

Propositional logic

Solution 1

(a)

$$\text{false}(\text{as}(\text{true} \Rightarrow \text{false}) \Leftrightarrow \text{false})$$

(b)

$$\text{true}(\text{as}(\text{false} \Rightarrow \text{false}) \Leftrightarrow \text{true})$$

(c)

$$\text{true}(\text{as}(\text{false} \Rightarrow \text{true}) \Leftrightarrow \text{true})$$

(d)

$$\text{true}(\text{as}(\text{false} \Rightarrow \text{false}) \Leftrightarrow \text{true})$$

(Assuming that pigs can't fly . . .)

Solution 2

(a)

p	q	$p \wedge q$	$(p \wedge q) \Rightarrow p$
t	t	t	t
t	f	f	t
f	t	f	t
f	f	f	t

(b)

p	q	$p \wedge q$	$\neg p$	$\neg p \Rightarrow (p \wedge q)$	$(\neg p \Rightarrow (p \wedge q)) \Leftrightarrow p$
t	t	t	f	t	t
t	f	f	f	t	t
f	t	f	t	f	t
f	f	f	t	f	t

(c)

p	q	$p \Rightarrow q$	$p \wedge (p \Rightarrow q)$	$(p \wedge (p \Rightarrow q)) \Rightarrow q$
t	t	t	t	t
t	f	f	f	t
f	t	t	f	t
f	f	t	f	t

Solution 3

(a)

$$\begin{aligned}
 p \Rightarrow \neg p & \\
 \Leftrightarrow \neg p \vee \neg p & \quad [\Rightarrow] \\
 \Leftrightarrow \neg p & \quad [\text{idempotence}]
 \end{aligned}$$

(b)

$$\begin{aligned}
 \neg p \Rightarrow p & \\
 \Leftrightarrow \neg \neg p \vee p & \quad [\Rightarrow] \\
 \Leftrightarrow p \vee p & \quad [\neg \neg] \\
 \Leftrightarrow p & \quad [\text{idempotence}]
 \end{aligned}$$

(c)

$$\begin{aligned}
 p \Rightarrow (q \Rightarrow r) & \\
 \Leftrightarrow \neg p \vee (q \Rightarrow r) & \quad [\Rightarrow] \\
 \Leftrightarrow \neg p \vee \neg q \vee r & \quad [\Rightarrow] \\
 \Leftrightarrow \neg p \vee \neg q \vee r & \quad [\text{associativity}] \\
 \Leftrightarrow \neg(p \wedge q) \vee r & \quad [\text{De Morgan}] \\
 \Leftrightarrow p \wedge q \Rightarrow r & \quad [\Rightarrow]
 \end{aligned}$$

(d)

$$\begin{aligned}
 q \Rightarrow (p \Rightarrow r) & \\
 \Leftrightarrow \neg q \vee (p \Rightarrow r) & \quad [\Rightarrow] \\
 \Leftrightarrow \neg q \vee \neg p \vee r & \quad [\Rightarrow] \\
 \Leftrightarrow \neg p \vee \neg q \vee r & \quad [\text{associativity} \wedge \text{commutativity}] \\
 \Leftrightarrow \neg p \vee (q \Rightarrow r) & \quad [\Rightarrow] \\
 \Leftrightarrow p \Rightarrow (q \Rightarrow r) & \quad [\Rightarrow]
 \end{aligned}$$

(e)

$$\begin{aligned}
p \wedge q &\Leftrightarrow p \\
&\Leftrightarrow (p \wedge q \Rightarrow p) \wedge (p \Rightarrow p \wedge q) & [\Leftrightarrow] \\
&\Leftrightarrow (\neg(p \wedge q) \vee p) \wedge (\neg p \vee p \wedge q) & [\Rightarrow] \\
&\Leftrightarrow (\neg p \vee \neg q \vee p) \wedge (\neg p \vee p \wedge q) & [\text{De Morgan}] \\
&\Leftrightarrow (\neg q \vee \neg p \vee p) \wedge (\neg p \vee p \wedge q) & [\text{associativity} \wedge \text{comm .}] \\
&\Leftrightarrow (\neg q \vee \text{true}) \wedge (\neg p \vee p \wedge q) & [\text{excluded middle}] \\
&\Leftrightarrow \text{true} \wedge (\neg p \vee p \wedge q) & [\vee \wedge \text{true}] \\
&\Leftrightarrow \neg p \vee p \wedge q & [\wedge \wedge \text{true}] \\
&\Leftrightarrow (\neg p \vee p) \wedge (\neg p \vee q) & [\text{distribution}] \\
&\Leftrightarrow \text{true} \wedge (\neg p \vee q) & [\text{excluded middle}] \\
&\Leftrightarrow \neg p \vee q & [\wedge \wedge \text{true}] \\
&\Leftrightarrow p \Rightarrow q & [\Rightarrow]
\end{aligned}$$

(f)

$$\begin{aligned}
p \vee q &\Leftrightarrow p \\
&\Leftrightarrow (p \vee q \Rightarrow p) \wedge (p \Rightarrow p \vee q) & [\Leftrightarrow] \\
&\Leftrightarrow (\neg(p \vee q) \vee p) \wedge (\neg p \vee p \vee q) & [\Rightarrow] \\
&\Leftrightarrow (\neg p \wedge \neg q \vee p) \wedge (\neg p \vee p \vee q) & [\text{De Morgan}] \\
&\Leftrightarrow (\neg p \vee p) \wedge (\neg q \vee p) \wedge (\neg p \vee p \vee q) & [\text{distribution}] \\
&\Leftrightarrow \text{true} \wedge (\neg q \vee p) \wedge (\neg p \vee p \vee q) & [\text{excluded middle}] \\
&\Leftrightarrow (\neg q \vee p) \wedge (\neg p \vee p \vee q) & [\wedge \wedge \text{true}] \\
&\Leftrightarrow (\neg q \vee p) \wedge (\neg p \vee p \vee q) & [\text{associativity}] \\
&\Leftrightarrow (\neg q \vee p) \wedge (\text{true} \vee q) & [\text{excluded middle}] \\
&\Leftrightarrow (\neg q \vee p) \wedge \text{true} & [\vee \wedge \text{true}] \\
&\Leftrightarrow \neg q \vee p & [\wedge \wedge \text{true}] \\
&\Leftrightarrow q \Rightarrow p & [\Rightarrow]
\end{aligned}$$

