Physics 562 - Computational Physics

Assignment 3: Chaotic Double Pendulum

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Abstract

This paper examine emergent chaos by analyzing a double pendulum. A double pendulum is a chaotic system that is nonlinear, deterministic, and highly sensitive to initial conditions. There are three states for the double pendulum to be in: no chaos, emerging chaos, and full chaos. These states correspond to a Lyapunov exponent that is negative, zero, and positive, respectively

1 Single Pendulum

A single pendulum is a very predictable system that undergoes simple harmonic motion. The kinetic energy of a pendulum is given by

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta})^2 \tag{1}$$

and the potential energy is given by

$$V = mgh = mgl(1 - cos(\theta)) \tag{2}$$

where l is the length of the pendulum, and m is the mass of the bob. The lagrangian is given by

$$\mathcal{L} = T - V = \frac{1}{2}m(l\dot{\theta})^2 - mgl(1 - cos(\theta)). \tag{3}$$

The euler-lagrange equations can be applied to the lagrangian in order to get the equations of motions, but instead we will use the hamiltonian. The hamiltonian is given by

$$\mathcal{H} = \Sigma p_{q_i}\dot{q}_i - \mathcal{L} = p_\theta\dot{\theta} - \frac{1}{2}m(l\dot{\theta})^2 + mgl(1 - cos(\theta)). \tag{4}$$

Furthurmore we know that

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \theta} = ml^2\dot{\theta} \tag{5}$$

which can be rewritten

$$\dot{\theta} = \frac{p_\theta}{ml^2} \tag{6}$$

so we can substitute into the hamiltonian to get it of the form

$$\mathcal{H} = \frac{p_\theta^2}{ml^2} - \frac{1}{2}ml^2(\frac{p_\theta}{ml^2})^2 + mgl(1 - cos(\theta)). \tag{7}$$

$$\mathcal{H} = \frac{p_\theta^2}{ml^2} - \frac{1}{2}\frac{p_\theta^2}{ml^2} + mgl(1 - cos(\theta)). \tag{8}$$

$$\mathcal{H} = \frac{1}{2}\frac{p_\theta^2}{ml^2} + mgl(1 - cos(\theta)). \tag{9}$$

We can then solve the hamiltonian equations of motion which are

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{ml^2} \tag{10}$$

$$\dot{p_\theta} = -\frac{\partial \mathcal{H}}{\partial \theta} = -mglsin\theta \tag{11}$$

2 Double Pendulum

The double pendulum is more complex. The equations of motion for the double pendulum is given by

$$\dot{\theta}_1 = \frac{\partial \mathcal{H}}{\partial p_1} = \frac{lp_1 - lp_2 cos(\theta_1 - \theta_2)}{l^3[m_1 + m_2 sin^2(\theta_1 - \theta_2)]} \tag{12}$$

$$\dot{\theta}_2 = \frac{\partial \mathcal{H}}{\partial p_2} = \frac{l(m_1 + m_2)p_2 - lm_2 p_1 cos(\theta_1 - \theta_2)}{l^3 m_2[m_1 + m_2 sin^2(\theta_1 - \theta_2)]} \tag{13}$$

$$\dot{p_1} = -\frac{\partial \mathcal{H}}{\partial \theta_1} = -(m_1 + m_2)glsin(\theta_1) - C_1 + C_2 \tag{14}$$

$$\dot{p_2} = -\frac{\partial \mathcal{H}}{\partial \theta_2} = -m_2 glsin(\theta_2) + C_1 - C_2 \tag{15}$$

where

$$C_1 \equiv \frac{p_1 p_2 sin(\theta_1 - \theta_2)}{l^2[m_1 + m_2 sin^2(\theta_1 - \theta_2)]} \tag{16}$$

$$C_2 \equiv \frac{l^2 m_2 p_1^2 + l^2(m_1 + m_2)p_2^2 - l^2 m_2 p_1 p_2 cos(\theta_1 - \theta_2)}{2l^4[m_1 + m_2 sin^2(\theta_1 - \theta_2)]^2} sin[2(\theta_1 - \theta_2)] \tag{17}$$

