Active Applied Discrete Structures

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Preface

Active Applied Discrete Structures is designed for use in a flipped class using *Applied Discrete Structures*. Each chapter is designed for a 75 minute class period. The class format that is presumed is for the instructor to assign students to read materials that will be covered in class n in time to submit a short assignment by the beginning of class $n-1$. Then at the time of class n, they work on more challenging problems in groups of 4-6 students.

A tricky thing about this format is getting started. On class 1, no prior reading can be expected. The binary representation of positive integers is discussed. This is mostly done through group work. The assumption is that since most students in the class are computer science majors, they have some familiarity with the topic and can help other students with the problems. Class 2 is also a bit of a rush. I ask student do do reading for that class and get responses back to me a couple of days before the second class. They are also asked to prepare for Class 3 and get responses to me by Class 2. Fortunately, the material in Chapter 1 isn't too difficult. The rest of the semester proceeds as described above.

The main web page for Applied Discrete Structures is http://discretemath.org

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Contents

Chapter 1

Binary Representation of Positive Integers

1.1 Reading

Since this is the first class meeting, there is no prior reading. Half of the class is devoted to explaining the way the class will be run. Then we will explore the binary representation of positive integers, which is in Section 1.4 of Applied Discrete Structures. A sheet with the base 10 numbers 1 through 64 and their corresponding binary representations is passed out. Students are asked to identify patterns.

1.2 Questions

1. What base 10 number is equal to 101000010_2?
2. What is the base 2 representation of 911?
3. An even number is an (integer) multiple of 2. For example, 12 is even because $12 = 6 \cdot 2$ but 13 is not even since $12 = \frac{13}{2} \cdot 2$. How can you quickly tell whether a number represented in base 10 is even? How can you quickly tell whether a number represented in base 2 is even?
4. How can you quickly tell whether a number represented in base 10 is a multiple of 5? Can you quickly tell whether a number represented in base 2 is a multiple of 5?
5. How can you quickly tell whether a number represented in base 10 is a multiple of 8? Can you quickly tell whether a number represented in base 2 is a multiple of 8?
6. How can you quickly tell whether a number represented in base 10 is a multiple of 9? Can you quickly tell whether a number represented in base 2 is a multiple of 9?

1.3 Handouts

Look for patterns in these two tables. The second gives the binary form of integers padded with 0's so as to contain exactly 4 bits.

Base 10	Base 2	Base 10	Base 2
1	1_2	33	100001_2
2	10_2	34	100010_2
3	11_2	35	100011_2
4	100_2	36	100100_2
5	101_2	37	100101_2
6	110_2	38	100110_2
7	111_2	39	100111_2
8	1000_2	40	101000_2
9	1001_2	41	101001_2
10	1010_2	42	101010_2
11	1011_2	43	101011_2
12	1100_2	44	101100_2
13	1101_2	45	101101_2
14	1110_2	46	101110_2
15	1111_2	47	101111_2
16	10000_2	48	110000_2
17	10001_2	49	110001_2
18	10010_2	50	110010_2
19	10011_2	51	110011_2
20	10100_2	52	110100_2
21	10101_2	53	110101_2
22	10110_2	54	110110_2
23	10111_2	55	110111_2
24	11000_2	56	111000_2
25	11001_2	57	111001_2
26	11010_2	58	111010_2
27	11011_2	59	111011_2
28	11100_2	60	111100_2
29	11101_2	61	111101_2
30	11110_2	62	111110_2
31	11111_2	63	111111_2
32	100000_2	64	1000000_2

n	padded binary n
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
14	1110
15	1111

Chapter 2

Sets and Operations on them

2.1 Reading

Before class, read Sections 1.1 and 1.2 of Applied Discrete Structures. Respond to the following question: How are the set operations union and intersection similar to the operations addition and multiplication on numbers, and how are they different?

Also, turn in solutions to these exercises: Section 1.1: #2, and Section 1.2: #2

2.2 In-Class Questions

1. Section 1.1 #4 (b), (c)

2. Section 1.2 #4 (b) and #6

3. Find two sets A and B for which $|A| = 5$, $|B| = 6$, and $|A \cup B| = 9$. What is $|A \cap B|$?

4. For any sets A and B, define $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ and $AB = \{ab \mid a \in A \text{ and } b \in B\}$. If $A = \{1, 2\}$ and $B = \{2, 3, 4\}$, what is $|A \times B|$? What is $|AB|$?

5. A common data structure for a software implementation of sets is a "bitmap." The way it works is if you want to work with subsets of a universe, U, with cardinality n you first establish an ordering of U when u_k is the kth element. A set A is then represented by a string of n bits $b_1b_2 \ldots b_n$ when b_k is 1 if $u_k \in A$ and is 0 otherwise. In the following questions, assume $U = \{1, 2, 3, 4, 5\}$ with the ordering as listed.

   (a) What are the bit strings for the empty set and for U?

   (b) What are the bit strings for $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$?

   (c) What are the general rules for determining the the bit strings for $A \cap B$ and $A \cup B$? What their bit strings in this particular case?

Chapter 3

Sets, Sums & Products

3.1 Reading

Read Sections 1.3 and 1.5 of Applied Discrete Structures.

