I wanted to investigate further by hand some small cases.

Case 1 $\boxed{A = T^m = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}}$ $\boxed{\sigma = (12)}$

Recall $[\ ,\ ]$ is determined by $\left[\begin{array}{c|c} 0 & A \\ \hline A & 0 \end{array}\right] = \Gamma$.

We care about $\operatorname{Im}(\Gamma - I) \subseteq \ker(\Gamma - I)^{\perp}$ and

$$\left[\begin{array}{c|c} -I & A \\ \hline A & -I \end{array}\right] \left[\begin{array}{c|c} I & A \\ \hline 0 & I \end{array}\right] = \left[\begin{array}{c|c} -I & 0 \\ \hline A & A^2 - I \end{array}\right] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & m & 0 & 2m \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

column-reduce for image

also for row reducing

$$\left[\begin{array}{c|c} I & 0 \\ \hline A & I \end{array}\right] \left[\begin{array}{c|c} -I & A \\ \hline A & -I \end{array}\right] = \left[\begin{array}{c|c} -I & A \\ \hline 0 & A^2 - I \end{array}\right] = \begin{bmatrix} -1 & 0 & 1 & m \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 2m \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

from which we see $\ker^{\perp}$ is only 3-dim $= \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}^{\perp} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\perp$ for our form

Explicitly, let $\mathcal{C} = \mathbb{C}$-span of all Γ-commutators.

Note $[x_1 x_2, x_1^{-1} x_2^{-1} y_1] = x_1 x_2 x_1^{-1} x_2^{-1} y_1 - x_1^{-1} x_2^{-1} y_1 \cdot (x_2 x_1) = y_1 - q y_1$

$= (1-q) y_1 \Rightarrow y_1 \in \mathcal{C}$.

similarly $y_2 \in \mathcal{C}$ i.e. $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \notin \ker^{\perp}$

And in fact (1) $y_1^{\alpha} y_2^{\beta} \in \mathcal{C}$ unless $\alpha + \beta = 0$ (i.e. $\begin{bmatrix} 0 \\ \alpha \\ 0 \\ \beta \end{bmatrix} \notin \ker^{\perp}$)

Also $x_1^u x_2^v y_1^{\alpha} y_2^{\beta} \in \mathcal{C}$ unless $\alpha+\beta=0$ and all $u, v \in \mathbb{Z}$

by just considering $[x_1 x_2, (x_1 x_2)^{-1} x_1^u x_2^v y_1^{\alpha} y_2^{\beta}] = (1 - q^{*}) x_1^u x_2^v y_1^{\alpha} y_2^{\beta}$

and $* \neq 0$ unless $\alpha + \beta = 0$

If $\alpha+\beta \neq 0$ note then $y_1^\alpha y_2^\beta,\ y_1^\beta y_2^\alpha \in C \Rightarrow$

$$y_1^\alpha y_2^\beta - y_1^\beta y_2^\alpha = (1-\sigma) y_1^\alpha y_2^\beta$$

Further $[x_1^\alpha x_2^\beta, 1] = x_1^\alpha x_2^\beta - x_2^\alpha x_1^\beta = (1-\sigma) x_1^\alpha x_2^\beta$ ②

so ~~taking~~ σ-coinvariants kills no extra x's, and we only need consider expressions involving $y_1 y_2^{-1}$ etc.

Observe $[y_1, y_2^{-1}] = y_1 y_2^{-1} - y_2^{-1}(x_2^m y_2) = y_1 y_2^{-1} - q^{-m} x_2^m$

$$[y_2, y_1^{-1}] = y_1^{-1} y_2 - q^{-m} x_1^m$$

so $y_1 y_2^{-1} - y_1^{-1} y_2 \equiv q^{-m}(x_2^m - x_1^m) \in C$ by ②

So we see all $(1-\sigma) x_1^u x_2^v y_1^\alpha y_2^\beta \in C$ for all $u, v, \alpha, \beta \in \mathbb{Z}$

even allowing $\alpha+\beta=0$.

Take care: $[y_1^\alpha, y_2^{-\alpha}] = y_1^\alpha y_2^{-\alpha} - y_2^{-\alpha}(x_2^m y_2)^\alpha$

$$= (y_1 y_2^{-1})^\alpha - y_2^{-\alpha}(\underbrace{x_2^m y_2 x_2^m y_2 \cdots x_2^m y_2}_{\alpha})$$

$$= (y_1 y_2^{-1})^\alpha - q^{*} x_2^m \quad \text{where } * \text{ is more complicated}$$

eg. $y_2^{-3} x_2^m y_2 x_2^m y_2 x_2^m y_2 = y_2^{-2} q^{-m} x_2^m x_2^m y_2 x_2^m y_2 = q^{-m} y_2^{-1} q^{-2m} x_2^{2m} x_2^m y_2$

$$= q^{-m} q^{-2m} q^{-3m} x_2^{3m}$$

So σ-coinvariants make no difference.

