The Dual Meaning of $\mathbb{R}^n$

Let $\mathbb{R}$ denote the real numbers and n be a positive integer. Initially, we define

$$\mathbb{R}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{R} \text{ for } i = 1, \cdots, n\}.$$

As such, $\mathbb{R}^2$ is typically viewed as the plane and $\mathbb{R}^3$ as space.

We shall also view $\mathbb{R}^n$ as being a set of vectors. To those who have taken MA 227, the process by which this is accomplished will appear familiar – since it agrees with the manner curves and surfaces in $\mathbb{R}^3$ are parameterized by vector-valued functions. Initially, however, we will view a vector in $\mathbb{R}^n$ as being represented by a directed line segment in $\mathbb{R}^n$ - which is as far as a student goes in MA 126.

If $P = (p_1, \cdots, p_n)$ and $Q = (q_1, \cdots, q_n)$ are two points in $\mathbb{R}^n$, then $\overrightarrow{PQ}$ denotes the directed line segment in $\mathbb{R}^n$ with tail at P and head at Q. The real number, $q_i - p_i$, is the i-th component of $\overrightarrow{PQ}$ - for $i = 1, \cdots, n$.

Recall that we said $\overrightarrow{PQ}$ represents a vector in $\mathbb{R}^n$. This is because a vector is a rather mobile creature. To be specific, we say that the function, $f: \mathbb{R}^n \to \mathbb{R}^n$, is a translation provided there exist $c_1, \cdots, c_n \in \mathbb{R}$ such that

$$f(x_1, \cdots, x_n) = (x_1 + c_1, \cdots, x_n + c_n).$$

If $P, Q, R, S \in \mathbb{R}^n$, then $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ represent the same vector precisely when there is a translation, $f: \mathbb{R}^n \to \mathbb{R}^n$, such that

$f(P) = R$ and $f(Q) = S$.

<u>Claim</u>: $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ represent the same vector in $\mathbb{R}^n$ if and only if the i-th component of $\overrightarrow{PQ}$ equals the i-th component of $\overrightarrow{RS}$ for $i = 1, \dots, n$.

<u>Proof in one direction</u>: Let $P = (p_1, \dots, p_n)$, $Q = (q_1, \dots, q_n)$, $R = (r_1, \dots, r_n)$ and $S = (s_1, \dots, s_n)$. If $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ represent the same vector there exists a translation, $f : \mathbb{R}^n \to \mathbb{R}^n$, such that $f(P) = R$ and $f(Q) = S$.

Suppose $f(x_1, \cdots, x_n) = (x_1 + c_1, \cdots, x_n + c_n)$.

Then $f(P) = R$ implies that

$$p_i + c_i = r_i \text{ or } c_i = r_i - p_i, \ i = 1, \cdots, n.$$

Also, $f(Q) = S$ implies that

$$q_i + c_i = s_i \text{ or } c_i = s_i - q_i, \ i = 1, \cdots, n.$$

It follows that

$$r_i - p_i = c_i = s_i - q_i \text{ or } q_i - p_i = s_i - r_i$$

for $i = 1, \cdots, n$. So the i-th component of $\overrightarrow{PQ}$ equals the i-th component of $\overrightarrow{RS}$ for $i = 1, \cdots, n$.

<u>HW #1</u>: Prove the opposite direction of the Claim.

We'll write $\vec{v} = [v_1, \cdots, v_n]$ to denote the vector whose i-th component is v_i, for $i = 1, \cdots, n$. The Claim tells us that $\vec{v}$ is completely determined by its components. Alternatively, if $\vec{w} = [w_1, \cdots, w_n]$ then $\vec{v} = \vec{w}$ if and only if $v_i = w_i$ for $i = 1, \cdots, n$. Let $O = (0, \cdots, 0)$, i.e. the origin in $\mathbb{R}^n$, and $V = (v_1, \cdots, v_n)$. Then the directed line segment, $\overrightarrow{OV}$, represents $\vec{v}$. When we want to view $\mathbb{R}^n$ as a set of vectors, we will associate $[v_1, \cdots, v_n]$ to the point $(v_1, \cdots, v_n)$ — but we will retain the notation $[v_1, \cdots, v_n]$.