In order to understand how the double pendulum behaves, the momentum will be plotted against the angle for each pendulum. In addition, the lyapunov exponent will be calculated for three different cases. These cases are

Listing 1: Module `Cases`

```

    !!Low Energy - No Chaos
    y(1)=1._dp*(pi/180) ! initial angle for 1st pendulum
    y(2)=0._dp          ! initial momentum for 1st pendulum
    y(3)=1._dp*(pi/180) ! initial angle for 2nd pendulum
    y(4)=0._dp          ! initial momentum for 2nd pendulum

    !!Medium Energy - Emerging Chaos But Stable Orbit
    y(1)=1._dp*(pi/180) ! initial angle for 1st pendulum
    y(2)=1._dp          ! initial momentum for 1st pendulum
    y(3)=1._dp*(pi/180) ! initial angle for 2nd pendulum
    y(4)=1.01_dp        ! initial momentum for 2nd pendulum
```

```

    !!High Energy - Full Chaos
    y(1)=1._dp*(pi/180) ! initial angle for 1st pendulum
    y(2)=1._dp         ! initial momentum for 1st pendulum
    y(3)=1._dp*(pi/180) ! initial angle for 2nd pendulum
    y(4)=1.3_dp        ! initial momentum for 2nd pendulum
```

The mass for the first bob is 1 kg and the mass of the second bob is 2 kg. The length of both pendulums is 1 meter. The lyapunov exponents are given by

$$|p_1 - p_2| = e^{\lambda \cdot t} \tag{18}$$

where P_1 and p_2 are the momenta for the two pendulums, and λ is the lyapunov exponent.

3 The Fortran95 code

The code solves the equation of motion using the Runga-Kutta method. First a module called `NumType` is created to store all my global parameters.

Listing 2: Module `NumType`

```

module NumType

    save
    integer, parameter :: dp = kind(1.d0)
    real(dp), parameter :: pi = 4*atan(1._dp), &
    e = exp(1._dp)
    complex(dp), parameter :: iic = (0._dp,1._dp)

end module NumType
```