Response Question: If A is a finite set, why is the number of elements in the power set of A a power of 2?

Also, turn in solutions to these exercises:

1. Let $B = \{0, 1\}$. List elements of $\mathcal{P}(B)$, $B \times B$ and $B \times B \times B$.
2. Calculate $\sum_{k=1}^{3}(2k-1)$, $\sum_{k=1}^{4}(2k-1)$, and $\sum_{k=1}^{5}(2k-1)$. Do you see a pattern?

3.2 In-Class Questions

1. Let $X = \{n \in \mathbb{N} \mid 10 \leq n < 20\}$. Find examples of sets with the properties below and very briefly explain why your examples work.
   (a) A set $A \subseteq \mathbb{N}$ with $|A| = 10$ such that $X - A = \{10, 12, 14\}$.
   (b) A set $B \in \mathcal{P}(X)$ with $|B| = 5$.
   (c) A set $C \subseteq \mathcal{P}(X)$ with $|C| = 5$.
   (d) A set $D \subseteq X \times X$ with $|D| = 5$.
   (e) A set $E \subseteq X$ such that $|E| \in E$.
2. Explain why there is no set A which satisfies $A = \{2, |A|\}$
3. Use summation or product notation to rewrite the following.
   (a) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{50}$
   (b) $1 + 5 + 9 + 13 + \cdots + 421$
   (c) $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \cdots \cdot \frac{99}{100}$
4. Are there sets A and B such that $|A| = |B|$, $|A \cup B| = 10$, and $|A \cap B| = 5$? Explain.
5. Consider the universe of postive integers greater than or equal to 2. Let A_2 be the set of all multiples of 2 except for 2. Let A_3 be the set of all multiples of 3 except for 3. And so on, so that A_n is the set of all multiple of n except for n, for any $n \geq 2$. Describe (in words) the set $(A_2 \cup A_3 \cup A_4 \cup \cdots)^c$.

Chapter 4

Counting: Product Rule and Permutations

4.1 Reading

Read Sections 2.1 and 2.2 of Applied Discrete Structures

Response Question: Suppose A and B are finite sets. Explain how the cardinality the Cartesian product $A \times B$ can be determined using the Rule of Products.

Also, turn in solutions to these exercises:

1. 2.1: #4
2. 2.2: How many ways can the letters in the word DRACUT be arranged? They don't have to form a real word.

4.2 In-Class Questions

1. How many of the integers from 100 to 999 have the property that the sum of their digits is even? For example, 561 would counted, but 214 would not be counted.
2. How many positive integers divide evenly into $67,500 = 2^2 3^3 5^4$?
3. The manager of a baseball team has decide on the batting order of his team. He has selected the nine batters already.
   (a) How many ways could he select a batting order?
   (b) He decides that the catcher must bat before the shortstop? How many ways can he select a batting order now?
   (c) In addition to the restriction about the catcher and shortstop, suppose he decides that the pitcher must bat immediately after the first baseman. How many ways can the manager select a batting order now?
4. How many ways can the letters in the word APPLE be arranged?

Chapter 5

Partitions and Combinations

5.1 Reading

Read Sections 2.3 and 2.4 of Applied Discrete Structures.

Response question: In mathematics, the word partition is used in two contexts. One is for partitions of sets, as described in Section 2.3. The other is for partitions of a positive integer. An example of a partition of 5 is $3+1+1$, a sum of positive integers equal to 5. It is customary to write the terms of the sum in non-increasing order since $1+3+1$ is considered the same partition of 5. The other partitions of 5 are 5, $4+1$, $3+2$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$. How might a listing of all partitions of an integer like 5 help in listing all partitions of a set with that many elements?

Exercises to do and turn in:

1. 2.3 #2
2. 2.4 #4

5.2 In-Class Questions

1. Section 2.3 #6
2. How many different partitions are there of the set $\{1, 2, 3, 4, 5\}$
3. How many ways can you arrange the letters in the word BOOKKEEPER?
4. Section 2.4 #12
5. Section 2.4 #5
6. Section 2.4 #6
7. Consider the set of lattice paths from $(0, 0)$ to $(8, 8)$. You should know one quick formula for the cardinality of that set. However, counting a different way can lead to an interesting identity involving binomial coefficients. Notice that any path goes through exactly one of the points $(0, 8), (1, 7), (2, 6), \ldots, (8, 0)$. Count the number of lattice paths that go through each of those 9 points - leave the expression in terms of binomial coefficients. Even more interesting is what you get if generalize to a destination of (n, n), $n \geq 1$.

5.3 Some Lattices

Here are a couple of lattices for you to doodle with.

