Proof by Contradiction Examples

Example 1 : Basic Contradiction

To prove p, assume not p and derive a contradiction (false).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ulcorner\neg\, p \wedge (q \wedge \neg\, q)\urcorner^{[1]}}{q \wedge \neg\, q}\ [\wedge\text{ -elim-2}]}{q}\ [\wedge\text{ -elim-1}]}{\neg\, q}\ [\wedge\text{ -elim-2}]}{\mathit{false}}\ [\text{contradiction}]}{p}\ [\text{false elim}]}{(\neg\, p \wedge (q \wedge \neg\, q)) \Rightarrow p}\ [\Rightarrow\text{ -intro}^{[1]}]$$

Once we derive false, we can conclude anything—here we conclude p by discharging assumption 1.

Example 2 : Simple Example - Proving q

Prove: $(p \wedge \neg\, p) \Rightarrow$ q

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ulcorner p \wedge \neg\, p\urcorner^{[1]}}{p}\ [\wedge\text{ -elim-1}]}{\neg\, p}\ [\wedge\text{ -elim-2}]}{\mathit{false}}\ [\text{contradiction}]}{q}\ [\text{false elim}]}{(p \wedge \neg\, p) \Rightarrow q}\ [\Rightarrow\text{ -intro}^{[1]}]$$

From a contradiction, we can derive anything (ex falso quodlibet).

Example 3 : Law of Non - Contradiction

Prove that p $\wedge \neg\, p$ leads to a contradiction.

$$\dfrac{\dfrac{\ulcorner p \wedge \neg\, p\urcorner^{[1]}}{\mathit{false}}\ [\text{contradiction}]}{p \wedge \neg\, p \Rightarrow \mathit{false}}\ [\Rightarrow\text{ -intro}^{[1]}]$$

This proves the law of non-contradiction - no proposition can be both true and false.

Example 4 : Indirect Proof

Prove: $(\neg\, p \Rightarrow \mathit{false}) \Rightarrow \neg\, \neg\, p$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathit{false}}\ [\Rightarrow\text{ elim}]}{\neg\, p}\ [\text{assumption}]}{\neg\, \neg\, p}\ [\neg\text{ -intro}^{[2]}]}{(\neg\, p \Rightarrow \mathit{false}) \Rightarrow \neg\, \neg\, p}\ [\Rightarrow\text{ -intro}^{[1]}]$$

This is the pattern for indirect proof: if assuming $\neg\, p$ leads to false, then $\neg$ not p holds.

Example 5 : Modus Tollens via Contradiction

Prove: $(p \Rightarrow q) \land \neg\, q \Rightarrow \neg\, p$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{q}\;[\Rightarrow \text{elim}] \qquad \dfrac{}{\mathit{false}}\;[\text{contradiction}]}{p}\;[\text{assumption}]}{\neg\, p}\;[\neg\text{ -intro}^{[2]}]}{((p \Rightarrow q) \land \neg\, q) \Rightarrow \neg\, p}\;[\Rightarrow\text{ -intro}^{[1]}]$$

Classic modus tollens pattern using contradiction.

Example 6 : Disjunction from Contradiction

Prove: $\neg\, p \Rightarrow (p \Rightarrow q)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\ulcorner p \urcorner^{[2]}}{\mathit{false}}\;[\text{contradiction}]}{q}\;[\text{false elim}]}{(p \Rightarrow q)}\;[\Rightarrow\text{ -intro}^{[2]}]}{\neg\, p \Rightarrow (p \Rightarrow q)}\;[\Rightarrow\text{ -intro}^{[1]}]$$

From $\neg\, p$, we can prove $p \Rightarrow q$ for any q.

Example 7 : Double Negation Elimination

Prove: $\neg\;\; \neg\, p \Rightarrow p(\mathit{requires}(\mathit{classical})(\mathit{logic}))$

$$\dfrac{\dfrac{\dfrac{\dfrac{\ulcorner \neg\, p \urcorner^{[2]}}{\mathit{false}}\;[\text{contradiction}]}{p}\;[\text{false elim}]}{\neg\,\neg\,\neg\, p}\;[\neg\text{ -intro}^{[2]}]}{\neg\,\neg\, p \Rightarrow \neg\,\neg\,\neg\, p}\;[\Rightarrow\text{ -intro}^{[1]}]$$

In classical logic, not $\neg\, p$ implies p. This requires excluded middle or equivalent axiom.

