Name: Jeremy M. Hein

ID: 810952267

CSCI 3104, Algorithms — **Charlie Carlson & Ewan Davies**

Problem Set 2 – Due Sept. 10, 2020 — **Fall 2020, CU-Boulder**

Advice 1: For every problem in this class, you must justify your answer: show how you arrived at it and why it is correct. If there are assumptions you need to make along the way, state those clearly.

Advice 2: Informal reasoning is typically insufficient for full credit. Instead, write a logical argument, in the style of a mathematical proof.

Instructions for submitting your solutions:

- The solutions **should be typed using** LaTeXand we cannot accept hand-written solutions. Here's a short intro to LaTeX.
- You should submit your work through the **class Canvas page** only.
- You may not need a full page for your solutions; pagebreaks are there to help Gradescope automatically find where each problem is. Even if you do not attempt every problem, please submit this template of at least 6 pages (or Gradescope has issues with it). **We will not accept submissions with fewer than 5 pages**.
- **You must CITE any outside sources you use, including websites and other people with whom you have collaborated. You do not need to cite a CA, TA, or course instructor.**
- **Posting questions to message boards or tutoring services including, but not limited to, Chegg, StackExchange, etc., is STRICTLY PROHIBITED. Doing so is a violation of the Honor Code.**

Quicklinks: 1 (2a) (2b) (2c) (2d) (2e) (2f)

Problem 1. Name (a) one advantage, (b) one disadvantage, and (c) one alternative to worst-case analysis. For (a) and (b) you should use full sentences.

Answer: One advantage to a worst-case analysis is that we know how our algorithm would perform under it's worst scenario. This can be very powerful to a programmer, because we may know that an algorithm is impossible to run before we write a single line of code (i.e. if an algorithm would take too long for a standard computer to run). One disadvantage is that the worst-case analysis doesn't give any insight into the average performance of the algorithm. Perhaps the worst-case is an edge-case that has significantly greater time complexity than non-edge cases. We would have no insight into this with only the worst-case analysis. One alternative may be to conduct both worst-case and best-case analyses to give us an upper and a lower bound, and then test the average run time.

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Problem 2. Put the growth rates in order, from slowest-growing to fastest. That is, if your answer is $f_1(n), f_2(n), \ldots, f_k(n)$, then $f_i(n) \leq O(f_{i+1}(n))$ for all i. If two adjacent ones have the same order of growth (that is, $f_i(n) = \Theta(f_{i+1}(n))$), you must specify this as well. Justify your answer (show your work).

- You may assume transitivity: if $f(n) \leq O(g(n))$ and $g(n) \leq O(h(n))$, then $f(n) \leq O(h(n))$, and similarly for little-oh, etc. Note that the goal is to order the growth rates, so transitivity is very helpful. We encourage you to make use of transitivity rather than comparing all possible pairs of functions, as using transitivity will make your life easier.
- You may also use the Limit Comparison Test (see Michael's Calculus Notes on Canvas). However, you **MUST** show all limit computations at the same level of detail as in Calculus I-II. Should you choose to use Calculus tools, whether you use them correctly will count towards your mastery score.
- You may **NOT** use heuristic arguments, such as comparing degrees of polynomials or identifying the "high order term" in the function.
- If it is the case that $g(n) = c \cdot f(n)$ for some constant c, you may conclude that $f(n) = \Theta(g(n))$ without using Calculus tools. You must clearly identify the constant c (with any supporting work necessary to identify the constant- such as exponent or logarithm rules) and include a sentence to justify your reasoning.

(2a) *Polynomials.*

$$3n+1, \qquad n^6, \qquad \frac{1}{n}, \qquad 1, \qquad n^2+3n-5, \qquad n^2, \qquad \sqrt{n}, \qquad 10^{100}.$$

Note I will be using the Limit Comparison Test in the form below to compare the polynomials:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)}$$

First we will compare $3n+1$ and n^6:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{3n+1}{n^6} = \frac{\infty}{\infty} \\
&= \lim_{n\to\infty} \frac{3}{6n^5} && \text{(L'Hopital's Rule)} \\
&= \frac{3}{\infty} && \text{(Taking the limit)} \\
&= 0 \\
3n+1 &\le O(n^6) && \text{(Limit Comparison Test)}
\end{aligned}
$$

