Lemma

Tuesday 1 April 2025 15:34

Let $\omega = (1)^{N_1} (2)^{N_2} \dots (s)^{N_s}$

$$N = \sum_{k=1}^{s} k N_k$$

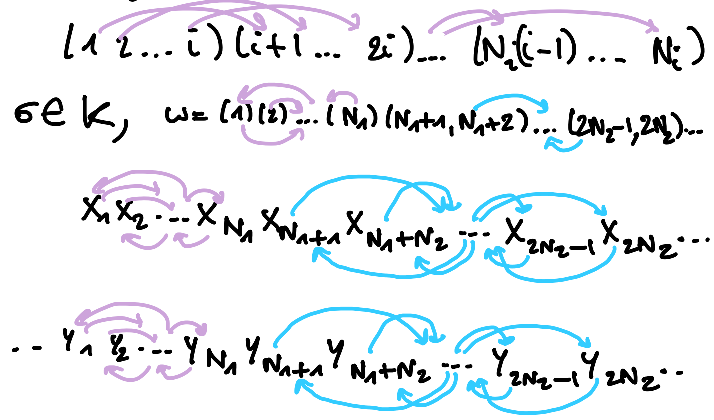
$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

$C_\omega \cong H \rtimes K,$

with $H \cong \mathbb{Z}_2^{N_2} \times \dots \times \mathbb{Z}_s^{N_s}$

$K \cong S_{N_1} \times S_{N_2} \times S_{N_s}$

where S_{N_i} is isomorphic to the group that permutes N_i-cycles of length i by conjugation.



$$M := Id - \gamma\omega = \begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & M_s \end{bmatrix}$$

$2N \times 2N$ matrix

$v \in \mathbb{Z}^{2n}$

$$v = \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \\ a_N \\ b_N \end{pmatrix} = x_1^{a_1} x_2^{a_2} \dots x_N^{a_N} y_1^{b_1} \dots y_N^{b_N}$$

where $M_i = \begin{bmatrix} P_i & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 \\ \vdots & & \ddots & 0 \\ 0 & & & P_i \end{bmatrix} \Big\} \begin{matrix} N_i \\ P_i \text{ blocks} \end{matrix}$

$1 \le i \le s$

where $P_i = \begin{bmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \dots & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} -a & -b \\ -c & -d \end{matrix} \\ \begin{matrix} -a & -b \\ -c & -d \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \vdots & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \ddots & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \vdots \end{bmatrix} \updownarrow 2i$

$1 < i \le s$

$$\left[\begin{array}{c|c|c|c|c} \vdots & \vdots & \vdots & \ddots & \begin{matrix}0&0\\0&0\end{matrix} \\ \hline \begin{matrix}0&0\\0&0\end{matrix} & \cdots & \begin{matrix}0&0\\0&0\end{matrix} & \begin{matrix}-a&-b\\-c&-d\end{matrix} & \begin{matrix}1&0\\0&1\end{matrix} \end{array}\right]$$

$$P_1 = \begin{bmatrix} 1-a & 0 \\ 0 & 1-b \end{bmatrix}.$$

Can do column/rows reduction on each blocks to get the Smith normal form.

$$\text{coker}(M) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\ldots\oplus\mathbb{Z} / a_1\mathbb{Z}\oplus a_2\mathbb{Z}\oplus\ldots\oplus a_{2N}\mathbb{Z}$$

generated by v_i's being the columns/a_i after Smith column reduction

$$M \overset{\text{column}}{\rightsquigarrow} M' = [a_1v_1 \; a_2v_2 \ldots a_{2N}v_{2N}] \overset{\text{row}}{\rightsquigarrow} \text{diag}(a_1,\ldots a_{2N})$$

upper triangular

We want to quotient also by the commutant relations $v_i - \sigma v_i = 0$ for $a_i \neq 0$

Smith column reduce of M

Smith column reductions of M_i

$$\tilde{M} = \left[\begin{array}{c|c|c|c} \tilde{M}_1 & 0 & 0 & 0 \\ \hline 0 & \tilde{M}_2 & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & \tilde{M}_S \end{array}\right]$$

$2N\times 2N$ matrix

$v \in \mathbb{Z}^{2n}$

$$v = \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \\ a_N \\ b_N \end{pmatrix} = x_1^{a_1} x_2^{a_2} \ldots x_N^{a_N} y_1^{b_1} \ldots y_N^{b_N}$$

$$\tilde{M}_i = \left[\begin{array}{c|c|c|c} \tilde{P}_i & 0 & 0 & 0 \\ \hline 0 & \tilde{P}_i & 0 & 0 \\ \hline \vdots & & \ddots & 0 \\ \hline 0 & & & \tilde{P}_i \end{array}\right] \begin{array}{l} \updownarrow z_i \\ \\ \\ \updownarrow z_i \end{array}$$

z_i z_i

, where $\tilde{P}_i$ is the column Smith reduced form of P_i

$$\tilde{P}_i = \left[c_1 \underset{\substack{\shortparallel \\ \omega_1}}{\begin{pmatrix} 1 \\ z \\ \vdots \\ z \end{pmatrix}} \quad c_2 \underset{\substack{\shortparallel \\ \omega_2}}{\begin{pmatrix} 0 \\ 1 \\ z \\ z \\ \vdots \\ z \end{pmatrix}} \ldots c_{2i} \underset{\omega_{2i}}{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}} \right]$$

$$\tilde{M}_i = [c_1\vec{w}_{11} \; c_2\vec{w}_{21} \cdots c_{2i}\vec{w}_{i1} \; c_1\vec{w}_{12} \; c_2\vec{w}_{22} \cdots c_{2i}\vec{w}_{i2} \cdots c_1\vec{w}_{1N_i} \; c_2\vec{w}_{2N_i} \cdots c_{2i}\vec{w}_{iN_i}]$$

where $\vec{w}_{kl} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vec{w}_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ← $\vec{w}_k$ in the lth $\mathbb{Z}^{2i}$ space