Listing 3: `rkf45.f95`

```

subroutine rkf45step(t,y,h)  !4-th order Runge-Kutta step

    use setup, only : dp, n_eq
    implicit none
```

```
   real(dp), intent(inout) :: t, h
   real(dp), dimension(n_eq), intent(inout) :: y
   real(dp), dimension(n_eq) :: k1, k2, k3, &
                           k4, k5, k6, y1, y2
   real(dp), parameter :: epsilon = 1.e-6_dp, &
                     tiny = 1.e-20_dp
   real(dp) :: rr, delta

   call deriv(t,        h,  y,     k1)
   call deriv(t+h/4,    h,  y+ k1/4,    k2)
   call deriv(t+3*h/8,  h,  y+ (3*k1+9*k2)/32,  k3)
   call deriv(t+12*h/13,h, y+ (1932*k1-7200*k2+7296*k3) &
                  2197 , k4)
   call deriv(t+h,h,  y+ (439*k1/216-8*k2+3680*k3/513 &
                  -845*k4/4104),    k5)
   call deriv(t+h/2,    h,  y+ (-8*k1/27 +2*k2-3544*k3/2565 + &
                  1859*k4/4104 -11*k5/40),    k6)

   y1 = y + 25*k1/216 + 1408*k3/2565 + 2197*k4/4104 &
    - k5/5
   y2 = y + 16*k1/135 + 6656*k3/12825 + 28561*k4/ &
   56430 - 9*k5/50 + 2*k6/55

  rr = sqrt(dot_product(y1-y2,y1-y2))/h + tiny

   if ( rr < epsilon ) then
       t = t + h
       y = y1
       delta = 0.92_dp * (epsilon/rr)**(0.2_dp)
       h = delta*h
       write (unit = 3,fmt='(3f20.10)') t, y(1)
       write (unit = 4,fmt='(3f20.10)') t, y(2)
       write (unit = 5,fmt='(3f20.10)') t, y(3)
       write (unit = 6,fmt='(3f20.10)') t, y(4)
       write (unit = 7,fmt='(3f20.10)') y(1), y(2)
       write (unit = 8,fmt='(3f20.10)') y(3), y(4)
       write (unit = 9,fmt='(3f20.10)') -sin(y(1)),-cos(y(1))
       write (unit = 10,fmt='(3f20.10)') -sin(y(1))-sin(y(3)) &
       ,-cos(y(1))-cos(y(3))
   else
```

```fortran
        delta = 0.92_dp * (epsilon/rr)**(0.25_dp)
        h = delta*h
    end if


    contains

        subroutine deriv(t,h,y,k)   ! derivative

            use setup, only : dp, n_eq, g, length, mass1, mass2
            implicit none
            real(dp), intent(in) :: t, h
            real(dp), dimension(n_eq), intent(in) :: y
            real(dp), dimension(n_eq) :: f, k
            real(dp) :: c1, c2

            c1 = (y(2)*y(4)*sin(y(1)-y(3)))/ &
                (length*length*(mass1+mass2*(sin(y(1)-y(3)))**2))
            c2 = (length**2*mass2*y(2)**2 + &
                length**2*(mass1+mass2)*y(4)**2- &
                length*length*mass2*y(2)*y(4)*cos(y(1)-y(3)))/ &
                (2*length**4*(mass1+mass2*(sin(y(1)-y(3)))**2)**2)* &
                sin(2*(y(1)-y(3)))

            f(1) = (length*y(2) - length*y(4)*cos(y(1)-y(2)))/ &
                    (length**3*(mass1+mass2*(sin(y(1)-y(2)))**2))
            f(2) = -(mass1+mass2)*g*length*sin(y(1)) - c1 + c2
            f(3) = (length*(mass1+mass2)*y(4) - length*&
                    mass2*y(2)*cos(y(1)-y(2)))/ &
                    (length**3*mass2*(mass1+mass2*(sin(y(1)-y(2)))**2))
            f(4) = -mass2*g*length*sin(y(3)) + c1 - c2

            k(1:n_eq) = h*f(1:n_eq)

        end subroutine deriv
```

The main program is `adpend` and it begins with its own module.

Listing 4: `adpend.f95`

```fortran


```

```fortran
module setup

    use NumType
    implicit none
    integer, parameter :: n_eq = 4
    real(dp), parameter:: g = 10.0_dp, length = 1.0_dp, &
     mass1 = 1.0_dp, mass2 = 2.0_dp
    real(dp) :: t, tmax, dt, lambda
    real(dp), dimension(n_eq) :: y

end module setup

program pendulum

    use setup
    implicit none

    !!initial conditions!!

    t = 0._dp                       ! time to start
    tmax = 100._dp                  ! time to exit
    dt = 0.001_dp                   ! time step
    lambda = 0._dp                  !intiate a value for lyapanov exponent


    !!Below are four cases - Make sure only one set of y is uncommented!!

    !!Low Energy - No Chaos
    y(1) = 1._dp*(pi/180)           ! initial angle 1
    y(2) = 0._dp                    ! initial momentum 1
    y(3) = 1._dp*(pi/180)           ! initial angle 2
    y(4) = 0._dp                    ! initial momentum 2

    !!Medium Energy - Emerging Chaos But Stable Orbit
!   y(1) = 1._dp*(pi/180)           ! initial angle 1
!   y(2) = 1._dp                    ! initial momentum 1
!   y(3) = 1._dp*(pi/180)           ! initial angle 2
!   y(4) = 1.01_dp                  ! initial momentum 2

    !!High Energy - Full Chaos
```

```
!   y(1) = 1._dp*(pi/180)        ! initial angle 1
!   y(2) = 1._dp                 ! initial momentum 1
!   y(3) = 1._dp*(pi/180)        ! initial angle 2
!   y(4) = 1.3_dp                ! initial momentum 2

    !!High Energy - Full Chaos
!   y(1) = 60._dp*(pi/180)       ! initial angle 1
!   y(2) = 1._dp                 ! initial momentum 1
!   y(3) = 60._dp*(pi/180)       ! initial angle 2
!   y(4) = 2._dp                 ! initial momentum 2


    !!open all the files that data will be written to!!

    open(unit = 3, file = 'pend_angle1.data', &
        action = 'write', status = 'replace')
    open(unit = 4, file = 'pend_mom1.data', &
        action = 'write', status = 'replace')
    open(unit = 5, file = 'pend_angle2.data', &
        action = 'write', status = 'replace')
    open(unit = 6, file = 'pend_mom2.data', &
        action = 'write', status = 'replace')
    open(unit = 7, file = 'angle_mom1.data', &
        action = 'write', status = 'replace')
    open(unit = 8, file = 'angle_mom2.data', &
        action = 'write', status = 'replace')
    open(unit = 9, file = 'pend_xy1.data', &
        action = 'write', status = 'replace')
    open(unit = 10, file = 'pend_xy2.data', &
        action = 'write', status = 'replace')
    open(unit = 11, file = 'lambda.data', &
        action = 'write', status = 'replace')

    !!calculate the momenta and angle of the pendulums!!

    do while ( t < tmax )
        if ( t + dt > tmax) dt = tmax -t
        call rkf45step(t,y,dt)
        !use the runga kutta method to calculate for theta and momentum
        lambda = lambda + log(abs(y(4)-y(2)))/t
```

```
        !update the summation of the lyapanov exponent
        write (unit = 11,fmt='(3f20.10)') lambda
        !write lamba to file
    end do

end program pendulum
```