Our current list is $3n+1$, n^6. Now we'll compare $\frac{1}{n}$:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{3n+1}{\frac{1}{n}} \\
&= \lim_{n\to\infty} n(3n+1) && \text{(Simplifying)} \\
&= \infty(3\infty+1) && \text{(Taking the limit)} \\
&= \infty \\
\frac{1}{n} &\le O(3n+1) && \text{(Limit Comparison Test)} \\
\frac{1}{n} &\le O(3n+1) \le O(n^6) && \text{(Transitivity)}
\end{aligned}
$$

Our current list is $\frac{1}{n}$, $3n+1$, n^6. Now we will look at our constants, 1 and 10^{100}:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{1}{10^{100}} \\
&= \frac{1}{10^{100}} && \text{(Limit Laws)} \\
1 &= \Theta(10^{100}) && \text{(Limit Comparison Test)}
\end{aligned}
$$

So our constants are equal in terms of growth rates, now we'll compare to $\frac{1}{n}$:

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{1}{\frac{1}{n}} \\
&= \lim_{n\to\infty} n && \text{(Simplifying)} \\
&= \infty && \text{(Taking the limit)} \\
\frac{1}{n} &\leq O(1) && \text{(Limit Comparison Test)}
\end{aligned}$$

So our constants are greater than $\frac{1}{n}$ in terms of growth rates. Now we'll compare to $3n+1$:

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{1}{3n+1} \\
&= \frac{1}{3\infty+1} && \text{(Taking the limit)} \\
&= \frac{1}{\infty} \\
&= 0 \\
1 &\leq O(3n+1) && \text{(Limit Comparison Test)} \\
\frac{1}{n} \leq O(1) = \Theta(10^{100}) &\leq O(3n+1) \leq O(n^6) && \text{(Transitivity)}
\end{aligned}$$

Our current list is $\frac{1}{n}$, 1, 10^{100}, $3n+1$, n^6. Now we'll compare n^2 and n^2+3n-5:

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{n^2}{n^2+3n-5} = \frac{\infty}{\infty} \\
&= \lim_{n\to\infty} \frac{2n}{2n+3} = \frac{\infty}{\infty} && \text{(L'Hopital's Rule)} \\
&= \lim_{n\to\infty} \frac{2}{2} && \text{(L'Hopital's Rule)} \\
&= 1 && \text{(Limit Laws)} \\
n^2 &= \Theta(n^2+3n-5) && \text{(Limit Comparison Test)}
\end{aligned}$$

So n^2 and n^2+3n-5 are equal in terms of growth rates. Now we'll compare to n^6:

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{n^2}{n^6} = \frac{\infty}{\infty} \\
&= \lim_{n\to\infty} \frac{2n}{6n^5} = \frac{\infty}{\infty} && \text{(L'Hopital's Rule)} \\
&= \lim_{n\to\infty} \frac{2}{30n^4} && \text{(L'Hopital's Rule)} \\
&= \frac{2}{30\infty} && \text{(Taking the limit)} \\
&= 0 \\
n^2 &\leq O(n^6) && \text{(Limit Comparison Test)}
\end{aligned}$$

So our n^2 terms are less than n^6 in terms of growth rates. Now we will compare to $3n+1$:

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{n^2}{3n+1} = \frac{\infty}{\infty} \\
&= \lim_{n\to\infty} \frac{2n}{3} && \text{(L'Hopital's Rule)} \\
&= \frac{2\infty}{3} && \text{(Taking the limit)} \\
&= \infty \\
3n+1 &< o(n^2) && \text{(Limit Comparison Test)} \\
\frac{1}{n} \leq O(1) = \Theta(10^{100}) \leq O(3n+1) &\leq O(n^2) = \Theta(n^2+3n-5) \leq O(n^6) && \text{(Transitivity)}
\end{aligned}$$

Our current list is $\frac{1}{n}$, 1, 10^{100}, $3n+1$, n^2, n^2+3n-5, n^6. Now we will compare $\sqrt{n}$ to 1:

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \\
&= \frac{1}{\infty} && \text{(Taking the limit)} \\
&= 0 \\
1 &\leq O(\sqrt{n}) && \text{(Limit Comparison Test)}
\end{aligned}$$

Last, we will compare $\sqrt{n}$ to $3n+1$:

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{\sqrt{n}}{3n+1} = \frac{\infty}{\infty} \\
&= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{3} && \text{(L'Hopital's Rule)} \\
&= \frac{\frac{1}{\infty}}{3} && \text{(Taking the limit)} \\
&= 0 \\
\sqrt{n} &\le O(3n+1) && \text{(Limit Comparison Test)}
\end{aligned}$$

Our final list is $\frac{1}{n}$, 1, 10^{100}, $\sqrt{n}$, $3n+1$, n^2, n^2+3n-5, n^6.

