# Propositional logic

## Solution 1

(a) *false* (as (*true* $\Rightarrow$ *false*) $\Leftrightarrow$ *false*)

(b) *true* (as (*false* $\Rightarrow$ *false*) $\Leftrightarrow$ *true*)

(c) *true* (as (*false* $\Rightarrow$ *true*) $\Leftrightarrow$ *true*)

(d) *true* (as (*false* $\Rightarrow$ *false*) $\Leftrightarrow$ *true*)

(Assuming that pigs *can't* fly . . . )

## Solution 2

(a)

| $p$ | $q$ | $p \wedge q$ | $(\mathbf{p} \wedge \mathbf{q}) \Rightarrow \mathbf{p}$ |
|---|---|---|---|
| $t$ | $t$ | $t$ | **t** |
| $t$ | $f$ | $f$ | **t** |
| $f$ | $t$ | $f$ | **t** |
| $f$ | $f$ | $f$ | **t** |

(b)

| $p$ | $q$ | $p \wedge q$ | $\neg\, p$ | $\neg\, p \Rightarrow (p \wedge q)$ | $(\neg\, \mathbf{p} \Rightarrow (\mathbf{p} \wedge \mathbf{q})) \Leftrightarrow \mathbf{p}$ |
|---|---|---|---|---|---|
| $t$ | $t$ | $t$ | $f$ | $t$ | **t** |
| $t$ | $f$ | $f$ | $f$ | $t$ | **t** |
| $f$ | $t$ | $f$ | $t$ | $f$ | **t** |
| $f$ | $f$ | $f$ | $t$ | $f$ | **t** |

(c)

| $p$ | $q$ | $p \Rightarrow q$ | $p \wedge (p \Rightarrow q)$ | $(\mathbf{p} \wedge (\mathbf{p} \Rightarrow \mathbf{q})) \Rightarrow \mathbf{q}$ |
|---|---|---|---|---|
| $t$ | $t$ | $t$ | $t$ | **t** |
| $t$ | $f$ | $f$ | $f$ | **t** |
| $f$ | $t$ | $t$ | $f$ | **t** |
| $f$ | $f$ | $t$ | $f$ | **t** |

**Solution 3**

(a)

$$\begin{aligned} & p \Rightarrow \neg p \\ & \quad \Leftrightarrow \neg p \vee \neg p && [\Rightarrow] \\ & \quad \Leftrightarrow \neg p && [\text{idempotence}] \end{aligned}$$

(b)

$$\begin{aligned} & \neg p \Rightarrow p \\ & \quad \Leftrightarrow \neg\neg p \vee p && [\Rightarrow] \\ & \quad \Leftrightarrow p \vee p && [\neg\,\neg] \\ & \quad \Leftrightarrow p && [\text{idempotence}] \end{aligned}$$

(c)

$$\begin{aligned} & p \Rightarrow (q \Rightarrow r) \\ & \quad \Leftrightarrow \neg p \vee (q \Rightarrow r) && [\Rightarrow] \\ & \quad \Leftrightarrow \neg p \vee (\neg q \vee r) && [\Rightarrow] \\ & \quad \Leftrightarrow (\neg p \vee \neg q) \vee r && [\text{associativity}] \\ & \quad \Leftrightarrow \neg (p \wedge q) \vee r && [\text{De Morgan}] \\ & \quad \Leftrightarrow (p \wedge q) \Rightarrow r && [\Rightarrow] \end{aligned}$$

(d)

$$\begin{aligned} & q \Rightarrow (p \Rightarrow r) \\ & \quad \Leftrightarrow \neg q \vee (p \Rightarrow r) && [\Rightarrow] \\ & \quad \Leftrightarrow \neg q \vee (\neg p \vee r) && [\Rightarrow] \\ & \quad \Leftrightarrow \neg p \vee (\neg q \vee r) && [\text{associativity and commutativity}] \\ & \quad \Leftrightarrow \neg p \vee (q \Rightarrow r) && [\Rightarrow] \\ & \quad \Leftrightarrow p \Rightarrow (q \Rightarrow r) && [\Rightarrow] \end{aligned}$$

(e)

$$
\begin{aligned}
&((p \wedge q) \Leftrightarrow p) \\
&\quad \Leftrightarrow ((p \wedge q) \Rightarrow p) \wedge (p \Rightarrow (p \wedge q)) && [\Leftrightarrow] \\
&\quad \Leftrightarrow (\neg(p \wedge q) \vee p) \wedge (\neg p \vee (p \wedge q)) && [\Rightarrow] \\
&\quad \Leftrightarrow ((\neg p \vee \neg q) \vee p) \wedge (\neg p \vee (p \wedge q)) && [\text{De Morgan}] \\
&\quad \Leftrightarrow (\neg q \vee (\neg p \vee p)) \wedge (\neg p \vee (p \wedge q)) && [\text{associativity and comm.}] \\
&\quad \Leftrightarrow (\neg q \vee \mathit{true}) \wedge (\neg p \vee (p \wedge q)) && [\text{excluded middle}] \\
&\quad \Leftrightarrow \mathit{true} \wedge (\neg p \vee (p \wedge q)) && [\vee \text{ and } \mathit{true}] \\
&\quad \Leftrightarrow \neg p \vee (p \wedge q) && [\wedge \text{ and } \mathit{true}] \\
&\quad \Leftrightarrow (\neg p \vee p) \wedge (\neg p \vee q) && [\text{distribution}] \\
&\quad \Leftrightarrow \mathit{true} \wedge (\neg p \vee q) && [\text{excluded middle}] \\
&\quad \Leftrightarrow \neg p \vee q && [\wedge \text{ and } \mathit{true}] \\
&\quad \Leftrightarrow p \Rightarrow q && [\Rightarrow]
\end{aligned}
$$

(f)

$$
\begin{aligned}
&((p \vee q) \Leftrightarrow p) \\
&\quad \Leftrightarrow ((p \vee q) \Rightarrow p) \wedge (p \Rightarrow (p \vee q)) && [\Leftrightarrow] \\
&\quad \Leftrightarrow (\neg(p \vee q) \vee p) \wedge (\neg p \vee (p \vee q)) && [\Rightarrow] \\
&\quad \Leftrightarrow ((\neg p \wedge \neg q) \vee p) \wedge (\neg p \vee (p \vee q)) && [\text{De Morgan}] \\
&\quad \Leftrightarrow ((\neg p \vee p) \wedge (\neg q \vee p)) \wedge (\neg p \vee (p \vee q)) && [\text{distribution}] \\
&\quad \Leftrightarrow (\mathit{true} \wedge (\neg q \vee p)) \wedge (\neg p \vee (p \vee q)) && [\text{excluded middle}] \\
&\quad \Leftrightarrow (\neg q \vee p) \wedge (\neg p \vee (p \vee q)) && [\wedge \text{ and } \mathit{true}] \\
&\quad \Leftrightarrow (\neg q \vee p) \wedge ((\neg p \vee p) \vee q) && [\text{associativity}] \\
&\quad \Leftrightarrow (\neg q \vee p) \wedge (\mathit{true} \vee q) && [\text{excluded middle}] \\
&\quad \Leftrightarrow (\neg q \vee p) \wedge \mathit{true} && [\vee \text{ and } \mathit{true}] \\
&\quad \Leftrightarrow (\neg q \vee p) && [\wedge \text{ and } \mathit{true}] \\
&\quad \Leftrightarrow q \Rightarrow p && [\Rightarrow]
\end{aligned}
$$

## Solution 4

(a) $(p \vee q) \;\Leftrightarrow\; ((\neg p \vee \neg q) \wedge q)$ is not a tautology. You might illustrate this via a truth table or via a chain of equivalences, showing that the proposition is *not* equivalent to *true*. Alternatively, you might try and find a combination of values for which the proposition is *false*. (In this case, the proposition is *false* when $p$ and $q$ are both *true*.)

(b) $(p \lor q) \Leftrightarrow ((\neg p \land \neg q) \lor q)$ is not a tautology. In this case, the proposition is *false* when $p$ is *true* and $q$ is *false*.

**Solution 5**

(a) $\exists\, d : \mathit{Dog} \bullet \mathit{gentle}\,(d) \land \mathit{well\_trained}\,(d)$

(b) $\forall\, d : \mathit{Dog} \bullet \mathit{neat}\,(d) \land \mathit{well\_trained}\,(d) \Rightarrow \mathit{attractive}(d)$

(c) $\exists\, d : \mathit{Dog} \bullet \mathit{gentle}\,(d) \Rightarrow \forall\, t : \mathit{Trainer} \bullet \mathit{groomed}\,(d, t)$

**Solution 6**

(a) This is a true proposition: whatever the value of $x$, the expression $x^2 - x + 1$ denotes a natural number. If we choose $y$ to be this natural number, we will find that $p$ is true.

