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Case $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $\sigma = (12)$

note $A - I = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$ $A^2 - I = \begin{bmatrix} 6 & 12 \\ 4 & 6 \end{bmatrix}$

Further $\overset{A-I}{\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$ $\overset{A^2-I}{\begin{bmatrix} 6 & 12 \\ 4 & 6 \end{bmatrix}} \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}$ (3)

So w/o σ-coinvariants $\mathbb{C}_q\langle x_1^{\pm} y_1^{\pm} x_2^{\pm} y_2^{\pm}\rangle / \mathbb{C}$ is spanned by

$$1 \quad x_2 \quad x_2^2 \quad x_2^3 \quad x_2^4 \quad x_2^5$$

$$y_2 \quad x_2 y_2 \quad x_2^2 y_2 \quad x_2^3 y_2 \quad x_2^4 y_2 \quad x_2^5 y_2$$

our explicit column reduction $\left[\begin{array}{c|c} -I & A \\ \hline A & -I \end{array}\right] \left[\begin{array}{c|c} I & A \\ \hline 0 & I \end{array}\right]$ tells us

$$\left[(x_1^2 y_1) \cdot x_2 ,\ (x_1^2 y_1 x_2)^{-1} \right]_r = 1 - q^* x_2^6 y_2^4$$

$$\left[(x_1^3 y_1^2) \cdot y_2 ,\ (x_1^3 y_1^2 y_2)^{-1} \right]_r = 1 - q^* x_2^{12} y_2^6$$

as in $\begin{bmatrix} 6 & 12 \\ 4 & 6 \end{bmatrix}$

But further (3) tells us from simplifying $(x_1^2 y_1 x_2)^{-2} (x_1^3 y_1^2 y_2)^{1}$, $\begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$

resp $(x_1^2 y_1 x_2)^{-3} (x_1^3 y_1^2 y_2)^{2}$

$$\left[x_1^{-1} x_2^{-2} y_2 ,\ x_1 x_2^2 y_2 \right] = q^2 y_2^2 - 1 \quad (*)$$

$$\left[x_2^{-3} y_1 y_2^2 ,\ (x_2^{-3} y_1 y_2^2)^{-1} \right] = 1 - \cancel{q^{-16}}\, x_2^6\, q^{-12} \quad (**)$$

Two important observations:

(1) Suppose B, C, D, E, M are monomials

and $[B, C]_q = D - E$. Then $[B, CM] = DM - q^{*}EM$

But $*$ need not be zero.

This means $\mathcal{C}$ is really a $\mathbb{C}$-span of commutators and

Not a $\mathbb{C}_q\langle x_1^{\pm} x_2^{\pm} y_1^{\pm} y_2^{\pm}\rangle$ - submodule.

(writing $\mathbb{C}_q\langle\ \rangle$ for $\mathbb{C}(q)\langle x_1^{\pm} x_2^{\pm} y_1^{\pm} y_2^{\pm}\rangle \Big/ \left\langle \begin{matrix} x_1x_2 = x_2x_1 \\ x_1y_2 = y_2x_1 \\ y_1y_2 = y_2y_1 \\ x_2y_1 = y_1x_2 \\ y_1x_1 = q\, x_1y_1,\ y_2x_2 = t\, x_2y_2 \end{matrix} \right\rangle$)

so all the linear algebra on exponents is sloppy about

coefficients.

In other words, it doesn't see if $\mathcal{C} \ni x_1 - qx_2$

and $x_1 - q^2 x_2$

(hypothetically) which would force $x_1, x_2 \in \mathcal{C}$.

So $\mathbb{C}_q\langle X, Y\rangle/\mathcal{C}$ can be smaller i.e. ≤ 12-dim BUT I think one can show this won't happen & dim $= 12$.

(2)

When taking σ-coinvariants we LOSE this

"almost-submodule" property from (1).

$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $\sigma = (12)$ Call $\mathcal{C}$ = span of all σ-twisted commutators

From # and ## can always reduce monomials $x_2^i y_1^j$ to have $0 \le i < 6$, $0 \le j < 2$

$$[x_1^{-1} x_2^{-2} y_2,\ x_1 x_2^2 y_2 M] = q^2 y_2^2 M - x_1 x_2^2 y_2 M y_2^{-1} x_2^{-2} x_1^{-1}$$

$$[x_2^{-3} y_1 y_2^2,\ (x_2^{-3} y_1 y_2^2)^{-1} N] = N - y_2^{-2} y_1^{-1} x_2^3 N x_2^3 y_1 y_2^2$$

Examine how to express $x_1^i y_1^j$ in x_2's, y_2's so can reduce any monomial mod $\mathcal{C}$.

Computations by hand yielded

$\mathcal{C} \ni$

$x_1 - x_2^2 y_2$ $y_1 - q^{-5} x_2^3$

$x_1^2 - q^2 x_2^4 y_2$ $x_1 y_1 - q^{-6} x_2^5 y_2$

$x_1^3 - q^5 y_2$ $x_1^2 y_1 - x_2$ 4

$x_1^3 y_1 - x_2^3 y_2$

$x_1^4 - q^4 x_2^2$ $x_1^4 y_1 - q^{-2} x_2^2$

$x_1^5 - q^5 x_2^4 y_2$ $x_1^5 y_1 - q^5 x_2 y_2$

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which may have errors.

Now let $\mathcal{C}(\sigma)$ be $\mathcal{C}$ enlarged by the span of all

$$x_1^i y_1^j x_2^k y_2^l - x_2^i y_2^j x_1^k y_1^l = x_1^i y_1^j x_2^k y_2^l - x_1^k y_1^l x_2^i y_2^j$$

If Smith normal form of $A - I = \begin{bmatrix} 2-1 & 3 \\ 1 & 2-1 \end{bmatrix}$ controls algebra$/\mathcal{C}(\sigma)$ (vector subspace quotient) then we expect mod $C(\sigma)$ any monomial should be equivalent to something in the span of 1 and y_2.

I don't see how to do this. Note:

(4) gives us $x_2 - x_2^2 y_2 \in C(\sigma)$ etc:

$$\boxed{1 \quad x_2 \quad x_2^2} \qquad \underbrace{x_2^3 = q^5 y_2 \qquad x_2^4 = q^4 x_2^2 \qquad x_2^5 = q^5 x_2^4 y_2 = q^3 x_2^2}$$

$$\boxed{1 \cdot y_2 \quad x_2 y_2} \qquad x_2^2 y_2 = x_2 \qquad \boxed{x_2^3 y_2} \qquad x_2^4 y_2 = q^{-2} x_2^2 \qquad x_2^5 y_2 = q^5 x_2 y_2$$

So I see all monomials are in the span of

$\{1, x_2, x_2^2, y_2, x_2 y_2, x_2^3 y_2\}$ mod $C(\sigma)$

but I don't see dependence relations.

Does anyone see how just using $C(\sigma)$ as a subvector space?