Diagonalization

We are now in a position to address the question: If $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, when is there an ordered basis for $\mathbb{R}^n$, $\mathcal{V}$ say, such that the matrix which represents T with respect to $\mathcal{V}$ is diagonal? Since this question is one of the focal points in MA 371 - where, more appropriately, the scalars are complex numbers rather than real numbers - here, we will only open the door to the study of this question.

To get things started, let's suppose that $\mathcal{V} = \{\vec{v}_1, \cdots, \vec{v}_n\}$ is an ordered basis for $\mathbb{R}^n$ and the matrix which represents T with respect to $\mathcal{V}$ is a diagonal matrix. I.e.

$$T([x_1, \cdots, x_n]_{\mathcal{V}}) = \left(\begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right)^t_{\mathcal{V}}.$$

Note then that, if $j \in \{1, \cdots, n\}$,

$$T(\vec{v}_j) = T\Big([0, \cdots, 0, \underset{\substack{\uparrow \\ j\text{-th coordinate}}}{1}, 0, \cdots, 0]_{\mathcal{V}} \Big)$$

$$= \left(\begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)_q^t \begin{matrix} \\ \\ \\ \leftarrow \text{j-th row} \\ \\ \\ \\ \end{matrix}$$

$$= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ d_j \\ 0 \\ \vdots \\ 0 \end{bmatrix}_q^t \begin{matrix} \\ \\ \\ \leftarrow \text{j-th row} \\ \\ \\ \\ \end{matrix}$$

$$= [0, \cdots, 0, \underset{\substack{\uparrow \\ \text{j-th coordinate}}}{d_j}, 0, \cdots, 0]_q$$

$$= d_j \, [0, \cdots, 0, \underset{\substack{\uparrow \\ \text{j-th coordinate}}}{1}, 0, \cdots, 0]_q$$

$$= d_j \vec{v}_j$$

We conclude that $T(\vec{v}_j) = d_j \vec{v}_j$, for

$j = 1, \cdots, n$. Therefore, if $\mathcal{V} = \{\vec{v}_1, \cdots, \vec{v}_n\}$ is an ordered basis for $\mathbb{R}^n$ such that the matrix representing T with respect to $\mathcal{V}$ is diagonal, there must exist $d_1, \cdots, d_n \in \mathbb{R}$ such that $T(\vec{v}_j) = d_j \vec{v}_j$ for $j = 1, \cdots, n$.

On the other hand, our method for constructing matrices which represent T immediately implies the following: Suppose $\mathcal{V} = \{\vec{v}_1, \cdots, \vec{v}_n\}$ is an ordered basis for $\mathbb{R}^n$ and there exist $d_1, \cdots, d_n \in \mathbb{R}$ such that $T(\vec{v}_j) = d_j \vec{v}_j$, for $j = 1, \cdots, n$. Then

$$T([x_1,\cdots,x_n]_{\mathcal{V}}) = \left(\begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right)^t_{\mathcal{V}}.$$

These observations establish the following Theorem.

<u>Theorem #17</u>: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then T can be represented by a diagonal matrix with respect to some ordered basis for $\mathbb{R}^n$ if and only if there exists some ordered basis, $\mathcal{V} = \{\vec{v}_1, \cdots, v_n\}$ say, for $\mathbb{R}^n$ and real numbers, $d_1, \cdots, d_n$, such that

$T(\vec{v}_j) = d_j \vec{v}_j$, for $j = 1, \cdots, n$.

Theorem #17 provides us with a path to follow - in order to understand the diagonalization question. We should focus our attention towards finding real numbers, λ, and nonzero vectors $\vec{v} \in \mathbb{R}^n$ such that $T(\vec{v}) = \lambda \vec{v}$. The terminology used here is: If $\lambda \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^n$ is a nonzero vector such that $T(\vec{v}) = \lambda \vec{v}$, then λ is called an eigenvalue for T and $\vec{v}$ is called an eigenvector for T associated to the eigenvalue λ.