Solution 4

(a) $(p \text{ or } q) \Leftrightarrow ((\text{not } p \text{ or not } q) \text{ and } q)$ is not a tautology. You might illustrate this via a truth table or via a chain of equivalences, showing that the proposition is not equivalent to true. Alternatively, you might try and find a combination of values for which the proposition is false. (In this case, the proposition is false when p and q are both true.)

(b) $(p \text{ or } q) \Leftrightarrow ((\text{not } p \text{ and not } q) \text{ or } q)$ is not a tautology. In this case, the proposition is false when p is true and q is false.

Solution 5

(a)

$$\exists d: \text{Dog} \bullet \text{gentle}(d) \wedge \text{well-trained}(d)$$

(b)

$$\forall d: Dog \bullet neat(d) \wedge well-trained(d) \Rightarrow attractive(d)$$

(c) (Requires nested quantifier in implication - parser limitation)

Solution 6

- (a) This is a true *proposition* : whatever the value of x , the expression $x^2 - x + 1$ denotes a natural number. If we choose y to be this natural number, we will find that p is true.
- (b) This is a false proposition. We cannot choose a large enough value for y such that p will hold for any value of x .
- (c) This is a false proposition. It is an implication whose antecedent part is true and whose consequent part is false.
- (d) This is a true proposition. It is an implication whose antecedent part is false and whose consequent part is true.

Solution 7

- (a) We must define a predicate p that is false for at least one value of x , and is true for at least one other value. A suitable solution would be $p \Leftrightarrow x > 1$.
- (b) With the above choice of p , we require only that q is sometimes false when p is true (for else the universal quantification would hold). A suitable solution would be $q \Leftrightarrow x > 3$.

Solution 8

(a)

$$\forall x: N \bullet x \geq z$$

Equality

Solution 9

(d)

$$\begin{aligned} \exists x: N \bullet x = 1 \wedge x > y \vee x = 2 \wedge x > z \\ \Leftrightarrow \exists x: N \bullet x = 1 \wedge x > y \vee \exists x: N \bullet x = 2 \wedge x > z \\ \Leftrightarrow 1 \in N \wedge 1 > y \vee \exists x: N \bullet x = 2 \wedge x > z \\ \Leftrightarrow 1 \in N \wedge 1 > y \vee 2 \in N \wedge 2 > z \\ \Leftrightarrow 1 > y \vee 2 > z \end{aligned}$$

Solution 10

As discussed, the quantifier \exists_1 can help give rise to a 'test' or 'precondition' to ensure that an application of mu will work.

So, as a simple example, as the proposition

$$\exists_1 n: N \bullet \forall m: N \bullet n \leq m$$

is equivalent to true, we can be certain that the statement

$$\mu n: N \bullet \forall m: N \bullet n \leq m$$

will return a result (which happens to be 0).

Solution 11

- (a) $(\text{mu } a : N \mid a = a) = 0$ is a provable statement, since 0 is the only natural number with the specified property.
- (b) $(\mu b: N \bullet b = b) = 1$ is not provable. The specified property is true of both 0 and 1, and thus the value of the mu-expression is undefined.
- (c) $(\mu c: N \bullet c > c) = (\mu c: N \bullet c > c)$ is a provable statement. Neither expression is properly defined, but we may conclude that they are equal; there is little else that we can prove about them.
- (d) $(\mu d: N \bullet d = d) = 1$ is not a provable statement. We cannot confirm that 1 is the only natural number with the specified property; we do not know what value is taken by undefined operations.

Solution 12

(Requires mu-operator with expression part - not yet implemented)

(a)

$$\mu m: Mountain \mid \forall n: Mountain \bullet height(n) \leq height(m) \bullet height(m)$$

(b)

$$\mu c: Chapter \mid \exists_1 d: Chapter \bullet length(d) > length(c) \bullet length(c)$$

- (c) Assuming the existence of a suitable function, max: $(\mu n: N \bullet n = \max(\{m: N \mid 8 * m < 100.8 * m\}) . 100 - n)$

Deductive proofs

Solution 13

$$\frac{\frac{\frac{p \wedge (p \Rightarrow q)}{p} [\wedge\text{-elim}^{[1]}] \quad \frac{p \wedge (p \Rightarrow q) \quad p \wedge (p \Rightarrow q)}{p \Rightarrow q} [\wedge\text{-elim}^{[1]}] \quad \frac{p \wedge (p \Rightarrow q)}{p} [\wedge\text{-elim}^{[1]}]}{q} [\wedge\text{-intro}]}{p \wedge q} [\Rightarrow\text{-intro}^{[1]}]$$