The code is run by typing `./pend`. Various data sets are plotted to different files for easy graphing.

4 Results

A single pendulum is very ordered and predictable.

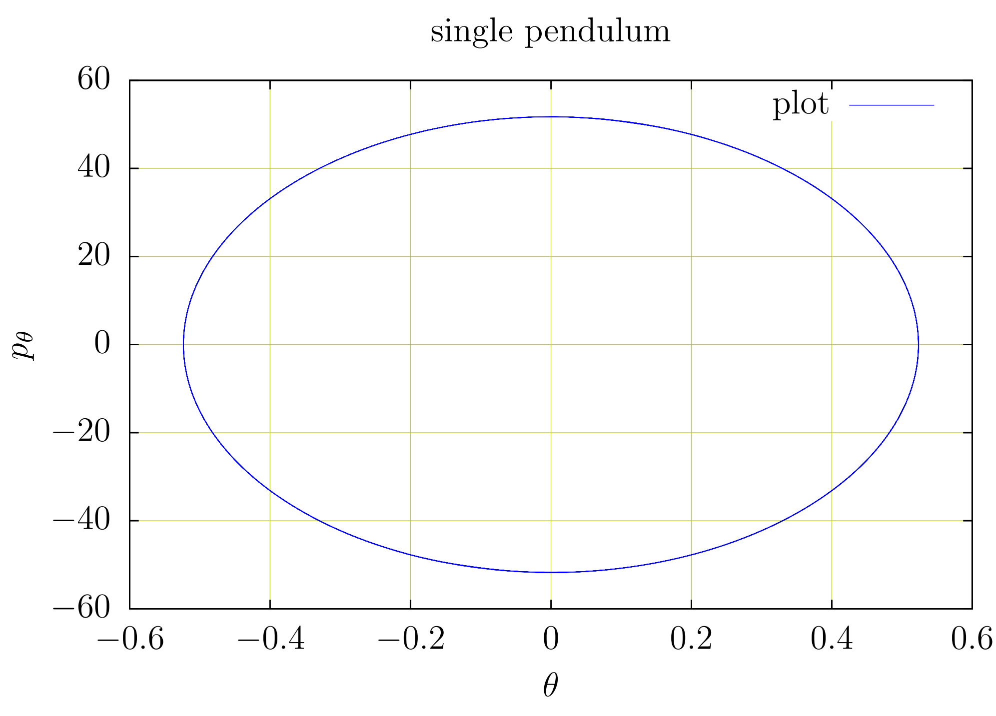


Figure 1: A simple pendulum.

