Law of Excluded Middle Examples

Example 1 : Statement of LEM

The law of excluded middle states: for any proposition p, either p is true or $\neg\, p$ is true.

$p \lor \neg\, p$
This is an axiom of classical logic but not of constructive logic.

Example 2 : Proof by Cases Using LEM

Prove: $(p \Rightarrow q) \Rightarrow ((\neg\, p \Rightarrow q) \Rightarrow q)$

$$\dfrac{\dfrac{\ulcorner p \Rightarrow q \urcorner^{[1]}}{p \lor \neg\, p}\,[\text{LEM}] \quad \dfrac{}{q}\,[\Rightarrow \text{elim with p} \Rightarrow \text{q}] \quad \dfrac{}{q}\,[\Rightarrow \text{elim with} \neg\, \text{p} \Rightarrow \text{q}] \quad \dfrac{\ulcorner p \Rightarrow q \urcorner^{[1]}}{q}\,[\lor\, \text{elim}]}{\dfrac{\neg\, p \Rightarrow q}{q}\,[\lor\, \text{elim}]}\,[\text{assumption}]$$

By LEM, we can case-analyze on p $\lor \neg\, p$. In both cases we derive q.

Example 3 : Double Negation Elimination Using LEM

Prove: $\neg\ \neg\, p \Rightarrow p$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathit{false}}\,[\text{contradiction with} \neg\, \neg\, \text{p}]}{p}\,[\text{false elim}]}{p}\,[\lor\, \text{elim}]}{p}\,[\lor\, \text{elim}]$$

LEM enables double negation elimination in classical logic.

Example 4 : De Morgan's Law Using LEM

Prove: $\neg\ (p \land q) \Rightarrow (\neg\, p \lor \neg\, q)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\, p}\,[\text{identity}]}{\neg\, p \lor \neg\, q}\,[\lor\, \text{intro left}]}{\neg\, p \lor \neg\, q}\,[\lor\, \text{elim}]}{\neg\, p \lor \neg\, q}\,[\lor\, \text{elim}]$$

This proof requires LEM to case-analyze on both p and q.

Example 5 : Decidability

A proposition is decidable if we can prove p or not p. In logical notation, we would $write$: $decidable(p) means$ $p(or)(\neg\, p)$.

LEM states that all propositions are decidable in classical logic.

Example 6 : Proof by Contradiction via LEM

Prove: $(\neg\, p \Rightarrow \mathit{false}) \Rightarrow p$

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\mathit{false}}\;[\Rightarrow \text{elim}]}{p}\;[\text{false elim}]}{p}\;[\vee \text{ elim}]}{p}\;[\vee \text{ elim}]$$

This shows that proof by contradiction follows from LEM.

Example 7 : Material Implication

Prove: $(p \Rightarrow q) \Leftrightarrow (\neg\, p \vee q)$

Left-to-right $(p \Rightarrow (q \Rightarrow \neg\, p \vee q))$:

$$\cfrac{\cfrac{\cfrac{}{\neg\, p \vee q}\;[\vee \text{ intro left}]}{\neg\, p \vee q}\;[\vee \text{ elim}]}{\neg\, p \vee q}\;[\vee \text{ elim}]$$

Right-to-left $(\neg\, p \vee q \Rightarrow (p \Rightarrow q))$:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\mathit{false}}\;[\text{contradiction}]}{q}\;[\text{false elim}] \qquad \cfrac{}{q}\;[\text{identity}] \qquad \cfrac{\ulcorner \neg\, p \vee q \urcorner^{[1]}}{q}\;[\vee \text{ elim}]}{p}\;[\text{assumption}]}{p \Rightarrow q}\;[\Rightarrow\text{-intro}^{[1]}]$$

Material implication equivalence uses LEM in the forward direction.

Example 8 : Pierce's Law

Prove: $((p \Rightarrow q) \Rightarrow p) \Rightarrow$ p

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ulcorner p \urcorner^{[2]}}{\mathit{false}}\ [\text{contradiction}]}{q}\ [\text{false elim}]}{p \Rightarrow q}\ [\Rightarrow\text{-intro}^{[2]}]}{p}\ [\Rightarrow \text{elim with premise}]}{p}\ [\lor \text{ elim}]}{p}\ [\lor \text{ elim}]$$

Pierce's law is equivalent to LEM in classical logic.

