Linear Independence

Definitions: Suppose m is a positive integer and $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$. Set $S = \{\vec{v}_1, \cdots, \vec{v}_m\}$. We say that S is a linearly independent set of vectors provided:

If $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = \vec{0}$, then $c_1 = c_2 = \cdots = c_m = 0$.

Put another way, if S is linearly independent, then $\sum_{i=1}^{m} c_i \vec{v}_i \neq \vec{0}$ if any one $c_j \neq 0$.

If S is not linearly independent, we say S is linearly dependent.

We have already encountered the notions of nonzero vector, non-parallel vectors and non-coplanar vectors. These are special cases of the notion of linear independence.

For instance, let $\vec{v} \in \mathbb{R}^n$ and $S = \{\vec{v}\}$. Consider the equation $c\vec{v} = \vec{0}$. If $\vec{v} = [v_1, \cdots, v_n]$ and $\vec{v} \neq \vec{0}$, then $v_k \neq 0$ for some $k \in \{1, \cdots, n\}$. Consequently, the k-th component of $c\vec{v}$, namely cv_k, can equal 0 only when $c = 0$. Thus, $S = \{\vec{v}\}$ is linearly independent if $\vec{v} \neq \vec{0}$. On the other hand, if $\vec{v} = \vec{0}$, then

$1\vec{v} = \vec{v} = \vec{0}$ - so $S = \{\vec{0}\}$ is linearly dependent. We have shown: If $S = \{\vec{v}\}$ then S is linearly independent precisely when $\vec{v} \neq \vec{0}$.

Next, suppose $\vec{v}, \vec{w} \in \mathbb{R}^n$. Set $S = \{\vec{v}, \vec{w}\}$. If S is linearly dependent there exist $c, d \in \mathbb{R}$ such that $c\vec{v} + d\vec{w} = \vec{0}$ where either $c \neq 0$ or $d \neq 0$. If $c \neq 0$, then $\vec{v} = -\frac{d}{c}\vec{w}$. If $d \neq 0$, then $\vec{w} = \frac{-c}{d}\vec{v}$. In either case, $\vec{v}$ and $\vec{w}$ are parallel vectors.

On the other hand, if $\vec{v}$ and $\vec{w}$ are parallel there is some $c \in \mathbb{R}$ such that either $\vec{w} = c\vec{v}$ or $\vec{v} = c\vec{w}$. If $\vec{w} = c\vec{v}$, then $(-c)\vec{v} + (1)\vec{w} = \vec{0}$ and $1 \neq 0$. If $\vec{v} = c\vec{w}$, then $(1)\vec{v} + (-c)\vec{w} = \vec{0}$ and $1 \neq 0$. Thus, in either case, $S = \{\vec{v}, \vec{w}\}$ is linearly dependent.

We have shown that $S = \{\vec{v}, \vec{w}\}$ is linearly dependent precisely when $\vec{v}$ and $\vec{w}$ are parallel. It follows that $S = \{\vec{v}, \vec{w}\}$ is linearly independent precisely when $\vec{v}$ and $\vec{w}$ are not parallel vectors.

The statement concerning non-coplanar vectors involves a little more work –

but the ideas involved are analogous to those we've already employed. Hopefully, you now have some intuition concerning the notion of linear independence.

<u>HW #17</u>: Prove that

$$S = \{ [1,0,0], [0,0,1], [1,0,3] \}$$

is a linearly dependent set of vectors in $\mathbb{R}^3$.

Linear Independence and Spans

The notions of linear independence and span work well as a team — each providing a key element to yield the notion of a basis. Bases will be discussed in the next section. Here, we prove some basic observations concerning the relationship between linear independence and span.

Throughout this section, m will be a positive integer and $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$. We will use, on occasion, the term "reindex" — applied to a set of vectors. This only effects the order in which the vectors are listed in the set. Since $\{1,2\}=\{2,1\}$, the set itself is unchanged.

