March 28, 2025

general $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$ $\qquad$ trace $A = a+d$

Note $ad-bc=1$ and $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Consider an $m$-cycle $\in \mathfrak{S}_m$ e.g. $\begin{bmatrix} 0 & 0 & 0 & & 1 \\ 1 & 0 & 0 & & 0 \\ & 1 & 0 & & 0 \\ & & 1 & & \vdots \\ & & & \ddots\ 1 & 0 \end{bmatrix}$

$$B = \begin{bmatrix} \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & & & & \\ & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & & & \\ & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & & \\ & & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \\ & & & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \end{bmatrix} \begin{bmatrix} & & & & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \\ \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & & & & \\ & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & & & \\ & & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & & \\ & & & \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & \end{bmatrix}$$

$$\begin{bmatrix} & & & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \\ \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & & & & \\ & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & & & \\ & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & & \\ & & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \end{bmatrix}$$

So $C = B - \mathrm{Id}_m = \begin{bmatrix} \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} & & & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \\ \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} & & & \\ & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} & & \\ & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} & \\ & & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \end{bmatrix}$

Let us column reduce $C$

$$\begin{bmatrix} \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} & & & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \\ \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} & & & \\ & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} & & \\ & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} & \\ & & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} & & & & \begin{smallmatrix} 0 & 0 \\ c & d \end{smallmatrix} \\ \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} & & & \begin{smallmatrix} a^2 & ab \\ ac & bc \end{smallmatrix} \\ & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} & & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} & \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \\ & & & \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \end{bmatrix} \rightsquigarrow$$

$$
\left[\begin{array}{cc|cc|cc|cc|cc}
-1 & 0 & & & & & & & 0 & 0 \\
0 & -1 & & & & & & & 0 & 0 \\
\hline
a & b & -1 & 0 & & & & & a^2+bc & ab+bd \\
c & d & 0 & -1 & & & & & ac+dc & bc+d^2 \\
\hline
 & & a & b & -1 & 0 & & & 0 & 0 \\
 & & c & d & 0 & -1 & & & 0 & 0 \\
\hline
 & & & & a & b & -1 & 0 & 0 & 0 \\
 & & & & c & d & 0 & -1 & 0 & 0 \\
\hline
 & & & & & & a & b & -1 & 0 \\
 & & & & & & c & d & 0 & -1
\end{array}\right]
=
\left[\begin{array}{c|c|c|c|c}
-I & & & & 0 \\
\hline
A & -I & & & A^2 \\
\hline
 & A & -I & & \\
\hline
 & & A & -I & \\
\hline
 & & & A & -I
\end{array}\right]
$$

since column reduce via right mult by

$$
\left[\begin{array}{c|c|c|c|c}
I & & & & A \\
\hline
 & I & & & \\
\hline
 & & I & & \\
\hline
 & & & I & \\
\hline
 & & & & I
\end{array}\right]
$$

Check:
$$
\begin{bmatrix}
-I & & & & A \\
A & -I & & & \\
 & A & -I & & \\
 & & A & -I & \\
 & & & A & -I
\end{bmatrix}
\begin{bmatrix}
I & & & & A \\
 & I & & & \\
 & & I & & \\
 & & & I & \\
 & & & & I
\end{bmatrix}
=
\begin{bmatrix}
-I & & & & 0 \\
A & -I & & & A^2 \\
 & A & -I & & \\
 & & A & -I & \\
 & & & A & -I
\end{bmatrix}
$$

repeat:
$$
\underbrace{\begin{bmatrix}
-I & & & & 0 \\
A & -I & & & A^2 \\
 & A & -I & & \\
 & & A & -I & \\
 & & & A & -I
\end{bmatrix}}_{\text{semi-column reduced}}
\underbrace{\begin{bmatrix}
I & & & & 0 \\
 & I & & & A^2 \\
 & & I & & \\
 & & & I & \\
 & & & & I
\end{bmatrix}}_{\text{matrix that column reduces}}
=
\begin{bmatrix}
-I & & & & 0 \\
A & -I & & & 0 \\
 & A & -I & & A^3 \\
 & & A & -I & \\
 & & & A & -I
\end{bmatrix}
$$

