Overview

This project asks what types of functions can appear in (well behaved) *representations of Lie superalgebras*. Specifically, do these functions have strong enough recursive behavior to qualify as *q-holonomic functions*?

We're motivated by the fact that these representations can be used to make quantum knot invariants, whose q-holonomicity is closely linked to the role they play in physics. Lie superalgebras are particularly interesting because they appear in recent constructions of physical theories known as Chern-Simons theories [MW15].

Main References:

Lie Superalgberas: We will focus on $\mathfrak{sl}(n|m)$ for $n \neq m$, and on specifically their typical representations. The classic references are [Kac77, Kac78].

q-Holonomic Systems: We'll want to use the closure properties.

- A good survey is [GL16].
- The classic is [Sab93], but it's in french.
- For the $\mathfrak{sl}(2|1)$ version, there's [BG24].
- For a proof of q-holonomicity for the coloured Jones polynomial, there's [GL05].

Details

We will focus on *typical representations* of the classical lie superalgebra $\mathfrak{sl}(n|m)$, where $n \neq m$. These are classified up to isomorphism by a tuple of complex parameters $(a_1, \ldots, a_{m+n+1})$ which must satisfy the following set of linear inequalities:

$$a_{m+1} \neq \sum_{k=m+2}^{j} a_k - \sum_{\ell=1}^{m} a_\ell - 2m - 2 + i + j \tag{1}$$

for $i = 1, \ldots, m+1$, $j = m+1, \ldots, m+n+1$, see [Kac78, Example 1, pg 620].

The parameters a_k are called the *weights* of the associated representation $V(\overline{a})$, which is characterised by having a *highest weight vector* $v \in V(\overline{a})$ such that

$$h_k v = a_k v, \quad \text{and} \quad E_k v = 0 \quad \text{for } k = 1, \ldots, m+n+1. \tag{2}$$

- We want to describe these representations in terms of explicit matrices (which will depend on the parameters a_k.)
- Then we want to prove that the coefficients of those matrices are q-holonomic functions.
- We'd like to also understand if the R-matrix has q-holonomic coefficients.

References

[BG24] Jennifer Brown and Nathan Geer. Quantum invariants arising from $u_h\mathfrak{sl}(2|1)$ are q-holonomic. (arXiv:2403.12882), March 2024. arXiv:2403.12882.

[GL05] Stavros Garoufalidis and Thang TQ Lê. The colored Jones function is q-holonomic. *Geometry & Topology*, 9(3):1253–1293, 2005.

[GL16] Stavros Garoufalidis and T. T. Q. Le. A survey of q-holonomic functions. *arXiv:1601.07487 [hep-th]*, September 2016. arXiv: 1601.07487.

[Kac77] Victor G Kac. Lie superalgebras. *Advances in mathematic*, 26(1), 1977.

[Kac78] V. Kac. Representations of classical lie superalgebras. In Konrad Bleuler, Axel Reetz, and Herbert Rainer Petry, editors, *Differential Geometrical Methods in Mathematical Physics II*, Lecture Notes in Mathematics, page 597–626, Berlin, Heidelberg, 1978. Springer.

[MW15] Victor Mikhaylov and Edward Witten. Branes and supergroups. *Communications in Mathematical Physics*, 340(2):699–832, December 2015.

[Sab93] Claude Sabbah. Systèmes holonomes d'èquations aux q-diffèrences. *D-modules and mircolocal geometry*, page 125–147, 1993.