<u>Proposition 9</u>: Suppose $S=\{\vec{v}_1,\cdots,\vec{v}_m\}$ is linearly independent and $\vec{w}\in\mathbb{R}^n$. Set $\tilde{S}=\{\vec{v}_1,\cdots,\vec{v}_m,\vec{w}\}$. Then $\tilde{S}$ is linearly independent if and only if $\vec{w}\notin \text{span}(S)$.

<u>Proof</u>: Initially, suppose $\vec{w}\notin\text{span}(S)$, and $\sum_{i=1}^{m} c_i\vec{v}_i + d\vec{w}=\vec{0}$. If $d\neq 0$, then $\sum_{i=1}^{m} c_i\vec{v}_i + d\vec{w}=\vec{0}$ implies that $\vec{w}=\sum_{i=1}^{m}\left(\frac{-c_i}{d}\right)\vec{v}_i$. But

$\sum_{i=1}^{m} \left(\frac{-c_i}{d}\right) \vec{v}_i \in \text{span}(S)$ — so $\vec{w} \in \text{span}(S)$.

This contradicts our assumption that $\vec{w} \notin \text{span}(S)$. This implies that $d = 0$.

Now, since $d = 0$, the equation $\sum_{i=1}^{m} c_i \vec{v}_i + d\vec{w} = \vec{0}$ can be written as $\sum_{i=1}^{m} c_i \vec{v}_i = \vec{0}$. However, S is assumed to be linearly independent — so $\sum_{i=1}^{m} c_i \vec{v}_i = \vec{0}$ implies that $c_1 = \cdots = c_m = 0$.

Thus, if $\vec{w} \notin \text{span}(S)$, then $\sum_{i=1}^{m} c_i \vec{v}_i + d\vec{w} = \vec{0}$ implies that $c_1 = \cdots = c_n = d = 0$ — so $\tilde{S}$ is also linearly independent.

Suppose now that $\vec{w} \in \text{span}(S)$. Then there exist $c_1, \ldots, c_m \in \mathbb{R}$ such that $\vec{w} = \sum_{i=1}^{m} c_i \vec{v}_i$. As such $\sum_{i=1}^{m} (-c_i)\vec{v}_i + (1)\vec{w} = \vec{0}$. Since $1 \neq 0$, it follows that $\tilde{S}$ is linearly dependent. We have shown that if $\vec{w} \in \text{span}(S)$ then $\tilde{S}$ must be linearly dependent. We conclude that if $\tilde{S}$ is linearly independent, then we must have that $\vec{w} \notin \text{span}(S)$.

Proposition 9 reflects the nonzero, nonparallel and non-coplanar nature of a set of linearly independent vectors. Indeed, as we shall see, since S is assumed to be linearly independent, span(S) will be a copy of $\mathbb{R}^m$ living in $\mathbb{R}^n$ and passing through the origin. When $\tilde{S}$ is linearly independent, span$(\tilde{S})$ will be a copy of $\mathbb{R}^{m+1}$ which contains span(S).

The following Proposition exhibits the nonzero, nonparallel, non-coplanar nature of linearly independent vectors from a slightly different perspective.

<u>Proposition</u> 10: Let $S = \{\vec{v}_1, \cdots, \vec{v}_m\}$ and suppose $\text{span}(S) \neq \{\vec{0}\}$. Then there is a nonempty subset, S' say, of S such that

1) S' is linearly independent, and

2) $\text{span}(S') = \text{span}(S)$.

<u>Note</u>: The hypothesis $\text{span}(S) \neq \{\vec{0}\}$ is essential here. Observe that if $\text{span}(S) = \{\vec{0}\}$, then $\vec{v}_i = \vec{0}$ for $i = 1, \cdots, m$, and no set of

vectors, having $\vec{0}$ as an element, can be linearly independent.

<u>Proof of Proposition 10</u>: If S is linearly independent, take $S' = S$ an we are done.

Suppose now that S is linearly dependent. We'll show how an element of S can be removed from S - without altering the span.