Using this terminology, Theorem #17 tells us that T can be realized by a diagonal matrix with respect to some ordered basis for $\mathbb{R}^n$ if and only if there is an ordered basis for $\mathbb{R}^n$ whose elements are eigenvectors for T.

As we shall see, the trick to finding eigenvectors is to initially find eigenvalues. To get things started, suppose $A = [a_{ij}]$ is the $n \times n$ matrix which represents T with respect to $\mathcal{S}_n = \{\vec{e}_1, \cdots, \vec{e}_n\}$ – the standard basis for $\mathbb{R}^n$. Then

$$T([x_1, \cdots, x_n]) = \left(A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right)^t .$$

Suppose now that $\vec{v} = [a_1, \cdots, a_n]$ is an eigenvector for T associated to the eigenvalue λ. Then $T(\vec{v}) = \lambda \vec{v}$ can be written as

$$\left(A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right)^t = \lambda [a_1, \cdots, a_n]$$

or

$$A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \lambda \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}.$$

Since $I_n \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, the last equation can be written as

$$A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \lambda I_n \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

or

$$\lambda I_n \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} - A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

or

$$(\lambda I_n - A) \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

so

$$\left((\lambda I_n - A) \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right)^t = [0, \cdots, 0] .$$

It follows that if $S: \mathbb{R}^n \to \mathbb{R}^n$ is the linear transformation defined by

$$S([x_1, \cdots, x_n]) = \left((\lambda I_n - A) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right)^t ,$$

then $S(\vec{v}) = \vec{0}_n$. Since $\vec{v}$ is an eigenvector,

$\vec{v} \neq \vec{0}_n$ – so $\ker(S) \neq \{\vec{0}\}$.

According to Theorem #13 on page 373, $\ker(S) \neq \{\vec{0}\}$ implies that

$$\det\left(\lambda I_n - A\right) = 0.$$

We conclude that if λ is an eigenvalue for T, then $\det(\lambda I_n - A) = 0$.

On the other hand, suppose $\lambda \in \mathbb{R}$ and $\det(\lambda I_n - A) = 0$. Then, according to Theorem #13, $\ker(S) \neq \{\vec{0}\}$ where

$$S([x_1, \cdots, x_n]) = \left((\lambda I_n - A)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right)^t.$$

$\ker(S) \neq \{\vec{0}\}$ implies there is a nonzero vector $\vec{v} = [a_1, \cdots, a_n]$ such that $S(\vec{v}) = \vec{0}$.

This, in turn, can be written as

$$\left((\lambda I_n - A)\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right)^t = [0, \cdots, 0].$$

Now, starting with this last equation, and reversing the steps on pages 399 and 400, we obtain

$$\left(A\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right)^t = \lambda\,[a_1, \cdots, a_n],$$

or $T(\vec{v}) = \lambda \vec{v}$. Since $\vec{v} \neq \vec{0}$, we conclude that λ is an eigenvalue for T. Thus, if $\lambda \in \mathbb{R}$ and $\det(\lambda I_n - A) = 0$, then λ is an eigenvalue for T.

We have established

Proposition #39: Suppose $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation which is represented by the matrix A with respect to the standard basis of $\mathbb{R}^n$. Then λ is an eigenvalue for T if and only if $\det(\lambda I_n - A) = 0$.

Proposition #39 leads us to the study of the characteristic polynomial for T (or A):

$$p(\lambda) = \det(\lambda I_n - A).$$

According to Proposition #39, the zeros of $p(\lambda)$ are precisely the eigenvalues for T.

Ex 1: We are finally in a position to see how I came up with the vectors in Example 2 on page 308. Recall, we were given the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T([x,y]) = [x+y, x-y].$$

Since $T(\vec{e}_1) = T([1,0]) = [1,1]$

and $T(\vec{e}_2) = T([0,1]) = [1,-1]$,

the matrix which represents T with respect to S_2 is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

So the characteristic polynomial for T is

$$p(\lambda) = \det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} \lambda-1 & -1 \\ -1 & \lambda+1 \end{bmatrix} \right)$$

$$= (\lambda-1)(\lambda+1) - 1$$

$$= \lambda^2 - 1 - 1$$

$$= \lambda^2 - 2 .$$

The zeros of $p(\lambda)$ are $\lambda = \pm\sqrt{2}$.