Figure 1 shows that you can predict the values of the momentum for any time t. The momentum is maximum when the pendulum is at the bottom of the arc and zero at the ends of the swing. Because of this, there is no chaos in the pendulums movement.

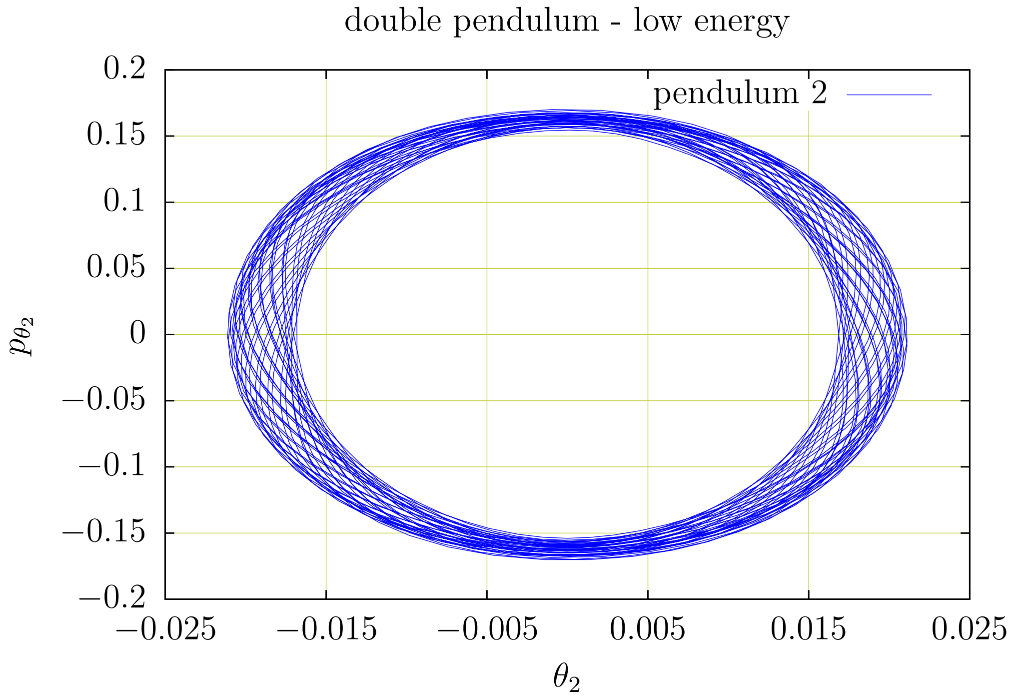


Figure 2: Plot for the second bob.

Figure 2 is similar to the single pendulum. The orbit has slight variation but it is a stable orbit. The calculated lyapanov exponent is a negative number, which confirms that the orbit is closed.