(b) This is a false proposition. We cannot choose a large enough value for $y$ such that $p$ will hold for any value of $x$.

(c) This is a false proposition. It is an implication whose antecedent part is true and whose consequent part is false.

(d) This is a true proposition. It is an implication whose antecedent part is false and whose consequent part is true.

**Solution 7**

(a) We must define a predicate $p$ that is false for at least one value of $x$, and is true for at least one other value. A suitable solution would be $p \Leftrightarrow x > 1$.

(b) With the above choice of $p$, we require only that $q$ is sometimes false when $p$ is true (for else the universal quantification would hold). A suitable solution would be $q \Leftrightarrow x > 3$.

**Solution 8**

(a) $\forall\, x : \mathbb{N} \bullet x \geq z$

(b) $\forall\, z : \mathbb{N} \bullet z \geq x + y$

(c) $x + 3 > 0 \land \forall\, z : \mathbb{N} \bullet z \geq x + 3$

## Equality

### Solution 9

(a)

$$\begin{aligned}
&\exists\, y : \mathbb{N} \bullet y \in \{0,1\} \land y \neq 1 \land x \neq y \\
&\quad \Leftrightarrow \exists\, y : \mathbb{N} \bullet y = 0 \land x \neq y && \text{[arithmetic]} \\
&\quad \Leftrightarrow 0 \in \mathbb{N} \land x \neq 0 && \text{[one-point rule]} \\
&\quad \Leftrightarrow x \neq 0
\end{aligned}$$

(b)

$$\begin{aligned}
&\exists\, x, y : \mathbb{N} \bullet x + y = 4 \land x < y \\
&\quad \Leftrightarrow \exists\, x, y : \mathbb{N} \bullet y = 4 - x \land x < y \\
&\quad \Leftrightarrow \exists\, x : \mathbb{N} \bullet 4 - x \in \mathbb{N} \land x < 4 - x \\
&\quad \Leftrightarrow \mathit{true}
\end{aligned}$$

The final equivalence holds because $0 \in \mathbb{N}$, $4 - 0 \in \mathbb{N}$, and $0 < 4$.

(c)

$$\begin{aligned}
&\forall\, x : \mathbb{N} \bullet \exists\, y : \mathbb{N} \bullet x = y + 1 \\
&\quad \Leftrightarrow \forall\, x : \mathbb{N} \bullet \exists\, y : \mathbb{N} \bullet y = x - 1 && \text{[arithmetic]} \\
&\quad \Leftrightarrow \forall\, x : \mathbb{N} \bullet x - 1 \in \mathbb{N} && \text{[one-point rule]} \\
&\quad \Leftrightarrow \mathit{false} && \text{[defns. of } - \text{ and } \mathbb{N}]
\end{aligned}$$

The final equivalence holds because $0 \in \mathbb{N}$ and yet $0 - 1 \notin \mathbb{N}$. We may assume that the subtraction operator is defined for all integers.

(d)

$$\begin{aligned}
&\exists\, x : \mathbb{N} \bullet (x = 1 \land x > y) \lor (x = 2 \land x > z) \\
&\quad \Leftrightarrow (\exists\, x : \mathbb{N} \bullet x = 1 \land x > y) \lor (\exists\, x : \mathbb{N} \bullet x = 2 \land x > z) \\
&\quad \Leftrightarrow (1 \in \mathbb{N} \land 1 > y) \lor (\exists\, x : \mathbb{N} \bullet x = 2 \land x > z) \\
&\quad \Leftrightarrow (1 \in \mathbb{N} \land 1 > y) \lor (2 \in \mathbb{N} \land 2 > z) \\
&\quad \Leftrightarrow 1 > y \lor 2 > z
\end{aligned}$$

### Solution 10

As discussed, the quantifier $\exists_1$ can help give rise to a ‘test’ or ‘precondition’ to ensure that an application of $\mu$ will work.

So, as a simple example, as the proposition

$$\exists_1 n : \mathbb{N} \bullet (\forall m : \mathbb{N} \bullet n \leq m)$$

is equivalent to *true*, we can be certain that the statement

$$(\mu\, n : \mathbb{N} \mid (\forall m : \mathbb{N} \bullet n \leq m))$$

will return a result (which happens to be 0).

### Solution 11

(a) $(\, \mu\, a : \mathbb{N} \mid a = a + a \,) \;=\; 0$ is a provable statement, since 0 is the only natural number with the specified property.

(b) $(\, \mu\, b : \mathbb{N} \mid b = b * b \,) \;=\; 1$ is not provable. The specified property is true of both 0 and 1, and thus the value of the $\mu$-expression is undefined.

(c) $(\, \mu\, c : \mathbb{N} \mid c > c + c \,) \;=\; (\, \mu\, c : \mathbb{N} \mid c > c + c \,)$ is a provable statement. Neither expression is properly defined, but we may conclude that they are equal; there is little else that we can prove about them.

(d) $(\, \mu\, d : \mathbb{N} \mid d = d \div d \,) \;=\; 1$ is not a provable statement. We cannot confirm that 1 is the only natural number with the specified property; we do not know what value is taken by $0 \div 0$.

### Solution 12

(a)
$$\begin{array}{l} (\, \mu\, m : \mathit{Mountain} \mid \\ \quad (\forall n : \mathit{Mountain} \bullet \mathit{height}(n) \leq \mathit{height}(m)) \bullet \\ \qquad \mathit{height}(m) \,) \end{array}$$

(b) $(\, \mu\, c : \mathit{Chapter} \mid (\exists_1 d : \mathit{Chapter} \bullet \mathit{length}(d) > \mathit{length}(c)) \bullet \mathit{length}(c) \,)$

(c) Assuming the existence of a suitable function, *max*:

$$(\, \mu\, n : \mathbb{N} \mid n = \max \{\, m : \mathbb{N} \mid 8 * m < 100 \bullet 8 * m \,\} \bullet 100 - n \,)$$

## Deductive proofs

### Solution 13

$$\dfrac{\dfrac{\dfrac{\lceil p \wedge (p \Rightarrow q) \rceil^{[1]}}{p}\,[\wedge-\text{elim}] \qquad \dfrac{\dfrac{\lceil p \wedge (p \Rightarrow q) \rceil^{[1]}}{p \Rightarrow q}\,[\wedge-\text{elim}] \quad \dfrac{\lceil p \wedge (p \Rightarrow q) \rceil^{[1]}}{p}\,[\wedge-\text{elim}]}{q}\,[\Rightarrow-\text{elim}]}{p \wedge q}\,[\wedge-\text{intro}]}{(p \wedge (p \Rightarrow q)) \Rightarrow (p \wedge q)}\,[\Rightarrow-\text{intro}^{[1]}]$$

**Solution 14**

In one direction:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\lceil (p \wedge q) \Leftrightarrow p \rceil^{[1]}}{p \Rightarrow p \wedge q}\;[\Leftrightarrow\!-\text{elim2}] \qquad \lceil p \rceil^{[2]}}{p \wedge q}\;[\Rightarrow\!-\text{elim}]}{q}\;[\wedge\!-\text{elim2}]}{p \Rightarrow q}\;[\Rightarrow\!-\text{intro}^{[2]}]}{((p \wedge q) \Leftrightarrow p) \Rightarrow (p \Rightarrow q)}\;[\Rightarrow\!-\text{intro}^{[1]}]
$$

and the other:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\lceil p \wedge q \rceil^{[2]}}{p}\;[\wedge\!-\text{elim1}]}{(p \wedge q) \Rightarrow p}\;[\Rightarrow\!-\text{intro}^{[2]}] \qquad \dfrac{\dfrac{\lceil p \rceil^{[3]} \qquad \dfrac{\lceil p \Rightarrow q \rceil^{[1]} \qquad \lceil p \rceil^{[3]}}{q}\;[\Rightarrow\!-\text{elim}]}{p \wedge q}\;[\wedge\!-\text{intro}]}{p \Rightarrow (p \wedge q)}\;[\Rightarrow\!-\text{intro}^{[3]}]}{(p \wedge q) \Leftrightarrow p}\;[\Leftrightarrow\!-\text{intro}]}{(p \Rightarrow q) \Rightarrow ((p \wedge q) \Leftrightarrow p)}\;[\Rightarrow\!-\text{intro}^{[1]}]
$$

We can then combine these two proofs with $\Leftrightarrow\!-$*intro*.

