

## Function Composition

[*Person, Char, B, Record, ValidatedRecord, ProcessedRecord, Output*]

### Example 1 : Basic Forward Composition

Forward composition applies f first, then g:

$$\left| \begin{array}{l} f : \mathbb{N} \rightarrow \mathbb{N} \\ g : \mathbb{N} \rightarrow \mathbb{N} \\ h : \mathbb{N} \rightarrow \mathbb{N} \end{array} \right| \quad \begin{array}{l} \hline \forall n : \mathbb{N} \bullet f(n) = n + 1 \\ \forall n : \mathbb{N} \bullet g(n) = n * 2 \\ h = f \circ g \end{array}$$

$$h(n) = g(f(n)) = g(n + 1) = (n + 1) * 2 = 2n + 2$$

### Example 2 : Backward Composition

Backward composition applies the second function first (using comp):

$$\left| \begin{array}{l} f2 : \mathbb{N} \rightarrow \mathbb{N} \\ g2 : \mathbb{N} \rightarrow \mathbb{N} \\ h2 : \mathbb{N} \rightarrow \mathbb{N} \end{array} \right| \quad \begin{array}{l} \hline \forall n : \mathbb{N} \bullet f2(n) = n + 1 \\ \forall n : \mathbb{N} \bullet g2(n) = n * 2 \\ h2 = f2 \circ g2 \end{array}$$

$$h2(n) = f2(g2(n)) = f2(n * 2) = (n * 2) + 1 = 2n + 1$$

### Example 3 : Comparison - Forward vs Backward

Forward composition  $f \circ g$ :

Input  $\rightarrow f \rightarrow g \rightarrow$  Output

$$(f \circ g)(x) = g(f(x))$$

Backward composition  $f \circ g$ :

Input  $\rightarrow g \rightarrow f \rightarrow$  Output

$$(f \circ g)(x) = f(g(x))$$

Notice the *reversal* : *o* applies *left-to-right*, *o* applies *right-to-left*.

## Example 4 : Three - Function Composition

$addOne : \mathbb{N} \rightarrow \mathbb{N}$
$double : \mathbb{N} \rightarrow \mathbb{N}$
$square : \mathbb{N} \rightarrow \mathbb{N}$
$pipeline : \mathbb{N} \rightarrow \mathbb{N}$
$\forall n : \mathbb{N} \bullet addOne(n) = n + 1$
$\forall n : \mathbb{N} \bullet double(n) = 2 * n$
$\forall n : \mathbb{N} \bullet square(n) = n * n$
$pipeline = addOne \circ double \circ square$

$pipeline(n) = square(double(addOne(n))) = square(double(n + 1)) = square(2(n + 1)) = (2(n + 1))^2$

## Example 5 : Identity Composition Laws

$identity : \mathbb{N} \rightarrow \mathbb{N}$
$anyFunc : \mathbb{N} \rightarrow \mathbb{N}$
$leftId : \mathbb{N} \rightarrow \mathbb{N}$
$rightId : \mathbb{N} \rightarrow \mathbb{N}$
$\forall n : \mathbb{N} \bullet identity(n) = n$
$\forall n : \mathbb{N} \bullet anyFunc(n) = n * n$
$leftId = identity \circ anyFunc$
$rightId = anyFunc \circ identity$

$leftId = anyFunc$  (identity is left identity for  $\circ$ )

$rightId = anyFunc$  (identity is right identity for  $\circ$ )

## Example 6 : Associativity

Composition is associative:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

You can group compositions in any order—the result is the same.

$f3 : \mathbb{N} \rightarrow \mathbb{N}$
$g3 : \mathbb{N} \rightarrow \mathbb{N}$
$h3 : \mathbb{N} \rightarrow \mathbb{N}$
$comp1 : \mathbb{N} \rightarrow \mathbb{N}$
$comp2 : \mathbb{N} \rightarrow \mathbb{N}$
$\forall n : \mathbb{N} \bullet f3(n) = n + 1$
$\forall n : \mathbb{N} \bullet g3(n) = n * 2$
$\forall n : \mathbb{N} \bullet h3(n) = n * n$
$comp1 = (f3 \circ g3) \circ h3$
$comp2 = f3 \circ (g3 \circ h3)$

$comp1 = comp2$  for all inputs.

## Example 7 : Composition with Partial Functions

$safeSqrt : \mathbb{N} \rightarrow \mathbb{N}$ $addTen : \mathbb{N} \rightarrow \mathbb{N}$ $composed : \mathbb{N} \rightarrow \mathbb{N}$
$\forall n : \mathbb{N} \bullet n * n = n \Rightarrow safeSqrt(n) = n$ $\forall n : \mathbb{N} \bullet addTen(n) = n + 10$ $composed = safeSqrt \circ addTen$

*composed* is only defined where *safeSqrt* is defined, then adds 10 to the result.

## Example 8 : Relation Composition

Composition works for relations too:

$parent : Person \leftrightarrow Person$ $grandparent : Person \leftrightarrow Person$
$grandparent = parent \circ parent$

*grandparent* relates a person to their grandparents via two parent links.