Since S is linearly dependent, there exist $c_1, \ldots, c_m \in \mathbb{R}$ such that $\sum_{i=1}^{m} c_i \vec{v}_i = \vec{0}$ and some $c_j \neq 0$.

If necessary, we can reindex the elements of S so that $c_m \neq 0$.

Note that $c_m \neq 0$ implies that

$$\vec{v}_m = \sum_{i=1}^{m-1} \left(\frac{-c_i}{c_m}\right) \vec{v}_i .$$

<u>Claim</u>: $\text{span}(\{\vec{v}_1, \cdots, \vec{v}_{m-1}\}) = \text{span}(S)$.

Since $S = \{\vec{v}_1, \cdots, \vec{v}_m\}$, it follows that $\vec{v}_i \in \text{span}(S)$ for $i = 1, \cdots, m-1$ – due to HW #15. Since $\text{span}(S)$ is a subspace of $\mathbb{R}^n$, hence a vector space in its own right, any linear combination of elements in $\text{span}(S)$ yields an element of $\text{span}(S)$ – consequently,

$$\text{span}(\{\vec{v}_1, \cdots, \vec{v}_{m-1}\}) \subset \text{span}(S).$$

Similarly, $\vec{v}_i \in \operatorname{span}(\{\vec{v}_1, \cdots, \vec{v}_{m-1}\})$ for $i = 1, \cdots, m-1$ by HW #15. Since

$$\vec{v}_m = \sum_{i=1}^{m-1} \left(\frac{-c_i}{c_m}\right) \vec{v}_i$$

and

$$\sum_{i=1}^{m-1} \left(\frac{-c_i}{c_m}\right) \vec{v}_i \in \operatorname{span}(\{\vec{v}_1, \cdots, \vec{v}_{m-1}\})$$

it follows that $S \subset \operatorname{span}(\{\vec{v}_1, \cdots, \vec{v}_{m-1}\})$. Since $\operatorname{span}(\{\vec{v}_1, \cdots, \vec{v}_{m-1}\})$ is a subspace of $\mathbb{R}^n$ it follows that

$$\operatorname{span}(S) \subset \operatorname{span}(\{\vec{v}_1, \cdots, \vec{v}_{m-1}\}).$$

Now, since

$$\operatorname{span}(\{\vec{v}_1, \cdots, \vec{v}_{m-1}\}) \subset \operatorname{span}(S)$$

and

$$\operatorname{span}(S) \subset \operatorname{span}(\{\vec{v}_1, \cdots, \vec{v}_{m-1}\})$$

we conclude that

$$\text{span}(\{\vec{v}_1, \cdots, \vec{v}_{m-1}\}) = \text{span}(S).$$

This establishes the Claim.

Now, if $\{\vec{v}_1, \cdots, \vec{v}_{m-1}\}$ is linearly independent, we set $S' = \{\vec{v}_1, \cdots, \vec{v}_{m-1}\}$ and we are done. If $\{\vec{v}_1, \cdots, \vec{v}_{m-1}\}$ is linearly dependent, we can apply the argument above to $\{\vec{v}_1, \cdots, \vec{v}_{m-1}\}$. After reindexing, if needed, we would obtain

$$\begin{aligned}\text{span}(\{\vec{v}_1, \cdots, \vec{v}_{m-2}\}) &= \text{span}(\{\vec{v}_1, \cdots, \vec{v}_{m-1}\}) \\ &= \text{span}(S).\end{aligned}$$

Continue this process. Note that

since $\text{span}(S) \neq \{\vec{0}\}$ this process must terminate in a nonempty subset, $S' \subset S$, such that S' is linearly independent and $\text{span}(S') = \text{span}(S)$.

<u>Note</u>: The essential observation made in Proposition 10 is: If $S = \{\vec{v}_1, \cdots, \vec{v}_m\}$, $m > 1$, and

$$\sum_{i=1}^{m} c_i \vec{v}_i = \vec{0}$$

where $c_j \neq 0$, then removing $\vec{v}_j$ from S will not alter the span.