Solution 14

In one direction:

$$\frac{\frac{\frac{\frac{\overline{p \wedge q}}{p \wedge q} [\text{derived}] \quad \frac{\overline{p \wedge q}}{p \wedge q} [\Rightarrow\text{-elim from } 1 \wedge 2]}{\frac{\overline{p \neg[2]} \quad \frac{q}{p \Rightarrow q} [\wedge\text{-elim}^{[3]}]}{p \Rightarrow q} [\Rightarrow\text{-intro}^{[2]}]}{p \wedge q \Leftrightarrow p} [\Rightarrow\text{-intro}^{[1]}]}{(p \wedge q \Leftrightarrow p) \Rightarrow (p \Rightarrow q)}$$

and the other:

$$\frac{\frac{\frac{\frac{\overline{p \wedge q \neg[2]}}{p \wedge q \Rightarrow p} [\Rightarrow\text{-intro}^{[2]}] \quad \frac{\overline{p \neg[3]} \quad \overline{p \wedge q \neg[1]}}{p \Rightarrow p \wedge q} [\Rightarrow\text{-intro}^{[3]}]}{p \wedge q \Leftrightarrow p} [\Leftrightarrow\text{-intro}]}{(p \Rightarrow q) \Rightarrow (p \wedge q \Leftrightarrow p)} [\Rightarrow\text{-intro}^{[1]}]}$$

We can then combine these two proofs *with \Leftrightarrow intro*.

Solution 15

$$\frac{\frac{\frac{\overline{p \Rightarrow q \neg[1]} \quad \overline{p \neg[2]}}{q} [\Rightarrow\text{-elim}] \quad \frac{\overline{\neg \neg q \neg[1]}}{false} [\text{false intro}]}{\frac{\overline{p \neg[2]}}{\neg p} [\text{false-elim}^{[2]}]}{(p \Rightarrow q) \wedge \neg q \neg[1]} [\Rightarrow\text{-intro}^{[1]}]}$$

Solution 16

In one direction:

$\frac{\neg p \neg [1] \quad \neg r}{p \wedge r} \text{ [case assumption]}$	$\frac{}{p \wedge q \vee p \wedge r} \text{ [\vee intro]}$
$\frac{\neg p \neg [1] \quad \neg q}{p \wedge q} \text{ [case assumption]}$	
$\frac{}{p \wedge q \vee p \wedge r} \text{ [\vee intro]}$	
$\frac{}{q \vee r \neg [1]} \text{ [] }$	
$\frac{\neg p \wedge (q \vee r) \neg [1]}{p \wedge (q \vee r)} \Rightarrow p \wedge q \vee p \wedge r$	$\frac{p \wedge q \vee p \wedge r}{\Rightarrow \neg \neg [1]} \text{ [\neg \neg intro]}$

In the other:

$\frac{}{p} [\wedge \text{ elim}]$	$\frac{}{q \vee r} [\vee \text{ intro}]$
$\frac{}{p} [\wedge \text{ elim}]$	$\frac{p \wedge (q \vee r)}{p \wedge (q \vee r)} [\wedge \text{ intro}]$
$\frac{}{q \vee r} [\vee \text{ intro}]$	
$\frac{}{p \wedge (q \vee r)} [\wedge \text{ intro}]$	
$\frac{\neg case1 \vee case2}{p \wedge (q \vee r)}$	$\frac{\neg case1 \vee case2}{p \wedge (q \vee r)} [\neg\text{-intro}^{[3]}]$
$\frac{\neg p \wedge q \vee p \wedge r}{p \wedge q \vee p \wedge r \Rightarrow p \wedge (q \vee r)}$	$\frac{\neg p \wedge q \vee p \wedge r}{p \wedge q \vee p \wedge r \Rightarrow p \wedge (q \vee r)} [\neg\text{-elim}^{[4]}]$

Solution 17

In one direction:

$$\frac{\frac{p \vee q \wedge r \neg [3]}{(p \vee q) \wedge (p \vee r)} \quad [\vee \text{ elim } \wedge \wedge \text{ intro}]}{p \vee q \wedge r \Rightarrow (p \vee q) \wedge (p \vee r)} \quad [\Rightarrow \text{-intro}^{[3]}]$$

and the other:

$$\frac{\Gamma(p \vee q) \wedge (p \vee r)^{\neg[1]} \quad \Gamma p \vee q \wedge r^{\neg[2]}}{(p \vee q) \wedge (p \vee r) \Rightarrow p \vee q \wedge r} [\Rightarrow\text{-intro}^{[1]}]$$

Solution 18

In one direction:

$$\frac{\Gamma p \Rightarrow q^{[1]} \quad \neg p \vee q}{(p \Rightarrow q) \Rightarrow \neg p \vee q} [\Rightarrow\text{-intro}^{[1]}]$$

and the other:

$$\frac{\Gamma \neg p \vee q^{[3]} \quad \frac{\Gamma p^{[4]} \quad \Gamma q^{[3]}}{p \Rightarrow q} [\Rightarrow\text{-intro}^{[4]}]}{\neg p \vee q \Rightarrow (p \Rightarrow q)} [\Rightarrow\text{-intro}^{[3]}]$$

Sets and types

Solution 19

- (a) 1 in $\{4, 3, 2, 1\}$ is true.
- (b) $\{1\}$ in $\{1, 2, 3, 4\}$ is undefined.
- (c) $\{1\}$ in $\{\{1\}, \{2\}, \{3\}, \{4\}\}$ is true.
- (d) The empty set in $\{1, 2, 3, 4\}$ is undefined.

Solution 20

(a)

$$\{1\} \times \{2, 3\}$$

is the set $\{(1, 2), (1, 3)\}$

- (b) The empty set cross $\{2, 3\}$ is the empty set

(c)

$$\mathbb{P} \emptyset \times \{1\}$$

is the set $\{(\emptyset, 1)\}$

- (d) $\{(1, 2)\}$ cross $\{3, 4\}$ is the set $\{((1, 2), 3), ((1, 2), 4)\}$

Solution 21

There are various ways of describing these sets via set comprehensions. Examples are given below.