**Solution 15**

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\lceil (p \Rightarrow q) \wedge \neg\, q \rceil^{[1]}}{p \Rightarrow q}\;[\wedge\!-\text{elim1}] \qquad \lceil p \rceil^{[2]}}{q}\;[\Rightarrow\!-\text{elim}] \qquad \dfrac{\lceil (p \Rightarrow q) \wedge \neg\, q \rceil^{[1]}}{\neg\, q}\;[\wedge\!-\text{elim2}]}{\mathit{false}}\;[\mathit{false}\!-\text{intro}]}{\neg\, p}\;[\mathit{false}\!-\text{elim1}^{[2]}]}{((p \Rightarrow q) \wedge \neg\, q) \Rightarrow \neg\, p}\;[\Rightarrow\!-\text{intro}^{[1]}]
$$

**Solution 16**

In one direction:

$$\dfrac{\dfrac{\lceil p \wedge (q \vee r) \rceil^{[1]}}{q \vee r}\,[\wedge\text{−elim2}] \qquad \dfrac{\dfrac{\dfrac{\lceil p \wedge (q \vee r) \rceil^{[1]}}{p}\,[\wedge\text{−elim1}] \quad \lceil q \rceil^{[2]}}{p \wedge q}\,[\wedge\text{−intro}]}{(p \wedge q) \vee (p \wedge r)}\,[\vee\text{−intro1}] \qquad \dfrac{\dfrac{\dfrac{\lceil p \wedge (q \vee r) \rceil^{[1]}}{p}\,[\wedge\text{−elim1}] \quad \lceil r \rceil^{[2]}}{p \wedge r}\,[\wedge\text{−intro}]}{(p \wedge q) \vee (p \wedge r)}\,[\vee\text{−intro2}]}{\dfrac{(p \wedge q) \vee (p \wedge r)}{(p \wedge (q \vee r)) \Rightarrow ((p \wedge q) \vee (p \wedge r))}\,[\Rightarrow\text{−intro}^{[1]}]}\,[\vee\text{−elim}^{[2]}]$$

In the other:

$$\dfrac{\lceil (p \wedge q) \vee (p \wedge r) \rceil^{[3]} \qquad \dfrac{\dfrac{\lceil p \wedge q \rceil^{[4]}}{p}\,[\wedge\text{−elim1}] \quad \dfrac{\dfrac{\lceil p \wedge q \rceil^{[4]}}{q}\,[\wedge\text{−elim2}]}{q \vee r}\,[\vee\text{−intro1}]}{p \wedge (q \vee r)}\,[\wedge\text{−intro}] \qquad \dfrac{\dfrac{\lceil p \wedge r \rceil^{[4]}}{p}\,[\wedge\text{−elim1}] \quad \dfrac{\dfrac{\lceil p \wedge r \rceil^{[4]}}{r}\,[\wedge\text{−elim2}]}{q \vee r}\,[\vee\text{−intro2}]}{p \wedge (q \vee r)}\,[\wedge\text{−intro}]}{\dfrac{p \wedge (q \vee r)}{((p \wedge q) \vee (p \wedge r)) \Rightarrow (p \wedge (q \vee r))}\,[\Rightarrow\text{−intro}^{[3]}]}\,[\vee\text{−elim}^{[4]}]$$

**Solution 17**

In one direction:

$$
\dfrac{\dfrac{\dfrac{\lceil p \vee (q \wedge r) \rceil^{[3]} \quad \dfrac{\lceil p \rceil^{[1]}}{p \vee q}\,[\vee\text{−intro1}] \quad \dfrac{\dfrac{\lceil q \wedge r \rceil^{[1]}}{q}\,[\wedge\text{−elim1}]}{p \vee q}\,[\vee\text{−intro2}]}{p \vee q}\,[\vee\text{−elim}^{[1]}] \qquad \dfrac{\lceil p \vee (q \wedge r) \rceil^{[3]} \quad \dfrac{\lceil p \rceil^{[2]}}{p \vee r}\,[\vee\text{−intro1}] \quad \dfrac{\dfrac{\lceil q \wedge r \rceil^{[2]}}{r}\,[\wedge\text{−elim2}]}{p \vee r}\,[\vee\text{−intro2}]}{p \vee r}\,[\vee\text{−elim}^{[2]}]}{(p \vee q) \wedge (p \vee r)}\,[\wedge\text{−intro}]}{(p \vee (q \wedge r)) \Rightarrow ((p \vee q) \wedge (p \vee r))}\,[\Rightarrow\text{−intro}^{[3]}]
$$

and the other:

$$
\dfrac{\dfrac{\dfrac{\lceil (p \vee q) \wedge (p \vee r) \rceil^{[1]}}{p \vee q}\,[\wedge\text{−elim1}] \quad \dfrac{\lceil p \rceil^{[2]}}{p \vee (q \wedge r)}\,[\vee\text{−intro2}] \quad \dfrac{\dfrac{\lceil (p \vee q) \wedge (p \vee r) \rceil^{[1]}}{p \vee r}\,[\wedge\text{−elim2}] \quad \dfrac{\lceil p \rceil^{[3]}}{p \vee (q \wedge r)}\,[\vee\text{−intro1}] \quad \dfrac{\dfrac{\lceil q \rceil^{[2]} \quad \lceil r \rceil^{[3]}}{q \wedge r}\,[\wedge\text{−intro}]}{p \vee (q \wedge r)}\,[\vee\text{−intro2}]}{p \vee (q \wedge r)}\,[\vee\text{−elim}^{[3]}]}{p \vee (q \wedge r)}\,[\vee\text{−elim}^{[2]}]}{((p \vee q) \wedge (p \vee r)) \Rightarrow (p \vee (q \wedge r))}\,[\Rightarrow\text{−intro}^{[1]}]
$$

**Solution 18**

In one direction:

$$
\dfrac{\dfrac{\dfrac{}{p \vee \neg p}\,[\text{excluded middle}] \quad \dfrac{\dfrac{\lceil p \Rightarrow q \rceil^{[1]} \quad \lceil p \rceil^{[2]}}{q}\,[\Rightarrow\text{−elim}]}{\neg p \vee q}\,[\vee\text{−intro2}] \quad \dfrac{\lceil \neg p \rceil^{[2]}}{\neg p \vee q}\,[\vee\text{−intro1}]}{(\neg p \vee q)}\,[\vee\text{−elim}^{[2]}]}{(p \Rightarrow q) \Rightarrow (\neg p \vee q)}\,[\Rightarrow\text{−intro}^{[1]}]
$$

and the other:

$$\cfrac{\cfrac{\cfrac{\cfrac{\lceil \neg p \vee q \rceil^{[3]} \quad \cfrac{\lceil p \rceil^{[4]} \quad \lceil \neg p \rceil^{[5]}}{\mathit{false}}\,[\text{false}-\text{intro}] \quad \cfrac{\lceil q \rceil^{[5]} \quad \lceil \neg q \rceil^{[6]}}{\mathit{false}}\,[\text{false}-\text{intro}]}{\mathit{false}}\,[\vee-\text{elim}^{[5]}]}{q}\,[\text{false}-\text{elim1}^{[6]}]}{p \Rightarrow q}\,[\Rightarrow-\text{intro}^{[4]}]}{(\neg p \vee q) \Rightarrow (p \Rightarrow q)}\,[\Rightarrow-\text{intro}^{[3]}]$$

## Sets and types

### Solution 19

(a) $1 \in \{4, 3, 2, 1\}$ is true.

(b) $\{1\} \in \{1, 2, 3, 4\}$ is undefined.

(c) $\{1\} \in \{\{1\}, \{2\}, \{3\}, \{4\}\}$ is true.

(d) $\emptyset \in \{1, 2, 3, 4\}$ is undefined.

### Solution 20

(a) $\{1\} \times \{2, 3\}$ is the set $\{(1, 2), (1, 3)\}$

(b) $\emptyset \times \{2, 3\}$ is the set $\emptyset$

(c) $(\mathbb{P}\,\emptyset) \times \{1\}$ is the set $\{(\emptyset, 1)\}$

(d) $\{(1, 2)\} \times \{3, 4\}$ is the set $\{((1, 2), 3), ((1, 2), 4)\}$

### Solution 21

There are various ways of describing these sets via set comprehensions. Examples are given below.