We know that if $S = \{\vec{v}_1, \cdots, \vec{v}_m\}$

is linearly independent, then

$\sum_{i=1}^{m} c_i \vec{v}_i = \vec{0}$ implies $c_1 = \cdots = c_m = 0$.

In other words, $\vec{0}$ is uniquely expressed as a linear combination of the elements in S. It turns out that $\vec{0}$ is not the only element of span(S) which enjoys this property.

<u>Proposition 11</u>: Suppose $S = \{\vec{v}_1, \cdots, \vec{v}_m\}$ is linearly independent and $\vec{w} \in \text{span}(S)$. Then $\vec{w}$ can be uniquely expressed as a linear combination of the elements of S.

HW #18: Prove Proposition 11.

Hint: If $\vec{w} = \sum_{i=1}^{m} c_i \vec{v}_i$ and $\vec{w} = \sum_{i=1}^{m} d_i \vec{v}_i$ consider $\vec{0} = \vec{w} - \vec{w}$.

The following Proposition will prove very useful in the next section.

Proposition 12: Set $S = \{\vec{v}_1, \cdots, \vec{v}_m\}$ and suppose S is linearly independent. In addition, suppose $\vec{w} \in \operatorname{span}(S)$ and $\vec{w} = \sum_{i=1}^{m} c_i \vec{v}_i$ where $c_m \neq 0$. Set $\tilde{S} = \{\vec{v}_1, \cdots, \vec{v}_{m-1}, \vec{w}\}$.

Then

1) $\tilde{S}$ is also linearly independent,

and

2) $\operatorname{span}(\tilde{S}) = \operatorname{span}(S)$.

<u>Proof of 1)</u>: Suppose that

$$\sum_{i=1}^{m-1} d_i \vec{v}_i + d_m \vec{w} = \vec{O}.$$

Since

$$\vec{w} = \sum_{i=1}^{m} c_i \vec{v}_i$$

we obtain

$$\sum_{i=1}^{m-1} d_i \vec{v}_i + d_m \sum_{i=1}^{m} c_i \vec{v}_i = \vec{O}$$

or

$$\sum_{i=1}^{m-1} (d_i + d_m c_i) \vec{v}_i + d_m c_m \vec{v}_m = \vec{O}.$$

Since S is linearly independent,

$$d_i + d_m c_i = 0 \quad \text{for } i = 1, \cdots, m-1$$

and

$$d_m c_m = 0 .$$

As $c_m \neq 0$ by hypothesis we must have that $d_m = 0$.

Now $d_m = 0$ combined with

$$d_i + d_m c_i = 0 \quad \text{for } i = 1, \cdots, m-1$$

yields $d_i = 0$ for $i = 1, \cdots, m-1$.

It follows that

$$d_1 = \cdots = d_m = 0 .$$

This implies that $\tilde{S}$ is linearly independent.

HW #19: Complete the proof of Proposition 12.

Bases and What Could/Should Be Called The Fundamental Theorem of Linear Algebra

Suppose W is a subspace of $\mathbb{R}^n$, m is a positive integer and $S = \{\vec{v}_1, \ldots, \vec{v}_m\}$ is a subset of W. We say that S is a basis for W provided

1) S is linearly independent, and
2) $\mathrm{span}(S) = W$.

Examples of bases are abundant. For instance $\{\vec{i} = [1,0], \vec{j} = [0,1]\}$ is a basis for $\mathbb{R}^2$ — called the standard basis for

$\mathbb{R}^2$. More generally, if

$$\vec{e}_k = \underset{\substack{\uparrow \\ k\text{-th component}}}{[0,\cdots,0,1,0,\cdots,0]} \in \mathbb{R}^n$$

for $k = 1, \cdots, n$, then

$$\mathcal{S}_n = \{\vec{e}_1, \cdots, \vec{e}_n\}$$

is the standard basis for $\mathbb{R}^n$.

In fact, if $S = \{\vec{v}_1, \cdots, \vec{v}_m\}$, $m \geq 1$, is a linearly independent set of vectors in $\mathbb{R}^n$, then S is a basis for $W = \text{span}(S)$.