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(2b) Prove that for any $a, b > 0$ where $a \neq 1$ and $b \neq 1$, that $\log_a(n) = \Theta(\log_b(n))$. Here, a and b do not depend on n. [**Hint:** Review the change of base formula.]

Proof. Again, I will use the Limit Comparison Test as noted in part 2(a):

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{\log_a(n)}{\log_b(n)} \\
&= \lim_{n\to\infty} \frac{\frac{\ln n}{\ln a}}{\frac{\ln n}{\ln b}} && \text{(Logarithm Rules - Change base } e\text{)} \\
&= \lim_{n\to\infty} \frac{\ln b \ln n}{\ln a \ln n} && \text{(Simplifying)} \\
&= \frac{\ln b}{\ln a} \lim_{n\to\infty} \frac{\ln n}{\ln n} && \text{(Limit Laws - } \frac{\ln b}{\ln a} \text{ is a constant)} \\
&= \frac{\ln b}{\ln a} \lim_{n\to\infty} \frac{1}{1} && \text{(Factoring)} \\
&= \frac{\ln b}{\ln a}
\end{aligned}$$

Since $0 < \frac{\ln b}{\ln a} < \infty$, we can say that $\log_a(n) = \Theta(\log_b(n))$.

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(2c) *Logarithms and related functions. [**Hint** Use part (2b).]*

$$(\log_3(n))^3 \qquad \log_5(n) \qquad \log_3(n) \qquad \sqrt[3]{n} \qquad \log_{2.5}(n) \qquad \log_5(n^2)$$

Referencing the proof in part $2(b)$, we know $\log_a(n) = \Theta(\log_b(n))$. Therefore:

$$\begin{aligned}
\log_5(n) &= \Theta(\log_3(n)) && \text{(Proof - 2}(b))\\
\log_5(n) &= \Theta(\log_{2.5}(n)) && \text{(Proof - 2}(b))\\
\log_3(n) &= \Theta(\log_{2.5}(n)) && \text{(Proof - 2}(b))
\end{aligned}$$

So our list is $\log_{2.5}(n)$, $\log_3(n)$, $\log_5(n)$. Now we'll look at $\log_5(n^2)$.

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{\log_5(n^2)}{\log_5(n)}\\
&= \lim_{n\to\infty} \frac{2\log_5(n)}{\log_5(n)} && \text{(Logarithm Rules)}\\
&= 2\lim_{n\to\infty} \frac{\log_5(n)}{\log_5(n)} && \text{(Limit Laws)}\\
&= 2\lim_{n\to\infty} \frac{1}{1} && \text{(Reducing Fraction)}\\
&= 2 && \text{(Taking the limit)}\\
\log_5(n^2) &= \Theta(\log_5(n) && \text{(Limit Comparison Test)}\\
\log_5(n^2) &= \Theta(\log_5(n)) = \Theta(\log_3(n)) = \Theta(\log_{2.5}(n)) && \text{(Transitivity)}
\end{aligned}$$

So our list is $\log_{2.5}(n)$, $\log_3(n)$, $\log_5(n)$, $\log_5(n^2)$. Now we'll look at $(\log_3(n))^3$:

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{(\log_3(n))^3}{\log_3(n)}\\
&= \lim_{n\to\infty} (\log_3(n))^2 && \text{(Reducing Fraction)}\\
&= (\log_3(\infty))^2 && \text{(Taking the limit)}\\
&= \infty\\
\log_3(n) &\leq O(\log_3(n))^3 && \text{(Limit Comparison Test)}
\end{aligned}$$

