Proof of final step

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Take $\omega = (1\,2\,3 \dots t)$, $A \in SL_2(\mathbb{Z})$ with $|\mathrm{tr} A| > 2$

claim $\left| \mathrm{Coher}(\mathrm{Id} - A\omega)^{\mathrm{tors}}_{C_{S_t}(\omega)} \right| = |\det(A - I)|$

Proof:

We write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Consider $M = \mathrm{Id} - A\omega$ column reduced to

$$\begin{pmatrix} -I & & & & \\ A & -I & & & \\ & A & \ddots & & \\ & & & -I & \\ & & & A & A^t - I \end{pmatrix}$$

where $A^t - I$ is column reduced to $\begin{pmatrix} \alpha & 0 \\ \gamma & \beta \end{pmatrix}$

~~Taking~~ Coher M we get $\mathbb{Z}^{2t}$ quotiented out by

$$\begin{cases} -X_i + aX_{i+1} + cY_{i+1} = 0 \\ -Y_i + bX_{i+1} + dY_{i+1} = 0 \\ \alpha X_t + \gamma Y_t = 0 \\ \beta Y_t = 0 \end{cases} \quad \text{for } i \in \{1, \dots, t-1\}$$

So we get $\mathrm{Coher} M \cong \mathbb{Z}/_{\alpha\mathbb{Z}} \oplus \mathbb{Z}/_{\beta\mathbb{Z}}$

(cause $\det(A^t - I) \neq 0$, $\alpha \neq 0$ and $\beta \neq 0$) (see appendix)

When we now take coinvariants $(\sigma v = v)$
we additionally quotient out by

$$\begin{cases} (i) & -X_i + aX_{i+1} + cY_{i+1} = -X_j + aX_{j+1} + cY_{j+1} \quad \forall i,j \\ (ii) & -Y_i + bX_{i+1} + dY_{i+1} = -Y_j + bX_{j+1} + dY_{j+1} \quad (t+1 \equiv 1) \\ (iii) & X_t + \frac{\gamma}{\alpha} Y_t = X_j + \frac{\gamma}{\alpha} Y_j \\ (iv) & Y_t = Y_j \end{cases} \quad \forall j$$

Note that (iii) and (iv) $\Longrightarrow \begin{cases} X_i = X_j \\ Y_i = Y_j \end{cases} \quad \forall i,j \quad (*)$
(holds cause $\det(A^t - I) \neq 0$)

Note we are working in Coher M so (i) and (ii) inside Coher M will give $0 = 0$ so no new relations except for when $j = t$. Then

$$\begin{cases} -X_t + aX_1 + cY_1 = 0 \\ -Y_t + bX_1 + dY_1 = 0 \end{cases} \overset{(*)}{\Longrightarrow} \begin{cases} (a-1)X_t + cY_t = 0 \\ bX_t + (d-1)Y_t = 0 \end{cases}$$

We can interpret this set of equations as quotienting out the (X_t, Y_t)-part of the lattice by $\mathrm{Im}(A - I)$. So similarly as before, we can column reduce $A - I$ to $\begin{pmatrix} r & 0 \\ t & s \end{pmatrix}$ and write the previous relations as

$$\begin{cases} rX_t + tY_t = 0 \\ sY_t = 0 \end{cases}$$

From now on we can focus on the (X_t, Y_t)-part of the lattice. For notational clarity, $X := X_t$ and $Y := Y_t$.

We quotient out this lattice by two sets of equations

$$\begin{cases} \alpha X + \gamma Y = 0 \\ \beta Y = 0 \end{cases} \quad \text{and} \quad \begin{cases} rX + tY = 0 \\ sY = 0 \end{cases}$$

coming from Coher — coming from coinvar

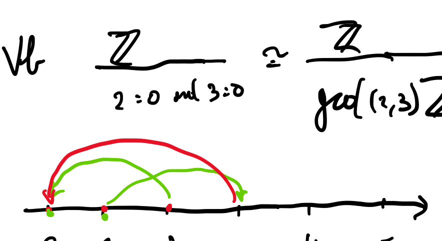
in 1D:

Consider $\begin{cases} \beta y = 0 \\ s\,y = 0 \end{cases} \overset{\beta > 0,\ s > 0}{\Longleftrightarrow} \begin{cases} (\beta + s)y = 0 \end{cases}$