One reason that bases are abundant is that they are not unique. Indeed, any nonzero vector in $\mathbb{R}^1$ is a basis for $\mathbb{R}^1$. What turns out to be true is that any two bases for the subspace W in $\mathbb{R}^n$ must have the same number of elements — a result of such far-reaching significance it could (perhaps should) be called the Fundamental Theorem of Linear Algebra.

<u>Theorem 8</u>: Suppose W is a subspace of $\mathbb{R}^n$ and both $\{\vec{v}_1, \cdots, \vec{v}_m\}$ and $\{\vec{w}_1, \cdots, \vec{w}_k\}$ are two bases for W. Then $m = k$.

<u>Proof</u>: We will assume that $m \neq k$ and arrive at a contradiction.

If $m \neq k$, then either $m > k$ or $k > m$. Without loss of generality, we'll suppose that $m > k$.

<u>Claim</u>: Assuming $m > k$, then, after possibly reindexing the elements in $\{\vec{v}_1, \dots, \vec{v}_m\}$, it follows that $\{\vec{w}_1, \dots, \vec{w}_k, \vec{v}_{k+1}, \dots, \vec{v}_m\}$ must be a basis for W.

If we accept the Claim for a moment, we can identify the contradiction. This being that $\{\vec{w}_1, \dots, \vec{w}_k, \vec{v}_{k+1}, \dots, \vec{v}_m\}$ cannot be linearly independent — so it cannot be a basis for W. The reasoning as to why $\{\vec{w}_1, \cdots, \vec{w}_k, \vec{v}_{k+1}, \cdots, \vec{v}_m\}$ cannot be linearly independent goes as follows:

Since $\{\vec{v}_1, \cdots, \vec{v}_m\}$ is a basis for W, $\vec{v}_j \in \text{span}(\{\vec{v}_1, \cdots, \vec{v}_m\}) = W$ for $j = 1, \cdots, m$ – by HW #15. On the other hand, $\{\vec{w}_1, \cdots, \vec{w}_k\}$ being a basis for W implies that $\text{span}(\{\vec{w}_1, \cdots, \vec{w}_k\}) = W$. Thus, after possibly reindexing the elements if $\{\vec{v}_1, \cdots, \vec{v}_m\}$,

$$\vec{v}_{k+1}, \cdots, \vec{v}_m \in \text{span}(\{\vec{w}_1, \cdots, \vec{w}_k\})$$

– so $\{\vec{w}_1, \cdots, \vec{w}_k, \vec{v}_{k+1}, \cdots, \vec{v}_m\}$ cannot be linearly independent according to Proposition 9.

Having identified the contradiction, the proof of the Theorem will be complete – once we have proven the Claim.

The proof of the Claim is an induction argument – where Proposition 12 is employed to replace $\vec{v}$'s by $\vec{w}$'s one step at a time.

Base Case: We wish to show that, after possibly reindexing $\{\vec{v}_1, \dots, \vec{v}_m\}_2$ the set $\{\vec{w}_1, \vec{v}_2, \dots, \vec{v}_m\}$ is a basis for W.

To accomplish this, note that since $\{\vec{w}_1, \cdots, \vec{w}_k\}$ is a basis for W, $\vec{w}_1 \in W$ and $\vec{w}_1 \neq \vec{0}$. Since $\{\vec{v}_1, \cdots, \vec{v}_m\}$ is also a basis for W, $\vec{w}_1 \in W = \text{span}\{\vec{v}_1, \cdots, \vec{v}_m\}$. Thus, $\vec{w}_1 = \sum_{i=1}^{m} c_i \vec{v}_i$, where some $c_j \neq 0$ – since $\vec{w}_1 \neq \vec{0}$. After reindexing $\{\vec{v}_1, \cdots, \vec{v}_m\}$, if needed, we may assume that $\vec{w}_1 = \sum_{i=1}^{m} c_i \vec{v}_i$ where $c_1 \neq 0$. Proposition 12 then implies that $\{\vec{w}_1, \vec{v}_2, \cdots, \vec{v}_m\}$ is linearly independent and

$$\text{span}(\{\vec{w}_1, \vec{v}_2, \cdots, \vec{v}_m\}) = \text{span}(\{\vec{v}_1, \cdots, \vec{v}_m\}) = W.$$

Note that Proposition 12 does indeed apply – due to $\{\vec{v}_1, \cdots, \vec{v}_m\}$ being a basis for W implying $\{\vec{v}_1, \cdots, \vec{v}_m\}$ is linearly independent.

