Brown University

Department of Economics

Senior Honors Thesis

Donating Strategically to Multiple Destinations

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Contents

1 Motivation

One framework for evaluating charitable causes that has gained popularity in recent years combines importance, tractability, and level of neglect [1, 7]. While high importance and solvability provide clear individual donation strategies, the goal of diverting resources from popular causes to other neglected ones leads to a coordination challenge, and is therefore more complicated from a game theory perspective. This paper develops the Multi-Destination Donations Game (MDG), in which players gain personal marginal utility by distributing wealth between a set of destinations, as a general model for analyzing the game-theoretic coordination properties of charitable donating. I then examine the Nash Equilibria and Social Welfare Maximizing (SWM) solutions of the model in two different environments.

In the first environment, the agents' marginal utility functions can be non-monotonic. I refer to this case as the General Environment (GE). While this environment can capture essentially all possibilities, very little can be said about the MDG under the GE. There is no guarantee of a pure-strategy Nash Equilibrium, and when a Nash Equilibrium is present, it is not necessarily unique or Pareto Optimal. Additionally, there is no guarantee of a unique SWM solution, and finding any SWM solution is NP-hard.

The literature on Congestion Games provides some helpful insight for how to generate cleaner results. Congestion Games have a finite number of players, each with a finite set of strategies. Each strategy consists of choosing a subset of a total set of resources; the utility derived from each resource is inversely proportional to the number of players that choose that resource [4]. To capture this "congestion criterion," I examine a second environment in which agents' marginal utility functions must be strictly decreasing. I refer to this case as the Decreasing Environment (DE). This environment eliminates scenarios where agents may have an incentive to start funding a destination only if others fund it as well due to internal increasing returns to funding for that destination. While this assumption will not adequately model all charity-related situations at all times, such as one in which a new charity requires overhead start-up costs and thus will have higher marginal returns from future donations, it does a good job at describing situations where donations to different charities complement each other. For example, if a society donates to both a global health charity and an economic development charity, it might gain more utility than it would have yielded by donating excessively to one or the other, since health and economic development support each other. This model can even be extended beyond charities, to any example where resources of any kind can be split between multiple causes that complement each other. For example, public investing in training doctors to prescribe medication and in training nurses to administer medication go hand in hand; investing excessively in one or the other will reach a point of very low returns to funding, whereas investing more equally between the two will yield higher returns to funding.

Milchtaich shows that Congestion Games always have a pure-strategy Nash Equilibrium [4], and indeed, enforcing a congestion criterion with the DE guarantees the existence of a pure-strategy Nash Equilibrium in the MDG. The congestion criterion also guarantees that the final wealth distribution across destinations will be the same for all Nash Equilibria. This result is significant from a technical standpoint because finding any one Nash Equilibrium reveals the final wealth distribution and thus the final utility for each player that would arise in any other Nash Equilibrium. This result is also significant from an intuitive standpoint because it implies that one agent in a congested environment cannot help shift the society to a different, potentially better total wealth distribution. Since the Nash Equilibria for the MDG in a DE are still not guaranteed to be Pareto Optimal, I also analyze the SWM solutions in the DE. Meyers and Schulz show that the most general congestion game problems are still NP-hard [3], and thus solving for the role of coordination in societies with thousands or millions of agents becomes impractical. However, in addition to showing that the DE leads to a unique SWM, this paper breaks new ground by applying the concept of congestion in a way that allows this SWM solution to be calculated in polynomial time. This makes the MDG in a DE a practical tool for affecting philanthropic decisions and public policy on a large scale.

Section 2 presents the model and game, which consists of a society of agents, independent wealth values for each agent, and destinations where the agents must donate their wealth. The agents also have marginal utility functions, which are based on the total wealth given to each destination by all agents combined. Section 3 examines the properties of the Nash Equilibria in the model, and Section 4 examines the properties of the SWM solutions for the model and presents the computational complexity results. Throughout the paper, results will be presented in parallel for the General Environment and the Decreasing Environment. Finally, Section 5 summarizes the results and recommend directions for further research.

2 The Model

2.1 The Environment

Definition 1 (General Environment)**.** *A General Environment (GE) contains a set of agents, a wealth for each agent, a set of destinations, and a marginal utility curve for each agent and destination pair. Formally, a GE ξ is a tuple (N, W, D, F), where*

- $N = \{1, 2, ..., n\}$ *is the set of agents;*
- $W = \{w_1, w_2, ..., w_n\}$ *is the vector of each agent's wealth;*
- $D = \{1, 2, ..., d\}$ *is the set of destinations;*
- $F = \{f_1, f_2, ..., f_n\}$ *is the vector of each agent's marginal utility curves;*
  - ∗ $f_i = \{f_{i1}, f_{i2}, ..., f_{id}\}$ *is agent i's vector of marginal utility curves for destinations 1 through d;*
  - ∗ $f_{ih} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *maps the total donations to destination h to a marginal utility for agent i, and must be continuous.*

Definition 2 (Decreasing Environment)**.** *A Decreasing Environment (DE) ξ' is the same as a GE ξ, with the added conditions that $\frac{\partial f_{ih}}{\partial x} < 0 \ \forall\ i \in N,\ h \in D$, and that all marginal utility curves must be invertible in polynomial time.*

2.2 Multi-Destination Donations Game

Definition 3 (Multi-Destination Donations Game)**.** *A Multi-Destination Donations Game (MDG) is a game played in either a GE or DE in which agents distribute all of their wealth between the destinations. Formally, an MDG ϕ is a tuple (ξ^*, A, U), where*

- $\xi^* = \{N, W, D, F\}$ *is either a GE ξ or a DE ξ';*
- $A = \{A_1, A_2, ..., A_n\}$ *is the vector of each agent's set of strategies;*
  - ∗ A_i *is agent i's set of strategies, where*

$$A_i = \{a_i = (a_{i1}, ..., a_{id}) \in \mathbb{R}^d_+ \mid \sum_{h \in D} a_{ih} = w_i\} \ \forall\ i \in N$$

The vector a_i thus dictates how agent i distributes their wealth among the d destinations. Agent i's strategy set is the set of all feasible distributions, or the set of all d-vectors where the sum of all the components equals that agent's wealth;

- $U(F, A) = \{u_1(f_1, A), u_2(f_2, A), ..., u_n(f_n, A)\}$ *is the vector of each agent's utility functions, which depend on that agent's marginal utility curves and the donations of all agents in the society. Formally,*

$$u_i(f_i, A) = \sum_{h=1}^{d} \int_0^{\delta_h} f_{ih}(x)dx$$

where $\delta_h = \sum_{i \in N} a_{ih}$. *In other words, agent i's utility from each destination is the integral of that agent's marginal utility curve for that destination from 0 to the total amount of wealth donated to that destination; i's total utility is the sum of the utility they gain from each destination.*

2.3 Additional Terminology

Definition 4 (Total Wealth Distribution)**.** *A total wealth distribution is a d-vector where the sum of all the components equals the total wealth of all the agents. Define WD_{ξ^*} to be the set of all total wealth distributions in a given environment $\xi^* = (N, W, D, F)$:*

$$WD_{\xi^*} = \{\delta = (\delta_1, ..., \delta_d) \in \mathbb{R}_+^d \mid \sum_{h \in D} \delta_h = \sum_{i \in N} w_i\}$$

Note that a total wealth distribution in environment ξ^* can equivalently be defined as the summation of individual actions from an MDG $\phi = (\xi^*, A, U)$ played in that environment:

$$WD_{\xi^*} = \{\sum_{i \in N} a_i \mid a_i \in A_i\}$$

3 Results: Nash Equlibria

3.1 Existence

Claim 5. *A pure-strategy Nash Equilibrium is not guaranteed for an MDG played in a GE.*

Proof. Consider the MDG $\phi = (\xi, A, U)$, where ξ is a GE with two destinations, two agents with one unit of wealth each, and marginal utility curves illustrated by Figure 1. For agent 1, both marginal utility curves are the same and decreasing. For agent 2, both marginal utility curves are the same and increasing. Agent 1 wants wealth to be distributed evenly between the two destinations, and therefore, any final wealth distribution $\delta \in WD_\xi$ except $(1, 1)$ will not be a Nash Equilibrium. However, if $\delta = (1, 1)$, agent 2 will not be best responding, so $(1, 1)$ must also not be a Nash Equilibrium. Therefore, this example has no pure-strategy Nash Equilibrium. □

Now we turn to MDGs played in a Decreasing Environment.