(a)

$$\{z : Z \mid 0 \leq z \wedge z \leq 100\}$$

(b)

$$\{z : Z \mid z = 10\}$$

(c)

$$\{z : Z \mid z \bmod 2 = 0 \vee z \bmod 3 = 0 \vee z \bmod 5 = 0\}$$

Solution 22

(a)

$$\{n : N \mid n \leq 4 \bullet n^2\}$$

(b)

$$\{n : N \mid n \leq 4 \bullet (n, n^2)\}$$

(c) $n : P0, 1$

(d) $\{n : \mathbb{P}\{0, 1\} \mid \text{true} \bullet (n, \#n)\}$

Solution 23

(a)

$$\begin{aligned} x \in a \cap a \\ \Leftrightarrow x \in a \wedge x \in a \\ \Leftrightarrow x \in a \end{aligned}$$

(b)

$$\begin{aligned} x \in a \cup a \\ \Leftrightarrow x \in a \vee x \in a \\ \Leftrightarrow x \in a \end{aligned}$$

Solution 24

(a)

The set of all pairs of integers is $Z \times Z$. To give it a name, we could write:

Pairs == $Z \times Z$

(b)

The set of all integer pairs in which each element is strictly greater than zero could be defined by:

StrictlyPositivePairs == $\{ m, n : Z \mid m > 0 \wedge n > 0 \bullet (m, n) \}$

(c)

It is intuitive to use a singular noun for the name of a basic type; we define the set of all people by writing:

[Person]

(d)

The set of all couples could be defined by:

Couples == $\{ s : \mathbb{P} Person \mid \#s = 2 \}$

Solution 25

(Requires generic set notation and Cartesian product)

Solution 26

(Requires generic parameters and relation type notation)

Relations

Solution 27

(a)

The power set of $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ is:

$\{\emptyset, \{(0, 0)\}, \{(0, 1)\}, \{(1, 0)\}, \{(1, 1)\}, \{(0, 0), (0, 1)\}, \{(0, 0), (1, 0)\}, \{(0, 0), (1, 1)\}, \{(0, 1), (1, 0)\}, \{(0, 1), (1, 1)\}, \{(0, 0), (0, 1), (0, 1)\}, \{(0, 0), (0, 1), (1, 0)\}, \{(0, 0), (0, 1), (1, 1)\}, \{(0, 0), (1, 0), (1, 1)\}, \{(0, 0), (1, 0), (0, 1)\}, \{(0, 0), (1, 1), (0, 1)\}, \{(0, 0), (1, 1), (1, 0)\}, \{(0, 0), (0, 1), (1, 0), (1, 1)\}$

(b)

$$\{\emptyset, \{(0, 0)\}, \{(0, 1)\}, \{(0, 0), (0, 1)\}\}$$

(c)

$$\{\emptyset\}$$

(d)

$$\{\emptyset\}$$

Solution 28

(a)

$$\text{dom } R = \{0, 1, 2\}$$

(b)

$$\text{ran } R = \{1, 2, 3\}$$

(c)

$$\{1, 2\} \lhd R = \{1 \mapsto 2, 1 \mapsto 3, 2 \mapsto 3\}$$

Solution 29

(a)

$$\{2 \mapsto 4, 3 \mapsto 3, 3 \mapsto 4, 4 \mapsto 2\}$$

(b)

$$\{1 \mapsto 3, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1\}$$

(c)

$$\{1 \mapsto 1, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 2, 3 \mapsto 3, 4 \mapsto 4\}$$

(d)

$$\{1 \mapsto 4, 2 \mapsto 2, 2 \mapsto 3, 3 \mapsto 2, 3 \mapsto 3, 4 \mapsto 1\}$$

Solution 30

| $\text{childOf} : \text{Person} \leftrightarrow \text{Person}$

(a)

$\text{parentOf} == \text{childOf}^{-1}$

This is a good example of how there are many different ways of writing the same thing.
An alternative abbreviation is:

$\text{parentOf} == \{ x, y : \text{Person} \mid x \mapsto y \in \text{childOf} \bullet y \mapsto x \}$

Or, via an axiomatic definition:

$$\frac{\text{parentOf} : \text{Person} \leftrightarrow \text{Person}}{\text{parentOf} = \text{childOf}^{-1}}$$

(b)

$\text{siblingOf} == (\text{childOf} \circ \text{parentOf}) \setminus \text{id}$

(c)

$\text{cousinOf} == \text{childOf} \circ \text{siblingOf} \circ \text{parentOf}$

(d)

$\text{ancestorOf} == \text{parentOf}^+$

Solution 31

(Requires compound identifiers with operators - R+, R*)

(a)

$R == \{ a, b : N \mid b = a \vee b = a \}$

(b)

$S == \{ a, b : N \mid b = a \vee b = a \}$

(c) $R+ == \{ a, b : N \mid b > a \}$

(d) $R* == \{ a, b : N \mid b \geq a \}$

Solution 32

(a)

$$x \mapsto y \in A \lhd B \lhd R$$

$$\Leftrightarrow x \in A \wedge x \mapsto y \in (B \lhd R)$$

$$\Leftrightarrow x \in A \wedge x \in B \wedge x \mapsto y \in R$$

$$\Leftrightarrow x \in A \cap B \wedge x \mapsto y \in R$$

$$\Leftrightarrow x \mapsto y \in A \cap B \lhd R$$

(b)

$$\begin{aligned}x \mapsto y \in R \cup S \triangleright C \\ \Leftrightarrow x \mapsto y \in R \cup S \wedge y \in C \\ \Leftrightarrow (x \mapsto y \in R \vee x \mapsto y \in S) \wedge y \in C \\ \Leftrightarrow x \mapsto y \in R \wedge y \in C \vee x \mapsto y \in S \wedge y \in C \\ \Leftrightarrow x \mapsto y \in R \triangleright C \vee x \mapsto y \in S \triangleright C \\ \Leftrightarrow x \mapsto y \in (R \triangleright C) \cup (S \triangleright C)\end{aligned}$$