## Example 9 : Function Pipeline Example

$toLowerCase : seq\ Char \rightarrow seq\ Char$ $trim : seq\ Char \rightarrow seq\ Char$ $reverse : seq\ Char \rightarrow seq\ Char$ $process : seq\ Char \rightarrow seq\ Char$
$process = toLowerCase \circ trim \circ reverse$

Process a string by converting to lowercase, trimming whitespace, then reversing.

## Example 10 : Mathematical Functions

$squareFunc : \mathbb{N} \rightarrow \mathbb{N}$ $tripleFunc : \mathbb{N} \rightarrow \mathbb{N}$ $combined : \mathbb{N} \rightarrow \mathbb{N}$
$\forall n : \mathbb{N} \bullet squareFunc(n) = n * n$ $\forall n : \mathbb{N} \bullet tripleFunc(n) = 3 * n$ $combined = squareFunc \circ tripleFunc$

*Combined* applies *squareFunc* first, then triples the result.

## Example 11 : Inverse via Composition

$encrypt : seq\ Char \rightarrow seq\ Char$ $decrypt : seq\ Char \rightarrow seq\ Char$ $identity\_check : seq\ Char \rightarrow seq\ Char$
$identity\_check = encrypt \circ decrypt$

If *encrypt* and *decrypt* are inverses, then *identity\_check* should be the identity function.

## Example 12 : Partial Composition

When composing  $f : A \rightarrow B$  and  $g : C \rightarrow D$ , we need  $\text{ran } f \subseteq \text{dom } g$ .

$$\left| \begin{array}{l} f4 : \mathbb{N} \rightarrow \mathbb{N} \\ g4 : \mathbb{N} \rightarrow \mathbb{N} \\ h4 : \mathbb{N} \rightarrow \mathbb{N} \end{array} \right| \begin{array}{l} f4 = \{1 \mapsto 10, 2 \mapsto 20\} \\ g4 = \{10 \mapsto 100, 20 \mapsto 200\} \\ h4 = f4 \circ g4 \end{array}$$

$h4 = \{1 \mapsto 100, 2 \mapsto 200\}$ . Composition is defined where ranges align.

## Example 13 : Monoid Structure

Functions from  $A \rightarrow A$  with composition form a monoid:

1. Composition is associative:  $(f \circ g) \circ h = f \circ (g \circ h)$
2. Identity *element* :  $\text{id}(x) = x$ ,  $\text{id} \circ f = f = f \circ \text{id}$
3. Closure: composing  $A \rightarrow A$  functions gives another  $A \rightarrow A$  function

## Example 14 : Practical Example - Data Processing Pipeline

In practice, you might compose functions to create data processing *pipelines* :  $\text{validate} \circ \text{process} \circ \text{format}$ . Each stage transforms data through a series of operations. The notation  $f \circ g \circ h$  means "apply f, then g, then h" which creates a pipeline from input to output.

## Example 15 : Composition Operator Precedence

Composition operators bind tighter than logical operators:

$f \circ g = h$  means  $(f \circ g) = h$ , not  $f \circ (g = h)$

Use parentheses to clarify:  $(f \circ g)$  or  $f \circ (g \circ h)$

## Example 16 : Self - Composition

$$\left| \begin{array}{l} \text{doubleIt} : \mathbb{N} \rightarrow \mathbb{N} \\ \text{quadrupleIt} : \mathbb{N} \rightarrow \mathbb{N} \end{array} \right| \begin{array}{l} \forall n : \mathbb{N} \bullet \text{doubleIt}(n) = 2 * n \\ \text{quadrupleIt} = \text{doubleIt} \circ \text{doubleIt} \end{array}$$

$\text{quadrupleIt}(n) = \text{doubleIt}(\text{doubleIt}(n)) = 2 * (2 * n) = 4n$

## Example 17 : Iterated Composition

$$\left| \begin{array}{l} \text{increment} : \mathbb{N} \rightarrow \mathbb{N} \\ \text{addFive} : \mathbb{N} \rightarrow \mathbb{N} \end{array} \right| \frac{\forall n : \mathbb{N} \bullet \text{increment}(n) = n + 1}{\text{addFive} = \text{increment} \circ \text{increment} \circ \text{increment} \circ \text{increment} \circ \text{increment}}$$

Composing increment 5 times gives addFive.

## Example 18 : Functional Programming Style

Composition enables a functional programming style:

Instead of  $result = h(g(f(x)))$

Use  $pipeline = f \circ g \circ h$ ;  $result = pipeline(x)$

This separates pipeline definition from application, improving modularity.

## Example 19 : Best Practices

When using composition:

1. Choose forward ( $\circ$ ) or backward ( $\circ$ ) consistently in your codebase
2. Document the data flow direction
3. Break complex pipelines into named intermediate steps
4. Type-check that domains and ranges align
5. Use composition to create reusable pipelines
6. Test individual functions before composing them
7. *Remember* :  $\circ$  reads left – to – right (natural for pipelines),  $\circ$  reads right-to-left (traditional math notation)