And representatives mod C are given by

$$1, x_2, x_2^2, \ldots, x_2^{m-1}$$

Case $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $\qquad \sigma = (12)$

note $A - I = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$ $\quad A^2 = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$ $\quad A^2 - I = \begin{bmatrix} 6 & 12 \\ 4 & 6 \end{bmatrix}$

Further $\underset{}{\overset{A-I}{\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}}} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}$ $\qquad \overset{A^2-I}{\begin{bmatrix} 6 & 12 \\ 4 & 6 \end{bmatrix}} \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & -2 \end{bmatrix}$ ③

So w/o σ-coinvariants $\mathbb{C}_q\langle x_1^{\pm} y_1^{\pm} x_2^{\pm} y_2^{\pm} \rangle / \mathbb{C}$ is spanned by

$$1 \quad x_2 \quad x_2^2 \quad x_2^3 \quad x_2^4 \quad x_2^5$$

$$y_2 \quad x_2 y_2 \quad x_2^2 y_2 \quad x_2^3 y_2 \quad x_2^4 y_2 \quad x_2^5 y_2$$

our explicit column reduction $\begin{bmatrix} -I & A \\ A & -I \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$ tells us

$$\left[(x_1^2 y_1) \cdot x_2 ,\ (x_1^2 y_1 x_2)^{-1} \right]_r = 1 - q^{*} x_2^{6} y_2^{4}$$

$$\left[(x_1^3 y_1^2) \cdot y_2 ,\ (x_1^3 y_1^2 y_2)^{-1} \right]_r = 1 - q^{*} x_2^{12} y_2^{6}$$

as in $\begin{bmatrix} 6 & 12 \\ 4 & 6 \end{bmatrix}$

But further ③ tells us from simplifying $(x_1^2 y_1 x_2)^{-2} (x_1^3 y_1^2 y_2)^{1}$, $\begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$

resp $(x_1^2 y_1 x_2)^{-3} (x_1^3 y_1^2 y_2)^{2}$

$$\left[x_1^{-1} x_2^{-2} y_2 ,\ x_1 x_2^2 y_2 \right] = q^2 y_2^2 - 1$$

$$\left[x_2^{-3} y_1 y_2^2 ,\ (x_2^{-3} y_1 y_2^2)^{-1} \right] = 1 - q^{-16} x_2^6$$

Two important observations:

① Suppose B, C, D, E, M are monomials

and $[B, C]_q = D - E$. Then $[B, CM] = DM - q^{*}EM$

But $*$ need not be zero.

This means $\mathcal{C}$ is really a $\mathbb{C}$-span of commutators and

NOT a $\mathbb{C}_q\langle x_1^{\pm} x_2^{\pm} y_1^{\pm} y_2^{\pm}\rangle$ - submodule.

(writing $\mathbb{C}_q\langle\ \rangle$ for $\mathbb{C}(q)\langle x_1^{\pm} x_2^{\pm} y_1^{\pm} y_2^{\pm}\rangle \Big/ \left\langle \begin{matrix} x_1x_2 = x_2x_1 \\ x_1y_2 = y_2x_1 \\ y_1y_2 = y_2y_1 \\ x_2y_1 = y_1x_2 \\ y_1x_1 = q\, x_1y_1,\ y_2x_2 = q\, x_2y_2 \end{matrix} \right\rangle$)

so all the linear algebra on exponents is sloppy about

coefficients.

In other words, it doesn't see if $\mathcal{C} \ni X_1 - qX_2$

and $X_1 - q^2X_2$

(hypothetically) which would force $X_1, X_2 \in \mathcal{C}$.

So $\mathbb{C}_q\langle X, Y\rangle / \mathcal{C}$ can be smaller i.e. ≤ 12-dim

② When taking σ-coinvariants we LOSE this

"almost-submodule" property from ①.

For instance – we quotient now by $X_1 - X_2$

but I see no reason to quotient by $X_1 M - q^* X_2 M$

for M = some monomial.

Thus I am revisiting relations for $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

being careful with q^*. Please check! Check all powers of q being careful about things like as $(X_1 y_1)^2 \neq X_1^2 y_1^2$ etc

$$[X_1, 1] = X_1 - X_2^2 y_2$$

$$[X_1^2, 1] = X_1^2 - q^2 X_2^4 y_2$$

$$[X_1^{-1} X_2^{-2} y_2, (y_2^{-1} X_2^2 X_1) X_2^4 y_2^2] = X_2^4 y_2^2 - q^{-6} X_2^4$$

$$[X_1^3, 1] = X_1^3 - q^6 X_2^6 y_2^3$$

$$[X_1^{-1} X_2^{-2} y_2, (y_2^{-1} X_2^2 X_1) y_2^3] = y_2^3 - q^{-5} y_2$$

$$C \ni q^9 y_2^3 - q^6 X_2^6 y_2^3$$

$$X_1^3 - q^9 y_2^3$$

$$X_1^3 - q^4 y_2$$

$$[y_2, 1] = y_2 - X_1^3 y_1^2$$

$$X_1^3 y_1^2 - q^{-5} X_1^3$$

$$y_2 - q^{-5} X_1^3$$

$$[X_2^{-3} y_1 y_2^2, y_2^{-2} y_1^{-1} X_2^3 \cdot y_2^3] = y_2^3 - q^{-3} X_2^6 y_2^3$$