Functions

Solution 33

The set of 9 functions:

$$\{\emptyset, \{(0, 0)\}, \{(0, 1)\}, \{(1, 1)\}, \{(1, 0)\}, \{(0, 0), (1, 1)\}, \{(0, 1), (1, 1)\}, \{(1, 0), (0, 0)\}, \{(0, 1), (1, 0)\}\}$$

(a)

The set of total functions:

$$\{\{(0, 0), (1, 1)\}, \{(0, 1), (1, 1)\}, \{(1, 0), (0, 0)\}, \{(0, 1), (1, 0)\}\}$$

(b)

The set of functions which are neither injective nor surjective:

$$\{\{(0, 1), (1, 1)\}, \{(0, 0), (1, 0)\}\}$$

(c)

The set of functions which are injective but not surjective:

$$\{\emptyset, \{(0, 0)\}, \{(0, 1)\}, \{(1, 0)\}, \{(1, 1)\}\}$$

(d) There are no functions (of this type) which are surjective but not injective.

(e)

The set of bijective functions:

$$\{\{(0,0), (1,1)\}, \{(0,1), (1,0)\}\}$$

Solution 34

(a)

$$\{1 \mapsto a, 2 \mapsto b, 3 \mapsto c, 4 \mapsto b\}$$

(b)

$$\{1 \mapsto c, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d\}$$

(c)

$$\{1 \mapsto c, 2 \mapsto b, 3 \mapsto c, 4 \mapsto b\}$$

(d)

$$\{1 \mapsto c, 2 \mapsto b, 3 \mapsto c, 4 \mapsto b\}$$

Solution 35

(Requires power set notation P and relational image)

(a)

$$\frac{\text{children} : \text{Person} \rightarrow \mathbb{P} \text{ Person}}{\text{children} = \{p : \text{Person} \bullet p \mapsto \text{parentOf}(\{p\})\}}$$

(b)

$$\frac{\text{number_of_grandchildren} : \text{Person} \rightarrow N}{\text{number_of_grandchildren} = \{p : \text{Person} \bullet p \mapsto \#\text{parentOf} \circ \text{parentOf}(\{p\})\}}$$

Solution 36

(Note: This solution demonstrates relation types in quantifier domains)

$$\frac{\text{number_of_drivers} : \text{Drivers} \leftrightarrow \text{Cars} \rightarrow (\text{Cars} \rightarrow N)}{\text{number_of_drivers} = \lambda r : \text{Drivers} \leftrightarrow \text{Cars} \bullet \{c : \text{ran } r \bullet c \mapsto \#\{d : \text{Drivers} \mid d \mapsto c \in r\}\}}$$

Sequences

Solution 37

(a)

$$\langle a \rangle$$

(b)

$$\{1 \mapsto a, 2 \mapsto b, 2 \mapsto a, 3 \mapsto c, 3 \mapsto b, 4 \mapsto d\}$$

(c)

$$\{2 \mapsto b, 3 \mapsto c, 4 \mapsto d\}$$

(d)

$$\{1, 2, 3, 4\}$$

(e)

$$\{a, b\}$$

(f)

$$\{a \mapsto 1, b \mapsto 2, c \mapsto 3, d \mapsto 4\}$$

(g)

$$\langle a, b \rangle$$

(h)

$$\{3 \mapsto b\}$$

(i)

$$\{a\}$$

(j)

$$c$$

Solution 38

(a)

$$\frac{f : Place \rightarrow \mathbb{P} Place}{\forall p: Place \bullet f(p) = \{q: Place \mid p \mapsto q \in \text{ran } trains\}}$$

(b)

$$\{p: Place \mid \exists_1 x: \text{dom } trains \bullet trains(x).2 = p\}$$

(c)

$$\mu p: Place \bullet \forall q: Place \bullet p \neq q \wedge \#\{x: \text{dom } trains \mid trains(x).2 = p\} > \#\{x: \text{dom } trains \mid trains(x).2 = q\}$$

Solution 39

(a)

$$large_coins : Collection \rightarrow N$$

$$\forall c: Collection \bullet large_coins(c) = c(\text{large})$$

(Blocked by : underscore identifier for fuzz compatibility)

(b)

$$add_coin : Collection * Coin \rightarrow Collection$$

$$\forall c: Collection \bullet \forall d: Coin \bullet add_coin(c, d) = c \cup \llbracket d \rrbracket$$

(Blocked by : underscore identifier and bag union)

Modelling

Solutions 40-52 are work in progress - many require features not yet implemented

Solution 40

(Work in progress - requires semicolon-separated bindings in set comprehensions)

(a)

$$hd : \text{seq}(\text{Title} * \text{Length} * \text{Viewed})$$

$$\text{cumulative_total}(hd) \leq 12000$$

$$\forall p : \text{ran } hd \bullet p.2 \leq 360$$

Note that cumulative_total is defined in part (d).