(a) $\{\, z : \mathbb{Z} \mid 0 \leq z \wedge z \leq 100 \,\}$

(b) $\{\, z : \mathbb{Z} \bullet 10 * z \,\}$

(c) $\{\, z : \mathbb{Z} \mid z \mathbin{\mathsf{mod}} 2 = 0 \vee z \mathbin{\mathsf{mod}} 3 = 0 \vee z \mathbin{\mathsf{mod}} 5 = 0 \,\}$

### Solution 22

(a) $\{n : \mathbb{N} \mid n \leq 4 \bullet n^2\}$

(b) $\{n : \mathbb{N} \mid n \leq 4 \bullet (n, n^2)\}$

(c) $\{n : \mathbb{P}\,\{0, 1\}\}$

(d) $\{n : \mathbb{P}\,\{0, 1\} \bullet (n, \#\, n)\}$

**Solution 23**

(a)

$$\begin{aligned} & x \in a \cap a \\ & \quad \Leftrightarrow (x \in a \wedge x \in a) && \text{[intersection]} \\ & \quad \Leftrightarrow (x \in a) && \text{[idempotence of } \wedge\text{]} \\ & \quad \Leftrightarrow x \in a \end{aligned}$$

(b)

$$\begin{aligned} & x \in a \cup a \\ & \quad \Leftrightarrow (x \in a \vee x \in a) && \text{[union]} \\ & \quad \Leftrightarrow (x \in a) && \text{[idempotence of } \vee\text{]} \\ & \quad \Leftrightarrow x \in a \end{aligned}$$

(c)

$$\begin{aligned} & x \in a \cap \emptyset \\ & \quad \Leftrightarrow x \in a \wedge x \in \emptyset && \text{[intersection]} \\ & \quad \Leftrightarrow x \in a \wedge \mathit{false} && \text{[property of } \emptyset\text{]} \\ & \quad \Leftrightarrow \mathit{false} && \text{[property of } \mathit{false}\text{]} \\ & \quad \Leftrightarrow x \in \emptyset && \text{[property of } \emptyset\text{]} \end{aligned}$$

(d)

$$\begin{aligned} & x \in a \cup \emptyset \\ & \quad \Leftrightarrow x \in a \vee x \in \emptyset && \text{[union]} \\ & \quad \Leftrightarrow x \in a \vee \mathit{false} && \text{[property of } \emptyset\text{]} \\ & \quad \Leftrightarrow x \in a && \text{[property of } \mathit{false}\text{]} \end{aligned}$$

(e)

$$
\begin{aligned}
& x \in a \cap (b \setminus a) \\
& \quad \Leftrightarrow x \in a \wedge x \in (b \setminus a) && \text{[intersection]} \\
& \quad \Leftrightarrow x \in a \wedge (x \in b \wedge x \notin a) && \text{[set minus]} \\
& \quad \Leftrightarrow x \in a \wedge (x \notin a \wedge x \in b) && \text{[commutativity]} \\
& \quad \Leftrightarrow (x \in a \wedge x \notin a) \wedge x \in b && \text{[associativity]} \\
& \quad \Leftrightarrow \mathit{false} \wedge x \in b && \text{[property of } \mathit{false}\text{]} \\
& \quad \Leftrightarrow \mathit{false} && \text{[property of } \mathit{false}\text{]} \\
& \quad \Leftrightarrow x \in \emptyset && \text{[property of } \emptyset\text{]}
\end{aligned}
$$

(f)

$$
\begin{aligned}
& x \in a \cup (b \setminus a) \\
& \quad \Leftrightarrow x \in a \vee x \in (b \setminus a) && \text{[union]} \\
& \quad \Leftrightarrow x \in a \vee (x \in b \wedge x \notin a) && \text{[set minus]} \\
& \quad \Leftrightarrow (x \in a \vee x \in b) \wedge (x \in a \vee x \notin a) && \text{[distribution]} \\
& \quad \Leftrightarrow (x \in a \vee x \in b) \wedge \mathit{true} && \text{[property of } \mathit{true}\text{]} \\
& \quad \Leftrightarrow (x \in a \vee x \in b) && \text{[property of } \mathit{true}\text{]} \\
& \quad \Leftrightarrow x \in a \cup b && \text{[union]}
\end{aligned}
$$

(g)

$$
\begin{aligned}
& x \in a \setminus (b \cup c) \\
& \quad \Leftrightarrow x \in a \wedge x \notin (b \cup c) && \text{[set minus]} \\
& \quad \Leftrightarrow x \in a \wedge \neg(x \in (b \cup c)) && \text{[property of } \notin\text{]} \\
& \quad \Leftrightarrow x \in a \wedge \neg(x \in b \vee x \in c) && \text{[union]} \\
& \quad \Leftrightarrow x \in a \wedge x \notin b \wedge x \notin c && \text{[De Morgan]} \\
& \quad \Leftrightarrow x \in a \wedge x \in a \wedge x \notin b \wedge x \notin c && \text{[idempotence of } \wedge\text{]} \\
& \quad \Leftrightarrow x \in a \wedge x \notin b \wedge x \in a \wedge x \notin c && \text{[commutativity]} \\
& \quad \Leftrightarrow (x \in a \setminus b) \wedge (x \in a \setminus c) && \text{[set minus]} \\
& \quad \Leftrightarrow x \in (a \setminus b) \cap (a \setminus c) && \text{[intersection]}
\end{aligned}
$$

(h)

$$
\begin{aligned}
& x \in a \setminus (b \cap c) \\
& \quad \Leftrightarrow x \in a \land x \notin (b \cap c) && \text{[set minus]} \\
& \quad \Leftrightarrow x \in a \land \neg(x \in (b \cap c)) && \text{[property of } \notin \text{]} \\
& \quad \Leftrightarrow x \in a \land \neg(x \in b \land x \in c) && \text{[intersection]} \\
& \quad \Leftrightarrow x \in a \land (x \notin b \lor x \notin c) && \text{[De Morgan]} \\
& \quad \Leftrightarrow (x \in a \land x \notin b) \lor (x \in a \land x \notin c) && \text{[distribution]} \\
& \quad \Leftrightarrow (x \in a \setminus b) \lor (x \in a \setminus c) && \text{[set minus]} \\
& \quad \Leftrightarrow x \in (a \setminus b) \cup (a \setminus c) && \text{[union]}
\end{aligned}
$$

## Definitions

### Solution 24

(a) The set of all pairs of integers is $\mathbb{Z} \times \mathbb{Z}$. To give it a name, we could write

$$Pairs == \mathbb{Z} \times \mathbb{Z}$$

(b) The set of all integer pairs in which each element is strictly greater than zero could be defined by

$$StrictlyPositivePairs == \{ m, n : \mathbb{Z} \mid m > 0 \land n > 0 \bullet (m, n) \}$$

or by

$$StrictlyPositivePairs == \{ p : Pairs \mid p.1 > 0 \land p.2 > 0 \}$$

(c) It is intuitive to use a singular noun for the name of a basic type; we define the set of all people by writing

$$[Person]$$

(d) The set of all couples could be defined by

$$Couples == \{ s : \mathbb{P}\, Person \mid \#s = 2 \}$$

or perhaps—depending upon the use of this set in the specification—with the additional structure of a tuple:

$$Couples == \{ a, b : Person \mid a \neq b \}$$

However, if we choose to represent couples as pairs, we must consider whether the couple $(a, b)$ is to be distinguished from the couple $(b, a)$.

(e) The set of all parties could be defined by

$$Parties == \{\, s : \mathbb{P}\, Person \mid \#s \geq 8 \,\}$$

**Solution 25**

(a) $\emptyset[\mathbb{N}] \in \emptyset[\mathbb{P}\,\mathbb{N}]$

(b) $\emptyset[(\mathbb{N} \times \mathbb{N})] \subseteq (\emptyset[\mathbb{N}] \times \emptyset[\mathbb{N}])$

(c) $(\emptyset[\mathbb{N}] \times \{\emptyset[\mathbb{N}]\}) \subseteq \emptyset[\mathbb{N} \times \mathbb{P}\,\mathbb{N}]$

**Solution 26**

We may define $\notin$ using generic abbreviation:

$$\_ \notin \_[X] == \{\, x : X;\ s : \mathbb{P}\, X \mid \neg\,(x \in s) \,\}$$

or a generic axiomatic definition:

$$\begin{array}{|l}
[X] \\
\hline
_ \notin _ : (X \leftrightarrow \mathbb{P}\ X) \\
\hline
\forall\, x : X;\ s : \mathbb{P}\, X \bullet \\
\qquad x \notin s \Leftrightarrow \neg\,(x \in s)
\end{array}$$

## Relations

**Solution 27**

(a) $\{\, \emptyset, \{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\},$
$\{(1,0),(1,1)\}, \{(0,0),(0,1)\}, \{(0,1),(1,1)\},$
$\{(0,1),(1,0)\}, \{(0,0),(1,1)\}, \{(0,0),(1,0)\},$
$\{(0,0),(1,0),(1,1)\}, \{(0,0),(0,1),(1,1)\},$
$\{(0,0),(0,1),(1,0)\}, \{(0,1),(1,0),(1,1)\},$
$\{(0,0),(0,1),(1,0),(1,1)\} \,\}$