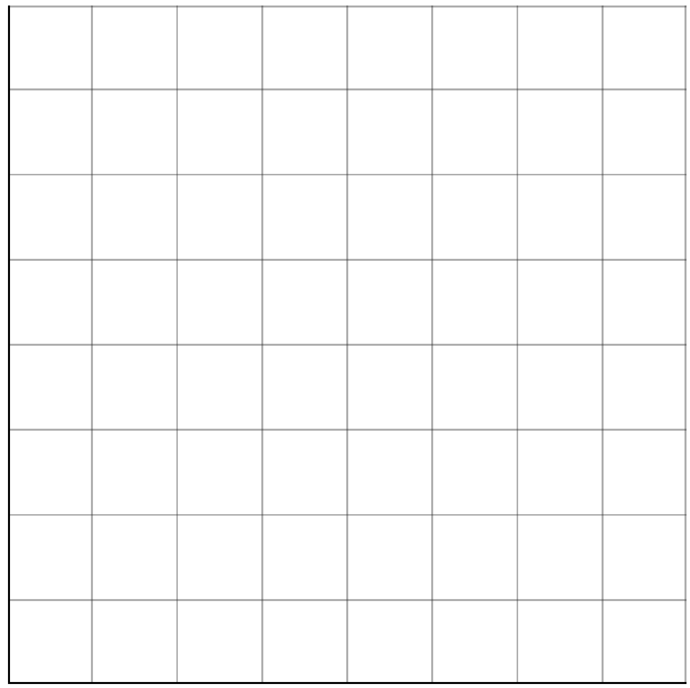

Figure 5.3.1

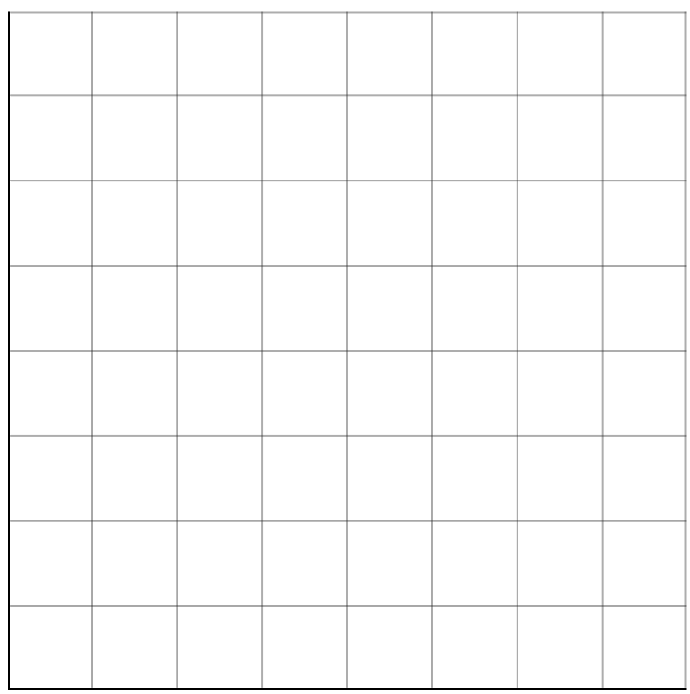

Figure 5.3.2

Chapter 6

Logic: Propositions and Truth Tables

6.1 Reading

Read sections 3.1 and 3.2 of Applied Discrete Structures.

Response Question: Suppose you were given a proposition generated by 100 propositional variables and you are asked whether there is at least one assignment of truth values that you could assign to these variables to make the proposition true. Why is constructing a truth table not practical. If you decided to examine all possible assignments of truth values and your computer could check one million cases per second, approximately how long would it take to check all cases?

Also, turn in solutions to these exercises:

- Section 3.1. #2
- Section 3.2 #2, parts (a) and (c)

6.2 In-Class Questions

1. Reword the following statements into "If...then" statements.
   (a) No resident of Chelmsford likes hot peppers.
   (b) For 3+7=10, it is necessary that cows fly.
   (c) For 3+7=10, it is sufficient that cows fly.
   (d) Lowell is the oldest city in Massachusetts unless mermaids exist.
   (e) I carry an umbrella when it rains.
2. Construct the truth table for $(p \vee q) \wedge (p \vee \neg q)$. Notice anything about the result?
3. Consider the statement "If Boris visits Hampton Beach, then he eats fried clams."
   (a) Write the converse of the statement.
   (b) Write the contrapositive of the statement.

(c) Is it possible for the contrapositive to be false? If it was, what would that tell you?

(d) Suppose the original statement is true, and that Boris eats fried clams. Can you conclude anything (about his travels)?

(e) Suppose the original statement is true, and that Boris does not eat fried clams. Can you conclude anything (about his travels)?

4. Consider the statement, "If a number is triangular or square, then it is not prime"

   (a) Make a truth table for the statement $(T \vee S) \to \neg P$.

   (b) If you believed the statement was false, what properties would a counterexample need to possess? Explain by referencing your truth table.

   (c) If the statement were true, what could you conclude about the number 5657, which is definitely prime? Again, explain using the truth table.

Chapter 7

Equivalence, Implication, and Laws of Logic

7.1 Reading

Read sections 3.3 and 3.4 of Applied Discrete Structures.

Response question: Explain why every proposition implies a tautology.

Also, turn in solutions to these exercises:

- 3.3: #2
- 3.4: #2

7.2 In-Class Questions

1. Find a proposition that is equivalent to $p \vee q$ and uses only conjunction and negation.

2. Frankie Fib was telling you what he consumed yesterday afternoon. He tells you, "I had either popcorn or raisins. Also, if I had cucumber sandwiches, then I had soda. But I didn't drink soda or tea." Of course you know that Frankie is the worlds worst liar, and everything he says is false. What did Frankie have to eat and drink?

