Some Examples Using Theorem 9

Here we discuss, in some detail, three examples which illustrate the algorithm provided in the proof of Theorem 9. As these examples employ the notation used in the proof of Theorem 9 - you should look over the proof of Theorem 9.

Ex 1: $A = A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \end{bmatrix}$

In this case, $j_1 = 2$ - since the first nonzero column of A_0 is column 2.

Also $i_1 = 2$ - since the nonzero entry in the 2^{nd}- column of A_0 is in the 2^{nd}-row of A_0.

Since $i_1 = 2 > 1$, A_1 is obtained by interchanging the rows of A_0 - so

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Since $a^{(1)}_{1j_1} = a^{(1)}_{12} = 1$, $A_2 = A_1$.

Also, since $a^{(2)}_{2j_1} = a^{(2)}_{22} = 0$, $A_3 = A_2$.

Thus,

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Since $m = 2$ in this example, A_3 plays the role of A_5 in the proof of the Base Case.

In this example

$$T_3 = [0\ 1\ 0\ 0\ 3]$$

and

$$B_3 = [0\ 0\ 0\ 0\ 2].$$

Observe that T_3 is a 1×5 matrix in RREF having a pivot in the j_1-st $= 2^{nd}$ column. Also the first $j_1 = 2$ columns of B_3 have only zero entries.

We now use the Inductive Step - with $l=1$.

j_2 = the first nonzero column of B_3

$= 5$.

Since there is a nonzero entry at the top of the j_2-nd $= 5^{th}$ column of B_3, $A_4 = A_3$.

Since $a^{(4)}_{2j_2} = a^{(4)}_{25} = 2 \neq 1$, A_5 is obtained by multiplying the second row of A_4 by $\frac{1}{2}$.

Thus,

$$A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We are now in the clean up process of the proof of the Inductive Step.

In this case, this means that we must replace the 3 appearing in the $a^{(5)}_{1j_2}$ entry, or the $a^{(5)}_{15}$ entry, with a 0. This is accomplished by adding -3 times the

2^{nd} row of A_5 to the 1^{st} row of A_5 - and yields

$$A_6 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

A_6 is in RREF.

Ex 2:

$$A = A_0 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

In this example, $j_1 =$ the 1^{st} nonzero column of A_0 is 1.

Since $a_{1j_1}^{(0)} \neq 0$, $A_1 = A_0$.

Since $a_{1j_1}^{(0)} = a_{11}^{(0)} = 1$, $A_2 = A_1$. Thus,

$$A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

and we now enter into the clean up portion of the Base Case.

Since $a^{(2)}_{2j_1} = a^{(2)}_{21} = 4$, we add -4 times the 1^{st} row of A_2 to the 2^{nd} row of A_2 - obtaining

$$A_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Since $a^{(3)}_{3j_1} = a^{(3)}_{31} = 7$, we now add -7 times the 1^{st} row of A_3 to the 3^{rd} row of A_3 - which yields

$$A_4 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}.$$

Since $m = 3$ in this example,

A_4 plays the role of A_5 in the proof of the Base Case.

As such,

$$T_4 = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

and

$$B_4 = \begin{bmatrix} 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}.$$

Note that T_4 is a 1×3 matrix in RREF, having a pivot in the $j_1 = 1$ column. Also, the first $j_1 = 1$ columns of B_4 are columns whose entries are all zeros.

We now enter into the Inductive Step. Note that $j_2 = 2$ – since the first nonzero column of B_4 is its 2^{nd} column. Since the top entry in the $j_2 = 2^{nd}$ column of B_4 is nonzero, $A_5 = A_4$. Since $a^{(5)}_{2j_2} = a^{(5)}_{22} = -3$, A_6 is obtained by multiplying the 2^{nd} row of A_5 by $-\frac{1}{3}$ - yielding

$$A_6 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -6 & -12 \end{bmatrix}.$$

We now enter into the clean up phase of the Inductive Step. Since

$$a^{(6)}_{3j_2} = a^{(6)}_{32} = -6,$$

A_7 is obtained by adding 6 times the 2^{nd} row of A_6 to the 3^{rd} row of A_6. Thus

$$A_7 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $a^{(7)}_{1j_2} = a^{(7)}_{12} = 2$, A_8 is obtained by adding -2 times the 2^{nd} row of A_7 to the 1^{st} row of A_7. So

$$A_8 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

This completes the 1^{st} application of the Inductive Step in this example. Note that

$$T_8 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

and

$$B_8 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

Observe that T_8 is a 2×3 matrix in RREF, having pivots in both the $j_1 = 1$ and $j_2 = 2$ columns. Also, the first $j_2 = 2$ columns of B_8 consist solely of zeros.

