Lemma

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Let $\omega = (1)^{N_1} (2)^{N_2} \dots (s)^{N_s}$

$$N = \sum_{k=1}^{s} k N_k$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

$C_\omega \cong H \rtimes K$,

with $H \cong \mathbb{Z}_2^{N_2} \times \dots \times \mathbb{Z}_s^{N_s}$

$K \cong S_{N_1} \times S_{N_2} \times S_{N_s}$

where S_{N_i} is isomorphic to the group that permutes N_i-cycles of length i by conjugation.

$(1\ 2 \dots i)(i+1 \dots 2i) \dots (N_i(i-1) \dots N_i)$

$\sigma \in K$, $\omega = (1)(2) \dots (N_1)(N_1+1, N_1+2) \dots (2N_2-1, 2N_2) \dots$

$x_1 x_2 \dots x_{N_1} x_{N_1+1} x_{N_1+N_2} \dots x_{2N_2-1} x_{2N_2} \dots$

$\dots y_1 y_2 \dots y_{N_1} y_{N_1+1} y_{N_1+N_2} \dots y_{2N_2-1} y_{2N_2} \dots$

$$M = Id - \gamma\omega = \begin{bmatrix} M_1 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & M_s \end{bmatrix}$$

$2N \times 2N$ matrix

$v \in \mathbb{Z}^{2n}$

$$v = \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \\ a_N \\ b_N \end{pmatrix} = x_1^{a_1} x_2^{a_2} \dots x_N^{a_N} y_1^{b_1} \dots y_N^{b_N}$$

where $M_i = \left.\begin{bmatrix} P_i & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 \\ \vdots & & \ddots & 0 \\ 0 & & & P_i \end{bmatrix}\right\}$ N_i P_i blocks

$1 \leq i \leq s$

where $P_i = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & -a & -b \\ 0 & 1 & 0 & 0 & & 0 & 0 & -c & -d \\ -a & -b & 1 & 0 & 0 & 0 & \vdots & 0 & 0 \\ -c & -d & 0 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & \vdots & \\ 0 & 0 & 0 & 0 & & 0 & 0 & & \end{bmatrix}$ $\updownarrow 2i$

$1 < i \leq s$

$$\left[\begin{array}{c|c|c|c|c} \vdots & \vdots & \vdots & \ddots & \begin{matrix}0&0\\0&0\end{matrix} \\ \hline \begin{matrix}0&0\\0&0\end{matrix} & \cdots & \begin{matrix}0&0\\0&0\end{matrix} & \begin{matrix}-a&-b\\-c&-d\end{matrix} & \begin{matrix}1&0\\0&1\end{matrix} \end{array}\right]$$

$$P_1 = \begin{bmatrix} 1-a & 0 \\ 0 & 1-b \end{bmatrix}.$$

Can do column/rows reduction on each blocks to get the Smith normal form.

$$\operatorname{coker}(M) \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\ldots\oplus\mathbb{Z} / a_1\mathbb{Z}\oplus a_2\mathbb{Z}\oplus\ldots\oplus a_{2N}\mathbb{Z}$$

generated by v_i's being the columns/a_i after Smith column reduction

$$M \overset{\text{column}}{\rightsquigarrow} M' = [a_1 v_1 \;\; a_2 v_2 \;\ldots\; a_{2N} v_{2N}] \overset{\text{row}}{\rightsquigarrow} \operatorname{diag}(a_1, \ldots, a_{2N})$$

$\leftarrow$ upper triangular

We want to quotient also by the commutant relations $v_i - \sigma v_i = 0$ for $a_i \neq 0$

Smith column reduct of M

Smith column reductions of M_i

$$\tilde{M} = \left[\begin{array}{c|c|c|c} \tilde{M}_1 & 0 & 0 & 0 \\ \hline 0 & \tilde{M}_2 & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & \tilde{M}_s \end{array}\right]$$

$2N\times 2N$ matrix

$v \in \mathbb{Z}^{2n}$

$$v = \begin{pmatrix} a_1 \\ b_1 \\ a_2 \\ b_2 \\ \vdots \\ a_N \\ b_N \end{pmatrix} = x_1^{a_1} x_2^{a_2} \ldots x_N^{a_N} y_1^{b_1} \ldots y_N^{b_N}$$

$$\tilde{M}_i = \left[\begin{array}{c|c|c|c} \tilde{P}_i & 0 & 0 & 0 \\ \hline 0 & \tilde{P}_i & 0 & 0 \\ \hline \vdots & & \ddots & 0 \\ \hline 0 & & & \tilde{P}_i \end{array}\right]$$

