

Solution 26

Relations

Solution 27

Solution 28

Solution 29

Solution 30

Solution 31

Solution 32

(a)

$$
\begin{aligned}
& x \mapsto y \in A \lhd B \lhd R \\
& \qquad \Leftrightarrow x \in A \wedge x \mapsto y \in (B \lhd R) \\
& \qquad \Leftrightarrow x \in A \wedge x \in B \wedge x \mapsto y \in R \\
& \qquad \Leftrightarrow x \in A \cap B \wedge x \mapsto y \in R \\
& \qquad \Leftrightarrow x \mapsto y \in A \cap B \lhd R
\end{aligned}
$$

(b)

$$
\begin{aligned}
&x \mapsto y \in R \cup S \rhd C \\
&\qquad \Leftrightarrow x \mapsto y \in R \cup S \land y \in C \\
&\qquad \Leftrightarrow (x \mapsto y \in R \lor x \mapsto y \in S) \land y \in C \\
&\qquad \Leftrightarrow x \mapsto y \in R \land y \in C \lor x \mapsto y \in S \land y \in C \\
&\qquad \Leftrightarrow x \mapsto y \in R \rhd C \lor x \mapsto y \in S \rhd C \\
&\qquad \Leftrightarrow x \mapsto y \in (R \rhd C) \cup (S \rhd C)
\end{aligned}
$$

Functions

Solution 33

Solution 34

Solution 35

Solution 36

Sequences

Solution 37

Solution 38

Solution 39

Modelling

Solution 40

Solution 41

Solution 42

Solution 43

Free types and induction

Solution 44

Solution 45

Solution 46

Solution 47

Supplementary material : assignment practice

Solution 48

Solution 49

Solution 50

Solution 51

Solution 52