Since $\{\vec{w}_1, \vec{v}_2, \cdots, \vec{v}_m\}$ is linearly independent and $\text{span}(\{\vec{w}_1, \vec{v}_2, \cdots, \vec{v}_m\}) = W$, we conclude that $\{\vec{w}_1, \vec{v}_2, \cdots, \vec{v}_m\}$ is a basis for W. This completes the Base Case.

<u>Inductive Step</u>: Suppose $\ell \in \{1, \cdots, k-1\}$ and, after possibly reindexing $\{\vec{v}_1, \cdots, \vec{v}_m\}$ we have that $\{\vec{w}_1, \cdots, \vec{w}_\ell, \vec{v}_{\ell+1}, \cdots, \vec{v}_m\}$ is a basis

for W. We must show that, after possibly having to further reindex the $\vec{v}$'s, that $\{\vec{w}_1, \cdots, \vec{w}_{\ell+1}, \vec{v}_{\ell+2}, \cdots, \vec{v}_m\}$ is a basis for W.

The proof of the inductive step is nearly identical to the proof of the Base Case. There is, however, a subtle element in the proof of the Inductive Step which does not occur in the Base Case's proof.

That $\vec{w}_{\ell+1} \in \text{span}(\{\vec{w}_1, \cdots, \vec{w}_\ell, \vec{v}_{\ell+1}, \cdots, \vec{v}_m\})$ follows from both $\{\vec{w}_1, \cdots, \vec{w}_k\}$ and $\{\vec{w}_1, \cdots, \vec{w}_k, \vec{v}_{\ell+1}, \cdots, \vec{v}_m\}$ being bases for W.

It follows that there exist $d_1, \cdots, d_m \in \mathbb{R}$ such that

$$\vec{w}_{\ell+1} = \sum_{i=1}^{\ell} d_i \vec{w}_i + \sum_{i=\ell+1}^{m} d_i \vec{v}_i .$$

Here is where we come to the new element in the Inductive Step's proof. In order to replace one of the $\vec{v}$'s with $\vec{w}_{\ell+1}$, using Proposition 12, we must show some $d_i \neq 0$ where $i \in \{\ell+1, \cdots, m\}$. In other words, we must show that some coefficient of a $\vec{v}$ is nonzero.

That some coefficient of a $\vec{v}$ must be nonzero proceeds as follows:

If $d_i = 0$ for $i = \ell+1, \cdots, m$, then

$$\vec{w}_{\ell+1} = \sum_{i=1}^{\ell} d_i \vec{w}_i .$$

This implies $\vec{w}_{\ell+1} \in \text{span}(\{\vec{w}_1, \cdots, \vec{w}_\ell\})$ – so $\{\vec{w}_1, \cdots, \vec{w}_{\ell+1}\}$ must be linearly dependent by Proposition 9. This, in turn, implies that $\{\vec{w}_1, \cdots, \vec{w}_k\}$ is linearly dependent – contradicting $\{\vec{w}_1, \cdots, \vec{w}_k\}$ being a basis for $\mathbb{W}$. This contradiction tells us that some $d_i \neq 0$ where $i \in \{\ell+1, \cdots, m\}$.