In passing, if $\mathbf{X} = (x_1, \cdots, x_n)$ and $\mathbf{Y} = (y_1, \cdots, y_n)$, then the distance from $\mathbf{X}$ to $\mathbf{Y}$ is:

$$\operatorname{dist}(\mathbf{X}, \mathbf{Y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Also, if $\vec{v} = [v_1, \cdots, v_n]$, then the length of $\vec{v}$ is:

$$|\vec{v}| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Note that the length of $\vec{v}$ is just $\operatorname{dist}(P, Q)$, where $\overrightarrow{PQ}$ represents $\vec{v}$.

Vector Arithmetic

If $\vec{v} = [v_1, \cdots, v_n]$ and $\vec{w} = [w_1, \cdots, w_n]$,

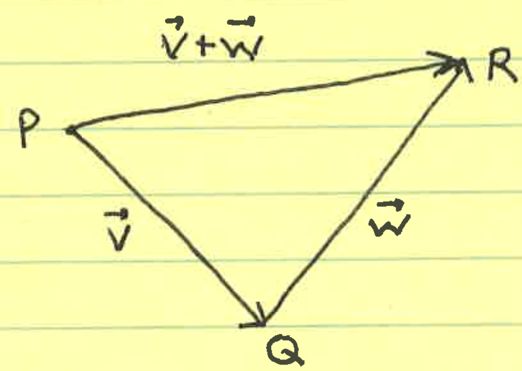
we define the sum, $\vec{v}+\vec{w}$, to be

$$\vec{v}+\vec{w} = [v_1+w_1, \cdots, v_n+w_n].$$

Geometrically, $\vec{v}+\vec{w}$ can be given

its usual resultant interpretation – even

in $\mathbb{R}^n$!



To see why this is the case, let

$P = (p_1, \cdots, p_n)$, $Q = (q_1, \cdots, q_n)$ and

$R = (r_1, \cdots, r_n)$. Since $\overrightarrow{PQ} = \vec{v} = [v_1, \cdots, v_n]$, it follows that $q_i - p_i = v_i$ for $i = 1, \cdots, n$ or $q_i = p_i + v_i$ for $i = 1, \cdots, n$. Since $\overrightarrow{QR} = \vec{w} = [w_1, \cdots, w_n]$, it follows that $r_i - q_i = w_i$ or $r_i = q_i + w_i$, for $i = 1, \cdots, n$. Recalling that $q_i = p_i + v_i$, we obtain $r_i = p_i + v_i + w_i$, for $i = 1, \cdots, n$. This implies that $r_i - p_i = v_i + w_i$, for $i = 1, \cdots, n$. So $\overrightarrow{PR} = [v_1 + w_1, \cdots, v_n + w_n]$.

If $r \in \mathbb{R}$ and $\vec{v} = [v_1, \cdots, v_n]$, we define scalar multiplication by

$$r\vec{v} = [rv_1, \cdots, rv_n].$$

Prior to discussing the geometric interpretation of scalar multiplication, we'll prove the following useful Proposition.

<u>Proposition 1</u>: Suppose $r \in \mathbb{R}$ and $\vec{v} = [v_1, \cdots, v_n]$, then $|r\vec{v}| = |r||\vec{v}|$.

<u>Proof</u>: $|r\vec{v}| = |[rv_1, \cdots, rv_n]|$

$$= \sqrt{(rv_1)^2 + \cdots + (rv_n)^2}$$

$$= \sqrt{r^2(v_1^2 + \cdots + v_n^2)}$$

$$= \sqrt{r^2}\sqrt{v_1^2 + \cdots + v_n^2}$$

$$= |r||\vec{v}|.$$

Now consider the subset S of $\mathbb{R}^n$ defined by $S = \{(tv_1, \cdots, tv_n) \mid t \in \mathbb{R}\}$.

Note that S is the set of points in

$\mathbb{R}^n$ associated to the set of vectors

$\{t\vec{v} \mid t \in \mathbb{R}\}$.

<u>Claim</u>: If $\vec{v} \neq \vec{0}$, then S is a line in $\mathbb{R}^n$ passing through the origin.