Now our list is $\log_{2.5}(n)$, $\log_3(n)$, $\log_5(n)$, $\log_5(n^2)$, $(\log_3(n))^3$. Lastly we will compare $\sqrt[3]{n}$ and $(\log_3(n))^3$:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{\sqrt[3]{n}}{(\log_3(n))^3} = \frac{\infty}{\infty} \\
&= \lim_{n\to\infty} \frac{\sqrt[3]{n}}{\frac{\log^3(n)}{\log^3(3)}} && \text{(Logarithm Rules)} \\
&= \lim_{n\to\infty} \frac{\frac{1}{3n^{2/3}}}{\frac{3\log^2(n)}{n\log^3(3)}} && \text{(L'Hopital's Rule)} \\
&= \lim_{n\to\infty} \frac{n\log^3(3)}{3n^{(2/3)}\log^2(n)} && \text{(Algebra)} \\
&= \lim_{n\to\infty} \frac{n^{1/3}\log^3(3)}{3\log^2(n)} && \text{(Factoring)} \\
&= \frac{\log^3(3)}{3} \lim_{n\to\infty} \frac{n^{1/3}}{\log^2(n)} = \frac{\infty}{\infty} && \text{(Removing Constants)} \\
&= \frac{\log^3(3)}{3} \lim_{n\to\infty} \frac{\frac{1}{3n^{2/3}}}{\frac{2\log(n)}{n}} && \text{(L'Hopital's Rule)} \\
&= \frac{\log^3(3)}{3} \lim_{n\to\infty} \frac{n}{3n^{(2/3)}2\log(n)} && \text{(Algebra)} \\
&= \frac{\log^3(3)}{3} \lim_{n\to\infty} \frac{n^{1/3}}{6\log(n)} && \text{(Factoring)} \\
&= \frac{\log^3(3)}{18} \lim_{n\to\infty} \frac{n^{1/3}}{\log(n)} = \frac{\infty}{\infty} && \text{(Removing Constants)} \\
&= \frac{\log^3(3)}{18} \lim_{n\to\infty} \frac{\frac{1}{3n^{2/3}}}{\frac{1}{n}} && \text{(L'Hopital's Rule)} \\
&= \frac{\log^3(3)}{18} \lim_{n\to\infty} \frac{n}{3n^{2/3}} && \text{(Algebra)} \\
&= \frac{\log^3(3)}{18} \lim_{n\to\infty} \frac{n^{1/3}}{3} && \text{(Factoring)} \\
&= \infty && \text{(Taking the limit)} \\
(\log_3(n))^3 &\leq O(\sqrt[3]{n}) && \text{(Limit Comparison Test)}
\end{aligned}
$$

Our final list is $\log_{2.5}(n)$, $\log_3(n)$, $\log_5(n)$, $\log_5(n^2)$, $(\log_3(n))^3$, $\sqrt[3]{n}$.

(2d) Construct specific functions $f(n)$ and $g(n)$ such that $f(n) = \Theta(g(n))$ but $2^{f(n)} \neq \Theta(2^{g(n)})$. Formally show that $2^{f(n)} \neq \Theta(2^{g(n)})$ here.

Let $f(n) = n$ and $g(n) = 5n$. Applying the Limit Comparison Test:

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{n}{5n} = \frac{\infty}{\infty} \\ &= \lim_{n\to\infty} \frac{1}{5} && \text{(L'Hopital's Rule)} \\ &= \frac{1}{5} \end{aligned}$$

Since $0 < \frac{1}{5} < \infty$, we can say that $n = \Theta(5n)$. Now looking at 2^n and 2^{5n}:

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{2^n}{2^{5n}} \\ &= \lim_{n\to\infty} \frac{1}{2^{4n}} && \text{(Factoring)} \\ &= \frac{1}{\infty} \\ &= 0 \\ 2^n &\leq O(2^{5n}) && \text{(Limit Comparison Test)} \end{aligned}$$

Since the limit above $= 0$, we can conclude $2^{f(n)} \neq \Theta(2^{g(n)})$.

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(2e) *Logarithms in exponents. [**Hint:** Review the logarithm change of base formula, as well as the rules of logarithms.]*

$$n^{\log_4(n)} \qquad n^{\log_5(n)} \qquad n^{1/\log_3(n)} \qquad n \qquad 1$$

We'll start with comparing $n^{1/\log_3(n)}$ and n. Specifically, we can look at the exponents since they are both of base n:

$1/\log_3(n)$ is decreasing as n gets large, so the term $n^{1/\log_3(n)}$ must have a slower growth rate than n. Specifically $n^{1/\log_3(n)} \leq O(n)$ for $n \geq 3$, since $log_3(3) = 1$ and $1/\log_3(n) \leq 1$ for $n \geq 3$.

Next we can compare $n^{\log_4(n)}$ and $n^{\log_5(n)}$. Again, we will compare the exponents since they are both of base n.

$\log_4(n) = \frac{lnn}{ln4}$ and $\log_5(n) = \frac{lnn}{ln5}$. We also know that $ln5 > ln4$, so the term $\frac{lnn}{ln5}$ must be less than or equal to $\frac{lnn}{ln4}$. This is true for $n \geq 1$, as $ln1 = 0$. Furthermore, this implies $n^{\log_5(n)} \leq O(n^{\log_4(n)})$ for $n \geq 1$.