(b) $\{\, \emptyset, \{(0,0)\}, \{(0,1)\}, \{(0,0),(0,1)\} \,\}$

(c) $\{\, \emptyset \,\}$

(d) $\{\, \emptyset \,\}$

**Solution 28**

(a) dom $R = \{0, 1, 2\}$

(b) ran $R = \{1, 2, 3\}$

(c) $\{1, 2\} \lhd R = \{1 \mapsto 2, 1 \mapsto 3, 2 \mapsto 3\}$

(d) $R \rhd \{1, 2\} = \{0 \mapsto 3, 1 \mapsto 3, 2 \mapsto 3\}$

**Solution 29**

(a) $\{2 \mapsto 4, 3 \mapsto 3, 3 \mapsto 4, 4 \mapsto 2\}$

(b) $\{1 \mapsto 3, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1\}$

(c) $\{1 \mapsto 1, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 2, 3 \mapsto 3, 4 \mapsto 4\}$

(d) $\{1 \mapsto 4, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 2, 3 \mapsto 3, 4 \mapsto 1\}$

**Solution 30**

(a) $\mathit{parentOf} == \mathit{childOf}^{\sim}$
This is a good example of how there are many different ways of writing the same thing. An alternative abbreviation is:

$$\mathit{parentOf} == \{x, y : \mathit{Person} \mid x \mapsto y \in \mathit{ChildOf} \bullet y \mapsto x\}$$

Or, via an axiomatic definition:

$$\begin{array}{|l} \mathit{parentOf} : \mathit{Person} \leftrightarrow \mathit{Person} \\ \hline \mathit{parentOf} = \mathit{childOf}^{\sim} \end{array}$$

The second version, via an axiomatic definition:

$$\begin{array}{|l} \mathit{parentOf} : \mathit{Person} \leftrightarrow \mathit{Person} \\ \hline \mathit{parentOf} = \{x, y : \mathit{Person} \mid x \mapsto y \in \mathit{ChildOf} \bullet y \mapsto x\} \end{array}$$

Another approach:

$$\begin{array}{|l} \mathit{parentOf} : \mathit{Person} \leftrightarrow \mathit{Person} \\ \hline \forall x, y : \mathit{Person} \bullet \\ \qquad x \mapsto y \in \mathit{parentOf} \Leftrightarrow y \mapsto x \in \mathit{ChildOf} \end{array}$$

(b) $\mathit{siblingOf} == (\mathit{childOf} \fatsemi \mathit{parentOf}) \setminus \mathrm{id}[\mathit{Person}]$

(c) $\mathit{cousinOf} == \mathit{childOf} \fatsemi \mathit{siblingOf} \fatsemi \mathit{parentOf}$

(d) $\mathit{ancestorOf} == \mathit{parentOf}^{+}$

**Solution 31**

(a) $R^{\mathrm{r}} = \{\, a, b : \mathbb{N} \mid b = a \lor b = a + 1 \,\}$

(b) $R^{\mathrm{s}} = \{\, a, b : \mathbb{N} \mid |\, a - b \,| = 1 \,\}$

(c) $R^{+} = \{\, a, b : \mathbb{N} \mid b > a \,\}$

(d) $R^{*} = \{\, a, b : \mathbb{N} \mid b \geq a \,\}$

**Solution 32**

(a)

$$
\begin{aligned}
& x \mapsto y \in A \lhd (B \lhd R) \\
& \quad \Leftrightarrow x \in A \land x \mapsto y \in (B \lhd R) && \text{[domain restriction]} \\
& \quad \Leftrightarrow x \in A \land x \in B \land x \mapsto y \in R && \text{[domain restriction]} \\
& \quad \Leftrightarrow x \in A \cap B \land x \mapsto y \in R && \text{[intersection]} \\
& \quad \Leftrightarrow x \mapsto y \in (A \cap B) \lhd R && \text{[domain restriction]}
\end{aligned}
$$

(b)

$$
\begin{aligned}
& x \mapsto y \in (R \cup S) \rhd C \\
& \quad \Leftrightarrow x \mapsto y \in (R \cup S) \land y \in C && \text{[range restriction]} \\
& \quad \Leftrightarrow (x \mapsto y \in R \lor x \mapsto y \in S) \land y \in C && \text{[union]} \\
& \quad \Leftrightarrow (x \mapsto y \in R \land y \in C) \lor (x \mapsto y \in S \land y \in C) && \text{[distribution]} \\
& \quad \Leftrightarrow x \mapsto y \in R \rhd C \lor x \mapsto y \in S \rhd C && \text{[range restriction]} \\
& \quad \Leftrightarrow x \mapsto y \in (R \rhd C) \cup (S \rhd C) && \text{[union]}
\end{aligned}
$$

(c)

$$
\begin{aligned}
& x \mapsto y \in (A \setminus B) \lhd R \\
& \qquad \Leftrightarrow x \in (A \setminus B) \wedge x \mapsto y \in R \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{[domain restriction]} \\
& \qquad \Leftrightarrow x \in A \wedge \neg\,(x \in B) \wedge x \mapsto y \in R \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{[set minus]} \\
& \qquad \Leftrightarrow x \in A \wedge x \mapsto y \in R \wedge \neg\,(x \in B) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{[properties of } \wedge\text{]} \\
& \qquad \Leftrightarrow x \in A \wedge x \mapsto y \in R \wedge (\neg\,(x \in B) \vee \neg\,(x \mapsto y \in R)) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{[since } P \wedge Q \Leftrightarrow P \wedge (Q \vee \neg\, P)\text{]} \\
& \qquad \Leftrightarrow x \mapsto y \in (A \lhd R) \wedge \neg\,(x \in B \wedge x \mapsto y \in R) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{[De Morgan]} \\
& \qquad \Leftrightarrow x \mapsto y \in (A \lhd R) \wedge \neg\,(x \mapsto y \in B \lhd R) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{[domain restriction]} \\
& \qquad \Leftrightarrow x \mapsto y \in (A \lhd R) \setminus (B \lhd R) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{[set minus]}
\end{aligned}
$$

## Functions

### Solution 33

The set of 9 functions:

$$
\begin{aligned}
\{\, & \emptyset, \{(0,0)\}, \{(0,1)\}, \{(1,1)\}, \\
& \{(1,0)\}, \{(0,0),(1,1)\}, \{(0,1),(1,1)\}, \\
& \{(1,0),(0,0)\}, \{(0,1),(1,0)\}\,\}
\end{aligned}
$$

(a) The set of total functions:

$$\{\, \{(0,0),(1,1)\}, \{(0,1),(1,1)\}, \{(1,0),(0,0)\}, \{(0,1),(1,0)\}\,\}$$

(b) The set of functions which are neither injective nor surjective:

$$\{\, \{(0,1),(1,1)\}, \{(0,0),(1,0)\}\,\}$$

(c) The set of functions which are injective but not surjective:

$$\{\, \emptyset, \{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(1,1)\}\,\}$$

(d) There are no functions (of this type) which are surjective but not injective

(e) The set of bijective functions:

$$\{\,\{(0,0),(1,1)\},\{(0,1),(1,0)\}\,\}$$

**Solution 34**

(a) $\{(1,a),(2,b),(3,c),(4,b)\}$

(b) $\{(1,c),(2,b),(3,c),(4,d)\}$

(c) $\{(1,c),(2,b),(3,c),(4,b)\}$

(d) $\{(1,c),(2,b),(3,c),(4,b)\}$

**Solution 35**

(a)

$$\begin{array}{|l} \mathit{children} : \mathit{Person} \rightarrow \mathbb{P}\ \mathit{Person} \\ \hline \mathit{children} = \{p : \mathit{Person} \bullet p \mapsto \mathit{parentOf}\ (\!|\ \{p\}\ |\!)\} \end{array}$$

(b)

$$\begin{array}{|l} \mathit{number\_of\_grandchildren} : \mathit{Person} \rightarrow \mathbb{N} \\ \hline \mathit{number\_of\_grandchildren} = \\ \qquad \{p : \mathit{Person} \bullet p \mapsto \#\,(\mathit{parentOf} \mathbin{;} \mathit{parentOf})\ (\!|\ \{p\}\ |\!)\} \end{array}$$

**Solution 36**

$$\begin{array}{|l} \mathit{number\_of\_drivers} : (\mathit{Drivers} \leftrightarrow \mathit{Cars}) \rightarrow (\mathit{Cars} \nrightarrow \mathbb{N}) \\ \hline \forall\, r : \mathit{Drivers} \leftrightarrow \mathit{Cars} \bullet \\ \qquad \mathit{number\_of\_drivers}\,(r) = \{c : \operatorname{ran}\ r \bullet c \mapsto \#\{d : \mathit{Drivers} \mid d \mapsto c \in r\}\} \end{array}$$

## Sequences

**Solution 37**

(a) $\langle a \rangle$

(b) $\{1 \mapsto a, 2 \mapsto b, 2 \mapsto a, 3 \mapsto c, 3 \mapsto b, 4 \mapsto d\}$