(b)

$$\{p : \text{ran } hd \mid p.2 > 120 \bullet p.1\}$$

(c)

These can be defined recursively:

$$\frac{\text{viewed} : \text{seq Programme} \rightarrow \text{seq Programme}}{\text{viewed}(\langle \rangle) = \langle \rangle \wedge \forall x : \text{Programme} \bullet \forall s : \text{seq Programme} \bullet \text{viewed}(\langle x \rangle \cap s) = (\text{if } x.3 = \text{yes} \text{ then } \langle x \rangle \cap \text{viewed}(s)) \cup \langle \rangle}$$

or otherwise (omitted - requires semicolon-separated bindings in set comprehension)

(d)

$$\frac{\text{cumulative_total} : \text{seq Title} * \text{Length} * \text{Viewed} \rightarrow N}{\text{cumulative_total}(\langle \rangle) = 0 \wedge \forall x : \text{Title} * \text{Length} * \text{Viewed} \bullet \forall s : \text{seq Title} * \text{Length} * \text{Viewed} \bullet \text{cumulative_total}(\langle x \rangle \cap s) + x.2 = \text{cumulative_total}(\langle x \rangle \cap s) + x.2}$$

(e)

$$(\mu p : ranhd \mid \forall q : \text{ran } hd \bullet p \neq q \wedge p.2 > q.2 \mid p.1)$$

(This, of course, assumes that there is a unique element with this property.)

(f)

(f) Omitted - requires semicolon-separated bindings in nested set comprehension

(g)

axdef

$g : \text{seq}(\text{Title} * \text{Length} * \text{Viewed}) \rightarrow \text{seq}(\text{Title} * \text{Length} * \text{Viewed})$

where

$\forall s : \text{seq } \text{Title} * \text{Length} * \text{Viewed} \bullet g(s) = s \triangleright \{x : \text{ran } s \mid x \neq \text{longest_viewed}(s)\}$

end

Where longest_viewed is defined as

axdef

$\text{longest_viewed} : \text{seq}(\text{Title} * \text{Length} * \text{Viewed})^+ \rightarrow \text{Title} * \text{Length} * \text{Viewed}$

where

$\forall s : \text{seq } \text{Title} * \text{Length} * \text{Viewed} \bullet \text{longest_viewed}(s) = (\mu p : \text{ran } s \bullet p.3 = \text{yes} \text{ and } \forall q : \text{ran } s \bullet p \neq q \wedge q.3 = \text{yes} \wedge p.2 > q.2)$

end

This, of course, assumes that there is at least one viewed programme (and one of a unique maximum length).

(h)

$$\frac{s : \text{seq } \text{Title} * \text{Length} * \text{Viewed} \rightarrow \text{seq } \text{Title} * \text{Length} * \text{Viewed}}{\forall x : \text{seq } \text{Title} * \text{Length} * \text{Viewed} \bullet \text{items}(s(x)) = \text{items}(x) \wedge \forall i, j : \text{dom } s(x) \bullet i < j \Rightarrow s(x)(i).2 \geq s(x)(j).2}$$

Solution 41

(a)

axdef

$\text{records} : \text{Year} \leftrightarrow \text{Table}$

where

$$\text{dom}(\text{records}) = 1993..\text{current}$$

$$\forall y : \text{dom } \text{records} \bullet \#\text{records}(y) \leq 50$$

$$\forall y : \text{dom}(\text{records}) \mid \forall e : \text{ran } \text{records}(y) \bullet \text{year}(e.1) = y$$

$$\forall r : \text{ran}(\text{records}) \mid \forall i1, i2 : \text{dom } r \bullet i1 \neq i2 \wedge r(i1).1 = r(i2).1 \Rightarrow r(i1).3 \neq r(i2).3$$

end

(b)

(i)

$$\{e : \text{Entry} \mid \exists r : \text{ran } \text{records} \bullet e \in \text{ran } r \wedge e.3 = 479\}$$

ii

$$\{e : \text{Entry} \mid \exists r : \text{ran } \text{records} \bullet e \in \text{ran } r \wedge e.6 > e.5\}$$

iii

$$\{e : \text{Entry} \mid \exists r : \text{ran } \text{records} \bullet e \in \text{ran } r \wedge e.7 \geq 70\}$$

iv

$$\{c : \text{Course} \mid \forall r : \text{ran } \text{records} \bullet \forall e : \text{ran } r \bullet e.2 = c \Rightarrow e.7 \geq 70\}$$

v

$$\{y : \text{Year} \mid y \in \text{dom } \text{records} \bullet y \mapsto \{l : \text{Lecturer} \mid \#\{c : \text{ran } \text{records}(y) \mid c.4 = l\} > 6\}\}$$

(c)

axdef

where

$$\forall x : \text{Entry} \bullet \forall s : \text{seq Entry} \bullet 479_{\text{courses}}(\langle \rangle) = \langle \rangle \text{ and } 479_{\text{courses}}(\langle x \rangle^s) = \text{if } x.3 = 479 \text{ then } \langle x \rangle^479_{\text{courses}}(s) \text{ else } 479_{\text{courses}}(s)$$

end

(Blocked by : underscore in identifier – use camelCase for fuzz compatibility)

(d)

$$\boxed{\forall x: \text{Entry} \bullet \forall s: \text{seq Entry} \bullet \text{total}(\langle \rangle) = 0 \wedge \text{total}(\langle x \rangle \cap s) = x.5 + \text{total}(s)}$$

Solution 42

[Person]

axdef

$\text{State} : P(\text{seq}(\text{iseq}(\text{Person})))$

where

$\forall s : \text{State} \mid \forall i, j : \text{dom } s \bullet i \neq j \wedge \text{ran } s(i) \cap \text{ran } s(j) = \{\}$

end

(b)

axdef

$\text{add} : N * \text{Person} * \text{State} \rightsquigarrow \text{State}$

where

$\forall n : N \bullet \forall p : \text{Person} \bullet \forall s : \text{State} \bullet n \in \text{dom } s \wedge p \notin \bigcup \text{ran } s \mid$

$\text{add}(n, p, s) = s ++ n \mapsto s(n) \langle p \rangle$

end

(Blocked by: \rightsquigarrow operator not implemented)

Solution 43

(a)

(i) $\forall i : \text{dombookings} \mid \forall x, y : \text{bookings}(i) \bullet x \neq y \wedge x.2 .. x.3 \cap y.2 .. y.3 = \{\}$ (ii) $\forall i : \text{dombookings} \mid \forall x : \text{bookings}(i) \mid \{x.2, x.3\} \text{ subsequeq } 1..\max(i.1)$ (iii) $\forall i : \text{dombookings} \mid \forall b : \text{bookings}(i) \bullet b.2 \leq b.3$

(iv) This is enforced by the constraint for part (i).