<u>Proof</u>: Initially suppose that $s<t$ and consider the two points $(sv_1, \cdots, sv_n)$ and $(tv_1, \cdots, tv_n)$. The directed line segment from $(sv_1, \cdots, sv_n)$ to $(tv_1, \cdots, tv_n)$ represents $[tv_1 - sv_1, \cdots, tv_n - sv_n] = (t-s)\vec{v}$. Proposition 1 implies that

$$\mathrm{dist}\big((sv_1, \cdots, sv_n), (tv_1, \cdots, tv_n)\big) = |t-s|\,|\vec{v}|$$

$$= (t-s)|\vec{v}| \quad \text{since } t>s.$$

Now, if $r<s<t$, it follows that

$$\mathrm{dist}\big((rv_1,\cdots,rv_n),(sv_1,\cdots,sv_n)\big)+\mathrm{dist}\big((sv_1,\cdots,sv_n),(tv_1,\cdots,tv_n)\big)$$

$$=\mathrm{dist}\big((rv_1,\cdots,rv_n),(tv_1,\cdots,tv_n)\big)$$

– since $(s-r)|\vec{v}|+(t-s)|\vec{v}|=(t-r)|\vec{v}|$. Note that this implies that any three points in S are colinear.

Suppose now that $t>0$. We'll show that the line segment joining $(-tv_1,\cdots,-tv_n)$ to $(tv_1,\cdots,tv_n)$ is contained in S. To this end, we have already observed that, if $-t<s<t$, then the point $(sv_1,\cdots,sv_n)$ lies on this line segment. Next observe that

$$\mathrm{dist}\big((-tv_1,\cdots,-tv_n),(tv_1,\cdots,tv_n)\big)=2t|\vec{v}|>0$$

– since $\vec{v}\neq\vec{0}$. Thus, if P is any point on the line segment from $(-tv_1, \dots, -tv_n)$ to $(tv_1, \dots, tv_n)$, then P is completely determined by its distance from $(-tv_1, \dots, -tv_n)$ and

$$0 \le \operatorname{dist}((-tv_1, \dots, -tv_n), P) \le 2t|\vec{v}|.$$

Now, if $\operatorname{dist}((-tv_1, \dots, -tv_n), P) = 0$, then $P = (-tv_1, \dots, -tv_n)$, and, if $\operatorname{dist}((-tv_1, \dots, -tv_n), P) = 2t|\vec{v}|$, then $P = (tv_1, \dots, tv_n)$ - due to uniqueness.

Suppose now that

$$\operatorname{dist}((-tv_1, \dots, -tv_n), P) = d,$$

where $0 < d < 2t|\vec{v}|$. Since

$$\operatorname{dist}((-tv_1, \dots, -tv_n), (sv_1, \dots, sv_n)) = (s+t)|\vec{v}|,$$

we set $(s+t)|\vec{v}| = d$. Solving for s yields

$$s = \frac{d}{|\vec{v}|} - t.$$

Now $0 < d < 2t|\vec{v}|$ implies $0 < \frac{d}{|\vec{v}|} < 2t$

or $-t < \frac{d}{|\vec{v}|} - t < t$. Thus, $-t < s < t$.

This implies there is an s such that

$-t < s < t$ and $\text{dist}((-tv_1, \cdots, -tv_n), (sv_1, \cdots, sv_n)) = d$.

So, by uniqueness, $P = (sv_1, \cdots, sv_n)$.

We have established that the line segment joining $(-tv_1, \cdots, -tv_n)$ to $(tv_1, \cdots, tv_n)$ is contained in S. Since this is true for any $t > 0$, no matter how large t is, we conclude that S is a line.

This completes the proof of the Claim.

We are now in a position to provide

a geometric interpretation for scalar multiplication. We now know that the set of all scalar multiples of the nonzero vector, $\vec{v}$, determine a line in $\mathbb{R}^n$ which passes through the origin. The proof of the Claim shows that $r\vec{v}$ is in the same direction as $\vec{v}$ — provided $r>0$. In this case, $r\vec{v}$ is shorter than $\vec{v}$ if $0<r<1$ while $r\vec{v}$ is longer than $\vec{v}$ if $r>1$. When $r<0$, $r\vec{v}$ and $\vec{v}$ have opposite directions.

The Claim suggests the following definition.

<u>Definition</u>: If $\vec{v}$ and $\vec{w}$ are two vectors in $\mathbb{R}^n$ we say $\vec{v}$ is parallel to $\vec{w}$ precisely when there is some $r \in \mathbb{R}$ such that either $r\vec{v} = \vec{w}$ or $r\vec{w} = \vec{v}$.

<u>Note</u>: Observe that $\vec{0}$ is parallel to every vector in $\mathbb{R}^n$.