(c) $\{2 \mapsto b, 3 \mapsto c, 4 \mapsto d\}$

(d) $\{1,2,3,4\}$

(e) $\{a,b\}$

(f) $\{a \mapsto 1, b \mapsto 2, c \mapsto 3, d \mapsto 4\}$

(g) $\langle a, b \rangle$

(h) $\{3 \mapsto b\}$

(i) $\{a\}$

(j) $c$

**Solution 38**

(a)

$$\begin{array}{|l} f : \mathit{Place} \rightarrow \mathbb{P}\,\mathit{Place} \\ \hline \forall\, p : \mathit{Place} \bullet f\ p = \{q : \mathit{Place} \mid p \mapsto q \in \mathrm{ran}\ \mathit{trains}\} \end{array}$$

(b) $\{p : \mathit{Place} \mid \exists_1\, x : \mathit{dom\ trains} \bullet (\mathit{trains}\ x).2 = p\}$

(c)
$$\begin{array}{l} (\mu\, p : \mathit{Place} \mid \\ \quad (\forall\, q : \mathit{Place} \mid p \neq q \bullet \\ \qquad \#\,\{x : \mathrm{dom}\ \mathit{trains} \mid (\mathit{trains}\ x).2 = p\} \\ \qquad > \\ \qquad \#\,\{x : \mathrm{dom}\ \mathit{trains} \mid (\mathit{trains}\ x).2 = q\})) \end{array}$$

**Solution 39**

(a)

$$\begin{array}{|l} \mathit{large\_coins} : \mathit{Collection} \rightarrow \mathbb{N} \\ \hline \forall\, c : \mathit{Collection} \bullet \\ \quad \mathit{large\_coins}\,(c) = c\,(\mathit{large}) \end{array}$$

(b)

$$\begin{array}{|l} \mathit{add\_coin} : \mathit{Collection} \times \mathit{Coin} \rightarrow \mathit{Collection} \\ \hline \forall\, c : \mathit{Collection};\ d : \mathit{Coin} \bullet \\ \quad \mathit{add\_coin}\,(c, d) = c \uplus [\![d]\!] \end{array}$$

## Modelling

### Solution 40

(a)

$$\begin{array}{|l} hd : \mathrm{seq}\,(Title \times Length \times Viewed) \\ \hline cumulative\_total\, hd \leq 12\,000 \\ \forall\, p : \mathrm{ran}\ hd \bullet p.2 \leq 360 \end{array}$$

Note that *cumulative_total* is defined in part (d).

(b) $\{p : \mathrm{ran}\ hd \mid p.2 > 120 \bullet p.1\}$

(c) These can be defined recursively:

$$\begin{array}{lll} viewed\,\langle\rangle = \langle\rangle & & \\ viewed\,\langle x\rangle \frown s = & \langle x\rangle \frown viewed\ s & \text{if}\, x.3 = yes \\ & viewed\ s & \text{otherwise} \end{array}$$

or otherwise:

$$\begin{array}{|l} not\_viewed : \mathrm{seq}\,(Title \times Length \times Viewed) \rightarrow \mathrm{seq}\,(Title \times Length \times Viewed) \\ \hline \forall\, s : \mathrm{seq}\,(Title \times Length \times Viewed) \bullet \\ \qquad not\_viewed\ s = s \upharpoonright \{t : Title;\ l : Length \bullet (t, l, no)\} \end{array}$$

(d)

$$\begin{array}{|l} cumulative\_total : \mathrm{seq}(Title \times Length \times Viewed) \rightarrow \mathbb{N} \\ \hline cumulative\_total\,(\langle\rangle) = 0 \\ \forall\, x : Title \times Length \times Viewed;\ s : \mathrm{seq}\,(Title \times Length \times Viewed) \bullet \\ \qquad cumulative\_total\,(\langle x\rangle \frown s) = x.2 + cumulative\_total\,(s) \end{array}$$

(e) $(\mu\, p : \mathrm{ran}\, hd \mid (\forall\, q : \mathrm{ran}\, hd \mid p \neq q \bullet p.2 > q.2) \bullet p.1)$ (This, of course, assumes that there is a unique element with this property.)

(f)

$$\begin{array}{|l} f : \mathrm{seq}\,(Title \times Length \times Viewed) \rightarrow (Title \nrightarrow Length) \\ \hline \forall\, s : \mathrm{seq}\,(Title \times Length \times Viewed) \bullet \\ \qquad f\ s = \{t : Title \mid (\exists\, p : \mathrm{ran}\ s \bullet p.1 = t) \bullet \\ \qquad\qquad\qquad t \mapsto cumulative\_total\,(s \upharpoonright \{l : Length;\ v : Viewed \bullet (t, l, v)\})\} \end{array}$$

(g)

$$\begin{array}{|l}
g : \mathrm{seq}\,(\mathit{Title} \times \mathit{Length} \times \mathit{Viewed}) \rightarrow \mathrm{seq}\,(\mathit{Title} \times \mathit{Length} \times \mathit{Viewed}) \\
\hline
\forall\, s : \mathrm{seq}\,(\mathit{Title} \times \mathit{Length} \times \mathit{Viewed}) \bullet \\
\qquad g\, s = s \upharpoonright \{x : \mathrm{ran}\ s \mid x \neq \mathit{longest_viewed}\, s\}
\end{array}$$

Where *longest_viewed* is defined as

$$\begin{array}{|l}
\mathit{longest_viewed} : \mathrm{seq}\,(\mathit{Title} \times \mathit{Length} \times \mathit{Viewed}) \nrightarrow \mathit{Title} \times \mathit{Length} \times \mathit{Viewed} \\
\hline
\forall\, s : \mathrm{seq}\,(\mathit{Title} \times \mathit{Length} \times \mathit{Viewed}) \bullet \\
\qquad \mathit{longest_viewed}\, s = (\mu\, p : \mathrm{ran}\ s \mid p.3 = \mathit{yes} \wedge \\
\qquad\qquad\qquad\qquad (\forall\, q : \mathrm{ran}\ s \mid p \neq q \wedge q.3 = \mathit{yes} \bullet p.2 > q.2))
\end{array}$$

This, of course, assumes that there is at least one viewed programme (and one of a unique maximum length).

(h)

$$\begin{array}{|l}
s : \mathrm{seq}\,(\mathit{Title} \times \mathit{Length} \times \mathit{Viewed}) \rightarrow \mathrm{seq}\,(\mathit{Title} \times \mathit{Length} \times \mathit{Viewed}) \\
\hline
\forall\, x : \mathrm{seq}\,(\mathit{Title} \times \mathit{Length} \times \mathit{Viewed}) \bullet \\
\qquad (\mathit{items}\,(s\, x) = \mathit{items}\,(x)) \\
\qquad \wedge \\
\qquad \forall\, i, j : \mathrm{dom}\,(s\, x) \bullet i < j \Rightarrow ((s\, x)\, i).2 \geq ((s\, x)\, j).2
\end{array}$$

**Solution 41**

(a)

$$\begin{array}{|l}
\mathit{records} : \mathit{Year} \nrightarrow \mathit{Table} \\
\hline
\mathrm{dom}\ \mathit{records} = 1993 \mathrel{..} \mathit{current} \\
\forall\, y : \mathrm{dom}\ \mathit{records} \bullet \#\,(\mathit{records}\ y) \leq 50 \\
\forall\, y : \mathrm{dom}\ \mathit{records} \bullet \forall\, e : \mathrm{ran}\,(\mathit{records}\ y) \bullet \mathit{year}\,(e.1) = y \\
\forall\, r : \mathrm{ran}\ \mathit{records} \bullet \\
\qquad (\forall\, i_1, i_2 : \mathrm{dom}\ r \bullet i_1 \neq i_2 \wedge (r\, i_1).1 = (r\, i_2).1 \Rightarrow (r\, i_1).3 \neq (r\, i_2).3)
\end{array}$$

(b) (i) $\{e : \mathit{Entry} \mid (\exists\, r : \mathrm{ran}\ \mathit{records} \bullet e \in \mathrm{ran}\ r \wedge e.3 = 479)\}$

(ii) $\{e : \mathit{Entry} \mid (\exists\, r : \mathrm{ran}\ \mathit{records} \bullet e \in \mathrm{ran}\ r) \wedge e.6 > e.5\}$

(iii) $\{e : \mathit{Entry} \mid (\exists\, r : \mathrm{ran}\ \mathit{records} \bullet e \in \mathrm{ran}\ r) \wedge e.7 \geq 70\}$

(iv) $\{c : \mathit{Course} \mid (\forall\, r : \mathrm{ran}\ \mathit{records} \bullet (\forall\, e : \mathrm{ran}\ r \bullet (e.2 = c \Rightarrow e.7 \geq 70)))\}$