Theorem 6. *A pure-strategy Nash Equilibrium is guaranteed for an MDG $\phi = (\xi', A, U)$, where ξ' is a DE with at least one agent and one destination.*

Proof. We can satisfy the conditions for Glicksberg's existence theorem [5]:

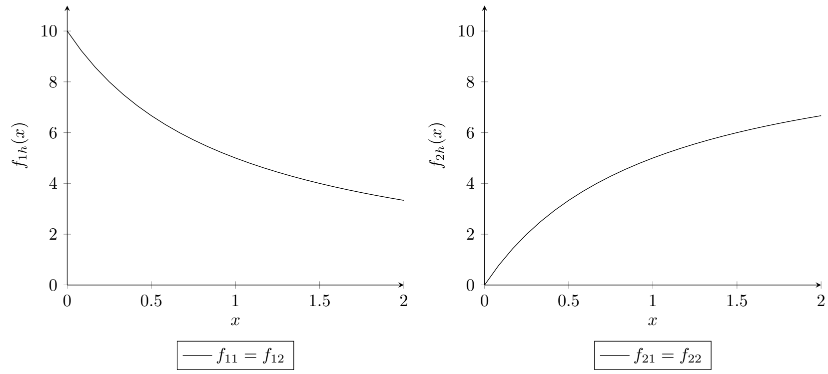

Figure 1

1. A_i is a simplex, which means it is both convex and compact [6].
2. $u_i(f_i, a_i, a_{-i})$ is continuous in a_i and a_{-i} since each marginal utility curve is continuous.
3. $u_i(f_i, a_i, a_{-i})$ is concave in a_i since the integral of a decreasing function is concave and the sum of concave functions is still concave.

Therefore, by Glicksberg's existence theorem, ϕ has a pure-strategy Nash Equilibrium. □

3.2 Uniqueness

Claim 7. *A unique Nash Equilibrium is not guaranteed for an MDG played in a DE or GE.*

Proof. Consider the MDG $\phi = (\xi', A, U)$, where ξ' is a DE with two destinations, two agents with one unit of wealth each, and marginal utility curves illustrated by Figure 2. Both agents have identical and decreasing marginal utility curves for both destinations, meaning that they both want wealth to be distributed evenly between the two destinations. Therefore, any combination of individual donations that produce the total wealth distribution of $(1, 1)$ will be a Nash Equilibrium. Therefore, a unique Nash Equilibrium is not guaranteed in a DE, and by extension, a unique Nash Equilibrium is not guaranteed in a GE. □

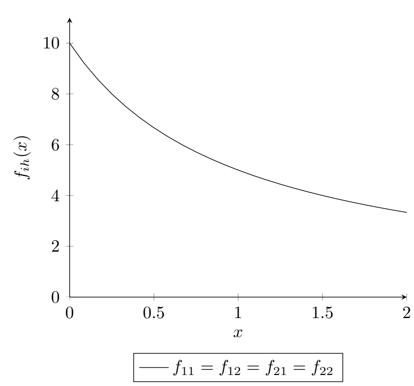

Figure 2

3.3 Equality of Total Wealth Distributions

Claim 8. *An MDG played in a GE can have Nash Equilibria with different total wealth distributions.*

Proof. Consider the MDG $\phi = (\xi, A, U)$, where ξ is a GE with two destinations, two agents with one unit of wealth each, and marginal utility curves illustrated by Figure 3. Since all the marginal utility curves are the same and increasing, both agents want all the wealth to be distributed to one destination or the other. This means that both $\{(1, 0), (1, 0)\}$ and $\{(0, 1), (0, 1)\}$ will be Nash Equilibria. The resulting total wealth distributions for these Nash Equilibria are $(2, 0)$ and $(0, 2)$, respectively. Therefore, an MDG played in a GE can have Nash Equilibria with different total wealth distributions. $\square$

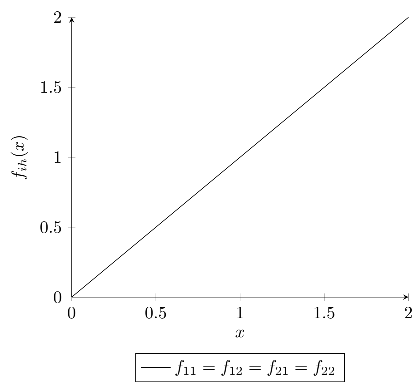

Figure 3

Now we turn our attention to MDGs played in a DE. To do this, we present a series of definitions and lemmas. For Definition 9 and Lemma 10, let δ be a total wealth distribution in a DE: $\delta = (\delta_1, ..., \delta_d) \in WD_{\xi'}$, where $\xi' = (N, W, D, F)$ is a DE.

Definition 9 (Maximum Set)**.** *Define the Maximum Set $M_i(\delta)$ for agent i as:*

$$M_i(\delta) = \{k \in D \mid f_{ik}(\delta_k) \geq f_{ih}(\delta_h) \ \forall \ h \in D\}$$

Note that agent i's marginal utility curve for destination h evaluated at funding level δ_h must be equal for all destinations in agent i's Maximum Set under wealth distribution δ. $M_i(\delta)$ therefore represents the set of candidate destinations given δ that could provide maximum utility to agent i from the next arbitrarily small wealth donation.

Lemma 10. *If δ corresponds to a Nash Equilibrium, agent i only funds destinations in $M_i(\delta)$.*

Proof. This is a proof by contradiction. For any Nash Equilibrium, take the corresponding total wealth distribution δ. Assume that, for some agent i, $\exists \ k \notin M_i(\delta)$ such that agent i gives some of their wealth to destination k. By the definition of $M_i(\delta)$, it must be that

$$\exists \ h \ \in D \ s.t. \ f_{ih}(\delta_h) > f_{ik}(\delta_k)$$

Since the marginal utility curves are continuous and decreasing, we have that in δ, $\exists \ \varepsilon > 0$ such that the next ε units of wealth given to destination h would yield more utility to agent i than the last ε units of wealth did when given to destination k. Thus, agent i would increase their utility by moving ε units of wealth from k to h. Thus, agent i is not currently best responding, which is a contradiction since δ corresponds to a Nash Equilibrium. Therefore, agent i does not donate any wealth to any destinations not contained in $M_i(\delta)$. This implies that, for any wealth distribution δ in a Nash Equilibrium, agent i only funds destinations in $M_i(\delta)$. $\square$

For Definition 11, Definition 12, and Lemma 13, let α and β be two unequal total wealth distributions in the same DE ξ': $\alpha = (\alpha_1, ..., \alpha_d) \in WD_{\xi'}$, $\beta = (\beta_1, ..., \beta_d) \in WD_{\xi'}$, $\alpha \neq \beta$.

