1

22 May 2025 Orbit counting

Prop'n

The # we are looking for where $\sigma = (123 \cdots m)$ is an m-cycle is the number of orbits of A acting on $\mathbb{Z}^2 / \mathrm{Im}(A^m - I)$ in the case $\det A^m - I \neq 0$. (Else we look at $\mathrm{Ker} / \mathrm{Im}(A^m - I)$)
? deal w/ this later.

Pf First, just focusing on one m-cycle, we proved the reduction

$$\left[\begin{array}{c|c|c} -I & & A \\ \hline A & -I & \\ \hline & A & -I \end{array}\right] \left[\begin{array}{c|c|c} I & & A \\ \hline & I & A^2 \\ \hline & & I \end{array}\right] = \left[\begin{array}{c|c|c} -I & 0 & 0 \\ \hline -A & -I & 0 \\ \hline 0 & A & A^m - I \end{array}\right]$$

means for this we need only consider span of monomials $x_m^i y_m^k$ with relations given by $A^m - I$. A1

(If $A^m - I = \begin{bmatrix} r & s \\ t & u \end{bmatrix}$ then $x_m^r y_m^t - g^*$, $x_m^s y_m^u - g^*$ are commutators & we equate w/ $x_m^0 y_m^0$. So if Smith column form of $A^m - I$ is $\begin{bmatrix} \alpha\beta & 0 \\ * & \alpha \end{bmatrix}$ we can specifically take

$1, x_m, x_m^2, \ldots, x_m^{\alpha\beta - 1}$

$y_m \quad x_m y_m \quad \cdots$

$\vdots$

$y_m^{\alpha-1} \quad \cdots \quad x_m^{\alpha\beta-1} y_m^{\alpha-1}$

and analogously if $\begin{bmatrix} \alpha & 0 \\ * & \alpha\beta \end{bmatrix}$)

But I don't think we need Smith for what follows or for SAGE (?).

What new relations do $\sigma = (12 \cdots m)$ give us?

$X_{m-1}^i y_{m-1}^k - X_m^i y_m^k$

but $[X_{m-1}^i y_{m-1}^k, 1] = X_{m-1}^i y_{m-1}^k - (X_m^a y_m^c)^i (X_m^b y_m^d)^k$

$= X_{m-1}^i y_{m-1}^k - q^* X_m^{ai+bk} y_m^{ci+dk}$

So $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ noting $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} i \\ k \end{bmatrix} = \begin{bmatrix} ai+bk \\ ci+dk \end{bmatrix}$

So if I renotate $q^* X_r^i y_r^k =: x^{\begin{bmatrix} i \\ k \end{bmatrix}}$ or $x^{\underline{v}}$ for $\underline{v} = \begin{bmatrix} i \\ k \end{bmatrix}$

good power as in $\tilde{m}_x$

then we get new relation $x_{m-1}^{\begin{bmatrix} i \\ k \end{bmatrix}} - x_m^{A\begin{bmatrix} i \\ k \end{bmatrix}} \Rightarrow$

via σ $x_m^{\underline{v}} - x_m^{A\underline{v}}$

Similarly $x_{m-2}^{\begin{bmatrix} i \\ k \end{bmatrix}} - x^{A^2\begin{bmatrix} i \\ k \end{bmatrix}}$ and so on.

So that σ sends $1 \to 2 \to \cdots \to m-1 \to m$ (and $m \to 1$) gives us

new relations $x_m^{v} \sim x_m^{Av} \sim x_m^{A^2v} \sim x_m^{A^3v} \sim \cdots \sim x_m^{A^m v}$.

Subclaim: $x_m^{A^m v} = x_m$ Pf (A1) say $x_m^{(A^m - I)\underline{v}} = x_m^{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

for all $\underline{v} \in \ker^{1}$ which we are assuming is $\mathbb{Z}^2$

this part doesn't need σ so we really have $x_m^{A^m v} \sim x_m^{v}$

3

So we are counting orbits

$$\begin{bmatrix} i \\ k \end{bmatrix} \bmod A^m - I \longrightarrow A\begin{bmatrix} i \\ k \end{bmatrix} \bmod A^m - I \to \cdots \to A^{m-1}\begin{bmatrix} i \\ k \end{bmatrix} \bmod A^m - I$$

generically an orbit has size m

but we see orbits size r with $r \mid m$ can happen.

I suspect sage can do this easily / fast (?)

Another way to approach is to see when is $\begin{bmatrix} i \\ k \end{bmatrix}$ a fixed point of A or of A^r for $r \mid m$.

ex'l/ $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $m = 2$

$A^2 = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$ $\qquad A^2 - I = \begin{bmatrix} 6 & 12 \\ 4 & 6 \end{bmatrix}$

When is $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} i \\ k \end{bmatrix} = \begin{bmatrix} i \\ k \end{bmatrix} \bmod \operatorname{span}\left\{\begin{bmatrix} 6 \\ 4 \end{bmatrix}\begin{bmatrix} 12 \\ 6 \end{bmatrix}\right\}$?

Clearly $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a fixed point.