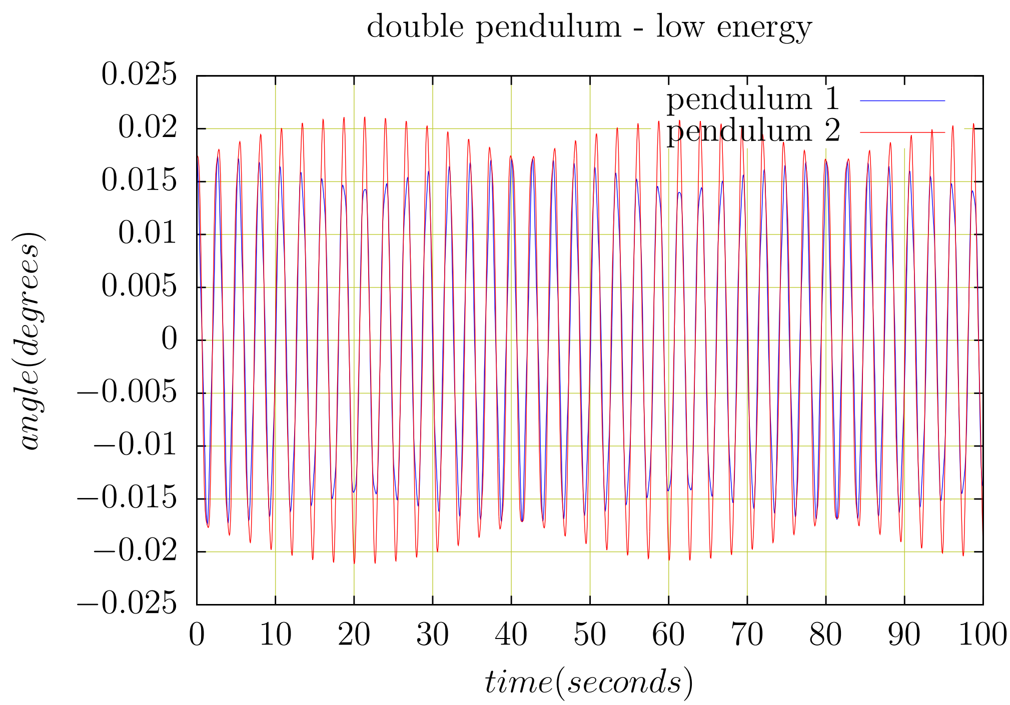


Figure 3: The angle of the two pendulums plotted over time