<u>Theorem 1</u>: Vector addition and scaler multiplication satisfy the following ten properties:

1) If $\vec{v}, \vec{w} \in \mathbb{R}^n$, then $\vec{v} + \vec{w} \in \mathbb{R}^n$.

2) If $\vec{v}, \vec{w} \in \mathbb{R}^n$, then $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.

3) If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, then $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.

4) If $\vec{v} \in \mathbb{R}^n$, then $\vec{v} + \vec{0} = \vec{v}$.

5) If $\vec{v} \in \mathbb{R}^n$, there is a $-\vec{v} \in \mathbb{R}^n$ such that $\vec{v} + (-\vec{v}) = \vec{0}$.

6) If $r \in \mathbb{R}$, $\vec{v} \in \mathbb{R}^n$, then $r\vec{v} \in \mathbb{R}^n$.

7) If $r \in \mathbb{R}$, $\vec{v}, \vec{w} \in \mathbb{R}^n$, then

$$r(\vec{v} + \vec{w}) = r\vec{v} + r\vec{w}.$$

8) If $r, s \in \mathbb{R}$, $\vec{v} \in \mathbb{R}^n$, then

$$(r+s)\vec{v} = r\vec{v} + s\vec{v}.$$

9) If $r, s \in \mathbb{R}$, $\vec{v} \in \mathbb{R}^n$, then

$$r(s\vec{v}) = (rs)\vec{v}.$$

10) If $\vec{v} \in \mathbb{R}^n$, then $1\vec{v} = \vec{v}$.

Proof of 2): Let $\vec{v} = [v_1, \cdots, v_n]$ and $\vec{w} = [w_1, \cdots, w_n]$. Then the i-th component of $\vec{v}+\vec{w}$ is v_i+w_i and the i-th component of $\vec{w}+\vec{v}$ is w_i+v_i, for $i=1,\cdots,n$. The result follows by noting that addition in $\mathbb{R}$ is commutative.

HW #2: Prove parts 3), 4), 5), 7), 8), 9) and 10) of Theorem 1.

HW #3: Suppose $\vec{v} \in \mathbb{R}^n$ and $\vec{v} \neq \vec{0}$. Prove that $\left|\frac{1}{|\vec{v}|}\vec{v}\right| = 1$. ($\frac{1}{|\vec{v}|}\vec{v}$ is known as the unit vector in the direction of $\vec{v}$.)

The Dot Product

If $\vec{v} = [v_1, \cdots, v_n]$ and $\vec{w} = [w_1, \cdots, w_n]$, we define the dot product of $\vec{v}$ and $\vec{w}$, denoted by $\vec{v} \cdot \vec{w}$, to be

$$\vec{v} \cdot \vec{w} = v_1 w_1 + \cdots + v_n w_n .$$

The following Theorem list some basic properties of the dot product.

Theorem 2 :

1) If $\vec{v} \in \mathbb{R}^n$, then $\vec{v} \cdot \vec{v} = |\vec{v}|^2$.

2) If $\vec{v}, \vec{w} \in \mathbb{R}^n$, then $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.

3) If $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, then

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} .$$

4) If $r \in \mathbb{R}$, $\vec{v}, \vec{w} \in \mathbb{R}^n$, then

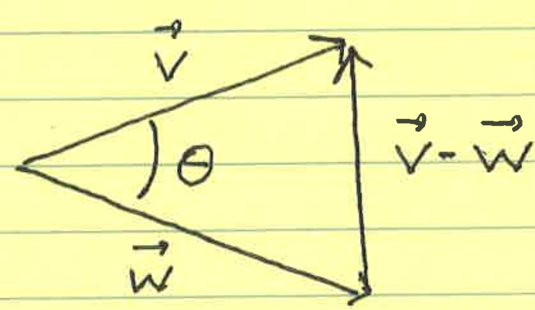
$$r(\vec{v} \cdot \vec{w}) = (r\vec{v}) \cdot \vec{w} = \vec{v} \cdot (r\vec{w}).$$

Proof of 1) : Let $\vec{v} = [v_1, \cdots, v_n]$.

Then
$$\vec{v} \cdot \vec{v} = v_1^2 + \cdots + v_n^2$$
$$= \left(\sqrt{v_1^2 + \cdots + v_n^2}\right)^2$$
$$= |\vec{v}|^2.$$

HW #4 : Complete the proof of Theorem 2.