Definition 11 (Comparative Sets)**.** *Define the Comparative Sets for α and β as the following:*

$$D_\alpha = \{h \in D \mid \alpha_h > \beta_h\}$$
$$D_\beta = \{h \in D \mid \alpha_h < \beta_h\}$$
$$D_\circ = \{h \in D \mid \alpha_h = \beta_h\}$$

Since α and β are both wealth distributions in the same game, the sum of all the donations in each distribution must be equal. Since the two vectors cannot be equal, we have that:

$$D_\alpha \neq \emptyset,\ D_\beta \neq \emptyset,\ D_\alpha \cap D_\beta = \emptyset$$

Definition 12 (Special Agents Set)**.** *Define the Special Agents Set S as the set of agents i for whom $M_i(\alpha)$ contains at least one element in the first Comparative Set D_α:*

$$S = \{i \in N \mid D_\alpha \cap M_i(\alpha) \neq \emptyset\}$$

Since Lemma 10 tells us that, in a Nash Equilibrium, agents will only fund destinations that are in their respective Maximum Sets, S intuitively represents the set of agents that fund any destinations in D_α in the Nash Equilibrium that corresponds to wealth distribution α. By extension, S also represents the complete set of agents from which the destinations in D_α receive all of their funding in the Nash Equilibrium corresponding to wealth distribution α.

Note that Lemma 10 guarantees that $S \neq \emptyset$. Since D_α contains only destinations that receive more funding in α than in β, all destinations in D_α must receive a positive amount of funding in α. This funding must come from some non-empty set of agents, and these agents will make up the set S. Now we present the fundamental lemma that provides the basis for Theorem 14.

Lemma 13. *$M_i(\beta) \subseteq D_\alpha\ \forall\ i \in S$.*

Proof. Since $S \neq \emptyset$, select an agent $i \in S$. Since $\frac{\partial f_{ih}}{\partial x} < 0\ \forall\ h \in D$ and $\alpha_h \leq \beta_h\ \forall\ h \in D \setminus D_\alpha$,

$$f_{ih}(\alpha_h) \geq f_{ih}(\beta_h) \quad \forall\ h \in D \setminus D_\alpha$$

For any destination $k \in D_\alpha \cap M_i(\alpha)$, $k \in M_i(\alpha)$ implies that

$$f_{ik}(\alpha_k) \geq f_{ih}(\alpha_h) \quad \forall\ h \in D$$

$k \in D_\alpha$ implies that $\alpha_k > \beta_k$. Since $\frac{\partial f_{ik}}{\partial x} < 0$,

$$f_{ik}(\beta_k) > f_{ik}(\alpha_k)$$

These three inequalities give the following result $\forall\ k \in D_\alpha \cap M_i(\alpha),\ h \in D \setminus D_\alpha$:

$$f_{ik}(\beta_k) > f_{ik}(\alpha_k) \geq f_{ih}(\alpha_h) \geq f_{ih}(\beta_h)$$

Thus, $\forall\ i \in S$, if a destination h is not in D_α, that destination cannot be in $M_i(\beta)$. Therefore:

$$M_i(\beta) \subseteq D_\alpha\ \forall\ i \in S$$

$\square$

Lemma 13 means that, in a Nash Equilibrium, all individuals will split their wealth (subject to other's donations) so that all the destinations they are funding look equally "attractive." When moving from wealth distribution α to wealth distribution β, any destination in D_α receives *less* funding, and therefore appears *more* attractive to all individuals because the marginal utility curves are decreasing. Similarly, any destination in $D \setminus D_\alpha$ will receive *equal or more* funding, and therefore appear *equally or less* attractive to all individuals.

Any agent i in the set S is already donating some of their wealth to some destinations in D_α. When we move from wealth distribution α to wealth distribution β, these destinations in D_α that i is already funding look more attractive, while all the destinations in $D \setminus D_\alpha$ look less attractive. Agent i therefore could not be funding destinations in $D \setminus D_\alpha$ in a Nash Equilibrium with wealth distribution β

Theorem 14 (Net Donations Equality Theorem)**.** *All Nash Equilibria of an MDG played in a DE will have the same total wealth distribution.*

Proof. This theorem will be proved by contradiction. Take an MDG $\phi = (\xi', A, U)$, where $\xi' = (N, W, D, F)$ is a DE. Assume this MDG has two Nash Equilibria with different total wealth distributions: $\alpha = (\alpha_1, ..., \alpha_d) \in WD_{\xi'}$, $\beta = (\beta_1, ..., \beta_d) \in WD_{\xi'}$, and $\alpha \neq \beta$. Definition 12 tells us that only agents $i \in S$ have destinations from the first Comparative Set D_α in their Maximum Set $M_i(\alpha)$. Lemma 10 thus guarantees that all funding for the destinations in D_α in wealth distribution α comes from just the agents in S. Since there may be other destinations not in D_α that are also receiving funding from agents in S, we have that the total funding of destinations in D_α under wealth distribution α is bounded *above* by the combined wealth of the agents in set S:

$$\sum_{h \in D_\alpha} \alpha_h \leq \sum_{i \in S} w_i$$

Lemmas 13 and 10 combined guarantee that, in wealth distribution β, agents in S give all of their wealth to destinations in D_α. Since it is possible that other agents not in S are also funding destinations in D_α in wealth distribution β, we have that the total funding of destinations in D_α under wealth distribution β is bounded *below* by the total wealth of the agents in set S:

$$\sum_{h \in D_\alpha} \beta_h \geq \sum_{i \in S} w_i$$

Combining the two inequalities above gives the following result:

$$\sum_{h \in D_\alpha} \beta_h \geq \sum_{h \in D_\alpha} \alpha_h$$

However, since $\alpha_h > \beta_h \ \forall \ h \in D_\alpha$, the following must hold:

$$\sum_{h \in D_\alpha} \alpha_h > \sum_{h \in D_\alpha} \beta_h$$

This is a contradiction. Therefore, there cannot be two Nash Equilibria in ϕ with different total wealth distributions, which means that all Nash Equilibria for an MDG played in a DE must have the same total wealth distribution. $\square$

Intuitively, the contradiction that leads to the Net Donations Equality Theorem can be considered as follows. If there are two Nash Equilibria corresponding to different total wealth distributions, there will always be a set of destinations that receive more funding in the second distribution. From the perspective of an agent under distribution 1, these destinations under distribution 2 look less attractive since they receive more funding and the marginal utility curves are decreasing. However, the fact that those destinations receive more funding also indicates that society considered them to be more attractive overall than distribution 1 indicated. While Theorem 14 is a clean result, it unfortunately does not imply efficiency in either environment.

3.4 Pareto Efficiency

Claim 15. *A Nash Equilibrium is not guaranteed to be Pareto Efficient for an MDG played in a DE or GE.*

Proof. Consider the MDG $\phi = (\xi', A, U)$, where ξ' is a DE with two destinations, three agents with one unit of wealth each, and marginal utility curves illustrated by Figure 4. Note that the marginal utility curves in the figure have an arbitrarily small negative slope. The action profile $\{(1, 0, 0), (0, 1, 0)\}$ will be a Nash Equilibrium for this MDG, since both agents are already funding the destination they like best and can't improve their welfare by moving their wealth. However, the action profile $\{(0, 0, 1), (0, 0, 1)\}$ would represent a strict improvement in utility for both agents. Therefore, a Pareto Efficient Nash Equilibrium is not guaranteed in a DE, and by extension, Pareto Efficiency is not guaranteed for Nash Equilibria in a GE. $\square$

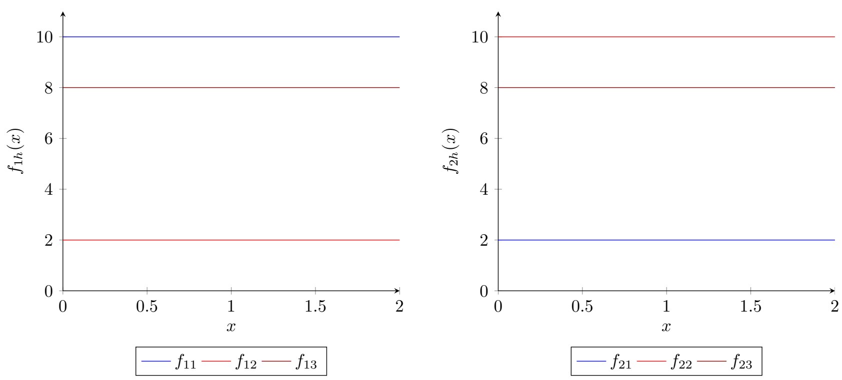

Figure 4

An immediate implication of this result is that Nash Equilibria in any MDG are also not guaranteed to maximize societal welfare, since any distribution that maximizes societal welfare must also be Pareto Efficient.