3. Construct the truth table for $(p \to q) \wedge (q \to r) \wedge (r \to p)$. Notice anything about the result?

4. The significance of the Sheffer Stroke is that it is a "universal" operation in that all other logical operations can be built from it.

   (a) Prove that $p|q$ is equivalent to $\neg(p \wedge q)$.

   (b) Prove that $\neg p \Leftrightarrow p|p$.

   (c) Build $\wedge$ using only the Sheffer Stroke.

   (d) Build $\vee$ using only the Sheffer Stroke.

7.3 The Sheffer Stroke

Another logical operation is the Sheffer Stroke, which is the subject of one of the exercises.

Table 7.3.1 Truth Table for the Sheffer Stroke

p	q	$p \mid q$
0	0	1
0	1	1
1	0	1
1	1	0

Chapter 8

Structured Proofs

8.1 Reading

Read section 3.5 of Applied Discrete Structures.

Response question: A proposition, P, generated by a set of propositional variables is said to be satisfiable if there is at least one way to assign truth values to all of the variables so that P Is true. Explain why P is satisfiable as long as $\neg P$ is not a tautology.

Also, turn in solutions to these exercises:

- Put the following into symbolic form and check its validity: If I am a good person, nothing bad will happen to me. Nothing happened to me. Therefore, I am a good person.
- Section 3.5: #4 (a)

8.2 In-Class Questions

1. Prove either directly or indirectly:

$$a \vee b, c \wedge d, a \rightarrow \neg c \Rightarrow b$$

2. In these two Lewis Carroll puzzles, you are given premises and are expected to form your own conclusion. In each of them, convert the premises to symbolic form, draw a conclusion, and then translate back to English.

   (a)
   - No bald creature needs a hairbrush.
   - No lizards have hair.

   (b)
   - Promise breakers are untrustworthy.
   - Wine drinkers are very communicative.
   - A man who keeps his promises is honest.
   - No teetotalers are pawnbrokers.
   - One can always trust a very communicative person.

3. There $n+1$, $n \geq 1$> people who want to go to a concert. All have different ages. You have three tickets: a back-stage pass and two regular (but distinguishable) tickets. Here are the rules for passing out the tickets:

- The backstage pass must go to the oldest person who gets a ticket.
- The person who gets the backstage pass can' t get either of the other two tickets, but the two regular tickets can both go to the same person.

How many ways can you give away the tickets? There are two ways to count. Find both and equate them.

8.3 Basic Logical Inferences

From section 3.4 of Applied Discrete Structures:

Table 8.3.1 Basic Logical Laws - Common Implications and Equivalences

Detachment (AKA Modus Ponens)	$(p \to q) \land p \Rightarrow q$
Indirect Reasoning (AKA Modus Tollens)	$(p \to q) \land \neg q \Rightarrow \neg p$
Disjunctive Addition	$p \Rightarrow (p \lor q)$
Conjunctive Simplification	$(p \land q) \Rightarrow p$ and $(p \land q) \Rightarrow q$
Disjunctive Simplification	$(p \lor q) \land \neg p \Rightarrow q$ and $(p \lor q) \land \neg q \Rightarrow p$
Chain Rule	$(p \to q) \land (q \to r) \Rightarrow (p \to r)$
Conditional Equivalence	$p \to q \Leftrightarrow \neg p \lor q$
Biconditional Equivalences	$(p \leftrightarrow q) \Leftrightarrow (p \to q) \land (q \to p) \Leftrightarrow (p \land q) \lor (\neg p \land \neg q)$
Contrapositive	$(p \to q) \Leftrightarrow (\neg q \to \neg p)$

Chapter 9

Mathematical Induction

9.1 Reading

Read Sections 3.6 and 3.7 of Applied Discrete Structures. It is only necessary to read 3.6 through Example 3.6.7.

Response question: You don't need induction to prove that the sum of the first n Positive integers equals $\frac{n(n+1)}{2}$. Google "Gauss sum of consecutive integers" and read about how you can do it even more simply. Explain what you read.

Also, turn in solutions to these exercises:

- Simplify the expressions
  - (a) $(\sum_{k=1}^{n+1} k^2) - (\sum_{k=1}^{n} k^2)$
  - (b) $\sum_{k=1}^{n}(\frac{1}{k} - \frac{1}{k+1})$
  - (c) $\frac{(n+2)!}{n!}$
- Prove that for $n \geq 0$, $\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$.

9.2 In-Class Questions

1. Prove that for $n \geq 1$,
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

2. Prove that it is possible to make up any postage of 28 cents or more using only five-cent and eight-cent stamps.

3. Suppose that a particular real number x has the property that $x + \frac{1}{x}$ is an integer. Prove that $x^n + \frac{1}{x^n}$ is an integer for all natural numbers n.

Chapter 10

Quantifiers and Proof Review

10.1 Reading

Read Sections 3.8 and 3.9 of Applied Discrete Structures

Response Question: In reviewing a certain local coffee roaster, a writer stated "...but all of its coffee is not fair trade." The writer was rebutting a claim by the roaster that "All of our coffee is fair trade." Explain why the reviewer's statement was incorrect.