repeat:
$$
\begin{bmatrix}
-I & & & & 0 \\
A & -I & & & 0 \\
 & A & -I & & A^3 \\
 & & A & -I & \\
 & & & A & -I
\end{bmatrix}
\begin{bmatrix}
I & & & & 0 \\
 & I & & & 0 \\
 & & I & & A^3 \\
 & & & I & \\
 & & & & I
\end{bmatrix}
=
\begin{bmatrix}
-I & & & & 0 \\
A & -I & & & 0 \\
 & A & -I & & 0 \\
 & & A & -I & A^4 \\
 & & & A & -I
\end{bmatrix}
$$

repeat

$$
\begin{bmatrix}
-I & & & & 0 \\
A & -I & & & 0 \\
 & A & -I & & 0 \\
 & & A & -I & A^4 \\
 & & & A & -I
\end{bmatrix}
\begin{bmatrix}
I & & & & 0 \\
 & I & & & 0 \\
 & & I & & 0 \\
 & & & I & A^4 \\
 & & & & I
\end{bmatrix}
=
\begin{bmatrix}
-I & & & & 0 \\
A & -I & & & 0 \\
 & A & -I & & 0 \\
 & & A & -I & 0 \\
 & & & A & A^5-I
\end{bmatrix}
$$

In this way, for an $m$-cycle we get

$$\begin{bmatrix} -I & & & \\ A & -I & & \\ & A & \ddots & \\ & & \ddots & A^m - I \end{bmatrix}$$

So for each $m$-cycle we only need understand Smith normal form of $A^m - I$

Check for known $A$:

$A = T^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ $\qquad A^m - I = T^{km} - I = \begin{bmatrix} 0 & km \\ & 0 \end{bmatrix}$

so each $m$-cycle contribs $km$

$A = S \overset{?}{=} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad S^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad S^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad S^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$m \equiv 0 \mod 4$ $\quad S^m - I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad$ no contrib

$m \equiv 1 \mod 4$ $\quad S^m - I = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & 0 \\ -1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \overset{\text{smith}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad$ contrib 2

$m \equiv 2 \mod 4$ $\quad S^m - I = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ $\quad$ contribs $-2, -2$

$m \equiv 3 \mod 4$ $\quad S^m - I = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \overset{\text{Smith}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad$ contribs 2

$A = ST = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$m \equiv 0 \bmod 3$ $\quad (ST)^m - I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ no contrib

$m \equiv 1 \bmod 3$ $\quad (ST)^m - I = \begin{bmatrix} -1 & 1 \\ -1 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -1 & 0 \\ -1 & -3 \end{bmatrix} \overset{\text{Smith}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ contrib 3

$m \equiv 2 \bmod 3$ $\quad (ST)^m - I = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -3 & -1 \\ 0 & -1 \end{bmatrix} \overset{\text{Smith}}{\rightsquigarrow} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ contrib 3

Doing it this way we can see each m-cycle is "independent" So this describes $HH_0(A; A_\omega)$.

To then get to $HH_0(A; A_\omega)_{Z(\omega)}$ use the "lemma".

Theorem

Suppose $\omega \in S_n$ has cycle type $1^{m_1} 2^{m_2} \cdots n^{m_n}$.

Then $HH_0(A; A_\omega) \cong$

$$\text{"}(\mathrm{Smith}(A - I))^{\times m_1} \times (\mathrm{Smith}(A^2 - I))^{m_2} \times \cdots$$

$$\cdots \times \mathrm{Smith}(A^k - I)^{m_k} \times \cdots \; \mathrm{Smith}(A^n - I)^{m_n}\text{"}$$

for each k-cycle

where by Smith B I mean compute and get $\approx \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$

and ignore $\beta, \alpha = 0$ or $1$ else contrib $\mathbb{Z}/\alpha\mathbb{Z} \times \mathbb{Z}/\beta\mathbb{Z}$