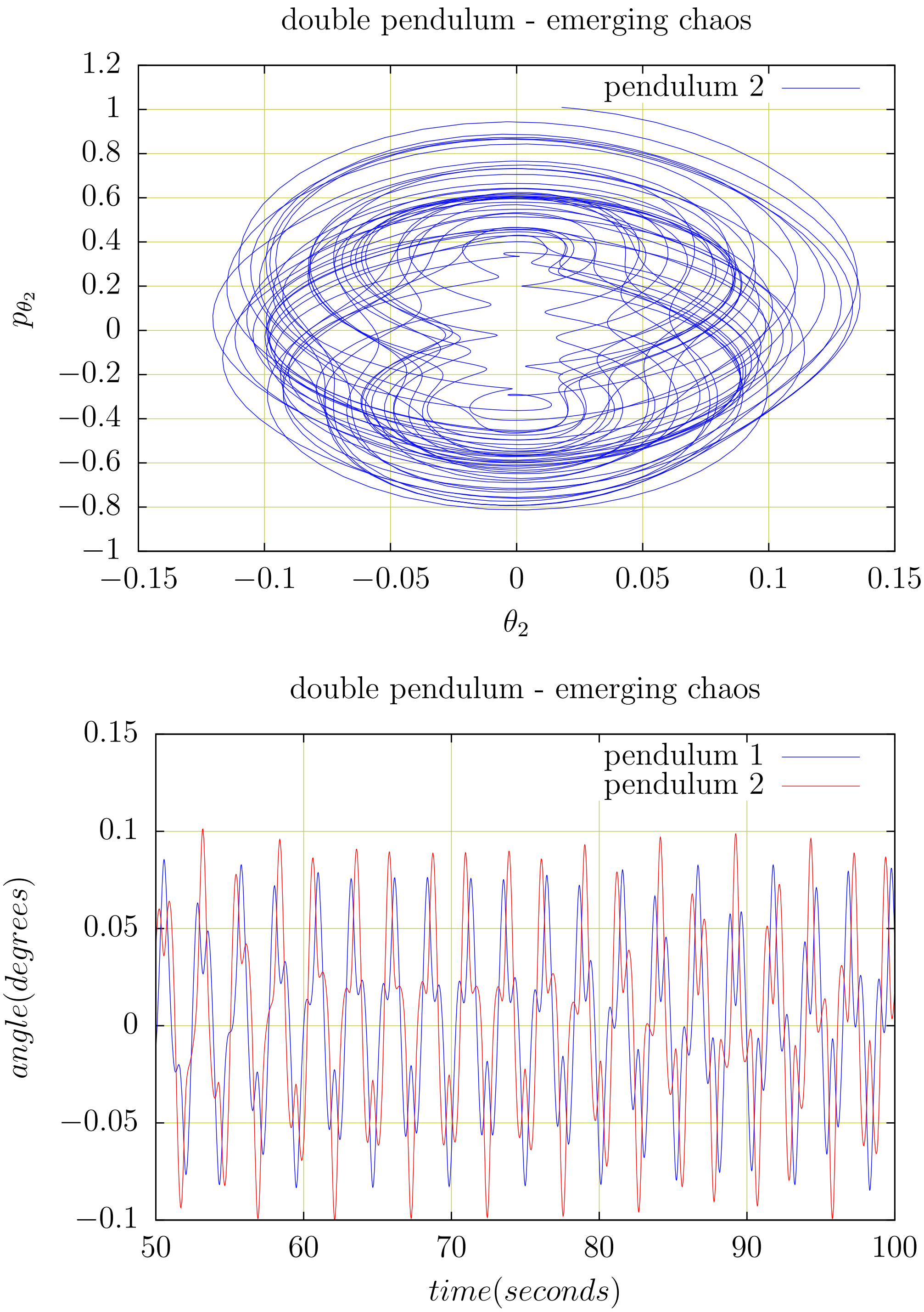
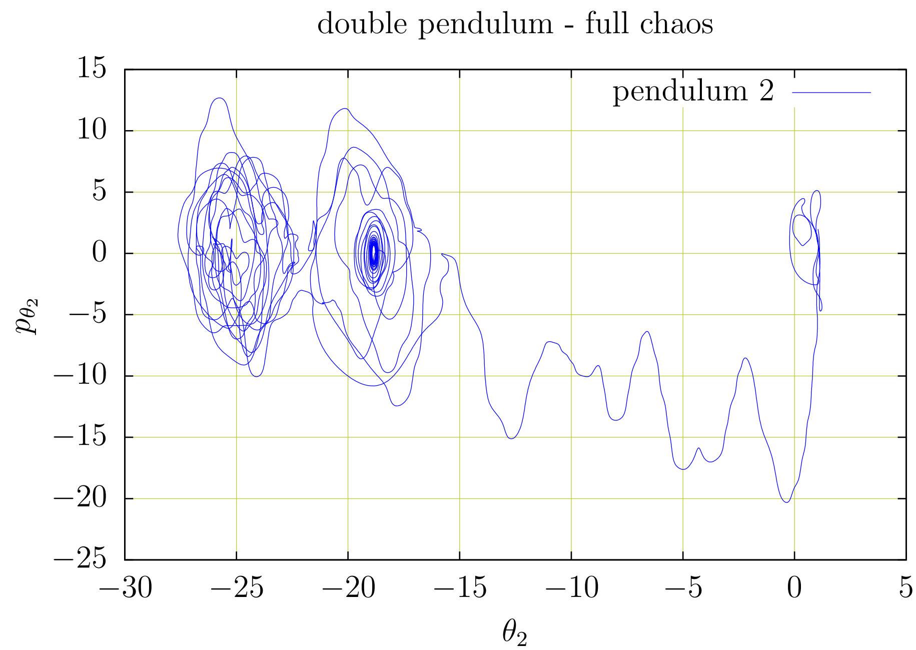