4 Results: Social Welfare Maximization

Being able to find a Social Welfare Maximizing (SWM) wealth distribution is of obvious interest to social planners and societies with cooperation more generally. The uniqueness or lack thereof of an SWM wealth distribution is relevant because a central planner may have varying abilities to achieve any given wealth distribution between destinations. Additionally, the practicality of finding this solution is relevant, since calculating an SWM wealth distribution using any algorithm that runs in super-polynomial time will rapidly become impractical when dealing with environments that have large numbers of agents and a large number of destinations that need to be coordinated, such as may be the case with a country that is evaluating how to distribute government revenue between different foreign aid options.

4.1 Uniqueness

Claim 16. *A unique SWM distribution is not guaranteed for an MDG played in a GE.*

Proof. Consider the example from Claim 8. Both Nash Equilibria in that example will be SWM distributions, even though the corresponding wealth distributions are not the same. □

Now we turn our attention to MDGs played in a DE. To motivate the proof of the primary result of this section (Theorem 20), we will first present a series of definitions and observations that can be made about SWM distributions in this case. For the rest of Section 4.1, let $\phi = \{\xi', A, U\}$ be an MDG where $\xi' = \{N, W, D, F\}$ is a DE.

Definition 17 (Societal Marginal Utility Curve)**.** *Define $f_h^s(x)$ be the Societal Marginal Utility Curve for destination h:*

$$f_h^s(x) = \sum_{i \in N} f_{ih}(x) \ \forall \ h \in D$$

This function takes in a donation amount x and returns the sum of all individual marginal utilities from destination h in a wealth allocation that gives x units of wealth to h. Thus, the integral of this curve from 0 to x represents the total utility gained by the entire society from donating a total of x units of wealth to destination h. Note that the Societal Marginal Utility Curves must be strictly decreasing in a DE, since each f_{ih} is strictly decreasing as well. The Societal Marginal Utility Curve is the metric by which a central planner could analyze the total utility provided by societal contributions to different destinations.

Definition 18 (Societal Maximum Set)**.** *Define the Societal Maximum Set $G(\delta)$ as:*

$$G(\delta) = \{k \in D \mid f_k^s(\delta_k) \geq f_h^s(\delta_h) \ \forall \ h \in D\}$$

where $\delta = (\delta_1, ..., \delta_d) \in WD_{\xi'}$.

Note that the Societal Maximum Set is the aggregate equivalent of the Maximum Set from Definition 9. $G(\delta)$ is the set of destinations that have the highest marginal utility value for all agents combined at donation vector δ. Since the linear transformation of dividing all $f_h^s(\delta_h)$ by N will not change which destinations are in $G(\delta)$, the Societal Maximum Set can also be thought of as the set of destinations that have the highest average marginal utility value for all agents at donation vector δ.

Since the Societal Marginal Utility Curves must be decreasing and continuous, the destinations in $G(\delta)$ would be the only valid candidates for the destination that would maximize societal welfare for the next arbitrarily small amount of wealth $\epsilon > 0$ donated to any destination in the game. Additionally, no amount of wealth given to any other destination will change this fact, since the marginal benefit from donations to other destinations will only continue to decrease. Thus, if a central planner were deciding where to allocate the next arbitrarily small amount of wealth to maximize societal utility, it should always go to a destination in $G(\delta)$.

Lemma 19. *$\delta = (\delta_1, ..., \delta_d) \in WD_{\xi'}$ is an SWM distribution if and only if $\delta_h > 0 \Rightarrow h \in G(\delta)$.*

Proof. First, we prove the "if" direction. Assume that $\delta_h > 0 \Rightarrow h \in G(\delta) \ \forall \ h \in D$. Choose a destination k with $\delta_k > 0$. Imagine taking any amount of wealth $\epsilon > 0$ away from destination k to donate instead to a different destination. Since all destinations currently receiving funding are in $G(\delta)$, they all have the same marginal utility under δ. Since the marginal utility curves are strictly decreasing, moving ϵ units of wealth from k to another destination in $G(\delta)$ will decrease overall utility. Similarly, all destinations not currently receiving funding must have y-intercepts of their Societal Marginal Utility Curves that are already equal to or lower than the current marginal utility value of the destinations in $G(\delta)$. Therefore, moving these ϵ units of wealth to an unfunded destination can also only decrease overall utility. This means that there is no way to increase societal utility by taking wealth away from a destination that has funding, which means that the current distribution δ is SWM. This proves the “if” direction.

Second, we will prove the “only if” direction. To do this, we will prove the contrapositive of the lemma by assuming that the conclusion is false. The contrapositive of the conclusion is:

$$h \notin G(\delta) \Rightarrow \delta_h = 0$$

So we will assume that $\exists\ h \notin G(\delta)\ s.t.\ \delta_h > 0$ and show that δ cannot be an SWM distribution.

If $\exists\ h \notin G(\delta)\ s.t.\ \delta_h > 0$, there must be destinations under wealth distribution δ that have higher societal marginal utility rates than destination h has. Since the Societal Marginal Utility Curves are strictly decreasing and continuous, a central planner could increase societal welfare by taking an arbitrarily small amount of wealth $\epsilon > 0$ away from destination h and giving it to a destination $k \in G(\delta)$. This means that δ is not an SWM distribution. This proves the “only if” direction. $\square$

Figure 5 gives a geometric interpretation to Lemma 19. In an SWM distribution, Lemma 19 says that all destinations that receive donations will all be in $G(\delta)$, which means that they all have the same societal marginal utility value at the level of their respective donations.

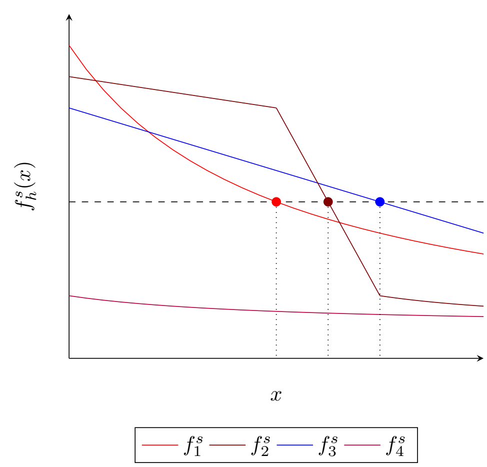

Figure 5

Figure 5 represents a potential SWM distribution in a DE with four destinations. The specific functional forms are unimportant, and the number of agents is also irrelevant since these curves are Societal Marginal Utility Curves. When we combine the result from Lemma 19 with Definition 18, we get that an SWM distribution can be pictured as a horizontal line cutting across each of the Societal Marginal Utility Curves.