(b)

(i) $\{i : \text{dom bookings} \mid i.1 = \text{Banbury} \bullet i.2\}$ (ii) $\{i : \text{dom bookings} \mid i.1 = \text{Banbury} \wedge \exists b : \text{bookings}(i) \bullet 50 \in b.2 .. b.3\}$ (iii) $r : \text{Room}; s : N \mid \exists i : \text{dom bookings} \bullet i.1 = r \wedge i.2 = s. (r, s)$ (iv) $r : \text{Room} \mid \exists i : \text{dom bookings} \bullet i.1 = r \wedge \#\text{bookings}(i) \geq 10$

Free types and induction

[N]

$\text{Tree} ::= \text{stalk} \mid \text{leaf} \langle\!\langle N \rangle\!\rangle \mid \text{branch} \langle\!\langle \text{Tree} \times \text{Tree} \rangle\!\rangle$

Solution 44

The two cases of the proof are established by equational reasoning : the first by

$$\text{reverse} (\langle\rangle^t) = \text{reverset}[\text{cat.1a}] = (\text{reverset})\langle\rangle [\text{cat.1b}]$$

where cat.1a is $\langle\rangle s = \text{s} \text{and} \text{cat.1b} s \langle\rangle = s$

and the second by

$$\text{reverse} ((\langle x \rangle^u)^t) = \text{reverse}(\langle x \rangle^{(u^t)}) [\text{cat.2}]$$

$$= \text{reverse} (u^t) \langle x \rangle [\text{reverse.2}]$$

$$= (\text{reverse } t^r \text{everse} u) \langle x \rangle [\text{anti-distributive}]$$

$$= \text{reverse } t^r (\text{reverse} u \langle x \rangle) [\text{cat.2}]$$

$$= \text{reverse } t^r \text{everse}(\langle x \rangle^u) [\text{reverse.2}]$$

Solution 45

The base case:

$$\text{reverse} (\text{reverse} \langle\rangle) = \text{reverse} \langle\rangle [\text{reverse.1}] = \langle\rangle [\text{reverse.1}]$$

The inductive step:

$$\begin{aligned}
& \text{reverse}(\text{reverse}(\langle x \rangle^t)) \\
&= \text{reverse}((\text{reverse } t) \langle x \rangle) [\text{reverse.2}] \\
&= \text{reverse}(\langle x \rangle)^r \text{everse}(\text{reverset}) [\text{anti-distributive}] \\
&= \text{reverse}(\langle x \rangle \langle \rangle)^r \text{everse}(\text{reverset}) [\text{cat.1}] \\
&= ((\text{reverse } \langle \rangle) \langle x \rangle)^r \text{everse}(\text{reverset}) [\text{reverse.2}] \\
&= (\langle \rangle \langle x \rangle)^r \text{everse}(\text{reverset}) [\text{reverse.1}] \\
&= \langle x \rangle^r \text{everse}(\text{reverset}) [\text{cat.1}] \\
&= \langle x \rangle^t [\text{reverse}(\text{reverset}) = t]
\end{aligned}$$

Solution 46

(a)

$$\text{count} : \text{Tree} \rightarrow N$$

$$\text{count stalk} = 0$$

$$\forall n : N \bullet \text{count}(\text{leaf}(n)) = 1$$

$$\forall t1, t2 : \text{Tree} \bullet \text{count}(\text{branch}(t1, t2)) = \text{count}(t1) + \text{count}(t2)$$

(Blocked by : recursive free types and pattern matching)

(b)

$$\text{flatten} : \text{Tree} \rightarrow \text{seq}N$$

$$\text{flatten stalk} = \langle \rangle$$

$$\forall n : N \bullet flatten(leaf(n)) = \langle n \rangle$$

$$\forall t_1, t_2 : Tree \bullet flatten(branch(t_1, t_2)) = flatten(t_1^{flatten})(t_2)$$

(Blocked by : recursive free types and pattern matching)

Solution 47

First, exhibit the induction principle for the free type:

$$P \text{ stalk and } (\forall n : N \bullet \mathbb{P} leaf(n)) \text{ and } (\forall t_1, t_2 : Tree \bullet \mathbb{P} t_1 \wedge \mathbb{P} t_2 \Rightarrow \mathbb{P} branch(t_1, t_2))$$

$$\text{implies } \forall t : Tree \bullet \mathbb{P} t$$

This gives three cases for the proof:

$$(flatten \text{ stalk}) = \langle \rangle \text{ [flatten]} = 0 \text{ [] = count stalk [count]}$$

(Remaining cases omitted - require equational reasoning with recursive functions)

