Orbit Count Patterns.

① First, as Matthias noticed, if for one m-cycle a matrix A yields f orbits then

$$\left(\frac{1}{1-t^m}\right)^f = 1 + f\,t^m + \binom{f}{2}t^{2m} + \cdots \binom{f+j-1}{j}t^{jm} + \cdots$$

\# orbits for cycle type $\underbrace{m\,m\,m \cdots m}_{j \text{ times}}$

② Can we see more structure in f or

$$f_m(A) = \frac{1}{m}\sum_{r|m} \varphi\left(\tfrac{m}{r}\right)\left|\det(A^r - I)\right| = \frac{1}{m}\sum_{p=1}^{m} \left|\det A^{\gcd(p,m)} - I\right|$$

$$= \frac{1}{m}\sum_{r|m} \varphi\left(\tfrac{m}{r}\right)\left|\operatorname{trace} A^r - 2\right| = -2 + \frac{1}{m}\sum_{r|m}\varphi\left(\tfrac{m}{r}\right)\operatorname{trace} A^r$$

in the case $\operatorname{tr} A > 2$

(in particular in the case $\det(A^r - I) \neq 0$ ever)

Since A is 2×2 and $\det A = 1$, $\det A^r$ and $\operatorname{tr} A^r$ are polynomials in $\operatorname{tr} A$. (independent of A so long as $\operatorname{tr} A > 2$ and similar formula for $\operatorname{tr} A < -2$)

e.g.

m	$f_m(A)$	
2	$\frac{1}{2}[(\operatorname{tr} A)^2 + \operatorname{tr} A - 2]$	-2
3	$\frac{1}{3}[(\operatorname{tr} A)^3 - \operatorname{tr} A]$	-2
4	$\frac{1}{4}[(\operatorname{tr} A)^4 - 3(\operatorname{tr} A)^2 + 2\operatorname{tr} A]$	-2
5	$\frac{1}{5}[(\operatorname{tr} A)^5 - 5(\operatorname{tr} A)^3 + 9\operatorname{tr} A]$	-2
⋮		

plug in $\operatorname{tr} A = 4$

$\frac{1}{2}(16+4-2)-2 = 7$

$\frac{1}{3}(4^3-4)-2 = 18$

$\frac{1}{4}(\quad)-2 = 52$

etc

or possibly one should expand in $\mathrm{tr}(A-I)$ instead of $\mathrm{tr}\,A$

LTR for you to do.

Question: In some OEIS kind of database what are these polynomials:

$$x^2+x-2 \qquad \text{or possibly} \qquad (x-2)^2+5(x-2)+4 \quad \text{etc.}$$

$$x^3-x$$

$$x^4-3x^2+2x$$

$$x^5-5x^3+9x$$

$$\vdots$$

?

Is there any known pattern?

3 In any case, once one finds $f_1(\gamma), f_2(\gamma), f_3(\gamma) \ldots$

the coeff of t^N in

$$\left(\frac{1}{1-t}\right)^{f_1(\gamma)} \left(\frac{1}{1-t^2}\right)^{f_2(\gamma)} \cdots \left(\frac{1}{1-t^m}\right)^{f_m(\gamma)} \cdots$$

Should count $\displaystyle\bigoplus_{\lambda \vdash N} \dim HH_0(A; A_{\omega,\gamma})_{\substack{Z(\omega) \\ \text{shape}\,\omega=\lambda}}$

since if $\lambda = (1^{k_1} 2^{k_2} \cdots m^{k_m} \cdots)$ the $\left(\frac{1}{1-t^m}\right)^{f_m(\gamma)}$ contribs $\binom{f_m+k_m-1}{k_m} t^{k_m \cdot m}$

So contrib $\binom{f_1+k_1-1}{k_1} t^{(k_1)1} \cdot \binom{f_2+k_2-1}{k_2} t^{(k_2)2} \cdots \binom{f_m+k_m-1}{k_m} t^{(k_m)m} \cdots \binom{f_N+k_N-1}{k_N} t^{(k_N)N}$

to $\boxed{}\, t^N$ as $\lambda \vdash N$ means $(k_1)1+(k_2)2+\cdots+(k_N)N = N$