Example 8 : Reductio ad Absurdum

Prove: $(\neg\, p \Rightarrow (q \land \neg\, q)) \Rightarrow p$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ulcorner \neg\, p \urcorner^{[2]}}{q \land \neg\, q}\;[\Rightarrow \text{elim}]}{q}\;[\land\text{ -elim-1}]}{\neg\, q}\;[\land\text{ -elim-2}]}{\mathit{false}}\;[\text{contradiction}]}{p}\;[\text{false elim}]}{\neg\,\neg\, p}\;[\neg\text{ -intro}^{[2]}]}{(\neg\, p \Rightarrow (q \land \neg\, q)) \Rightarrow \neg\,\neg\, p}\;[\Rightarrow\text{ -intro}^{[1]}]$$

Reductio ad absurdum: if assuming not p leads to absurdity, then p holds.

Example 9 : Contradiction with Universal Quantifier

Prove: $(\forall\, x \bullet \mathbb{P}\; x) \land \exists\, x \bullet \neg\; \mathbb{P}\; x$ is contradictory.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ulcorner(\forall\, x \bullet \mathbb{P}\; x) \land (\exists\, x \bullet \neg\; \mathbb{P}\; x)\urcorner^{[1]}}{\forall\, x \bullet \mathbb{P}\; x}\,[\land\text{ -elim-1}]}{\exists\, x \bullet \neg\; \mathbb{P}\; x}\,[\land\text{ -elim-2}]}{\neg\; \mathbb{P}\; a}\,[\exists\,\text{elim, fresh a}]}{\mathbb{P}\; a}\,[\forall\,\text{elim}]}{\mathit{false}}\,[\text{contradiction}]}{((\forall\, x \bullet \mathbb{P}\; x) \land (\exists\, x \bullet \neg\; \mathbb{P}\; x)) \Rightarrow \mathit{false}}\,[\Rightarrow\text{ -intro}^{[1]}]$$

If something holds for all x, it cannot fail for some x.

Example 10 : Proving Uniqueness by Contradiction

Prove: if f is injective, then f(x) = f(y) $\Rightarrow x = y$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\ulcorner x \neq y\urcorner^{[2]}}{f(x) \neq f(y)}\,[\text{injective property}]}{\mathit{false}}\,[\text{contradiction with f(x) = f(y)}]}{x = y}\,[\neg\text{ -intro}^{[2]}]}{(\mathit{injective}(f) \land f(x) = f(y)) \Rightarrow x = y}\,[\Rightarrow\text{ -intro}^{[1]}]$$

Uses contradiction to prove equality.

Example 11 : Case Analysis Leading to Contradiction

Prove: p $\lor$ q, $\neg\; p$, $\neg\; q \Rightarrow \mathit{false}$

$$\dfrac{\dfrac{\dfrac{}{\mathit{false}}\,[\text{contradiction with} \neg\text{ p}] \qquad \dfrac{}{\mathit{false}}\,[\text{contradiction with} \neg\text{ q}]}{\mathit{false}}\,[\lor\text{ elim from p} \lor \text{q}]}{((p \lor q) \land \neg\; p \land \neg\; q) \Rightarrow \mathit{false}}\,[\Rightarrow\text{ -intro}^{[1]}]$$

Both cases lead to contradiction, so the premises are inconsistent.

Example 12 : Best Practices for Contradiction Proofs

When using proof by contradiction:

1. Clearly mark the assumption you're contradicting with [assumption N]
2. Show explicitly where false is derived
3. Use [not intro from N] to discharge the assumption
4. Document the contradiction (what conflicts with what)
5. In natural deduction, false elim lets you conclude anything
6. *Remember* : *contradiction* is a classical technique (not constructive)