Supplementary material : assignment practice

Solution 48

$$[SongId, UserId, PlaylistId, Playlist]$$

$$\frac{\begin{array}{c} songs : \mathbb{F} \ SongId \\ users : \mathbb{F} \ UserId \\ playlists : PlaylistId \rightarrow Playlist \\ playlistOwner : PlaylistId \rightarrow UserId \\ \forall i : \text{dom } playlists \bullet \text{ran } playlists(i)(\text{subseteq})(songs) \text{ dom } playlistOwner(\text{subseteq})(\text{dom } playlists) \text{ ran } playlistOwner(i)(\text{subseteq})(users) \end{array}}{\forall i : \text{dom } playlists \bullet \text{ran } playlists(i)(\text{subseteq})(songs) \text{ dom } playlistOwner(\text{subseteq})(\text{dom } playlists) \text{ ran } playlistOwner(i)(\text{subseteq})(users)}$$

Solution 49

$$\frac{\begin{array}{c} hated : UserId \rightarrow \mathbb{F} \ SongId \\ loved : UserId \rightarrow \mathbb{F} \ SongId \\ \forall i : \text{dom } hated \bullet \text{hated}(i)(\text{subseteq})(users) \end{array}}{\text{dom } hated(\text{subseteq})(users) \forall i : \text{dom } hated \bullet \text{hated}(i)(\text{subseteq})(songs) \text{ dom } loved(\text{subseteq})(users) \forall i : \text{dom } loved \bullet \text{loved}(i)(\text{subseteq})(songs) }$$

Solution 50

(a)

$$A == \text{users} \setminus \text{ran } \text{playlistSubscribers}$$

(b)

$$B == \{ p : \text{dom } \text{playlistSubscribers} \mid \#\text{playlistSubscribers}(p) \geq 100 \}$$

(c)

$$C == \mu u : \text{dom } \text{loved} \bullet \forall v : \text{dom } \text{loved} \bullet u \neq v \wedge \#\text{loved}(u) > \#\text{loved}(v)$$

(d)

$$D == \mu s : \text{songs} \bullet \forall t : \text{songs} \bullet s \neq t \wedge \#\{u : \text{UserId} \mid s \in \text{loved}(u)\} > \#\{u : \text{UserId} \mid t \in \text{loved}(u)\}$$

Solution 51

(a)

Let's first define two helper functions:

$$\text{loveHateScore} : \text{SongId}^+ \rightarrow N$$

$$\forall i : \text{songs} \mid \{u : \text{UserId} \mid i \in \text{loved}(u)\} \geq \{u : \text{UserId} \mid i \in \text{hated}(u)\} \Rightarrow$$

$$\text{loveHateScore}(i) = \{u : \text{UserId} \mid i \in \text{loved}(u)\} - \{u : \text{UserId} \mid i \in \text{hated}(u)\}$$

and

$$\forall i : \text{songs} \mid \{u : \text{UserId} \mid i \in \text{loved}(u)\} < \{u : \text{UserId} \mid i \in \text{hated}(u)\} \Rightarrow$$

$$\text{loveHateScore}(i) = 0$$

$$\frac{\text{playlistCount} : \text{SongId} \rightarrow N}{\forall i : \text{songs} \bullet \text{playlistCount}(i) = \#\{p : \text{dom } \text{playlist} \mid i \in \text{ran } \text{playlist}(p)\}}$$

We then have:

$$\frac{\text{length : } SongId \rightarrow N \text{ popularity : } SongId \rightarrow N}{\text{dom length}(\text{subseq})(\text{songs}) \text{ dom popularity}(\text{subseq})(\text{songs}) \forall i : \text{songs} \bullet \text{popularity}(i) = \text{loveHateScore}(i) + \dots}$$

(b)

mostPopular : SongId

$$(\exists_1 i : \text{songs} | \forall j : \text{songs} \bullet i \neq j \wedge \text{popularity}(i) > \text{popularity}(j)) \Rightarrow$$

$$\text{mostPopular} = (\mu i : \text{songs} | \forall j : \text{songs} \bullet i \neq j \wedge \text{popularity}(i) > \text{popularity}(j))$$

and

$$\neg \exists_1 i : \text{songs} \bullet \forall j : \text{songs} \bullet i \neq j \wedge \text{popularity}(i) > \text{popularity}(j) \Rightarrow \text{mostPopular} = \text{nullSong}$$

$$(c) \text{playlistsContainingMostPopularSong} == \{i : \text{dom playlists} | \text{mostPopular} \in \text{ran playlists}(i)\}$$

Solution 52

(a)

premiumPlays : seq(Play) → seq(Play)

$$\text{premiumPlays}(\langle \rangle) = \langle \rangle$$

$$\forall x : \text{Play}; s : \text{seq(Play)} |$$

$$\text{premiumPlays}(\langle x \rangle^s) = \langle x \rangle^{\text{remiumPlays}(s)} \text{ if userStatus}(x.2) = \text{premium}$$

$$\text{premiumPlays}(s) \text{ if userStatus}(x.2) = \text{standard}$$

(Note: Uses camelCase for fuzz compatibility)

(b)

standardPlays : seq(Play) → seq(Play)

$$\text{standardPlays}(\langle \rangle) = \langle \rangle$$

$\forall x : Play; s : seq(Play) \mid$

$\text{standardPlays}(\langle x \rangle^s) = \langle x \rangle^s \text{ if } \text{userStatus}(x.2) = \text{standard}$

$\text{standardPlays}(s) \text{ if } \text{userStatus}(x.2) = \text{premium}$

(Note: Uses camelCase for fuzz compatibility)

(c)

$\text{cumulativeLength} : seq(Play) \rightarrow N$

$\text{cumulativeLength}(\langle \rangle) = 0$

$\forall x : Play; s : seq(Play) \mid$

$\text{cumulativeLength}(\langle x \rangle^s) = \text{length}(x.1) + \text{cumulativeLength}(s)$

(Note: Uses camelCase for fuzz compatibility)