We can now reindex $\{\vec{v}_1, \cdots, \vec{v}_m\}$, if needed, to arrive at

$$\vec{w}_{\ell+1} = \sum_{i=1}^{\ell} d_i \vec{w}_i + \sum_{i=\ell+1}^{m} d_i \vec{v}_i$$

where $d_{\ell+1} \neq 0$. That $\{\vec{w}_1, \cdots, \vec{w}_{\ell+1}, \vec{v}_{\ell+2}, \cdots, \vec{v}_m\}$ is a basis for W is now a consequence of Proposition 12.

This completes the proof of the Inductive Step — so the Claim is established. As previously noted, the proof of the Claim was all that remained to complete the proof of the Theorem.

Some Consequences of Theorem 8

Definition: Suppose W is a nonzero subspace of $\mathbb{R}^n$ and $\{\vec{v}_1, \cdots, \vec{v}_m\}$ is a basis for W. Then we say that the dimension of W is m, written $\dim(W) = m$. For completeness, we say $\dim(\{\vec{0}\}) = 0$.

Thus, $\dim(W)$ is just the number of elements in any basis for W. That is number is unique is due to Theorem 8. Since the standard basis for $\mathbb{R}^n$ has n elements, it follows that $\dim(\mathbb{R}^n) = n$.

<u>Proposition 13</u>: Suppose W is a nonzero subspace of $\mathbb{R}^n$ of dimension m and $\{\vec{w}_1, \cdots, \vec{w}_k\}$ is a linearly independent subset of W. Then $\{\vec{w}_1, \cdots, \vec{w}_k\}$ can be extended to a basis for W.

Consequently, $k \le m$.

<u>Proof</u>: If $\mathrm{span}(\{\vec{w}_1, \cdots, \vec{w}_k\}) = W$, then $\{\vec{w}_1, \cdots, \vec{w}_k\}$ is a basis for W – so $k = m$.

Suppose now that

$$\mathrm{span}(\{\vec{w}_1, \cdots, \vec{w}_k\}) \subsetneq W.$$

Let $\{\vec{v}_1, \cdots, \vec{v}_m\}$ be a basis for W.

Observe that if $\vec{v}_i \in \mathrm{span}(\{\vec{w}_1, \cdots, \vec{w}_k\})$

for $i = 1, \cdots, m$, then

$$\text{span}(\{\vec{v}_1, \cdots, \vec{v}_m\}) \subset \text{span}(\{\vec{w}_1, \cdots, \vec{w}_k\}).$$

Since $\text{span}(\{\vec{v}_1, \cdots, \vec{v}_m\}) = \mathbb{W}$, we arrive at the contradiction

$$\mathbb{W} \subset \text{span}(\{\vec{w}_1, \cdots, \vec{w}_k\}) \subsetneq \mathbb{W}.$$

It follows that some $\vec{v}_j$ is not an element in $\text{span}(\{\vec{w}_1, \cdots, \vec{w}_k\})$. If needed, $\{\vec{v}_1, \cdots, \vec{v}_m\}$ can be reindexed so that $\vec{v}_1 \notin \text{span}(\{\vec{w}_1, \cdots, \vec{w}_k\})$. Proposition 9 now implies that $\{\vec{w}_1, \cdots, \vec{w}_k, \vec{v}_1\}$ is a linearly independent set.

Since $\{\vec{w}_1, \cdots, \vec{w}_k\} \subset \mathbb{W}$ and $\vec{v}_1 \in \mathbb{W}$, it follows that $\{\vec{w}_1, \cdots, \vec{w}_k, \vec{v}_1\}$ is a linearly independent subset of $\mathbb{W}$.

If $\{\vec{w}_1, \cdots, \vec{w}_k, \vec{v}_1\}$ spans $\mathbb{W}$, i.e. $\operatorname{span}(\{\vec{w}_1, \cdots, \vec{w}_k, \vec{v}_1\}) = \mathbb{W}$, then $\{\vec{w}_1, \cdots, \vec{w}_k, \vec{v}_1\}$ is a basis for $\mathbb{W}$.