Example 9 : Tertium Non Datur

Tertium non datur (Latin: "no third possibility") is another name for LEM.

For any proposition p, there is no third option besides p and $\neg\, p$.

$\forall\, p \bullet p \lor \neg\, p$
This principle distinguishes classical from intuitionistic logic.

Example 10 : Constructive vs Classical

In constructive (intuitionistic) logic, LEM is not provable as a general principle.

Constructive: To prove p or not p, you must either:

- Construct a proof of p, or

- Construct a proof of not p (a proof that p leads to contradiction)

$\mathit{Classical} : \mathit{LEM}$ is an axiom—every proposition is assumed decidable without construction.

Example 11 : Example Where LEM Is Not Needed

Some propositions are constructively decidable:

$\forall\, n : \mathbb{N} \bullet (n = 0) \lor \neg\, (n = 0)$
We can decide this by checking if n equals 0.

But general propositions may not be decidable constructively:

$\forall\, p \bullet p \lor \neg\, (p - -(\mathit{requires}(\mathit{LEM}))) \in \mathit{general}(\mathit{case})$

Example 12 : Proof Strategy with LEM

Common proof strategy using LEM:

To prove Q:

1. Invoke p or $\neg\, p$ for some relevant p

2. [case] p prove Q assuming p

3. [case] $\neg\, p$ prove Q assuming $\neg\, p$

4. Conclude Q by or elim

This technique is called "proof by cases" or "case analysis."

Example 13 : LEM land Indirect Proof

Indirect proof pattern using LEM:

To prove p:

1. Assume not p [assumption 1]

2. Derive false

3. Conclude $\neg\ \neg\, p$ [$\neg$ intro from 1]

4. Apply $LEM : p \vee \neg\, p$

5. [case] p p

6. [case] $\neg\, p$ contradicts $\neg\ \neg\, p$, derive p by false elim

7. Conclude p [or elim]

LEM enables the final step (double negation elimination).

Example 14 : Drinker Paradox

Prove: There $\exists$ a person x such that if x drinks, then everyone drinks.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a}{Pick(any)(person)}}{a(drinks)}\,[\text{from p}]}{\forall\, y \bullet y(drinks)}\,[\text{from p}]}{a(drinks) \Rightarrow (\forall\, y \bullet y(drinks))}\,[\Rightarrow \text{intro, trivially true}]}{\exists\, x \bullet x(drinks) \Rightarrow (\forall\, y \bullet y(drinks))}\,[\exists\,\text{intro with x = a}] \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\exists\, b \bullet \neg\,(b(drinks))}\,[\text{negation of}\,\forall]}{\neg\,(b(drinks))}\,[\exists\,\text{elim}]}{b(drinks)}\,[\text{assumption}]}{b(drinks) \Rightarrow (\forall\, y \bullet y(drinks))}\,[\Rightarrow \text{-intro}^{[2]}]}{\exists\, x \bullet x(drinks) \Rightarrow (\forall\, y \bullet y(drinks))}\,[\exists\,\text{intro with x = b}]}{\dfrac{\exists\, x \bullet x(drinks) \Rightarrow (\forall\, y \bullet y(drinks))}{\exists\, x \bullet x(drinks) \Rightarrow (\forall\, y \bullet y(drinks))}\,[\vee\ \text{elim}]}\,[\vee\ \text{elim}]$$

This classical proof uses LEM crucially.

Example 15 : When to Use LEM

Use LEM when:

1. You need to prove something by case analysis on an arbitrary proposition

2. You're working in classical logic (not constructive)

3. You need double negation elimination

4. Your proof strategy requires considering both p $\wedge \neg p$

Avoid LEM when:

1. Working in constructive/intuitionistic logic

2. The proposition has a decidable, constructive proof

3. You want algorithms that compute witnesses (constructive proofs)

Example 16 : Relationship to Other Principles

In classical logic, these are equivalent:

- Law of excluded $middle : p \vee \neg p$

- Double negation elimination: $\neg \neg p \Rightarrow p$

- Pierce's law: $((p \Rightarrow q) \Rightarrow p) \Rightarrow$ p

- Reductio ad absurdum: $(\neg p \Rightarrow \mathit{false}) \Rightarrow p$

Accepting any one of these principles gives you classical logic.

In intuitionistic logic, none of these hold in general—you must construct proofs explicitly.