Overview

This project asks what types of functions can appear in (certain well behaved) *representations of quantised enveloping algebras of Lie superalgebras*. Specifically, do these functions have strong enough recursive behavior to qualify as *q-holonomic functions*?

We're motivated by the fact that these representations can be used to make quantum knot invariants, whose q-holonomicity is closely linked to the role they play in physics. Lie superalgebras are particularly interesting because they appear in recent constructions of physical theories known as Chern-Simons theories [MW15].

Main References:

Lie Superalgberas: We will focus on $\mathfrak{sl}(n|m)$ for $n \neq m$, and on specifically their typical representations.

- The classic references are [Kac77, Kac78].
- For a textbook, see [Mus12].

The Quantum Group $U_h\mathfrak{sl}(n|m)$ - we will focus on representations that come from those of $\mathfrak{sl}(n|m)$, but need the *quantum Serre relations* involving $q = e^{h/2}$ that appear in the quantisation in order to get interesting q-holonomic functions.

- Introduced in [Yam94], see especially Theorem 10.5.1 for a presentation.
- See also [Gee06], and [Gee07, §3.3] for some more brief treatments.
- A textbook for the non-super case is [Lus10].
- See [Gee07, Theorem 1.2] for the relationship between $\mathfrak{sl}(n|m)$ and $U_h\mathfrak{sl}(n|m)$ representations.
- For a in depth look at $\mathfrak{sl}(2|1)$, see [BG24].

q-Holonomic Systems: We'll want to use the closure properties.

- A good survey is [GL16].
- The classic is [Sab93], but it's in french.
- For the $\mathfrak{sl}(2|1)$ version, there's [BG24].
- For a proof of q-holonomicity for the coloured Jones polynomial, there's [GL05].

Details

We will focus on *typical representations* of the classical lie superalgebra $\mathfrak{sl}(n|m)$, where $n \neq m$. These have standard deformations to $U_h\mathfrak{sl}(n|m)$ representations and are classified up to isomorphism by a tuple of complex parameters $(a_1, \ldots, a_{m+n+1})$ which must satisfy the following set of linear inequalities:

$$a_{m+1} \neq \sum_{k=m+2}^{j} a_k - \sum_{\ell=1}^{m} a_\ell - 2m - 2 + i + j \tag{1}$$

for $i = 1, \ldots, m+1$, $j = m+1, \ldots, m+n+1$, see [Kac78, Example 1, pg 620].

The parameters a_k are called the *weights* of the associated representation $V(\overline{a})$, which is characterised by having a *highest weight vector* $v \in V(\overline{a})$ such that

$$h_k v = a_k v, \quad \text{and} \quad E_k v = 0 \quad \text{for } k = 1, \ldots, m+n+1. \tag{2}$$

Process:

- We want to describe the associated $U_h\mathfrak{sl}(n|m)$-representations in terms of explicit matrices (which will depend on the parameters n, m and a_k.)
- Then we want to prove that the coefficients of those matrices are q-holonomic functions.
- We'd like to also understand if the R-matrix has q-holonomic coefficients.

References

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