Also rewrite $(A-I)\begin{bmatrix} i \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \bmod \begin{matrix}\text{column} \\ \text{span}\end{matrix} (A^2 - I)$

Recall $(A-I)(A+I)\begin{bmatrix} r \\ s \end{bmatrix} = (A^2-I)\begin{bmatrix} r \\ s \end{bmatrix}$

So if $\begin{bmatrix} i \\ k \end{bmatrix} = (A+I)\begin{bmatrix} r \\ s \end{bmatrix} \notin$ col span $(A^2 - I)$ this will give a fixed pt.

4

And/or

One can run through all

$$\begin{bmatrix}0\\0\end{bmatrix} \begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}2\\0\end{bmatrix} \begin{bmatrix}3\\0\end{bmatrix} \begin{bmatrix}4\\0\end{bmatrix} \begin{bmatrix}5\\0\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} \begin{bmatrix}1\\1\end{bmatrix} \begin{bmatrix}2\\1\end{bmatrix} \begin{bmatrix}3\\1\end{bmatrix} \begin{bmatrix}4\\1\end{bmatrix} \begin{bmatrix}5\\1\end{bmatrix}$$

as $\begin{bmatrix}i\\k\end{bmatrix}$ & check orbits reducing mod $\text{span}\left\{\begin{bmatrix}6\\4\end{bmatrix}, \begin{bmatrix}12\\6\end{bmatrix}\right\}$

$= \text{span}\left\{\begin{bmatrix}6\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\right\}$

$$(A+I)\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3&3\\1&3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$$

and $(A-I)\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}1&3\\1&1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = \underbrace{\begin{bmatrix}6\\4\end{bmatrix}}_{}\sim\begin{bmatrix}0\\0\end{bmatrix}$ col span $A^m - I$

For $m=3$ $\sigma = (1\,2\,3)$ orbits are size 3 or size 1.

now $A^3 = \begin{bmatrix}26&45\\15&26\end{bmatrix}$, $A^3 - I = \begin{bmatrix}25&45\\15&25\end{bmatrix} \overset{\text{Smith}}{\sim} \begin{bmatrix}5&0\\5&10\end{bmatrix}$

so out of 50 terms $\begin{bmatrix}i\\k\end{bmatrix}$ $\begin{matrix}0\le i<5\\0\le k<10\end{matrix}$

w/ relations

we look at orbits

$$\underset{v}{\begin{bmatrix}i\\k\end{bmatrix}} \to \underset{Av}{\begin{bmatrix}2i+3k\\i+2k\end{bmatrix}} \to \underset{A^2v}{\begin{bmatrix}7i+12k\\4i+7k\end{bmatrix}} \to \begin{bmatrix}i\\k\end{bmatrix}$$

and solve for fixed points.

Can look at $\text{Im}(I+A+A^2)/\text{Im}(A^3-I)$ to find them

$$\begin{bmatrix}10&15\\5&10\end{bmatrix} \overset{\text{Smith}}{\sim} \begin{bmatrix}5&0\\0&5\end{bmatrix}$$

$$\begin{bmatrix}5\\0\end{bmatrix} \to \begin{bmatrix}2&3\\1&2\end{bmatrix}\begin{bmatrix}5\\0\end{bmatrix} = \begin{bmatrix}10\\5\end{bmatrix} = \begin{bmatrix}5\\0\end{bmatrix} + \begin{bmatrix}5\\5\end{bmatrix} \sim \begin{bmatrix}5\\0\end{bmatrix}$$

fixed

or $\begin{bmatrix}1&3\\1&1\end{bmatrix}\begin{bmatrix}5\\0\end{bmatrix} = \begin{bmatrix}5\\5\end{bmatrix} \sim \begin{bmatrix}0\\0\end{bmatrix}$

$$\begin{bmatrix}0\\5\end{bmatrix} \to \begin{bmatrix}2&3\\1&2\end{bmatrix}\begin{bmatrix}0\\5\end{bmatrix} = \begin{bmatrix}15\\10\end{bmatrix} = 3\begin{bmatrix}5\\5\end{bmatrix} - \begin{bmatrix}0\\10\end{bmatrix} + \begin{bmatrix}0\\5\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}1&3\\1&1\end{bmatrix}\begin{bmatrix}0\\5\end{bmatrix} = \begin{bmatrix}15\\5\end{bmatrix}$$

$$\sim \begin{bmatrix}0\\5\end{bmatrix} \qquad\qquad = \begin{bmatrix}15\\15\end{bmatrix} - \begin{bmatrix}0\\10\end{bmatrix} \sim \begin{bmatrix}0\\0\end{bmatrix}$$

But $\begin{bmatrix}0\\5\end{bmatrix} = \begin{bmatrix}5\\0\end{bmatrix} - \begin{bmatrix}5\\5\end{bmatrix} + \begin{bmatrix}0\\10\end{bmatrix}$ so we already counted it.

Similar for $\begin{bmatrix}10\\0\end{bmatrix}$ etc in image of $I + A + A^2$ as we expect as $25 \cdot 2 = 50$

So we get fixed points $\begin{bmatrix}0\\0\end{bmatrix} \sim 1$

$$\begin{bmatrix}5\\0\end{bmatrix} \sim X_3^5$$

Then # orbits are

$$\frac{1}{3}\Big(\underset{\text{id}}{50} + \underset{(123)}{2} + \underset{(132)}{2} \Big) = \frac{54}{3} = 18$$

oops I wrote 28 before — I meant 18

$m=4$ is more complicated since τ^2 can have fixed points.