Also, turn in solutions to these exercises:

- Section 3.8: #2
- Section 3.9: #2

10.2 In-Class Questions

1. Translate the following statement over the positive integers into symbols. Use $E(x)$ for "x is even" and $O(x)$ for "x is odd" in the first three parts.
   (a) No number is both even and odd.
   (b) One more than any even number is an odd number.
   (c) There is prime number that is even.
   (d) Between any two numbers there is a third number.
   (e) There is no number between a number and one more than that number.
2. Use quantifiers to state that for every positive integer, there is a larger positive integer.
3. One of the following is true and the other is false. Identify the true one says and explain why the other one is false.
$$(\exists b)_{\mathbb{Z}}((\forall a)_{\mathbb{Z}}(a+b=0))$$
$$(\forall a)_{\mathbb{Z}}((\exists b)_{\mathbb{Z}}(a+b=0))$$
4. Prove that the sum of of an odd integer and and even integer is odd.
5. Prove that if you divide 4 into a perfect square, $1, 4, 9, 16, \ldots$, the remainder will be either 0 or 1.
6. Prove that the cube root of 2 is an irrational number.

Chapter 11

Set Theory Logic

11.1 Reading

Read Sections 4.1 and 4.2 of Applied Discrete Structures

Response Question: Compare the Laws of Set Theory in Section 4.2 of Applied Discrete Structures with the Basic Laws of Logic in Section 3.5 of Applied Discrete Structures. Focus on any two different laws of set theory that you choose and discuss how they are similar to two logic laws.

Also, turn in solutions to these exercises:

- Section 4.1 #2.
- Section 4.2 #2 (b) and (c) only.

11.2 In-Class Questions

1. What can one say about the sets A and B if we know the following? Back up your answers with proofs.
   (a) $A \cup B = A$
   (b) $A \cap B = A$
   (c) $A – B = A$
   (d) $A \cap B = B \cap A$
   (e) $A – B = B – A$
2. (a) Given the following sets of integers, A, B, C, find the set of elements that belong to exactly one of the three sets.

$$\begin{aligned} A &= \{2, 4, 6, 8, 10, 12, 14, 16, 18\} \\ B &= \{2, 3, 5, 7, 11, 13, 17, 19\} \\ C &= \{3, 6, 9, 12, 15, 18\} \end{aligned}$$

   (b) Prove that for any three sets, A, B, C,

$$(A \cup B \cup C) \cap ((A^c \cap B^c) \cup (A^c \cap C^c) \cup (B^c \cap C^c))$$

   is the set of all elements that belong to exactly one of the three sets. Verify this fact first with the example in the previous part.

(c) Find a similar expression for the set of elements that belong to exactly one of any four sets A, B, C, D.

3. Recall that the power set of any set A is the set of all subsets of A and is denoted $\mathcal{P}(A)$. Which of the following are true?

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(b)$$
$$\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(b)$$

If either is not true, can you replace the equals sign with $\subseteq$ or $\supseteq$ to get a true statement?

Chapter 12

Minsets and Duality

12.1 Reading

Read Sections 4.3 and 4.4 of Applied Discrete Structures

Response Question: To what extend is there any duality in arithmetic of numbers with addition and multiplication? How does it break down where is doesn't in set theory?

Also, turn in solutions to these exercises:

- Consider the subsets $A = \{1, 3, 5\}$, $B = \{2, 3, 4\}$, where $U = \{1, 2, 3, 4, 5\}$. List the nonempty minsets generated by A and B.
- What is the dual of $A \cap (B \cap (A \cap B)^c) = \emptyset$?

12.2 In-Class Questions

1. A common way to denote a particular minset generated by a collection of subsets is as follows. If there are k subsets, $B_1, B_2, \ldots, B_k$, and $b = b_1 b_2 \cdots b_k$ is any string of k bits, then

$$M_b = M_{b_1 b_2 \cdots k} = D_1 \cap D_2 \cap \cdots \cap D_k,$$

where D_i is either B_i or B_i^c. If $b_i = 1$ then $D_i = B_i$ and if $b_i = 0$ then $D_i = B_i^c$. For example, if $k = 4$, $M_{0110} = B_1^c \cap B_2 \cap B_3 \cap B_4^c$.

   (a) Suppose $U = \{1, 2, 3, 4, 5\}$, $k = 2$, $B_1 = \{1, 2\}$, and $B_2 = \{2, 3, 4\}$. List the minsets generated by B_1 and B_2 using "M_b" notation. Notice that they form a partition of U.

   (b) How does this notation make help us see how many distinct minsets there could be that are generated by k subsets of a universe.

2. (a) Partition $\{1, 2, \ldots, 8\}$ into the minsets generated by $B_1 = \{1, 2\}$, $B_2 = \{1, 3, 5, 8\}$, and $B_3 = \{2, 3, 4, 6\}$.

   (b) How many different subsets of $\{1, 2, \ldots, 8\}$ can you create using B_1, B_2, and B_3 with the standard set operations?

   (c) Do there exist subsets C_1, C_2, C_3 with which you can generate every subset of $\{1, 2, ..., 8\}$? If so, can you find such a collection of subsets? If not, why? You might find the Venn diagram below useful for thinking about this problem.