(v) $\{y : \mathit{Year} \mid y \in \mathrm{dom}\ \mathit{records} \bullet$
$y \mapsto \{l : \mathit{Lecturer} \mid \#\,\{c : \mathrm{ran}\,(\mathit{records}\ y) \mid c.4 = l\} > 6\}\}$

(c)

$$\begin{array}{l}\forall\, x : \mathit{Entry};\ s : \operatorname{seq} \mathit{Entry} \bullet \\ \quad 479\_\mathit{courses}\,(\langle\rangle) = \langle\rangle \\ \quad \wedge \\ \quad 479\_\mathit{courses}\,(\langle x\rangle \frown s) = \begin{cases} \langle x\rangle \frown 479\_\mathit{courses}\,(s) & \text{if } x.3 = 479 \\ 479\_\mathit{courses}\,(s) & \text{otherwise} \end{cases}\end{array}$$

(d)

$$\begin{array}{l}\forall\, x : \mathit{Entry};\ s : \operatorname{seq} \mathit{Entry} \bullet \\ \quad \mathit{total}\,(\langle\rangle) = 0 \\ \quad \wedge \\ \quad \mathit{total}\,(\langle x\rangle \frown s) = x.5 + \mathit{total}\,(s)\end{array}$$

**Solution 42**

$[\mathit{Person}]$

(a)

$$\begin{array}{|l} \mathit{State} : \mathbb{P}(\operatorname{seq}(\operatorname{iseq} \mathit{Person})) \\ \hline \forall\, s : \mathit{State} \bullet \\ \quad (\forall\, i, j : \operatorname{dom} s \mid i \neq j \bullet \\ \qquad \operatorname{ran}(s\ i) \cap \operatorname{ran}(s\ j) = \emptyset) \end{array}$$

(b)

$$\begin{array}{|l} \mathit{add} : \mathbb{N} \times \mathit{Person} \times \mathit{State} \nrightarrow \mathit{State} \\ \hline \forall\, n : \mathbb{N};\ p : \mathit{Person};\ s : \mathit{State} \mid \\ \quad n \in \operatorname{dom} s \wedge p \notin \operatorname{ran}\,(\bigcup\,(\operatorname{ran}\ s)) \bullet \\ \qquad \mathit{add}\,(n, p, s) = s \oplus \{n \mapsto (s\ n) \frown \langle p\rangle\} \end{array}$$

**Solution 43**

(a)
i. $\forall\, i : \operatorname{dom}\ \mathit{bookings} \bullet$
$(\forall\, x, y : \mathit{bookings}\ i \mid x \neq y \bullet (x.2 \mathrel{..} x.3) \cap (y.2 \mathrel{..} y.3) = \emptyset)$
ii. $\forall\, i : \operatorname{dom}\ \mathit{bookings} \bullet (\forall\, x : \mathit{bookings}\ i \bullet \{x.2, x.3\} \subseteq 1 \mathrel{..} \max\ i.1)$
iii. $\forall\, i : \operatorname{dom}\ \mathit{bookings} \bullet (\forall\, b : \mathit{bookings}\ i \bullet b.2 \leq b.3)$

iv. This is enforced by the constraint for part (i).

(b) i. $\{i : \mathrm{dom}\ \mathit{bookings} \mid i.1 = \mathit{Banbury} \bullet i.2\}$

ii. $\{c : \mathit{Cinema};\ f : \mathit{Film} \mid$
$(\exists\, i : \mathrm{dom}\ \mathit{bookings} \bullet i.1 = c \land i.2 = f) \bullet$
$(f, (c, \#\,\{d : \mathit{Date} \mid (\exists\, j : \mathrm{dom}\ \mathit{bookings} \bullet j.1 = c \land j.2 = f \land j.3 = d)\}))\}$

## Free types and induction

### Solution 44

The two cases of the proof are established by equational reasoning: the first by

$$\begin{aligned}
&\mathit{reverse}\,(\langle\rangle \frown t) \\
&\quad = \mathit{reverse}\ t && [\text{cat.1a}] \\
&\quad = (\mathit{reverse}\ t) \frown \langle\rangle && [\text{cat.1b}]
\end{aligned}$$

where cat.1a is

$$\langle\rangle \frown s = s$$

and cat.1b is

$$s \frown \langle\rangle = s$$

and the second by

$$\begin{aligned}
&\mathit{reverse}\,((\langle x\rangle \frown u) \frown t) \\
&\quad = \mathit{reverse}\,(\langle x\rangle \frown (u \frown t)) && [\text{cat.2}] \\
&\quad = \mathit{reverse}\,(u \frown t) \frown \langle x\rangle && [\text{reverse.2}] \\
&\quad = (\mathit{reverse}\ t \frown \mathit{reverse}\ u) \frown \langle x\rangle \\
& && [\mathit{reverse}\,(u \frown t) = (\mathit{reverse}\ t \frown \mathit{reverse}\ u)] \\
&\quad = \mathit{reverse}\ t \frown (\mathit{reverse}\ u \frown \langle x\rangle) && [\text{cat.2}] \\
&\quad = \mathit{reverse}\ t \frown \mathit{reverse}\,(\langle x\rangle \frown u) && [\text{reverse.2}]
\end{aligned}$$

**Solution 45**

The base case:

$$\begin{aligned}
&\mathit{reverse}\,(\mathit{reverse}\,\langle\rangle) \\
&\quad = \mathit{reverse}\,\langle\rangle && [\text{reverse.1}] \\
&\quad = \langle\rangle && [\text{reverse.1}]
\end{aligned}$$

The inductive step:

$$\begin{aligned}
&\mathit{reverse}\,(\mathit{reverse}\,(\langle x\rangle \frown t)) \\
&\quad = \mathit{reverse}\,((\mathit{reverse}\,t) \frown \langle x\rangle) && [\text{reverse.2}] \\
&\quad = \mathit{reverse}\,(\langle x\rangle) \frown \mathit{reverse}\,(\mathit{reverse}\,t) && [\text{anti-distributive}] \\
&\quad = \mathit{reverse}\,(\langle x\rangle \frown \langle\rangle) \frown \mathit{reverse}\,(\mathit{reverse}\,t) && [\text{cat.1}] \\
&\quad = ((\mathit{reverse}\,\langle\rangle) \frown \langle x\rangle) \frown \mathit{reverse}\,(\mathit{reverse}\,t) && [\text{reverse.2}] \\
&\quad = (\langle\rangle \frown \langle x\rangle) \frown \mathit{reverse}\,(\mathit{reverse}\,t) && [\text{reverse.1}] \\
&\quad = \langle x\rangle \frown \mathit{reverse}\,(\mathit{reverse}\,t) && [\text{cat.1}] \\
&\quad = \langle x\rangle \frown t && [\mathit{reverse}\,(\mathit{reverse}\,t) = t]
\end{aligned}$$

**Solution 46**

(a)

$$\begin{array}{|l}
\mathit{count} : \mathit{Tree} \to \mathbb{N} \\
\hline
\mathit{count}\,\mathsf{stalk} = 0 \\
\forall\, n : \mathbb{N} \bullet \mathit{count}\,(\mathsf{leaf}\,n) = 1 \\
\forall\, t_1, t_2 : \mathit{Tree} \bullet \mathit{count}\,(\mathsf{branch}\,(t_1, t_2)) = \mathit{count}\,t_1 + \mathit{count}\,t_2
\end{array}$$

(b)

$$\begin{array}{|l}
\mathit{flatten} : \mathit{Tree} \to \mathrm{seq}\,\mathbb{N} \\
\hline
\mathit{flatten}\,\mathsf{stalk} = \langle\rangle \\
\forall\, n : \mathbb{N} \bullet \mathit{flatten}\,(\mathsf{leaf}\,n) = \langle n\rangle \\
\forall\, t_1, t_2 : \mathit{Tree} \bullet \mathit{flatten}\,(\mathsf{branch}\,(t_1, t_2)) = \mathit{flatten}\,t_1 \frown \mathit{flatten}\,t_2
\end{array}$$

**Solution 47**

First, we exhibit the following induction principle for our free type:

$$\frac{P\,\mathsf{stalk} \quad \forall\, n : \mathbb{N} \bullet P\,(\mathsf{leaf}\,n) \quad \forall\, t_1, t_2 : \mathit{Tree} \bullet P\,t_1 \land P\,t_2 \Rightarrow P\,(\mathsf{branch}\,(t_1, t_2))}{\forall\, t : \mathit{Tree} \bullet P\,t}$$