In many cases we would have to repeat the Inductive Step - starting with A_8 - to improve the 3^{rd} row. However, in this case B_8 is a zero matrix - so A_8 is in RREF.

Ex 3:

$$A = A_0 = \begin{bmatrix} 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 6 & 0 & 0 & 4 \end{bmatrix}$$

Here $j_1 = 1$ - since the 1^{st} colum of A_0 is nonzero. However $a^{(0)}_{1j_1} = a^{(0)}_{11} = 0$. In fact $i_1 = 3$ - since this is the highest nonzero entry in the 1^{st} column. Thus, A_1 is obtained by interchanging the 1^{st} and 3^{rd} rows of A_0. So

$$A_1 = \begin{bmatrix} 1 & 6 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \end{bmatrix}$$

Since $a^{(1)}_{1j_1} = 1$, $A_2 = A_1$.

This takes us to the clean up phase of the Base Case. However, since

$a^{(2)}_{2j_1} = 0$, $A_3 = A_2$. Since

$a^{(3)}_{3j_1} = 0$, $A_4 = A_3$. So

$$A_4 = \begin{bmatrix} 1 & 6 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \end{bmatrix}.$$

Since $m = 3$, A_4 plays the role of A_s in the Base Case. Note that

$$T_4 = \begin{bmatrix} 1 & 6 & 0 & 0 & 4 \end{bmatrix}$$

and

$$B_4 = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 & 6 \end{bmatrix}.$$

Observe further that T_4 is a 1×5 matrix in RREF, having a pivot in the $j_1 = 1$ column. Also, the first $j_1 = 1$ columns of B_4 only have zeros for entries.

We now apply the Inductive Step to A_4. Note that $j_2 = 3$ - since the first nonzero column of B_4 is its 3rd column. Note further that $i_2 = 3$ in this case - due to $a^{(4)}_{2j_2} = a^{(4)}_{23} = 0$. Thus, A_5 is obtained by interchanging the 2nd and 3rd rows of A_4. Thus,

$$A_5 = \begin{bmatrix} 1 & 6 & 0 & 0 & 4 \\ 0 & 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

Since $a^{(5)}_{2j_2} = 2 \neq 1$, A_6 is obtained by multiplying the 2nd row of A_5 by $\frac{1}{2}$. This yields

$$A_6 = \begin{bmatrix} 1 & 6 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

We now enter the clean up phase of the Inductive Step. However, since $a^{(6)}_{3j_2} = a^{(6)}_{33} = 0$, $A_7 = A_6$. Also, since $a^{(7)}_{1j_2} = a^{(7)}_{13} = 0$, $A_8 = A_7$. Thus, we have

$$A_8 = \begin{bmatrix} 1 & 6 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

and we have completed the clean up phase of the Inductive Step.

Note that

$$T_8 = \begin{bmatrix} 1 & 6 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

and

$$B_8 = [0\ 0\ 0\ 1\ 2].$$

Observe that T_8 is a 2×5 matrix in RREF, having a pivot in both its $j_1 = 1$ and $j_2 = 3$ columns. Also, the first $j_2 = 3$ columns of B_8 consist of zeros.

We now have to apply the Inductive Step one more time - this time starting with A_8.

Note that $j_3 = 4$ - since this is the first nonzero column of B_8. Also $i_3 = 3$ - since the top entry in the $j_3 = 4$ th column of B_8 is nonzero.

As such, $A_9 = A_8$ - since no interchanging of rows is needed.

Since $a^{(9)}_{3j_3} = a^{(9)}_{34} = 1$, $A_{10} = A_9$.

So

$$A_{10} = \begin{bmatrix} 1 & 6 & 0 & 0 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

and we enter into the clean up phase of the inductive step.

Since $a^{(10)}_{1j_3} = a^{(10)}_{14} = 0$, $A_{11} = A_{10}$.