Figure 4: The second bob begins to show chaotic motion, but is in a stable orbit



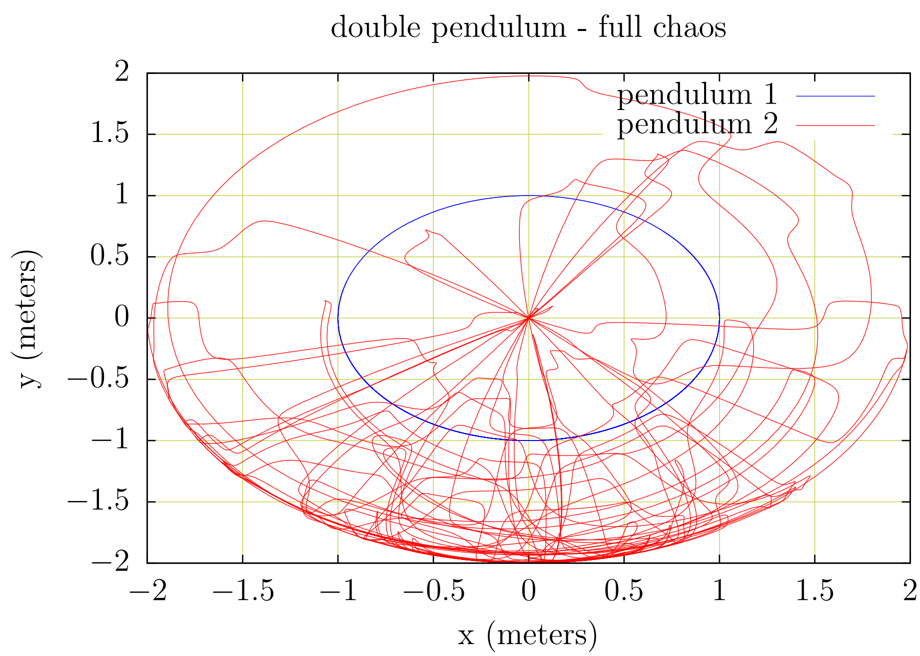


Figure 5: The second bob does not have a stable orbit. A graph of the x and y positions of the bobs are plotted so the chaotic motion can be visualized.

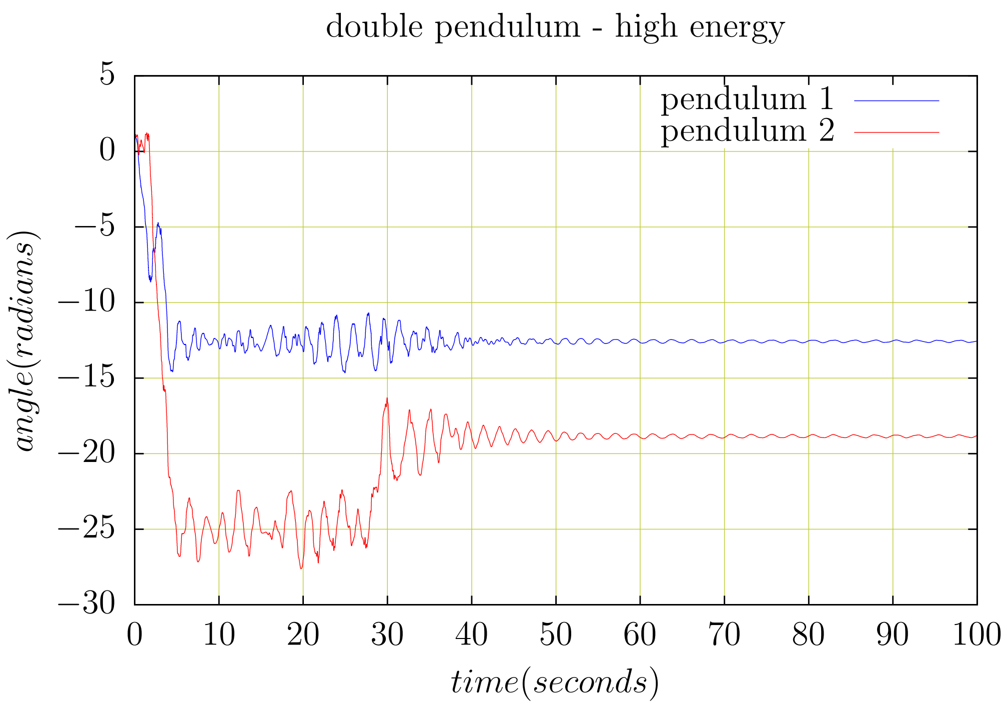


Figure 6: The chaotic behavior appears to die out after 40 seconds.

5 Summary and conclusions

A single pendulum is a non-chaotic, predictable system. A double pendulum is an inherently chaotic system, where the chaotic nature becomes more apparent as the difference between the initial conditions gets larger.

References

[1] M. Metcalf, J. Reid and M. Cohen, *Fortran 95/2003 explained*. Oxford University Press, 2004.