3. What is the dual of a minset? These sets are called “maxsets” Find the maxsets generated by the two sets in part (a) of the first problem. Why do you suppose they are called maxsets?

4. The descriptions of duality in Section 4.4 is not complete. If you expand expressions involving subsets, such as the expression $A \cap B \subseteq A$, which is a true statement in set theory. What should be the dual? How should we treat the subset symbol?

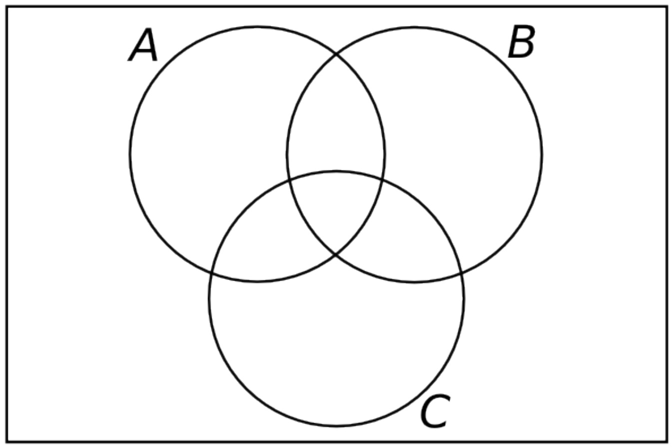

Figure 12.2.1 A three set Venn diagram

Chapter 13

Matrix Operations

13.1 Reading

Read Sections 5.1 and 5.2 of Applied Discrete Structures.

Response Question: Let $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. Select any 2 by 2 matrix with nonzero entries and call it B. Compute the products AB and BA What effect does A have on B in each case?

Also, turn in solutions to these exercises:

- Let $A = \left(\begin{array}{cc} 1 & -1 \\ 2 & 3 \end{array}\right)$ and $B = \left(\begin{array}{cc} 0 & 1 \\ 3 & -5 \end{array}\right)$

  (a) Compute AB and BA.

  (b) Compute $A + B$ and $B + A$.

- For the given matrices A find A^{-1} if it exists and verify that $AA^{-1} = A^{-1}A = I$. If A^{-1} does not exist explain why.

  (a) $A = \left(\begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array}\right)$

  (b) $A = \left(\begin{array}{cc} 2 & 1 \\ 4 & 2 \end{array}\right)$

There is a short video on matrix multiplication at https://youtu.be/zt-IU1lXFzs

13.2 In-Class Questions

1. Let $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$. Compute the product AB. Based on this result, what is A^{-1}.

2. If A is a an $m \times n$ matrix, we define the transpose of A to be the $n \times m$ matrix whose rows are the columns of A. For example, the transpose of

$$\left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right) \text{ is } \left(\begin{array}{cc} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{array}\right).$$

   The notation A^t is used for the transpose of A.

(a) If A is an $m \times n$ matrix, are the products AA^t and A^tA defined? What are the orders of the products that are defined?

(b) Given the following matrix, what useful information might you get from the products AA^t or A^tA.?

$$A = \begin{pmatrix} 16 & 11 & 4 & 3 & 15 \\ 16 & 17 & 13 & 12 & 6 \end{pmatrix}$$

3. If

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

show that $AX = B$ is a way of expressing the system $\begin{aligned} 2x_1 + x_2 &= 3 \\ x_1 - x_2 &= 1 \end{aligned}$ using matrices.

Express the following systems of equations using matrices:

(a) $\begin{aligned} 2x_1 - x_2 &= 4 \\ x_1 + x_2 &= 0 \end{aligned}$

(b) $\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ x_1 - x_2 + x_3 &= -1 \\ x_1 + 3x_2 + x_3 &= 5 \end{aligned}$

4. Prove by induction that for $n \geq 1$, $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$.

5. In this exercise, we propose to show how matrix multiplication is a natural operation. Suppose a bakery produces bread, cakes and pies every weekday, Monday through Friday. Based on past sales history, the bakery produces various numbers of each product each day, summarized in the 5×3 matrix D. It should be noted that the order could be described as "number of days by number of products." For example, on Wednesday (the third day) the number of cakes (second product in our list) that are produced is $d_{3,2} = 4$.

$$D = \begin{pmatrix} 25 & 5 & 5 \\ 14 & 5 & 8 \\ 20 & 4 & 15 \\ 18 & 5 & 7 \\ 35 & 10 & 9 \end{pmatrix}$$

The main ingredients of these products are flour, sugar and eggs. We assume that other ingredients are always in ample supply, but we need to be sure to have the three main ones available. For each of the three products, The amount of each ingredient that is needed is summarized in the 3×3, or "number of products by number of ingredients" matrix P. For example, to bake a cake (second product) we need $P_{2,1} = 1.5$ cups of flour (first ingredient). Regarding units: flour and sugar are given in cups per unit of each product, while eggs are given in individual eggs per unit of each product.

$$P = \begin{pmatrix} 2 & 0.5 & 0 \\ 1.5 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

These amounts are "made up", so don't used them to do your own baking!

(a) How many cups of flour will the bakery need every Monday? Pay close attention to how you compute your answer and the units of each number.

(b) How many eggs will the bakery need every Wednesday?