Since $a^{(11)}_{2j_3} = a^{(11)}_{24} = 2$, A_{12} is obtained by adding -2 times the 3^{rd} row of A_{11} to the 2^{nd} row of A_{11}. This leaves us with

$$A_{12} = \begin{bmatrix} 1 & 6 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

This completes the second application of the Inductive Step. Observe that $T_3 = A_{12}$ is in RREF and T_3 has pivots in the $j_1 = 1$, $j_2 = 3$ and $j_3 = 4$ columns.

<u>HW #22</u>: Transform the following matrices into RREF.

1) $$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

2) $$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

3) $$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

4) $$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Bases From Spanning Sets

Our ability to transform a mxn matrix into a mxn matrix in RREF is an extremely powerful tool. Our first application will be to construct a basis for a nonzero subspace of $\mathbb{R}^n$ - provided we are given a set of spanning vectors for the subspace. You might recall that every set of spanning vectors for a nonzero subspace of $\mathbb{R}^n$ contains a basis for the subspace. This fact is certainly useful, from the standpoint of theory, buts proves to be a bit tedious for computational purposes. It should also be noted that bases obtained using RREF tend to be very easy to work with.

Suppose W is a nonzero subspace of $\mathbb{R}^n$ and $\text{span}(\{\vec{v}_1, \cdots, \vec{v}_m\}) = W$. We can perform three elementary operations on the spanning set $\{\vec{v}_1, \cdots, \vec{v}_m\}$ which yield a new spanning set $\{\vec{w}_1, \cdots, \vec{w}_m\}$ for W.

<u>Operation I</u> : Fix some $j \in \{1, \cdots, m\}$ and let $r \in \mathbb{R}$, $r \neq 0$. Set

$\vec{w}_i = \vec{v}_i$ if $i \neq j$ and

$\vec{w}_j = r\vec{v}_j$.

Operation II: Fix $k, \ell \in \{1, \cdots, m\}$, where $k \neq \ell$ and let $s \in \mathbb{R}$.

Set $\vec{w}_i = \vec{v}_i$ for $i \neq \ell$ and

$$\vec{w}_\ell = \vec{v}_\ell + s\vec{v}_k .$$

Operation III: $\{\vec{w}_1, \cdots, \vec{w}_m\}$ is obtained by reindexing $\{\vec{v}_1, \cdots, \vec{v}_m\}$ – so, as sets,

$$\{\vec{w}_1, \cdots, \vec{w}_m\} = \{\vec{v}_1, \cdots, \vec{v}_m\}.$$

Note that if $\{\vec{w}_1, \cdots, \vec{w}_m\}$ denotes the output of any of these three operations, it isn't difficult to see that $\{\vec{w}_1, \cdots, \vec{w}_m\} \subset \text{span}(\{\vec{v}_1, \cdots, \vec{v}_m\})$.

So $\text{span}(\{\vec{w}_1, \cdots, \vec{w}_m\}) \subset \text{span}(\{\vec{v}_1, \cdots, \vec{v}_m\})$.

On the other hand, each of these operations is easily reversed. From this, it follows that

$$\{\vec{v}_1, \cdots, \vec{v}_m\} \subset \text{span}(\{\vec{w}_1, \cdots, \vec{w}_m\}).$$

So $\text{span}(\{\vec{v}_1, \cdots, \vec{v}_m\}) \subset \text{span}(\{\vec{w}_1, \cdots, \vec{w}_m\})$.

We conclude that each of these three elementary operations yields another spanning set for W.

I suspect you have noticed that these three elementary operations are remarkably similar to the three elementary row operations. To implement this analogy, we introduce the notion of the row space of a matrix.

Let $A = [a_{ij}]$ denote a $m \times n$ matrix. For $i = 1, \cdots, m$, set

$$\vec{v}_i = [a_{i1}, a_{i2}, \cdots, a_{in}].$$

Thus, $\vec{v}_i$ is just the i-th row of A - viewed as an element of $\mathbb{R}^n$. The row space of A is defined to be $\text{span}(\{\vec{v}_1, \cdots, \vec{v}_m\})$. In words, the row space of A is just the span of the rows in A - viewed as vectors in $\mathbb{R}^n$.