So $\pm\sqrt{2}$ are the eigenvalues for T. To find the eigenvectors associated to the eigenvalue λ - we solve the equation

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

When $\lambda = \sqrt{2}$, this equation becomes the system

$$x_1 + x_2 = \sqrt{2}\, x_1$$

$$x_1 - x_2 = \sqrt{2}\, x_2 .$$

Since any nonzero scalar multiple of an eigenvector will be an eigenvector — we set $x_2 = 1$ in the second equation — yielding $x_1 = \sqrt{2} + 1$. You can check that $[\sqrt{2}+1, 1]$ is also a solution to the first equation. Thus $\vec{v}_1 = [\sqrt{2}+1, 1]$ is an eigenvector for T associated to the eigenvalue $\sqrt{2}$.

When $\lambda = -\sqrt{2}$, we obtain the system

$$x_1 + x_2 = -\sqrt{2}\, x_1$$

$$x_1 - x_2 = -\sqrt{2}\, x_2 .$$

Letting $x_2 = 1$ once again, equation two implies $x_1 = 1 - \sqrt{2}$. Again, you can check that $[1-\sqrt{2}, 1]$ is a solution to the first equation - thus $\vec{v}_2 = [1-\sqrt{2}, 1]$ is an eigenvector for T associated to the eigenvalue $-\sqrt{2}$.

<u>HW #49</u>: Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ via $T([x,y]) = [y,x]$. T is the linear transformation studied in HW #41

on page 334. Use the methods of this section to obtain the ordered basis $\mathcal{V}$ occurring in part (iii) of HW #41.

Ex 2: Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ via

$$T([x,y]) = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right)^t.$$

The characteristic polynomial of T is

$$p(\lambda) = \det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} \lambda-1 & -1 \\ 0 & \lambda-1 \end{bmatrix} \right) = (\lambda-1)^2.$$

Thus, the only eigenvalue for T is $\lambda = 1$.

To construct the eigenvectors associated to $\lambda = 1$, we must solve the system

$$\begin{aligned} x+y &= x \\ y &= y \end{aligned} .$$

The first equation of this system implies $y=0$ – so, up to nonzero multiple, the only eigenvector associated to $\lambda = 1$ is $\vec{v} = [1,0]$.

As such, the T in this example cannot be represented by a diagonal matrix with respect to some ordered basis of $\mathbb{R}^2$.

<u>Ex 3</u>: Consider the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T([x,y]) = \left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right)^t .$$

Observe that if $[x,y]$ is a unit vector, then $[x,y] = [\cos t, \sin t]$ for some $t \in [0, 2\pi]$ and

$$T([\cos t, \sin t]) = \left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \right)^t$$

$$= [\cos t \cos\theta - \sin t \sin\theta, \cos t \sin\theta + \sin t \cos\theta]$$

$$= [\cos(t+\theta), \sin(t+\theta)].$$

This tells us that T is a rotation of $\mathbb{R}^2$ through θ radians.

The characteristic polynomial for T is

$$p(\lambda) = \det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} \lambda - \cos\theta & \sin\theta \\ -\sin\theta & \lambda - \cos\theta \end{bmatrix} \right)$$

$$= (\lambda - \cos\theta)^2 + \sin^2\theta$$

$$= \lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta$$

$$= \lambda^2 - 2\cos\theta\,\lambda + 1 \quad .$$

Using the quadratic formula, the zeros for $p(\lambda)$ are

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} \quad .$$

Note that $4\cos^2\theta - 4 < 0$ unless $\cos^2\theta = 1$.