Suppose now that $\vec{v}$ and $\vec{w}$ are two nonparallel vectors in $\mathbb{R}^n$. If we represent $\vec{v}$ and $\vec{w}$ by their position vectors, then $\vec{v}$ and $\vec{w}$ form two sides of a triangle. Using our geometric interpretation of vector addition, the third side of this triangle can be viewed as a directed line segment which represents $\vec{v}-\vec{w}$. Let $\theta \in (0,\pi)$ denote the angle between $\vec{v}$ and $\vec{w}$ in this triangle. This information is depicted below.



Applying the Law of Cosines to this triangle, we obtain:

$$|\vec{v}-\vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos\theta.$$

Note that

$$\begin{aligned} |\vec{v}-\vec{w}|^2 &= (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w}) \\ &= (\vec{v}-\vec{w})\cdot\vec{v} - (\vec{v}-\vec{w})\cdot\vec{w} \\ &= \vec{v}\cdot\vec{v} - \vec{w}\cdot\vec{v} - \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w} \\ &= |\vec{v}|^2 - 2\vec{v}\cdot\vec{w} + |\vec{w}|^2. \end{aligned}$$

It follows that

$$|\vec{v}|^2 - 2\vec{v}\cdot\vec{w} + |\vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}||\vec{w}|\cos\theta.$$

This last equation simplifies to yield

$$\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}|\cos\theta.$$

<u>HW #5</u>: Verify that the formula,

$$\vec{v}\cdot\vec{w} = |\vec{v}||\vec{w}| \cos\theta,$$

remains valid when $\vec{v}$ and $\vec{w}$ are nonzero, parallel vectors and $\theta \in [0, \pi]$.

Hint: Consider the cases when $\vec{v}$ and $\vec{w}$ have the same direction and $\vec{v}$ and $\vec{w}$ have opposite directions separately.

Theorem 3: Suppose $\vec{v}$ and $\vec{w}$ are two nonzero vectors in $\mathbb{R}^n$ and $\theta \in [0, \pi]$ is the angle between $\vec{v}$ and $\vec{w}$. Then

$$\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos\theta.$$

If $\theta \in [0, \pi]$, then $\cos\theta = 0$ precisely when $\theta = \pi/2$. This motivates the following definition.

Definition: If $\vec{v}$ and $\vec{w}$ are two vectors in $\mathbb{R}^n$, we say $\vec{v}$ and $\vec{w}$ are perpendicular or orthogonal if and only if $\vec{v} \cdot \vec{w} = 0$.

Note that $\vec{0}$ is perpendicular to any vector in $\mathbb{R}^n$.

Linear Combinations and Induction

Suppose m is a positive integer and $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_m \in \mathbb{R}^n$. If, in addition, $c_1, c_2, \cdots, c_m \in \mathbb{R}$, we say $\sum_{i=1}^{m} c_i \vec{v}_i$ is a linear combination of the vectors $\vec{v}_1, \cdots, \vec{v}_m$. If $\vec{w} \in \mathbb{R}^n$, we say that $\vec{w}$ is a linear combination of $\vec{v}_1, \cdots, \vec{v}_m$ if there exist $c_1, \cdots, c_m \in \mathbb{R}$ such that $\vec{w} = \sum_{i=1}^{m} c_i \vec{v}_i$.

To get a feeling for this construction, let's consider $\mathbb{R}^2$. Let $\vec{i} = [1,0]$ and $\vec{j} = [0,1]$. A generic vector in $\mathbb{R}^2$ can be written as $[x,y]$.

Note that

$$[x,y] = [x,0] + [0,y]$$
$$= x[1,0] + y[0,1]$$
$$= x\vec{i} + y\vec{j}.$$

Thus, every vector in $\mathbb{R}^2$ can be written as a linear combination of the two vectors $\vec{i}$ and $\vec{j}$.

If $k \in \{1, \cdots, n\}$, let $\vec{e}_k$ denote the vector in $\mathbb{R}^n$ whose i-th component is 0 if $i \neq k$ and 1 if $i = k$. I.e.

$$\vec{e}_1 = [1, 0, \cdots, 0],$$
$$\vec{e}_2 = [0, 1, 0, \cdots, 0],$$
$$\vdots$$

$$\vdots$$

$$\vec{e}_n = [0, \cdots, 0, 1].$$

Then $[x_1, \cdots, x_n] = \sum_{i=1}^{n} x_i \vec{e}_i$.

Thus, every vector in $\mathbb{R}^n$ can be written as a linear combination of $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n$.

