Behavior of $A^t - I$

Suppose A has eigenvalues $\alpha, \beta \in \mathbb{C}$.
Since $A \in SL(2, \mathbb{Z})$, $\alpha\beta = 1$ and $\alpha + \beta \in \mathbb{Z}$

If $\alpha \neq \beta$ then
A^t has eigenvalues α^t, β^t. Still $\alpha^t \beta^t = (\alpha\beta)^t = 1$
but also $\alpha^t + \beta^t \in \mathbb{Z}$

Let's not use t for our t-cycle and exponent going forward
as A^t looks like transpose(A).

Case I $\alpha \neq \beta$ so $\mathbb{C}^2$ has basis of eigenvectors for A

$A\underline{v} = \alpha \underline{v}$, $A\underline{w} = \beta \underline{w}$.

Then we see $A - I$ has eigenvalues $\alpha - 1, \beta - 1$

$A^t - I$ has eigenvalues $\alpha^t - 1, \beta^t - 1$

$A^t - I$ is invertible iff all its eigenvalues are nonzero.

Note: if $\alpha^t - 1 = 0$ then $\alpha^t = 1 \Rightarrow \alpha = e^{i\theta}, \beta = e^{-i\theta}$

so also $\beta^t - 1 = 0$

$\text{Trace}(A) = 2\cos\theta \in \mathbb{Z} \Rightarrow$

$\cos\theta \in \{ 0, 1, -1, \frac{1}{2}, -1/2 \}$

$\theta \in \{ \pm \pi/2, 0, \pi, \pm \pi/3, \pm 2\pi/3 \}$

$e^{i\theta} \in \{ \pm i, 1, -1, \pm 1/2 \pm i\sqrt{3}/2 \}$

So both eigenvalues of $A^t - I$ are 0, both eigenvalues of A^t are 1

and eigenvalues of A are complex conjugate pairs on list above. In particular trace $A \in \{ 0, +1, -1 \}$

$i + -i$ $\quad \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) + \left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$ $\quad \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$

but not $2, -2$ else $\alpha = \beta = 1$, $\alpha = \beta = -1$

So in the case $\alpha \neq \beta$ we get $A^t - I$ is invertible

unless $\text{tr}\, A \in \{0, +1, -1\}$

and here we understand the specific t.

Case 2 $\alpha=\beta$

Then $\alpha\beta=1 \Rightarrow \alpha^2=1 \Rightarrow \alpha=\pm 1 \Rightarrow \operatorname{tr} A \in \{2,-2\}$

characteristic polynomial of A is $x^2 \pm 2x + 1 = (x\pm 1)^2$

case 2a minimal polynomial of A is $(x\pm 1)$

$\Rightarrow A = I$ or $-I$

$\Rightarrow A^t$ is I or $-I$ $\Rightarrow A^t - I$ is $\underline{0}$ or $-2I$

more often

so we get $A^t - I$ invertible when $A=-I$, t odd

Case 2b minimal polynomial of A is $(x\pm 1)^2$.

i.e. A is not diagonalizable

Case 2b (i) trace $A = 2$ trace$(A-I) = 0$

$\dim(\text{nullspace}(A-I)) = 1$, we know $\det(A-I)=0$

char poly $A-I = x^2 =$ min poly $A-I$. since $A-I\neq \underline{0}$

Since $(A-I)(A^{t-1}+\cdots+A^2+A+I) = A^t - I$ we see $\det(A^t-I)=0$

so $A^t - I$ is never invertible

Case 2b (ii) trace $A = -2$ trace$(A-I) = -4$

Since $A-I \neq -2I$

$\min_{A-I}(y) = y^2+4y+4$ since $A-I$ satisfies $(x+1)^2 = 0 = ((x-1)+2)^2$

$= (x-1)^2 + 4(x-1) + 4$

$\parallel$

$\operatorname{char}_{A-I}(y)$

So $\det A-I = 4$. Thus $A-I$ is invertible.

Case 2b(ii) I

t even: $A^t - I = ((-A)^t - I)$ But $\mathrm{tr}(-A) = +2$ falls into case 2b(i)

so is never invertible

Case 2b(ii) II

t odd: Note $x^t + 1 = 1 - (-x)^t = (1-(-x))(1 + (-x) + (-x)^2 + \dots + (-x)^{t-1})$

$= (1+x)(1 - x + x^2 \dots + x^{t-1})$

Since A satisfies $(1+x)^2 = 0$, A also satisfies $(x^t+1)^2 = 0$

$\parallel$

$(x^t-1)^2 + 4(x^t-1) + 4$

so $A^t - I$ satisfies poly $y^2 + 4y + 4 = (y+2)^2 = 0$

so either this is its characteristic polynomial showing $\det(A^t - I) = 4 \neq 0$

or $(A^t - I + 2I) = \underline{0} \Rightarrow A^t = -I \Rightarrow A^t - I = -2I$

so still $\det = 4$ and invertible

To recap -

$A^t - I$ is not invertible in the cases:

trace $A = -2$, t even

trace $A = 2$ all t

trace $A \in \{0, 1, -1\}$ specific t (this is distinct eigenvalue case)

: evals $\alpha = i, \beta = -i$ $t \equiv 0 \bmod 4$ get A^t has evals $1, 1$ so $\det(A^t - I) = 0$

evals $\alpha = e^{i\pi/3}, \beta = e^{-i\pi/3}$ $t \equiv 0 \bmod 6$ "

$\alpha = e^{i2\pi/3}, \beta = e^{-i2\pi/3}$ $t \equiv 0 \bmod 3$ "