(c) Compute the matrix product DP. What do you notice?

(d) Suppose the costs of ingredients are \$0.12 for a cup of flour, \$0.15 for a cup of sugar and \$0.19 for one egg. How can this information be put into a matrix that can meaningfully be multiplied by one of the other matrices in this problem?

Chapter 14

Matrix Laws and Oddities

14.1 Reading

Read Sections 5.3 and 5.4 of Applied Discrete Structures

Response Question: Compare Matrix Law (15), The Inverse of Product Rule, with the fact that although you put your socks on before your shoes, you take your shoes off before taking off your socks.

Also, turn in solutions to these exercises:

- Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Compute A^2, A^3, A^4, and A^{-1}.
- Find at least three 2×2 matrices, A, such that $A^2 = A$.

14.2 In-Class Questions

1. Let A and B be $n \times n$ matrices of real numbers. Is $A^2 - B^2 = (A - B)(A + B)$? Explain.
2. Write each of the following systems in the form $AX = B$, and then solve the systems using matrices.

   (a) $\begin{array}{r} 4x_1 - 6x_2 = 20 \\ 3x_1 + 5x_2 = -6 \end{array}$ (b) $\begin{array}{r} 5x_1 - 1x_2 = 11 \\ -16x_1 + 5x_2 = 12 \end{array}$

3. Suppose that A, P, and B are all $m \times m$ matrices, $m \geq 2$, and $A = P^{-1}BP$. Prove that $A^n = P^{-1}B^nP$ for all $n \geq 1$.
4. Let $M_{n\times n}(\mathbb{R})$ be the set of real $n \times n$ matrices. Let $P \subseteq M_{n\times n}(\mathbb{R})$ be the subset of matrices defined by $A \in P$ if and only if $A^2 = A$. Let $Q \subseteq P$ be defined by $A \in Q$ if and only if $\det A \neq 0$.

   (a) Determine the cardinality of Q.

   (b) Consider the special case $n = 2$ and prove that a sufficient condition for $A \in P \subseteq M_{2\times 2}(\mathbb{R})$ is that A has a zero determinant (i.e., A is singular) and $tr(A) = 1$ where $tr(A) = a_{11} + a_{22}$ is the sum of the main diagonal elements of A.

   (c) Is the condition of part b a necessary condition?

Chapter 15

Relations

15.1 Reading

Read Sections 6.1 and 6.2 of Applied Discrete Structures

Response Question: Although any subset of a cartesian product of a set with itself can be a relation on that set, in the long run we are most concerned with a few important ones. Three examples of very important relations are

- Less than or equal to, $\leq$, on the integers,
- Set containment, $\subseteq$, on the power set of a set,
- Logical implication, $\Rightarrow$, on any set of propositions.

Discuss any similarities you see between these three relations.

Also, turn in solutions to these exercises:

1. Consider the two relations on people: M, where aMb if a's mother is b; and S, where aSb if a and b are siblings. Describe, in words, the two relations MS and SM.
2. Let $A = \{1, 2, 3, 4, 6, 12\}$. Draw a digraph for the relation "divides" on A.

15.2 In-Class Questions

1. Let S be the set of "spaces" in the floor of your classroom. Draw a digraph of the relation c, where $s_1 c s_2$ if and only if s_1 is connected to s_2 with at least on doorway.
2. Given s and t, relations on $\mathbb{Z}$, $s = \{(1, n) : n \in \mathbb{Z}\}$ and $t = \{(n, 1) : n \in \mathbb{Z}\}$, what are st and ts? Hint: Even when a relation involves infinite sets, you can often get insights into them by drawing partial graphs.
3. Let A be the set of strings of 0's and 1's of length 3 or less.
   (a) Define the relation of w on A by xwy if x has the same number of 1's as y. For example, $01w100$, but $01w101$ is false. Draw a digraph for this relation.
   (b) Do the same for the relation p defined by xpy if x is a prefix of y. For example, $10p101$, but $01p101$ is false.

4. Consider logical implication, $\Rightarrow$, on the set of propositions $\{0, 1, p, q, p \vee q, p \wedge q, p \wedge p\}$. Draw a digraph of this relation.

Chapter 16

Properties of Relations

16.1 Reading

Read Section 6.3 of Applied Discrete Structures

Response Question: Recall that in geometry, two triangles are similar if and only if their corresponding angles have the same measure. What kind of relation is this on the set of all triangles on the plane?

Also, turn in solutions to these exercises:

- Prove that congruence modulo m is a transitive relation on the set of integers. Do this by assuming that $a \equiv_m b$ and $b \equiv_m c$, and applying the definition for $\equiv_m$ to conclude that $a \equiv_m c$
- Draw the ordering diagram for the relation "divides" on the divisors of $40 = 2^3 \cdot 5$.