The term $n^{\log_5(n)}$ must also be greater than n, as we know the exponent $\log_5(n)$ is increasing as n gets large. Specifically, $\log_5(n) \geq 1$ for $n \geq 5$, since $\log_5(5) = 1$. Thus, $n \leq O(n^{\log_5(n)})$ for $n \geq 5$.

That means our current list is $n^{1/\log_3(n)}$, n, $n^{\log_5(n)}$, $n^{\log_4(n)}$. The constant 1 must be lower than all of them, as all of the other terms are growing faster than the constant, which is stagnant. So the final list is 1, $n^{1/\log_3(n)}$, n, $n^{\log_5(n)}$, $n^{\log_4(n)}$.

(2f) *Exponentials. [**Hint:** Recall the Ratio and Root Tests from Michael's Calculus Notes.]*

$$n! \qquad 3^n \qquad 3^{5n} \qquad 3^{n\log_4(n)} \qquad 3^{n+13}$$

We'll start by comparing 3^n and 3^{5n}:

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{3^n}{3^{5n}} \\
&= \lim_{n\to\infty} \frac{1}{3^{4n}} && \text{(Factoring)} \\
&= \frac{3}{\infty} && \text{(Taking the limit)} \\
&= 0 \\
3^n &\leq O(3^{5n}) && \text{(Limit Comparison Test)}
\end{aligned}$$

Our current list is 3^n, 3^{5n}. Now we'll compare 3^n and $3^{n\log_4(n)}$:

$$\begin{aligned}
L &= \lim_{n\to\infty} \frac{3^n}{3^{n\log_4(n)}} \\
&= \lim_{n\to\infty} \frac{1}{3^{\log_4(n)}} && \text{(Factoring)} \\
&= \frac{1}{\infty} && \text{(Taking the limit)} \\
&= 0 \\
3^n &\leq O(3^{n\log_4(n)}) && \text{(Limit Comparison Test)}
\end{aligned}$$

Now we'll compare 3^{5n} and $3^{n\log_4(n)}$:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{3^{5n}}{3^{n\log_4(n)}} \\
&= \lim_{n\to\infty} \frac{3^{4n}}{3^{\log_4(n)}} && \text{(Factoring)} \\
&= \lim_{n\to\infty} 3^{4n-\frac{lnn}{ln4}} && \text{(Logarithm / Exponent Rules)} \\
&= 3^{\infty} && (4n > lnn \text{ for all } n \geq 1) \\
&= \infty \\
3^{n\log_4(n)} &\leq O(3^{5n}) && \text{(Limit Comparison Test)}
\end{aligned}
$$

In the above proof, note that when $n = 1$, $4n = 4$, and $ln(1) = 0$. When $n = 2$, $4n = 8 > ln(2)$. When $n = 3$, $4n = 12 > ln(3)$. When $n = n$, $4n > ln(n)$, so we know the exponent must be increasing as n gets large for $n \geq 1$.

Our current list is 3^n, $3^{n\log_4(n)}$, 3^{5n}. Now we'll compare 3^n and 3^{n+13}:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{3^{n}}{3^{n+13}} \\
&= \lim_{n\to\infty} \frac{3^{n}}{3^{n}3^{13}} && \text{(Exponent Rules)} \\
&= \lim_{n\to\infty} \frac{1}{3^{13}} && \text{(Factoring)} \\
&= \frac{1}{3^{13}} \\
3^n &= \Theta(3^{n+13}) && \text{(Limit Comparison Test)}
\end{aligned}
$$

By transitivity, our current list is 3^n, 3^{n+13}, $3^{n\log_4(n)}$, 3^{5n}. Now we'll compare $n!$ to 3^{5n}:
Consider the following series:

$$\sum_{n=0}^{\infty} \frac{n!}{3^{5n}}$$

Here, $a_n = \frac{n!}{3^{5n}}$. Now consider:

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{(n+1)!3^{5n}}{n!3^{5(n+1)}} \\
&= \lim_{n\to\infty} \frac{(n+1)3^{5n}}{3^{5n}3^5} && \text{(Factoring)} \\
&= \lim_{n\to\infty} \frac{n+1}{3^5} && \text{(Factoring)} \\
&= \infty
\end{aligned}
$$

Therefore, by the Ratio Test, we know that the series diverges, and also that

$$\lim_{n\to\infty} \frac{n!}{3^{5n}} = \infty$$

Then by the Limit Comparison Test, $3^{5n} \leq O(n!)$. Our final list is then 3^n, 3^{n+13}, $3^{n\log_4(n)}$, 3^{5n}, $n!$.