If $\{\vec{w}_1, \cdots, \vec{w}_k, \vec{v}_1\}$ doesn't span $\mathbb{W}$, it would follow that some

$$\vec{v}_j \notin \operatorname{span}(\{\vec{w}_1, \cdots, \vec{w}_k, \vec{v}_1\})$$

where $j \in \{2, \cdots, m\}$. After possibly reindexing the $\vec{v}$'s, we could assume

$$\vec{v}_2 \notin \operatorname{span}(\{\vec{w}_1, \cdots, \vec{w}_k, \vec{v}_1\}).$$

Then, using Proposition 9 once again, it follows that $\{\vec{w}_1, \cdots, \vec{w}_k, \vec{v}_1, \vec{v}_2\}$

is a linearly independent subset of W.

Since $\operatorname{span}(\{\vec{v}_1, \cdots, \vec{v}_m\}) = W$, this process of adding elements of $\{\vec{v}_1, \cdots, \vec{v}_m\}$ to $\{\vec{w}_1, \cdots, \vec{w}_k\}$ must terminate in a linearly independent set, $\{\vec{w}_1, \cdots, \vec{w}_k, \vec{v}_1, \cdots, \vec{v}_\ell\}$ say, which spans W. $\{\vec{w}_1, \cdots, \vec{w}_k, \vec{v}_1, \cdots, \vec{v}_\ell\}$ would then be the desired extension of $\{\vec{w}_1, \cdots, \vec{w}_k\}$ to a basis for W.

Note that when $\{\vec{w}_1, \cdots, \vec{w}_k\}$ doesn't span W, as above, $k+\ell = m$ where $\ell > 0$. Thus $k < m$ in this case.

Proposition 14: Suppose W is a m-dimensional subspace of $\mathbb{R}^n$, where $m > 0$, and $\{\vec{w}_1, \cdots, \vec{w}_\ell\}$ is a subset of W such that $\text{span}(\{\vec{w}_1, \cdots, \vec{w}_\ell\}) = W$. Then some nonempty subset of $\{\vec{w}_1, \cdots, \vec{w}_\ell\}$ is a basis for W. Consequently, $\ell \geq m$.

Note that Proposition 14 is just a re-statement of Proposition 10 employing the terms basis and dimension – which were not available when Proposition 10 was stated.

Proposition 15: Suppose W is a nonzero subspace of $\mathbb{R}^n$. Then W has a basis. Moreover, this basis has no more than n elements.

Proof: We'll address the moreover statement first – since it provides the key idea as to why W must have a basis. This key idea is that any basis for W would, in turn, have to be a linearly independent set of vectors in $\mathbb{R}^n$. Thus, Proposition 13 tells us that a basis for W could not have more than $n = \dim(\mathbb{R}^n)$ elements –

since any basis for W can be extended to a basis for $\mathbb{R}^n$. In other words, any linearly independent set of vectors in $\mathbb{R}^n$ has no more than n elements.

We now show that W has a basis. Initially observe that, since $W \neq \{\vec{0}\}$, there is some $\vec{w}_1 \in W$ such that $\vec{w}_1 \neq \vec{0}$. Then $\{\vec{w}_1\}$ is a linearly independent subset of W. If $\{\vec{w}_1\}$ spans W, then $\{\vec{w}_1\}$ is a basis for W.

If $\{\vec{w}_1\}$ doesn't span W, there must be a $\vec{w}_2 \in W$ such that $\vec{w}_2 \notin \operatorname{span}(\{\vec{w}_1\})$.

Proposition 9 implies that $\{\vec{w}_1, \vec{w}_2\}$ is a linearly independent subset of W. If $\{\vec{w}_1, \vec{w}_2\}$ spans W, then $\{\vec{w}_1, \vec{w}_2\}$ is a basis for W.

If $\{\vec{w}_1, \vec{w}_2\}$ doesn't span W, we continue the process of adding vectors in W to $\{\vec{w}_1, \vec{w}_2\}$ - so as to obtain a linearly independent subset of W. The key point is that this process must terminate in a basis for W - for otherwise we would be able to construct a linearly independent subset of $\mathbb{R}^n$ having more than n elements, which is impossible.