$\sigma \in K_1$ or more precisely its S_{N_i} part will permute the $\vec{w}_k$ position

$$\sigma \vec{w}_{kl} = \vec{w}_{kl'}$$

So the commutant relations will identify

$$\vec{w}_{kl} = \vec{w}_{kl'} \quad \forall\, 1 \le l, l' \le N_i$$

we had $\operatorname{coker} M_i = (\operatorname{coker} P_i)^{\oplus N_i}$

$$\cong \left(\frac{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}{c_1\mathbb{Z} \oplus c_2\mathbb{Z} \oplus \cdots \oplus c_{2i}\mathbb{Z}} \right)^{\oplus N_i}$$

$$\cong \left(\frac{\overset{\vec{w}_{1,1}}{\mathbb{Z}} \oplus \overset{\vec{w}_{2,1}}{\mathbb{Z}} \oplus \cdots \oplus \overset{\vec{w}_{2i,1}}{\mathbb{Z}}}{\underset{c_1\vec{w}_{1,1}}{c_1\mathbb{Z}} \oplus \underset{c_2\vec{w}_{2,1}}{c_2\mathbb{Z}} \oplus \cdots \oplus \underset{c_{2i}\vec{w}_{2i,1}}{c_{2i}\mathbb{Z}}} \right) \oplus \left(\frac{\overset{\vec{w}_{1,2}}{\mathbb{Z}} \oplus \overset{\vec{w}_{2i,2}}{\mathbb{Z}} \oplus \cdots \oplus \overset{\vec{w}_{2i,l}}{\mathbb{Z}}}{\underset{c_1\vec{w}_{1,2}}{c_1\mathbb{Z}} \oplus \underset{c_2\vec{w}_{2,2}}{c_2\mathbb{Z}} \oplus \cdots \oplus \underset{c_{2i}\vec{w}_{2i,2}}{c_{2i}\mathbb{Z}}} \right) \oplus$$

$$\cdots \left(\frac{\overset{\vec{w}_{1,N_i}}{\mathbb{Z}} \oplus \underset{\vec{w}_{2,N_i}}{\mathbb{Z}} \oplus \cdots \oplus \overset{\vec{w}_{2i,N_i}}{\mathbb{Z}}}{\underset{c_1\vec{w}_{1,N_i}}{c_1\mathbb{Z}} \oplus \underset{c_2\vec{w}_{2,N_i}}{c_2\mathbb{Z}} \oplus \cdots \oplus \underset{c_{2i}\vec{w}_{2i,N_i}}{c_{2i}\mathbb{Z}}} \right)$$

these spaces are identified by the σ's $\in S_{N_i}$

$$\Rightarrow \operatorname{coker} M_i \Big/ \substack{\text{commutant} \\ \text{rel}^{o} \in S_{N_i}} \cong \operatorname{coker} P_i$$

$$\Rightarrow \operatorname{coker} M \Big/ \substack{\text{commutant} \\ \text{relations for} \\ \sigma \in k} \cong \operatorname{coker} \bar{M}$$

$$\text{with } \bar{M} = \begin{bmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_s \end{bmatrix} = Id - \gamma \bar{\omega}$$

$2\bar{N} \times 2\bar{N}$ matrix

with $\bar{\omega} = (1)^{\bar{N}_1} (2)^{\bar{N}_2} \dots (s)^{\bar{N}_s}$, $\bar{N}_i = \begin{cases} 0 \text{ if } N_i = 0 \\ 1 \text{ otherwise} \end{cases}$

$\bar{N} = \sum_{k=1}^{s} k \bar{N}_k$

Now, $C_w \cong K \rtimes H$. We saw the action of k but let us look at $\underset{\substack{\uparrow \\ \text{power subgroup.}}}{H} \cong \mathbb{Z}_2^{N_2} \times \mathbb{Z}_3^{N_3} \times \dots \mathbb{Z}_s^{N_s}$

$$\forall\ 1 < i \leq s,\ \mathbb{Z}_i^{N_i} \cong \langle \underbrace{(12\dots i),(i+1 \dots 2i), \dots (N_i(i-1) \dots N_i i)}_{\text{acting on the } i^{th} \text{ subspace of } X_k Y_k} \rangle$$

Since the Subspaces of size i are already identified by K, it is enough to do the identification only on one of the subspaces, i.e., we just need one $\mathbb{Z}_i$.

$C_\omega = (C_{N_1} \times C_{N_2} \times \dots \times C_{N_S}) / N(\mathbb{Z}_1 \times \dots \mathbb{Z}_S)$

$C_{\bar{\omega}} \cong \mathbb{Z}_1^{\bar{N}_1} \times \mathbb{Z}_2^{\bar{N}_2} \times \dots \mathbb{Z}_S^{\bar{N}_S}$

$$\Rightarrow \text{coker}\,(\overset{\in 2N \times 2N}{\text{Id} - \gamma_\omega}) \Big/ C_\omega \text{ relations} \cong \text{coker}\,(\overset{\in 2\bar{N} \times 2\bar{N}}{\text{Id} - \gamma_{\bar{\omega}}}) \Big/ C_{\bar{\omega}} \text{ rel}^{\circ}$$