Linear combinations are one of the most frequently employed constructions in Linear Algebra. As such, we need to develop some machinery to deal with them. For instance, $\sum_{i=1}^{m} r_i \vec{v}_i$ and $\sum_{i=1}^{m} s_i \vec{v}_i$ are two linear combinations of $\vec{v}_1, \cdots, \vec{v}_m$, their sum is $\sum_{i=1}^{m} (r_i + s_i)\vec{v}_i$, i.e. $\sum_{i=1}^{m} r_i \vec{v}_i + \sum_{i=1}^{m} s_i \vec{v}_i = \sum_{i=1}^{m} (r_i + s_i) \vec{v}_i$.

Why is this formula valid?

To a degree, Theorem 1 is helpful. For instance, Theorem 1 tells us that

$$r_1\vec{v}_1 + s_1\vec{v}_1 = (r_1+s_1)\vec{v}_1$$

– which is our formula in the case $m=1$. However, using only Theorem 1 in the case $m=2$ we have

$$(r_1\vec{v}_1 + r_2\vec{v}_2) + (s_1\vec{v}_1 + s_2\vec{v}_2)$$

$$= \left((r_1\vec{v}_1 + r_2\vec{v}_2) + s_1\vec{v}_1 \right) + s_2\vec{v}_2$$

$$= \left(r_1\vec{v}_1 + (r_2\vec{v}_2 + s_1\vec{v}_1) \right) + s_2\vec{v}_2$$

$$= \left(r_1\vec{v}_1 + (s_1\vec{v}_1 + r_2\vec{v}_2) \right) + s_2\vec{v}_2$$

$$= \left((r_1\vec{v}_1 + s_1\vec{v}_1) + r_2\vec{v}_2 \right) + s_2\vec{v}_2$$

$$= (r_1\vec{v}_1 + s_1\vec{v}_1) + (r_2\vec{v}_2 + s_2\vec{v}_2)$$

$$= (r_1+s_1)\vec{v}_1 + (r_2+s_2)\vec{v}_2 \quad .$$

This computation suggests the limitations imposed using only Theorem 1.

What we really need is induction. Induction is a proof technique which is well suited for establishing the validity of formulas — such as the formula we want to establish. In spirit, induction is like videos of cascading dominoes — where dominoes are stood on end in such a way that when the first domino is tipped over, it tips the second domino over, which tips the third domino over, etc. — in the end all the dominoes are tipped over. Abstractly, the idea behind the success of these cascading domino videos is that two basic steps are satisfied:

Step 1: The first domino is tipped over.

Step 2: If the j-th domino tips over, then the $(j+1)$-st domino tips over.

Let's now prove the formula using induction. Step 1 for dominoes becomes the base case in an induction proof.

Base Case: Here $m=1$ and we must show that $r_1\vec{v}_1 + s_1\vec{v}_1 = (r_1+s_1)\vec{v}_1$. As we have already noted, the Base Case is valid – due to Theorem 1.

Step 2 for dominoes corresponds to the Inductive Step in an induction proof. For our formula, the Induction Step becomes:

Inductive Step: Suppose $j \in \{1, \cdots, m-1\}$.

Suppose that $\sum_{i=1}^{j} r_i \vec{v}_i + \sum_{i=1}^{j} s_i \vec{v}_i = \sum_{i=1}^{j} (r_i + s_i) \vec{v}_i$.

We must show $\sum_{i=1}^{j+1} r_i \vec{v}_i + \sum_{i=1}^{j+1} s_i \vec{v}_i = \sum_{i=1}^{j+1} (r_i + s_i) \vec{v}_i$.

To accomplish this, we'll write

$$\sum_{i=1}^{j+1} r_i \vec{v}_i = \sum_{i=1}^{j} r_i \vec{v}_i + r_{j+1} \vec{v}_{j+1} \quad \text{and}$$

$$\sum_{i=1}^{j+1} s_i \vec{v}_i = \sum_{i=1}^{j} s_i \vec{v}_i + s_{j+1} \vec{v}_{j+1}.$$

Note that repeated applications of Theorem 1, as in the case $m=2$, implies that

$$\left(\sum_{i=1}^{j} r_i\vec{v}_i + r_{j+1}\vec{v}_{j+1}\right) + \left(\sum_{i=1}^{j} s_i\vec{v}_i + s_{j+1}\vec{v}_{j+1}\right)$$
$$= \left(\sum_{i=1}^{j} r_i\vec{v}_i + \sum_{i=1}^{j} s_i\vec{v}_i\right) + \left(r_{j+1}\vec{v}_{j+1} + s_{j+1}\vec{v}_{j+1}\right).$$