An intersection between the line and a curve, as is the case with curves 1, 2, and 3, indicates that the destination corresponding to that curve receives funding equal to the x-coordinate of that intersection. If there is no intersection between the line and a given curve, as is the case with curve 4, that destination will not be funded in an SWM distribution. If the game contains more total wealth, we can lower the black horizontal line and shift all intersections to the right as each of funded destination gets additional funding. If the black line is lowered far enough so that it hits the y-intercept of f_4^s, destination 4 will also receive funding in an SWM distribution.

Theorem 20. *There is a unique SWM distribution for an MDG played in a DE.*

Proof. Assume there are two different SWM distributions for ϕ. Call these distributions α and β. That is, $\alpha = (\alpha_1, ..., \alpha_d) \in WD_{\xi'}$, $\beta = (\beta_1, ..., \beta_d) \in WD_{\xi'}$, and $\alpha \neq \beta$. Since both α and β must distribute all of the wealth in the game, there must be at least one destination h such that $\alpha_h > \beta_h$ and at least one destination k such that $\alpha_k < \beta_k$. Clearly, $\alpha_h > 0$, and so by Lemma 19, we have that $h \in G(\alpha)$. This means that $f^s_h(\alpha_h) \geq f^s_k(\alpha_k)$. Similarly, $\beta_k > 0$, and so $k \in G(\beta)$, which means that $f^s_k(\beta_k) \geq f^s_h(\beta_h)$. Since the Societal Marginal Utility Curves are strictly decreasing, we also have that $f^s_h(\alpha_h) < f^s_h(\beta_h)$ and $f^s_k(\alpha_k) > f^s_k(\beta_k)$. Combining these inequalities gives the following result:

$$f^s_h(\alpha_h) \geq f^s_k(\alpha_k) > f^s_k(\beta_k) \geq f^s_h(\beta_h) > f^s_h(\alpha_h)$$

$$\Rightarrow f^s_h(\alpha_h) > f^s_h(\alpha_h)$$

This yields a contradiction, and so there cannot be two distinct wealth distributions that both maximize societal welfare. □

4.2 Computational Complexity

Theorem 21. *Calculating an SWM distribution for an MDG played in a GE is NP-hard.*

Take an arbitrary MDG $\phi = (\xi, A, U)$, where $\xi = (N, W, D, F)$ is a GE. Label the amount of total utility achieved in ϕ with wealth distribution $\delta \in WD_\xi$ as Ψ_δ. Label the maximum possible amount of utility that can be produced in ϕ as $\Psi_\phi = \max_\delta \Psi_\delta$, and label the total wealth in ϕ as $w = \sum_{i \in N} w_i$. Define the language MAX as the set of all such MDGs where the maximum number of utils achievable in that game is equal to the total number of units of wealth in the game:

$$MAX = \{\langle \phi \rangle \mid \Psi_\phi = w\}$$

Determining if $\phi \in MAX$ is a problem of calculating Ψ_ϕ. We can show that MAX is NP-hard with a reduction from the NP-complete problem $SUBSET\_SUM$, where S is a set of binary integers, t is a binary integer, and

$$SUBSET\_SUM = \{\langle S, t \rangle \mid S = \{s_1, ..., s_d\} \textit{ and } \exists \textit{ subset } T \subseteq S \textit{ s.t. } \sum_{s_h \in T} s_h = t\}$$

To show that $SUBSET\_SUM \leq_p MAX$, we must describe a polynomial-time procedure that takes in a set of binary integers S and a binary integer t and outputs an MDG $\phi = (\xi, U, A)$ where $\xi = (N, W, D, F)$ is a GE, such that $\Psi_\phi = w \Leftrightarrow \exists\, T \subseteq S \textit{ s.t. } \sum_{s_h \in T} s_h = t$.

Define the function ρ as $\rho(\langle S, t \rangle) = \langle \xi, A, U \rangle$ with the following steps:

Input: $\langle\{s_1, ..., s_d\}, t\rangle$, where t and all $s_i, i \in \{1, ..., d\}$ are binary integers.

1. $N = \{1\}$
2. $W = \{t\}$
3. $D = \{1, ..., d\}$
4. $F = f_1 = \{f_{11}(x), ..., f_{1d}(x)\}$ where each $f_{1h}(x)$ is constructed as follows, where $\lambda = s_h - \frac{1}{t+1}$:

$$f_{1h}(x) = \begin{cases} 0 & \text{for} \quad 0 \leq x < \lambda \\ 4(t+1)^2(s_h)(x - s_h + \frac{1}{t+1}) & \text{for} \quad \lambda \leq x < \frac{s_h-\lambda}{2} \\ 4(t+1)^2(s_h)(s_h - x) & \text{for} \quad \frac{s_h-\lambda}{2} \leq x \leq s_h \\ 0 & \text{for} \quad x > s_h \end{cases}$$

5. Output $\langle\xi, A, U\rangle$ where $\xi = (N, W, D, F)$ and A and U come from Definition 3.

While the marginal utility functions this procedure creates look messy, they have a very clean geometric interpretation. $f_{1h}(x)$ takes the form of a spike starting at the point $(\lambda, 0)$, peaking half way between λ and s_h, then decreasing again to the point $(s_h, 0)$. Before and after this spike, the function is zero-valued. There are four important features of these functions for any destination $h \in D$:

1. $f_{1h}(x)$ is continuous.
2. $f_{1h}(x) \geq 0 \ \forall \ x$.
3. $f_{1h}(x) = 0 \ \forall \ x \notin [\lambda, s_h]$.
4. $\int_\lambda^{s_h} f_{1h}(x)dx = \int_0^{s_h} f_{1h}(x)dx = s_h$.

Features 1 and 2 make these functions valid marginal utility curves for a GE. Features 3 and 4 are crucial for the correctness of the proof. Visually, since the spike for every marginal utility curve has the same base width, features 3 and 4 mean that the height of the spike increases as s_h increases. Since *any* function that satisfies these four features would satisfy the needs of this proof, the tedious calculation steps needed to demonstrate that the provided functional form satisfies these features will not be shown here. Instead, the function provided in Step 4 serves as a demonstration that $f_{1h}(x)$ can be generated as a function of the inputs S and t.

Below is a visual representation of this reduction procedure to make the intuition of the proof easier to follow. Take the following sample element of $SUBSET\_SUM$:

$$\langle S, t\rangle = \langle\{1, 3, 4\}, 1\rangle \in SUBSET\_SUM$$

In this case, the function $\rho(\langle S, t\rangle)$ outputs an MDG $\phi = (\xi, A, U)$, where $\xi = (\{1\}, \{1\}, \{1, 2, 3\}, \{f_{11}, f_{12}, f_{13}\})$ is a GE, and where the marginal utility curves can be represented in Figure 6.

Note that $\rho(\langle S, t\rangle) \in MAX$ because the maximum utility in this game is achieved when the agent donates all their wealth to destination 1 (since one unit of wealth is not enough to reach the non-zero part of the marginal utility functions for the other two destinations). By construction, $\int_0^1 f_{11}(x)dx = 1$, and so the maximum utility in this game is equal to the wealth of the agent.