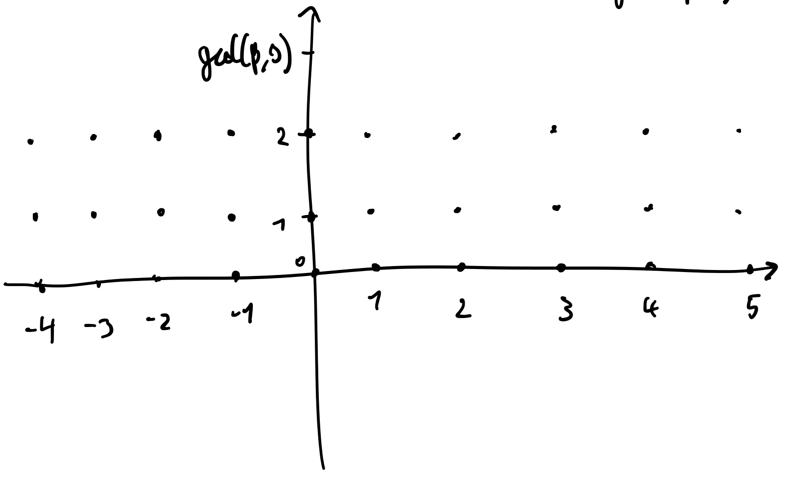
Hence $\dfrac{\mathbb{Z}}{\langle \beta = 0 \text{ and } s = 0 \rangle} \cong \dfrac{\mathbb{Z}}{\langle \beta + s = 0 \rangle} \cong \dfrac{\mathbb{Z}}{(\beta + s)\mathbb{Z}}$

$\beta, s > 0 \qquad \cong \dfrac{\mathbb{Z}}{\gcd(\beta, s)\mathbb{Z}}$

Vb $\dfrac{\mathbb{Z}}{2 = 0 \text{ and } 3 = 0} \cong \dfrac{\mathbb{Z}}{\gcd(2,3)\mathbb{Z}}$



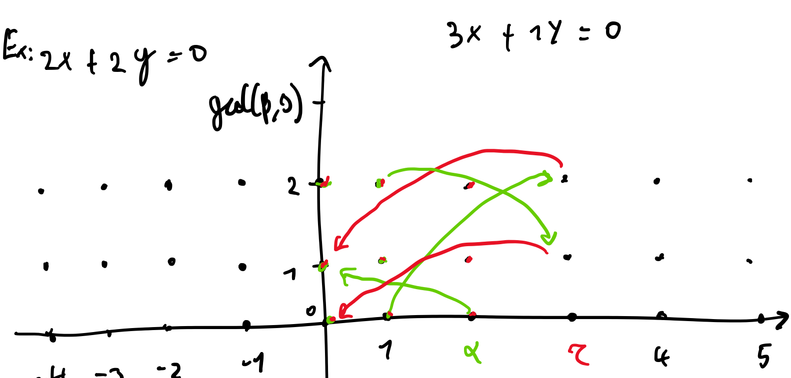
so quotienting out by $\beta y = 0$ and $sy = 0$ we have reduced the lattice to $\mathbb{Z} \oplus \mathbb{Z}/_{\gcd(\beta, s)}$



Now consider

$\alpha X + \gamma Y = 0$ and $rX + tY = 0$

Ex: $2x + 2y = 0$ $\qquad 3x + 1Y = 0$



The point is that we can use the same arguments as in the 1D case, now however we'll make some jumps in the y-direction but doesn't matter in determining which of the points "survive" hence we will end up with a rectangular grid of dimensions $\gcd(\alpha, r) \times \gcd(\beta, s)$.

Hence we have that

$$\left| \mathrm{Coher}(\mathrm{Id} - A\omega)^{\mathrm{tors}}_{C_{S_t}(\omega)} \right| = \gcd(\alpha, r)\gcd(\beta, s)$$

We have $A^t - I = (A - I)(A^{t-1} + \dots + I)$
hence the invariant factor a_i of $A - I$ divide the invariant factors c_i of $A^t - I$. So $a_i | c_i$.

Hence $\gcd(\alpha, r) = |r|$ and $\gcd(\beta, s) = |s|$

Hence

$$\left| \mathrm{Coher}(\mathrm{Id} - A\omega)^{\mathrm{tors}}_{C_{S_t}(\omega)} \right| = \gcd(\alpha, r)\gcd(\beta, s) = |r||s| = |\det(A - I)|$$

$\square$

Appendix

$$\boxed{\text{Lemma: } \det(A^t - I) \neq 0 \quad \forall t \in \mathbb{N}_0}$$

Proof: Consider the eigenvalues of A.

$$\det(A - \lambda I) = \lambda^2 - \mathrm{tr}A.\lambda + \det A = \lambda^2 - \mathrm{tr}A.\lambda + 1$$

Discriminant $= (\mathrm{tr}A)^2 - 4 > 0$ cause $|\mathrm{tr}A| > 2$

Hence A has two distinct real eigenvalues. Moreover, $\det(A) = 1$ so we can write these eigenvalues as $\{\lambda, \lambda^{-1}\}$ for some $\lambda \in \mathbb{R}$.

Hence A is diagonalisable (in $\mathbb{R}$)

$$PAP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

then $P(A^t - I)P^{-1} = \begin{pmatrix} \lambda^t & 0 \\ 0 & \lambda^{-t} \end{pmatrix} - I$

Hence $\det(A^t - I) = (\lambda^t - 1)(\lambda^{-t} - 1)$.

Because λ is real and $\lambda \neq \lambda^{-1}$ both factors are nonzero and hence $\det(A^t - I) \neq 0$.