Now let A denote a $m \times n$ matrix and let B denote a $m \times n$ matrix which results from performing an elementary row operation on A. Let $\{\vec{v}_1, \ldots, \vec{v}_m\}$ denote the rows in A - viewed as vectors in $\mathbb{R}^n$ and, similarly, let $\{\vec{w}_1, \ldots, \vec{w}_m\}$ denote the rows of B - viewed as vectors in $\mathbb{R}^n$. Note that whichever elementary row operation is applied to A so as to obtain B corresponds to performing the same roman numeral operation to $\{\vec{v}_1, \cdots, \vec{v}_m\}$ to obtain $\{\vec{w}_1, \cdots, \vec{w}_m\}$. Thus

$$\operatorname{span}(\{\vec{v}_1, \cdots, \vec{v}_m\}) = \operatorname{span}(\{\vec{w}_1, \cdots, \vec{w}_m\})$$

or

$$\text{row space of } A = \text{row space of } B.$$

We have established the following.

<u>Proposition 16</u>: Suppose A and B are two row equivalent $m \times n$ matrices. Then

$$\text{row space of } A = \text{row space of } B.$$

The following Proposition provides us with a computational punch.

<u>Proposition 17</u>: Suppose B is a nonzero $m \times n$ matrix in RREF. Then the nonzero rows in B, viewed as vectors in $\mathbb{R}^n$, form a basis for the row space of B.

<u>Proof</u>: Since B is assumed to be a nonzero matrix and a zero vector only contributes a zero vector to a span, the nonzero rows of B, when viewed as vectors in $\mathbb{R}^n$, span the row space of B — due to the definition of the row space of a matrix.

Suppose now that the nonzero rows of B are rows $1, \cdots, k$. Let $\vec{v}_i$ denote the element of $\mathbb{R}^n$ associated to the i-th row of B, for $i = 1, \cdots, k$. We must show $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is a linearly independent set of vectors.

To this end, note that B must have exactly k pivots – in columns $j_1, j_2, \cdots, j_k$ say. Let $\ell \in \{1, \cdots, k\}$. Note that since B is in RREF and $\vec{v}_i$ is the i-th row of B we must have that the j_ℓ-th component of $\vec{v}_i$ is 0 if $i \neq \ell$ and

1 if $i=\ell$. This is because each pivot is a 1 and, since B is in RREF, the only nonzero entry in any column of B containing a pivot is the pivot itself.

Suppose now that $c_1, \cdots, c_k \in \mathbb{R}$ and $\vec{w} = \sum_{i=1}^{k} c_i \vec{v}_i$. The last paragraph tells us that the j_ℓ'th component of $\vec{w}$ must be c_ℓ - due to the only $\vec{v}_i$ with a nonzero j_ℓ-th component is $\vec{v}_\ell$ and the j_ℓ-th component of $\vec{v}_\ell$ is 1. This immediately implies that if $\vec{w} = \vec{0}$, then

$c_\ell = 0$ for $\ell = 1, \cdots, k$. It follows that $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is linearly independent.

Since $\{\vec{v}_1, \cdots, \vec{v}_k\}$ spans the row space of B and is linearly independent – it is a basis for the row space of B.

Note: The proof that $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is linearly independent, as given above, shows that bases obtained from nonzero matrices in RREF are easy to work with. The next example illustrates this.

Ex: Let $W = \text{span}(\{[1,2,3],[4,5,6],[7,8,9]\})$.

Find a basis for W and determine whether $[1,1,1] \in W$ or $[1,1,1] \notin W$.

Note that

$$W = \text{row space of } \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

By Example 2 on page 183,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{ is row equivalent to}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Proposition 16 tells us that

$$W = \text{row space of} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Proposition 17 now tells us that $\{[1,0,-1], [0,1,2]\}$ is a basis for W.

To determine whether or not $[1,1,1] \in W$ boils down to whether or not $[1,1,1]$ can be written as a linear combination of $[1,0,-1]$ and $[0,1,2]$ — since $W = \text{span}(\{[1,0,-1], [0,1,2]\})$.

Note that

$$x[1,0,-1] + y[0,1,2] = [x, y, -x+2y].$$

So, if $[1,1,1] = x[1,0,-1] + y[0,1,2]$,

then $x = y = 1$. Now

$$[1,0,-1] + [0,1,2] = [1,1,1]$$

- so the answer is

$$[1,1,1] \in W.$$