16.2 In-Class Questions

1. Let $A = \{a, b, c, d\}$. Draw the graphs of relations on A where:
   (a) The first relation is reflexive, symmetric, but not transitive.
   (b) The second relation is transitive, but not symmetric and not reflexive.
   (c) The third relation is both an equivalence relation and a partial ordering.
2. Let $A = \{0, 1, 2, 3\}$ and let

$$r = \{(0,0),(1,1),(2,2),(3,3),(1,2),(2,1),(3,0),(0,3)\}$$

   (a) Verify that r is an equivalence relation on A.
   (b) Let $a \in A$ and define $c(a) = \{b \in A \mid arb\}$. $c(a)$ is called the **equivalence class of a under r**. Find $c(a)$ for each element $a \in A$.
   (c) Show that $\{c(a) \mid a \in A\}$ forms a partition of A for this set A.
   (d) Let r be an equivalence relation on an arbitrary set A. Prove that the set of all equivalence classes under r constitutes a partition of A.

3. Describe the equivalence classes under the relation congruence modulo 10 on the integers.

4. Let A be the set of strings of 0's and 1's of length 3 or less; and let B be the set of strings of 0's and 1's of length 3. What properties do the following relations have?

   (a) Define the relation of w on A by xwy if x has the same number of 1's as y. For example, $01w100$, but $01w101$ is false.

   (b) Define the the relation d on B defined by xdy if x differs from y in exactly one position. For example, $100d101$, but $100d111$ is false.

   (c) Define the the relation c defined on A by xcy if x is contained within y. For example, $10c101$, but $11c101$ is false.

   For any of these relations that are partial orderings, draw the Hasse diagram for that relation. For any of them that is an equivalence relation, identify the equivalence classes.

16.3 Congruence Modulo n

This is a fundamental relation on the set of integers.

Definition 16.3.1 Congruence Modulo m. Let m be a positive integer, $m \geq 2$. We define **congruence modulo m** to be the relation $\equiv_m$ defined on the integers by

$$a \equiv_m b \Leftrightarrow m \mid (a - b)$$

$\Diamond$

Chapter 17

Relation Matrices and Closure

17.1 Reading

Read Sections 6.4 and 6.5 of Applied Discrete Structures

Response Question: Let p be the relation on people where xpy if y is either x's mother or father. What is $\{z \mid xp^{+}z\}$?

Also, turn in solutions to this exercise:

Consider the relation, s, defined by the graph in Figure 17.1.1.

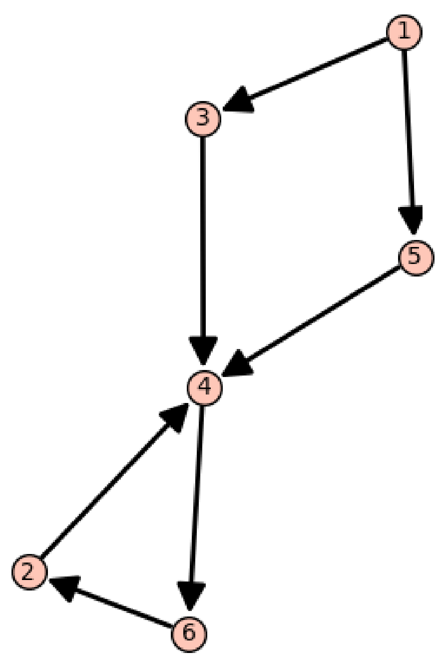

Figure 17.1.1 Digraph of s

(a) Determine the adjacency matrix of s.

(b) Use the matrix you have constructed, find the matrix of s^2 using matrix multiplication.

(c) Draw the graph of defined by the matrix product and verify that it is the graph of s^2.

(d) Determine the matrix of the transitive closure of s.

17.2 In-Class Questions

1. Let D be the set of weekdays, Monday through Friday, let W be a set of employees $\{1, 2, 3\}$ of a tutoring center, and let V be a set of computer languages for which tutoring is offered, $\{A(PL), B(asic), C(++), J(ava), L(isp), P(ython)\}$. We define s (schedule) from D into W by dsw if w is scheduled to work on day d. We also define r from W into V by wrl if w can tutor students in language l. If s and r are defined by matrices

$$S = \begin{array}{c} \\ M \\ T \\ W \\ R \\ F \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \end{array} \quad \text{and } R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & J & L & P \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

   (a) compute SR using Boolean arithmetic and give an interpretation of the relation it defines, and

   (b) compute SR using regular arithmetic and give an interpretation of what the result describes.

2. Let $A = \{a, b, c, d\}$. Let r be the relation on A with adjacency matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

   (a) Explain why r is a partial ordering on A.

   (b) Draw its Hasse diagram.

3. What common relations on $\mathbb{Z}$ are the transitive closures of the following relations?

   (a) aSb if and only if $a + 1 = b$.

   (b) aRb if and only if $|a - b| = 2$.

4. (a) Prove that if r is a transitive relation on a set A, then $r^2 \subseteq r$.

   (b) Find an example of a transitive relation for which $r^2 \neq r$.

References

Problems, particulary in-class problems, have come from several sources. I've attempted to identify any distinctive problems and the sources are listed here.

[1] Doerr, A, and K. Levasseur, *Applied Discrete Structures*

[2] David Pengelley, *From Lecture to Active Learning: Rewards for All, and Is It Really So Difficult?*, The College Math Journal, January 2020, **51** no. 1, 13–24, doi.org/10.1080/07468342.2020.1680228
The idea of teaching in an active learning environment had been on my radar for a while and I'd experimented with some aspects of the format. This article was the final impetus for launching this project.

Index