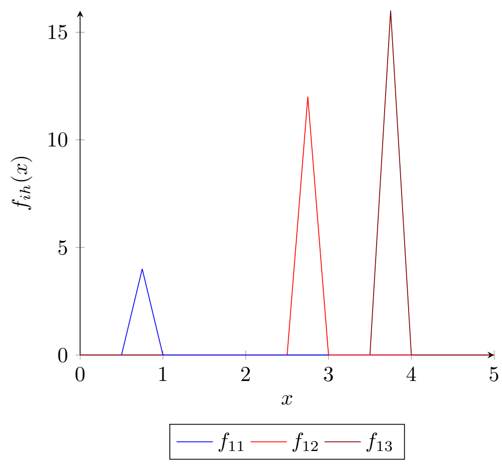

Figure 6

The intuition of this procedure is straightforward. Each number s_h in the integer list in an element of $SUBSET\_SUM$ is transformed into a destination h. This destination h has a marginal utility curve that is a sharp spike occurring right before the associated number s_h with a total area of s_h. The spike is so narrow that the one agent in the game cannot increase their utility by only funding a part of a spike and saving the extra money to donate somewhere else, because this extra money will not be enough to reach the non-zero-valued spike of any other curve. Therefore, if the agent wants to get any utility from destination h, they will fund the entire spike (by donating s_h units) and then stop, since they cannot get any more utility from donating past s_h to destination h. This action will give the agent s_h units of utility. The agent's utility will only be equal to their wealth if there is a subset of destinations where the associated integers s_h for each destination add up exactly to the wealth of the agent, which is the same as saying that there is a subset of integers that add up exactly to the target in the original $SUBSET\_SUM$ problem.

To demonstrate correctness of this procedure, three claims must be supported. However, before we prove these claims, we will present two Lemmas that will simplify those proofs. For proofs of these Lemmas, see the Appendix. For everything that follows through the end of the proof of Theorem 21, let $\phi = \{\xi, A, U\}$ where $\xi = (N, W, D, F)$ be the output of $\rho(\langle S, t \rangle)$ for some element $\langle S, t \rangle$. ϕ will always have only one agent, and therefore any wealth distribution must be created exclusively by that agent's actions:

$$\delta \in WD_\xi \Leftrightarrow \delta \in A_1$$

Thus, $\delta = (\delta_1, ..., \delta_d) \in WD_\xi$ is both a total wealth distribution and an action of agent 1, and it follows that δ_h refers to both the total wealth donated to destination h and also to agent 1's donations to destination h. We now partition all possible wealth distributions into two sets.

Definition 22. *Define I to be the set of all wealth distributions that contain only integer-valued donations to all destinations:*

$$I = \{\delta = (\delta_1, ..., \delta_d) \in WD_\xi \mid \delta_h \in \mathbb{Z} \ \forall \ h \in D\}$$

Definition 23. *Define NI to be the set of all wealth distributions that contain a non-integer-valued donation to any destination:*

$$NI = \{\delta = (\delta_1, ..., \delta_d) \in WD_\xi \mid \exists \ k \in D \ s.t. \ \delta_k \notin \mathbb{Z}\}$$

Note that these two sets form a partition of WD_ξ. We can now introduce the three results to facilitate the proof of Theorem 21. For the proofs of these three results, see the Appendix.

Lemma 24. $\Psi_{\beta \in I} = w \Leftrightarrow \beta_h \in \{0, s_h\} \ \forall \ h \in D.$

Corollary 25. $\max\{\Psi_{\beta \in I}\} \leq w.$

The intuition behind Lemma 24 and Corollary 25 is simple. When looking only at wealth distributions that have only integer-valued donations to all destinations, Lemma 24 says that the only way to get w units of utility is if the agent donates either 0 or s_h units of wealth to every destination h. Examining the picture in the example above should make this clear. Any integer donation to destination h less than s_h will produce 0 units of utility for the agent, and any integer donation to destination h greater than s_h will still only produce s_h units of utility for the agent. Thus, when considering destination h, the agent must either not donate at all or only make a donation of magnitude s_h if they are to achieve a one to one return of utility to wealth. Corollary 25 simply says that it is impossible for an integer-valued wealth distribution to get greater than a one to one return of utility to wealth, so the utility from a particular wealth distribution in I can never be greater than w.

Lemma 26. $\Psi_\phi = \max\{\Psi_{\beta \in I}\}.$

The intuition from this lemma follows from the construction of the marginal utility curves. As the agent gets more wealth, the curves are constructed to be more and more narrow. By keeping the curves narrow enough, it becomes impossible for the agent to increase their utility by only funding a spike part way through. This is because the amount of wealth they would save by doing that would be so small that they would be unable to use that wealth to reach the non-zero-valued spike part of the marginal utility function for any other destination. This construction means that, if an agent is funding a spike part way through, it will always make sense for them to finish funding the spike all the way through. This means that the maximum utility in the game can be achieved through some integer-valued wealth distribution.

At this point, the intuition behind the entire reduction should be clear. It will now be formalized in the following proof.

Proof of Theorem 21. To show that the reduction presented above is correct, we must prove the three following claims:

1. ρ runs in polynomial time.
   Proof. The only super-linear step in this procedure is the construction of the marginal utility functions, which involves arithmetic operations and thus runs in polynomial time. Therefore, ρ runs in polynomial time.

2. $\langle S,t\rangle \in SUBSET\_SUM \Rightarrow \phi \in MAX$.
   Proof.
   $$\langle S,t\rangle \in SUBSET\_SUM \Rightarrow \exists \textit{ subset } T \subseteq S \textit{ s.t.} \sum_{s_h \in T} s_h = t$$
   This means that ϕ will have a destination $h \;\forall\; s_h \in T$. By construction, the single agent will have exactly enough wealth to donate s_h units of wealth to each destination h that corresponds to an $s_h \in T$ without any wealth left over. Call this wealth distribution $\beta^* \in I$. Since the agent is either donating s_h units of wealth or no wealth to each destination h, $\beta^*_h \in \{0, s_h\} \;\forall\; h \in D$. By Lemma 24, $\Psi_{\beta^*} = w$. By Corollary 25, it must also be that $\Psi_{\beta^*} = \max\{\Psi_{\beta\in I}\}$. By Lemma 26, $\Psi_{\beta^*} = \Psi_\phi$. Therefore $\Psi_\phi = w$, and so $\phi \in MAX$.

3. $\langle S,t\rangle \notin SUBSET\_SUM \Rightarrow \phi \notin MAX$.
   Proof.
   $$\langle S,t\rangle \notin SUBSET\_SUM \Rightarrow \nexists \textit{ subset } T \subseteq S \textit{ s.t.} \sum_{s_h \in T} s_h = t$$
   This means that there is no set of destinations in ϕ that allows the single agent to distribute exactly s_h units of wealth to each destination h in that set with no wealth left over. This means that for any distribution of wealth $\beta \in I$, $\exists\; k \in D$ *s.t.* $\beta_k \notin \{0, s_k\}$. By Lemma 24, $\Psi_\beta \neq w \;\forall\; \beta \in I$. By Corollary 25, it must also be that $\max\{\Psi_{\beta\in I}\} < w$. Combined with Lemma 26, this means that $\Psi_\phi < w$, and so $\phi \notin MAX$.

Therefore, we have constructed a polynomial time procedure that takes an arbitrary instance of a $SUBSET\_SUM$ problem and turns it into a specific instance of a MAX problem. If we were able to calculate an SWM distribution in polynomial time, we could solve any insteance of MAX in polynomial time. We would then be able to invert the procedure above and generate a polynomial time algorithm for $SUBSET\_SUM$, which is not possible. Therefore, calculating an SWM distribution for an MDG played in a GE is NP-hard. □

We now turn our attention to the time complexity of calculating the SWM distribution for an MDG played in a Decreasing Environment. For the rest of this section, let $\phi = \{\xi', A, U\}$ be an MDG where $\xi' = \{N, W, D, F\}$ is a DE.

Theorem 27. *An SWM distribution for an MDG played in a DE can be calculated in polynomial time.*

To prove this result, we built off the definitions in Section 4.1, and then construct a polynomial time algorithm that finds the SWM distribution for an MDG played in a DE.