So T has no real eigenvalues unless θ is an integer multiple of π. Since $\sin\theta = 0$ if θ is an integer multiple of π, T can be represented by a diagonal matrix, with respect to some ordered basis for $\mathbb{R}^n$, if and only if $\theta = k\pi$ where k is an integer.

<u>Proposition #40</u>: Suppose $T: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $\lambda_1, \cdots, \lambda_k$ are distinct eigenvalues for T with associated eigenvectors $\vec{v}_1, \cdots, \vec{v}_k$, respectively. Then $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is a linearly independent set of vectors.

<u>Proof</u>: Suppose, to the contrary, that $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is linearly dependent. Since each $\vec{v}_i$ is an eigenvector, $\vec{v}_i \neq \vec{0}$ for $i = 1, \cdots, k$. It follows that

$\operatorname{span}(\{\vec{v}_1,\dots,\vec{v}_k\}) \neq \{\vec{0}\}$ — so some subset of $\{\vec{v}_1,\dots,\vec{v}_k\}$ is a basis for $\operatorname{span}(\{\vec{v}_1,\dots,\vec{v}_k\})$. After possibly reindexing $\{\vec{v}_1,\dots,\vec{v}_k\}$, we may assume $\{\vec{v}_1,\dots,\vec{v}_\ell\}$ is a basis for $\operatorname{span}(\{\vec{v}_1,\dots,\vec{v}_k\})$, where $1 \leq \ell < k$. Thus, $\vec{v}_k \in \operatorname{span}(\{\vec{v}_1,\dots,\vec{v}_\ell\})$ and therefore

$$\vec{v}_k = c_1\vec{v}_1 + \cdots + c_\ell\vec{v}_\ell$$

for some $c_1,\dots,c_\ell \in \mathbb{R}$.

Observe that

$$\begin{aligned}\lambda_k\vec{v}_k = T(\vec{v}_k) &= T(c_1\vec{v}_1 + \cdots + c_\ell\vec{v}_\ell) \\ &= \lambda_1 c_1\vec{v}_1 + \cdots + \lambda_\ell c_\ell\vec{v}_\ell .\end{aligned}$$

But
$$\begin{aligned}\lambda_k\vec{v}_k &= \lambda_k(c_1\vec{v}_1 + \cdots + c_\ell\vec{v}_\ell) \\ &= \lambda_k c_1\vec{v}_1 + \cdots + \lambda_k c_\ell\vec{v}_\ell .\end{aligned}$$

It follows that

$$\lambda_k c_1 \vec{v}_1 + \cdots + \lambda_k c_\ell \vec{v}_\ell = \lambda_1 c_1 \vec{v}_1 + \cdots + \lambda_\ell c_\ell \vec{v}_\ell$$

or

$$(\lambda_k - \lambda_1) c_1 \vec{v}_1 + \cdots + (\lambda_k - \lambda_\ell) c_\ell \vec{v}_\ell = \vec{0} .$$

Now, since $\{\vec{v}_1, \cdots, \vec{v}_\ell\}$ is a basis for $\text{span}(\{\vec{v}_1, \cdots, \vec{v}_k\})$, $\{\vec{v}_1, \cdots, \vec{v}_\ell\}$ must be linearly independent - so

$$(\lambda_k - \lambda_1) c_1 = 0$$
$$\vdots$$
$$(\lambda_k - \lambda_\ell) c_\ell = 0 .$$

Since the λ_i's are distinct, $\lambda_k - \lambda_i \neq 0$ for $i = 1, \cdots, \ell$, so $c_1 = \cdots = c_\ell = 0$. But this implies $\vec{v}_k = \vec{0}$ - which is a contradiction.

Since this contradiction results from our assuming that $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is linearly dependent, we conclude that $\{\vec{v}_1, \cdots, \vec{v}_k\}$ must be linearly independent.

Proposition #40 immediately implies the following

Theorem #18: Suppose the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ has n distinct eigenvalues. Then T can be represented by a diagonal matrix, with respect to some ordered basis for $\mathbb{R}^n$.