$2i$ $2i$ $2i$ $2i$

, where $\tilde{P}_i$ is the column Smith reduced form of P_i

$$\tilde{P}_i = \left[c_1 \underbrace{\begin{pmatrix} 1 \\ * \\ \vdots \\ \vdots \\ * \end{pmatrix}}_{w_1} \; c_2 \underbrace{\begin{pmatrix} 0 \\ 1 \\ * \\ * \\ \vdots \\ * \end{pmatrix}}_{w_2} \;\cdots\; c_{2i} \underbrace{\begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{w_{2i}} \right]$$

$$\tilde{M}_i = [c_1 \vec{w}_{11} \; c_2 \vec{w}_{21} \cdots c_{2^i} \vec{w}_{i1} \; c_1 \vec{w}_{12} \; c_2 \vec{w}_{22} \cdots c_{2^i} \vec{w}_{i2} \cdots c_1 \vec{w}_{1N_i} \; c_2 \vec{w}_{2N_i} \cdots c_{2^i} \vec{w}_{iN_i}]$$

where $\vec{w}_{k\ell} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vec{w}_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\vec{w}_k$ in the ℓth $\mathbb{Z}^{2^i}$ space

$\sigma \in K$, or more precisely its S_{N_i} part will permute the $\vec{w}_k$ position

$$\sigma \vec{w}_{k\ell} = \vec{w}_{k\ell'}$$

So the commutant relations will identify

$$\vec{w}_{k\ell} = \vec{w}_{k\ell'} \quad \forall\, 1 \le \ell, \ell' \le N_i$$

we had $\operatorname{coker} M_i = (\operatorname{coker} P_i)^{\oplus N_i}$

$$\cong \left(\frac{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}{c_1\mathbb{Z} \oplus c_2 \mathbb{Z} \oplus \cdots \oplus c_{2^i}\mathbb{Z}} \right)^{\oplus N_i}$$

$$\cong \left(\frac{\mathbb{Z}_{\vec{w}_{1,1}} \oplus \mathbb{Z}_{\vec{w}_{2,1}} \oplus \cdots \oplus \mathbb{Z}_{\vec{w}_{2^i,1}}}{c_1\mathbb{Z}_{c_1\vec{w}_{1,1}} \oplus c_2 \mathbb{Z}_{c_2\vec{w}_{2,1}} \oplus \cdots \oplus c_{2^i}\mathbb{Z}_{c_{2^i}\vec{w}_{2^i,1}}} \right) \oplus \left(\frac{\mathbb{Z}_{\vec{w}_{1,2}} \oplus \mathbb{Z}_{\vec{w}_{2,2}} \oplus \cdots \oplus \mathbb{Z}_{\vec{w}_{2^i,\ell}}}{c_1\mathbb{Z}_{c_1\vec{w}_{1,2}} \oplus c_2 \mathbb{Z}_{c_2\vec{w}_{2,2}} \oplus \cdots \oplus c_{2^i}\mathbb{Z}_{c_{2^i}\vec{w}_{2^i,2}}} \right) \oplus \cdots \left(\frac{\mathbb{Z}_{\vec{w}_{1,N_i}} \oplus \mathbb{Z}_{\vec{w}_{2,N_i}} \oplus \cdots \oplus \mathbb{Z}_{\vec{w}_{2^i,N_i}}}{c_1\mathbb{Z}_{c_1\vec{w}_{1,N_i}} \oplus c_2 \mathbb{Z}_{c_2\vec{w}_{2,N_i}} \oplus \cdots \oplus c_{2^i}\mathbb{Z}_{c_{2^i}\vec{w}_{2^i,N_i}}} \right)$$

these spaces are identified by the σ's $\in S_{N_i}$

$$\Rightarrow \operatorname{coker} M_i \Big/ \begin{array}{l}\text{commutant}\\ \text{rel}^{\circ} \in S_{N_i}\end{array} \cong \operatorname{coker} P_i$$

$$\Rightarrow \operatorname{coker} M \Big/ \begin{array}{l}\text{commutant}\\ \text{relations for}\\ \sigma \in K\end{array} \cong \operatorname{coker} \bar{M}$$

$$\text{with } \bar{M} = \begin{bmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_s \end{bmatrix} = Id - \gamma_{\bar{\omega}}$$

$2\bar{N} \times 2\bar{N}$ matrix

with $\bar{\omega} = (1)^{\bar{N}_1} (2)^{\bar{N}_2} \dots (s)^{\bar{N}_s}$, $\bar{N}_i = \begin{cases} 0 & \text{if } N_i = 0 \\ 1 & \text{otherwise} \end{cases}$

$$\bar{N} = \sum_{k=1}^{s} \bar{N}_k$$

$$\Rightarrow \operatorname{coker} \underbrace{(Id - \gamma_w)}_{\in 2N \times 2N} \Big/ C_w \text{ relations} \cong \operatorname{coker} \underbrace{(Id - \gamma_{\bar{\omega}})}_{\in 2\bar{N} \times 2\bar{N}} \Big/ C_{\bar{\omega}} \text{ rel}^{\circ}$$