Definition 28 (Total Funding Function)**.** *Define $v(x, D')$ for some set of destinations $D' \subseteq D$ to be the Total Funding Function:*

$$v(x, D') = \sum_{h\in D'} f^s_h(x)^{-1}$$

where each $f^s_h(x)$ is the Societal Marginal Utility Curve for destination h.

Since each $f^s_h(x)$ takes in a donation value and returns the marginal utility for society at that value, $f^s_h(x)^{-1}$ takes in a marginal utility value for society and returns the amount of wealth that must be donated to donation h to achieve that level of marginal societal utility. These functions must be invertible since they are strictly decreasing, and it is assumed that this inverse can be calculated in polynomial time.

The Total Funding Function $v(x, D')$ sums these inverses, and so it takes in a marginal utility value for society and returns the sum of the required donations to each of the destinations in D' for each of those destinations to achieve that level of marginal societal utility.

Definition 29 (Best Level Function)**.** *Define $\mu(x, D')$ for some set of destinations $D' \subseteq D$ to be the Best Level Function:*

$$\mu(x, D') = v(x, D')^{-1}$$

where $v(x, D')$ is the Total Funding Function from Definition 28.

This function takes in an amount of wealth and returns the marginal utility value for society that can be achieved for all destinations in D' with that amount of wealth. It essentially returns the value of the black dashed line from Figure 5 while only considering the destinations in D'.

Definition 30 (Ordered Destinations Set)**.** *Define $D_{order} = \{d_1, ..., d_d\}$ as the Ordered Destinations Set, which contains all destinations in D in order from highest to lowest y-intercept on the Societal Marginal Utility Curves. Thus, $\forall\ i < j,\ f^s_{d_i}(0) \geq f^s_{d_j}(0)$.*

The Algorithm.
Input: MDG $\phi = (\xi', A, U)$ where $\xi' = (N, W, D, F)$ is a DE.

1. Construct $D_{order} = \{d_1, ..., d_d\}$.
2. Initialize $D' = \{d_1\}$, and let $m = |D'|$.
3. If $\mu(w, D') < f^s_{d_{m+1}}(0)$, add d_{m+1} to D' and repeat. Otherwise, let $x^* = \mu(w, D')$ and continue.
4. Return $\delta = \{\delta_1, ..., \delta_d\}$ where $\delta_h = f^s_h(x^*)^{-1}\ \forall\ h \in D'$, and where $\delta_k = 0\ \forall\ k \in D \setminus D'$.

Intuition Behind the Algorithm

Step 1 orders the destinations from highest to lowest Societal Marginal Utility Curve y-intercept. This is an ordering of how good each destination looks before any donations have been given, and destinations earlier in this set should receive some amount of funding before destinations later in this set receive any funding. Step 2 initializes D', which can be thought of as the "funding set," to contain just the destination with the highest y-intercept.

Step 3 calculates the Best Level Function for the destinations in D'. If that line is lower than the highest y-intercept of the destinations that are not currently being funded, the destination with the next highest y-intercept is added to the funding set and the Best Level Function is recalculated. Since the same amount of wealth is being split between more destinations now and the societal marginal utility curves are decreasing, the output of the Best Level Function will be higher. Step 3 continues to add the next best destination that isn't currently being funded until the output of the Best Level Function is higher than the y-intercepts of all the destinations that are not being funded. This is the line that is drawn in Figure 5.

It is important to note here that, when Step 3 terminates on a societal utility level x^*, the highest possible marginal utility value for all the destinations not in the funding set is still lower than $f^s_h(x^*)^{-1}$ for any destination h that is in the funding set. This means that, in the wealth distribution δ that the algorithm outputs in Step 4, the Societal Maximum Set $G(\delta)$ will only contain destinations that are being funded. This means that if a destination is not in $G(\delta)$, it is receiving no funding, which means that δ must be an SWM distribution by Lemma 19.

Proof of Correctness. The above text shows that the algorithm returns the SWM distribution.

The second claim that must be proven is that the algorithm runs in polynomial time. Step 1 can be implemented with any standard polynomial time sorting algorithm plus polynomial time steps to calculate the y-intercept of each Societal Marginal Utility Curve before the sorting begins. Step 2 takes linear time. Step 4 in the worst case has to perform d functional inverses and d arithmetic calculations, and is thus also polynomial time. This leaves Step 3.

First, the Best Level Function $\mu(w, D')$ can be calculated in polynomial time for any wealth and set of destinations. Since all the individual marginal utility curves are strictly decreasing and polynomial-time invertible (PTI), constructing each Societal Marginal Utility Curve takes polynomial time, and these curves are also strictly decreasing and PTI. Thus, the creation of the Total Funding Function $v(x, D')$ takes polynomial time, and this curve is also strictly monotonic and PTI. Therefore, the creation of $\mu(w, D')$ takes polynomial time. Additionally, finding the destination that has the marginal utility curve with the next highest y-intercept and running arithmetic comparisons take polynomial time. Finally, the repetitions of Step 3 is a bounded by a linear function of number of destinations. Therefore, the entire algorithm runs in polynomial time. This completes the proof of Theorem 27. □

5 Conclusion and Open Questions

This paper defines the Multi-Destination Donations Game as a model to analyze the coordination challenges provided by charitable donations, and studies its Nash Equilibria and Social Welfare Maximizing Solutions. In the General Environment, where the marginal utility functions can be non-monotonic, we demonstrate through a series of examples that a pure-strategy Nash Equilibrium is not guaranteed, and that when one does exist, the Nash Equilibria are not necessarily unique, Pareto Efficient, or similar in their final allocation of wealth between destinations. Additionally, there can be multiple total wealth distributions that maximize social welfare, and we use a reduction from $SUBSET\_SUM$ to show that finding one of these Social Welfare Maximizing distributions is NP-hard.

When we introduce the congestion factor in the Decreasing Environment, we find a number of more well-behaved results. While a unique Nash Equilibrium is still not guaranteed, and while Nash Equilibria are still not guaranteed to be Pareto Efficient, we prove that a pure-strategy Nash Equilibrium always exists, and that all Nash Equilibria have the same final allocation of wealth between destinations. Additionally, we show that there is a unique total wealth distribution that maximizes social welfare, and present a polynomial-time algorithm for finding that allocation of wealth between destinations.

This work leaves an important open question for Multi-Destination Donations Games played in the Decreasing Environment. After showing that there is a unique wealth distribution between destinations for all Nash Equilibria and that there is a unique wealth distribution between destinations that maximizes social welfare, the natural extension is to examine under what conditions these two wealth distributions are equivalent. Knowing these conditions may be very useful for understanding the situations in which central organization may increase societal welfare by coordinating charitable donations and other forms of resource distribution. Measuring the difference in utility between the Nash Equilibrium distribution and the Social Welfare Maximizing solution in cases where they do differ will help demonstrate the potential quantitative gains of this central coordination. This would shed light on the potential value of charity evaluators that collect donations, perform research on charity options, and distribute those donations accordingly.

In the field of implementation theory, another direction for future work is to develop a different game that models strategic donations to multiple destinations where Nash Equilibria always have social welfare maximizing properties. Since the total wealth distribution that results in a Nash Equilibria of an MDG is not guaranteed to be the same as the total wealth distribution that maximizes social welfare, Multi-Destination Donations Games do not have ideal mechanism implementation properties.

On the empirical end, this paper provides a motivation for generating estimates of the marginal utility curves that are basis of analysis for the Multi-Destination Donations Game. There is significant room for improvement in the efficiency of how developed countries donate their wealth, and establishing groundwork for the functional forms of the marginal utility curves for actual people and agencies may allow some of the results from this paper to be applied to charity donation policy and coordination.