This gives rise to three cases for the proof, which we consider in turn:

$$
\begin{aligned}
&\#(\mathit{flatten}\,\mathsf{stalk}) \\
&\quad = \#\langle\rangle && \text{[flatten]} \\
&\quad = 0 && \text{[\#]} \\
&\quad = \mathit{count}\,\mathsf{stalk} && \text{[count]}
\end{aligned}
$$

For any $n \in \mathbb{N}$:

$$
\begin{aligned}
&\#(\mathit{flatten}\,\mathsf{leaf}\,n) \\
&\quad = \#\langle n\rangle && \text{[flatten]} \\
&\quad = 1 && \text{[\#]} \\
&\quad = \mathit{count}\,(\mathsf{leaf}\,n) && \text{[count]}
\end{aligned}
$$

The inductive step is a generalisation of the following result:

$$
\begin{aligned}
&\#(\mathit{flatten}\,\mathsf{branch}\,(t_1, t_2)) \\
&\quad = \#(\mathit{flatten}\,t_1 \frown \mathit{flatten}\,t_2) && \text{[flatten]} \\
&\quad = \#\mathit{flatten}\,t_1 + \#\mathit{flatten}\,t_2 && \text{[\# is distributive]} \\
&\quad = \mathit{count}\,t_1 + \mathit{count}\,t_2 && [P\ t_1 \wedge P\ t_2] \\
&\quad = \mathit{count}\,\mathsf{branch}\,(t_1, t_2) && \text{[count]}
\end{aligned}
$$

## Supplementary material: assignment practice

### Solution 48

$$
\begin{array}{|l}
\mathit{songs} : \mathbb{F}\ \mathit{SongId} \\
\mathit{users} : \mathbb{F}\ \mathit{UserId} \\
\mathit{playlists} : \mathit{PlaylistId} \nrightarrow \mathit{Playlist} \\
\mathit{playlist\_owner} : \mathit{PlaylistId} \nrightarrow \mathit{UserId} \\
\mathit{playlist\_subscribers} : \mathit{PlaylistId} \nrightarrow \mathbb{F}_1\ \mathit{UserId} \\
\hline
\forall\, i : \mathrm{dom}\ \mathit{playlists} \bullet \mathrm{ran}\,(\mathit{playlists}\ i) \subseteq \mathit{songs} \\
\mathrm{dom}\ \mathit{playlist\_owner} \subseteq \mathrm{dom}\ \mathit{playlists} \\
\mathrm{ran}\ \mathit{playlist\_owner} \subseteq \mathit{users} \\
\mathrm{dom}\ \mathit{playlist\_subscribers} \subseteq \mathrm{dom}\ \mathit{playlists} \\
\forall\, i : \mathrm{dom}\ \mathit{playlist\_subscribers} \bullet \mathit{playlist\_subscribers}\ i \subseteq \mathit{users} \\
\forall\, i : \mathrm{dom}\ \mathit{playlists} \bullet (\mathit{playlist\_owner}\ i) \in \mathit{playlist\_subscribers}\ i
\end{array}
$$

**Solution 49**

$$
\begin{array}{|l}
hated : UserId \nrightarrow \mathbb{F}\, SongId \\
loved : UserId \nrightarrow \mathbb{F}\, SongId \\
\hline
\mathrm{dom}\ hated \subseteq users \\
\forall\, i : \mathrm{dom}\ hated \bullet (hated\ i) \subseteq songs \\
\mathrm{dom}\ loved \subseteq users \\
\forall\, i : \mathrm{dom}\ loved \bullet (loved\ i) \subseteq songs \\
\forall\, i : \mathrm{dom}\ hated \cup \mathrm{dom}\ loved \bullet hated\ i \cap loved\ i = \emptyset
\end{array}
$$

**Solution 50**

(a) $A == users \setminus \bigcup(\mathrm{ran}\, playlist\_subscribers)$

(b) $B == \{p : \mathrm{dom}\, playlist\_subscribers \mid \#(playlist\_subscribers\ p) \geq 100\}$

(c) $$C == (\quad \mu\, u : \mathrm{dom}\ loved \mid (\forall\, v : \mathrm{dom}\ loved \mid u \neq v \bullet \#\,(loved\ u) > \#(loved\ v)))$$

(d) $$D == (\quad \mu\, s : songs \mid (\forall\, t : songs \mid s \neq t \bullet \quad \#\,\{u : UserId \mid s \in loved\ u\} > \#\,\{u : UserId \mid t \in loved\ u\}))$$

**Solution 51**

(a) Let's first define two helper functions:

$$
\begin{array}{|l}
love\_hate\_score : SongId \nrightarrow \mathbb{N} \\
\hline
\forall i : songs \bullet \\
\quad \# \{u : UserId \mid i \in loved\ u\} \\
\quad \geq \\
\quad \# \{u : UserId \mid i \in hated\ u\} \\
\quad \Rightarrow \\
\quad\quad love\_hate\_score\ i = \\
\quad\quad\quad \# \{u : UserId \mid i \in loved\ u\} \\
\quad\quad\quad - \\
\quad\quad\quad \# \{u : UserId \mid i \in hated\ u\} \\
\wedge \\
\forall i : songs \bullet \\
\quad \# \{u : UserId \mid i \in loved\ u\} \\
\quad < \\
\quad \# \{u : UserId \mid i \in hated\ u\} \Rightarrow \\
\quad\quad love\_hate\_score\ i = 0
\end{array}
$$

$$
\begin{array}{|l}
playlist\_count : SongId \nrightarrow \mathbb{N} \\
\hline
\forall i : songs \bullet \\
\quad playlist\_count\ i = \# \{p : \mathrm{dom}\ playlist \mid i \in \mathrm{ran}\ playlist\ p\}
\end{array}
$$

We then have

$$
\begin{array}{|l}
length : SongId \pfun \mathbb{N} \\
popularity : SongId \pfun \mathbb{N} \\
\hline
\mathrm{dom}\ length \subseteq songs \\
\mathrm{dom}\ popularity \subseteq songs \\
\forall\, i : songs \bullet \\
\qquad popularity\ i = love\_hate\_score\ i + playlist\_count\ i
\end{array}
$$

(b)

$$
\begin{array}{|l}
most\_popular : SongId \\
\hline
(\exists_1\, i : songs \bullet (\forall\, j : songs \mid i \neq j \bullet popularity\ i > popularity\ j)) \\
\Rightarrow \\
most\_popular = \\
\qquad (\mu\, i : songs \mid (\forall\, j : songs \mid i \neq j \bullet popularity\ i > popularity\ j)) \\
\wedge \\
\neg\, (\exists_1\, i : songs \bullet (\forall\, j : songs \mid i \neq j \bullet popularity\ i > popularity\ j)) \\
\Rightarrow \\
most\_popular = null\_song
\end{array}
$$

(c)

$$
\begin{array}{l}
playlists\_containing\_most\_popular\_song == \\
\qquad \{ i : \mathrm{dom}\ playlists \mid most\_popular \in \mathrm{ran}\, playlists\ i \}
\end{array}
$$

**Solution 52**

(a)

$$
\begin{array}{|l}
premium\_plays : \mathrm{seq}\, Play \rightarrow \mathrm{seq}\, Play \\
\hline
premium\_plays\ \langle\rangle = \langle\rangle \\
\forall\, x : Play;\ s : \mathrm{seq}\, Play \bullet \\
\qquad premium\_plays\ (\langle x \rangle \frown s) = \\
\qquad\quad \langle x \rangle \frown (premium\_plays\ s) \\
\qquad\qquad \text{if } user\_status\ (x.2) = premium \\
\qquad\quad premium\_plays\ s \\
\qquad\qquad \text{if } user\_status\ (x.2) = standard
\end{array}
$$

(b)

$$
\begin{array}{|l}
standard\_plays : \mathrm{seq}\, Play \rightarrow \mathrm{seq}\, Play \\
\hline
standard\_plays\, \langle\rangle = \langle\rangle \\
\forall\, x : Play;\ s : \mathrm{seq}\, Play \bullet \\
\qquad standard\_plays\, (\langle x \rangle \frown s) = \\
\qquad\qquad \langle x \rangle \frown (standard\_plays\, s) \\
\qquad\qquad\qquad \text{if } user\_status\, (x.2) = standard \\
\qquad\qquad standard\_plays\, s \\
\qquad\qquad\qquad \text{if } user\_status\, (x.2) = premium
\end{array}
$$

(c)

$$
\begin{array}{|l}
cumulative\_length : \mathrm{seq}\, Play \rightarrow \mathbb{N} \\
\hline
cumulative\_length\, \langle\rangle = 0 \\
\forall\, x : Play;\ s : \mathrm{seq}\, Play \bullet \\
\qquad cumulative\_length\, (\langle x \rangle \frown s) = \\
\qquad\qquad length\, (x.1) + cumulative\_length\, (s)
\end{array}
$$