The induction hypothesis implies that

$$\sum_{i=1}^{j} r_i\vec{v}_i + \sum_{i=1}^{j} s_i\vec{v}_i = \sum_{i=1}^{j} (r_i+s_i)\vec{v}_i$$

and Theorem 1 implies that

$$r_{j+1}\vec{v}_{j+1} + s_{j+1}\vec{v}_{j+1} = (r_{j+1}+s_{j+1})\vec{v}_{j+1}.$$

Putting these observations together yields

$$\sum_{i=1}^{j+1} r_i\vec{v}_i + \sum_{i=1}^{j+1} s_i\vec{v}_i$$
$$= \left(\sum_{i=1}^{j} r_i\vec{v}_i + r_{j+1}\vec{v}_{j+1}\right) + \left(\sum_{i=1}^{j} s_i\vec{v}_i + s_{j+1}\vec{v}_{j+1}\right)$$
$$= \left(\sum_{i=1}^{j} r_i\vec{v}_i + \sum_{i=1}^{j} s_i\vec{v}_i\right) + \left(r_{j+1}\vec{v}_{j+1} + s_{j+1}\vec{v}_{j+1}\right)$$
$$= \sum_{i=1}^{j} (r_i+s_i)\vec{v}_i + (r_{j+1}+s_{j+1})\vec{v}_{j+1}$$
$$= \sum_{i=1}^{j+1} (r_i+s_i)\vec{v}_i .$$

This completes the proof of the Inductive Step. Consequently, the formula is valid for any positive integer m.

We will use induction periodically this semester. I should note that a variation of the induction technique used above can be employed to prove a generalized commutativity and associativity law for vector addition. The upshot of this is that the sum of any finite number of vectors - no matter the order in which they are written or added together – remains unchanged. We won't prove this – but we will use it. In fact, we already have in a sense – since, in writing $\sum_{i=1}^{m} c_i \vec{v}_i$, I didn't specify a precise order in which these vectors should be added together.

Let's now use induction to prove

Proposition 2: Suppose W is a nonempty subset of $\mathbb{R}^n$ such that

1) if $\vec{w}_1, \vec{w}_2 \in W$, then $\vec{w}_1 + \vec{w}_2 \in W$

and

2) if $r \in \mathbb{R}$ and $\vec{w} \in W$, then

$r\vec{w} \in W$.

If $r_1, \cdots, r_m \in \mathbb{R}$ and $\vec{w}_1, \cdots, \vec{w}_m \in W$,

then $\sum_{i=1}^{m} r_i \vec{w}_i \in W$.

Proof:

Base Case ($m=1$): We must show that $r_1 \vec{w}_1 \in W$. But this follows immediately from hypothesis 2) for W.

Inductive Step: Suppose $j \in \{1, \cdots, m-1\}$ and $\sum_{i=1}^{j} r_i \vec{w}_i \in W$. We must show that $\sum_{i=1}^{j+1} r_i \vec{w}_i \in W$.

To accomplish this, write $\sum_{i=1}^{j+1} r_i \vec{w}_i$

as $\sum_{i=1}^{j} r_i \vec{w}_i + r_{j+1} \vec{w}_{j+1}$. The inductive hypothesis tells us that $\sum_{i=1}^{j} r_i \vec{w}_i \in W$. On the other hand, Property 2) for W implies that $r_{j+1} \vec{w}_{j+1} \in W$. Finally, Property 1) for W now implies that

$$\sum_{i=1}^{j} r_i \vec{w}_i + r_{j+1} \vec{w}_{j+1} \in W.$$

This establishes the Inductive Step – which completes the proof of Proposition 2.

<u>HW #6</u>: Suppose $r_1, \ldots, r_m \in \mathbb{R}$ and $\vec{v}_1, \ldots, \vec{v}_m, \vec{w} \in \mathbb{R}^n$. Establish the formula: $\left(\sum_{i=1}^{m} r_i \vec{v}_i\right) \cdot \vec{w} = \sum_{i=1}^{m} r_i (\vec{v}_i \cdot \vec{w})$.

Hint: Use Induction.