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References

[1] Broad, G. M. (2018). Effective animal advocacy: effective altruism, the social economy, and the animal protection movement. Agriculture and Human Values, 1-13.

[2] Gibbons, R. (1997). An introduction to applicable game theory. Journal of Economic Perspectives, 11(1), 127-149.

[3] Meyers, C. A., & Schulz, A. S. (2012). The complexity of welfare maximization in congestion games.

[4] Milchtaich, I. (1996). Congestion games with player-specific payoff functions. *Games and economic behavior, 13*(1), 111-124.

[5] Glicksberg, I. L. (1952). A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points. Proceedings of the American Mathematical Society, 3(1), 170-174.

[6] Li, S. (2014). Brouwer's fixed point theorem: the Walrasian auctioneer.

[7] Milam, P. E. Is tobacco control a "best-buy" for the developing world?.

7 Appendix

This Appendix presents the proofs for Lemma 24, Corollary 25, and Lemma 26.

Lemma 24. $\Psi_{\beta \in I} = w \Leftrightarrow \beta_h \in \{0, s_h\} \; \forall \; h \in D$.

Proof. For each destination h, there are four cases for the possible integer levels of donations in any wealth distribution $\beta \in I$:

1. $\beta_h = 0$. No wealth is wasted in this case, so the maximum utility to wealth ratio is not affected.

2. $1 \leq \beta_h \leq s_h - 1$. Since $\int_0^{s_h - 1} f_{1h}(x)dx = 0 < \beta_h$, the ratio of utility to wealth donated for β_h is less than 1.

3. $\beta_h = s_h$. Since $\int_0^{\beta_h} f_{1h}(x)dx = \int_0^{s_h} f_{1h}(x)dx = s_h$, the ratio of utility to wealth donated for β_h is equal to 1.

4. $\beta_{1h} \geq s_h + 1$. Since $\int_0^{\beta_h} f_{1h}(x)dx = s_h < \beta_h$, the ratio of utility to wealth donated for β_h is less than 1.

First, assume that for some $\beta \in I$, $\beta_h \in \{0, s_h\} \; \forall \; h \in D$. Then all wealth yields a utility to wealth ratio of 1, and $\Psi_\beta = w$. This proves the "if" direction.

Now, assume that $\exists \; k \in D \; s.t. \; \beta_k \notin \{0, s_k\}$ for some $\beta \in I$. Then there is at least some wealth being donated that returns less than its magnitude in utility. Since no donations can produce more than their magnitude in utility, the total utility to wealth ratio must be less than 1. Therefore, $\Psi_\beta < w \Rightarrow \Psi_\beta \neq w$. This proves the "only if" direction. □

Corollary 25. $\max\{\Psi_{\beta \in I}\} \leq w$.

Proof. Lemma 24 shows that the maximum ratio of utility to wealth donated is 1 in any distribution $\beta \in I$. This means that $\max\{\Psi_{\beta \in I}\} \leq w$. □

Lemma 26. $\Psi_\phi = \max\{\Psi_{\beta \in I}\}$.

Proof. For any wealth distribution $\delta \in WD_\xi$, let P_δ be the list of all destinations that are being funded in non-integer amounts:

$$P_\delta = \{h \in D \mid \delta_h \; mod \; 1 \neq 0\}$$

Note that $P_\delta = \emptyset \; \forall \; \delta \in I$, and that $P_\delta \neq \emptyset \; \forall \; \delta \in NI$. Now examine any distribution $\alpha \in NI$. Let p be the number of elements in P_α, and define α'_h to be the amount over the next lowest integer that the agent is funding destination h:

$$\alpha'_h = \alpha_h - \lfloor \alpha_h \rfloor$$

Since $0 < \alpha'_h < 1 \; \forall \; h \in P_\alpha$, and since the original wealth is an integer and must be distributed completely between the destinations, we have that:

$$\sum_{h \in P_\alpha} \alpha'_h \in \{1, 2, ..., p-1\}$$

Therefore, the average value of the elements in P_α, $\overline{P_\alpha}$, must be less than or equal to $\frac{p-1}{p}$. Since the sum of all donations is also bounded by the agent's wealth $w_1 = t$, we also have that $p - 1 \leq t$. This means that:

$$\overline{P_\alpha} \leq \frac{t}{t+1} \Rightarrow \exists\ k \in P_\alpha\ s.t.\ \alpha'_k \leq \frac{t}{t+1}$$

Since the marginal utility curve for every destination is zero-valued until the last $\frac{1}{t+1}$ portion before any given integer, the last α'_k units of wealth currently being donated to destination k give the agent no utility. This means that, as long as $P \neq \emptyset$, we can always find a destination $k \in P$ such that the last α'_k units of wealth being donated to destination k are giving no utility to the agent.

For any distribution $\alpha \in NI$, we can now construct a corresponding distribution $\alpha^* \in I$ such that $\Psi_{\alpha^*} \geq \Psi_\alpha$ by using the following procedure:

Input: Wealth distribution $\alpha \in NI$.

1. Construct the set P_α.
2. Choose a destination $k \in P_\alpha$ where the last α'_k units of wealth being donated to destination k are providing no utility to the agent.
3. Take α'_k units of wealth away from destination k and donate it instead to the lowest indexed destination $j \in P_\alpha,\ j \neq k$.
4. If the resulting wealth distribution is not in I, relabel it as α and go back to Step 1.
5. Output the resulting wealth distribution, and call it α^*.

This procedure must halt by the following logic. Notice that every iteration through this procedure decreases the number of elements in P_α by at least 1, since by construction we are finding a destination $k \in P_\alpha$ that is being funded in a non-integer amount, removing the excess α'_k units of wealth to turn its funding into an integer amount, and adding that wealth to a destination that is already being funded in a non-integer amount. The number of elements in P_α must therefore continue to decrease until P_α has two elements. At this point, when we run through the procedure, we must take wealth away from one destination in P_α and add it to other destination in P_α. Since the agent's wealth is an integer, there cannot be only one destination that is being funded in a non-integer amount. Therefore, at the end of this iteration, the resulting wealth distribution will be in I.

It is clear that each iteration in the above procedure provides a weak gain in utility for the agent, since Step 3 is specifically designed so that taking that amount of wealth away from that particular destination will cost the agent no utility, while adding the wealth somewhere else has the potential to increase the agent's utility. Therefore, $\forall\ \alpha \in NI$, $\exists\ \alpha^* \in I\ s.t.\ \Psi_{\alpha^*} \geq \Psi_\alpha$.

Now examine a distribution $\beta^* \in I$ such that $\Psi_{\beta^*} = max\{\Psi_{\beta \in I}\} \Rightarrow \Psi_{\beta^*} \geq \Psi_\beta\ \forall\ \beta \in I$. It can be seen by contradiction that $\Psi_{\beta^*} \geq \Psi_\alpha\ \forall\ \alpha \in NI$. This is because if we could find a distribution $\alpha \in NI$ such that $\Psi_\alpha > \Psi_\beta$, we could then use the procedure above to find an $\alpha^* \in I$ such that $\Psi_{\alpha^*} \geq \Psi_\alpha \Rightarrow \Psi_{\alpha^*} > \Psi_\beta$ which would contradict the assumption that $\Psi_{\beta^*} \geq \Psi_\beta\ \forall\ \beta \in I$. Therefore, it must be that $\Psi_{\beta^*} \geq \Psi_\alpha\ \forall\ \alpha \in I\ \cup\ NI$, which means that $\Psi_{\beta^*} = \Psi_\phi$. $\square$