The Determinant of a Product

This section extends HW #45. The main idea goes as follows: Suppose A is a $n\times n$ matrix and A is row equivalent to B, where B is in RREF. Then there exist elementary matrices, $E_1, \cdots, E_m$ such that $A = E_1 \cdots E_m B$. By HW #45,

$$\begin{aligned}
\det(A) &= \det(E_1)\det(E_2 \cdots E_m B) \\
&= \det(E_1)\det(E_2)\det(E_3 \cdots E_m B) \\
&\;\;\vdots \\
&= \det(E_1)\cdots\det(E_m)\det(B).
\end{aligned}$$

To see the usefulness of this computation – as you realized when doing HW #45, the determinant of an elementary matrix is nonzero. Thus, $\det(A) \neq 0$ if and only if $\det(B) \neq 0$. Since B is a $n \times n$ matrix in RREF, B either has a row of zeros or $B = I_n$. Thus, $\det(B) \neq 0$ if and only if $B = I_n$. It now follows that $\det(A) \neq 0$ if and only if A is row equivalent to I_n. Since A is invertible if and only if A is row equivalent to I_n — we have established

<u>Theorem #12</u>: Let A be a $n \times n$ matrix. Then A is invertible if and only if $\det(A) \neq 0$.

The following Theorem is a mild variation of the ideas leading to Theorem #12. It will prove useful in the near future.

<u>Theorem #13:</u> Let A be a $n \times n$ matrix and define $T : \mathbb{R}^n \to \mathbb{R}^n$ via

$$T([x_1, \cdots, x_n]) = \left(A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right)^t.$$

Then $\ker(T) \neq \{\vec{0}\}$ if and only if $\det(A) = 0$.

<u>Proof:</u> Recall that $\ker(T) \neq \{\vec{0}\}$ precisely when A is row equivalent to B, where B is in RREF and at least one column of B does not contain a pivot. Since B is $n \times n$

and in RREF, one column of B not containing a pivot is equivalent to one row of B not containing a pivot or, put another way, B has a row of zeros. Thus, $\ker(T) \neq \{\vec{0}\}$ if and only if $\det(B)=0$. But this means that $\ker(T) \neq \{\vec{0}\}$ if and only if $\det(A)=0$.

We now continue our discussion of determinants of products by proving

<u>Theorem #14</u>: Suppose A and B are two $n \times n$ matrices. Then $\det(AB) = \det(A)\det(B)$.

<u>Proof</u>: Suppose A is row equivalent to $\tilde{A}$

where $\tilde{A}$ is in RREF. Then there exist elementary matrices, $E_1, \cdots, E_m$, such that $A = E_1 \cdots E_m \tilde{A}$. By the remarks in the opening paragraph of this section we have

$$\det(AB) = \det(E_1) \cdots \det(E_m) \det(\tilde{A}B).$$

There are two cases.

<u>Case 1</u>: $\tilde{A}$ has a row of zeros.

As we have already noted, if $\tilde{A}$ has a row of zeros, then $\det(A) = 0$. On the other hand, if $\tilde{A}$ has a row of zeros, then $\tilde{A}B$ has a row of zeros –

which implies that $\det(AB) = 0$.

Thus, in this case both $\det(AB)$ and $\det(A)\det(B)$ equal zero.

Case 2: $\tilde{A}$ has no row of zeros - i.e.

$$\tilde{A} = I_n.$$

In this case, since $\det(I_n) = 1$, we have

$$\begin{aligned}
\det(AB) &= \det(E_1)\cdots\det(E_m)\det(I_n B) \\
&= \det(E_1)\cdots\det(E_m)\det(B) \\
&= \det(E_1)\cdots\det(E_m)\det(I_n)\det(B) \\
&= \det(E_1\cdots E_m I_n)\det(B) \\
&= \det(A)\det(B).
\end{aligned}$$

HW #46: Suppose A is an invertible $n \times n$ matrix. Prove

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

HW #47: Suppose A is a $n \times n$ matrix and $AA^t = I_n$. Prove that $\det(A) = 1$ or $\det(A) = -1$.

Cofactors and Related Topics

In this, the last section dedicated to computing determinants, we have gathered together a number of results related to cofactors in one way or another. Proofs will be omitted.

Notation and Terminology.

Let $A=[a_{ij}]$ denote a $n\times n$ matrix. If $r,s\in\{1,\cdots,n\}$, then A_{rs} will denote the $(n-1)\times(n-1)$ submatrix of A obtained by removing the r-th row and s-th column from A. The

(r,s)-minor of A, denoted by M_{rs}, is $M_{rs} = \det(A_{rs})$. The (r,s)-cofactor of A, denoted by C_{rs}, is $C_{rs} = (-1)^{r+s} M_{rs}$.

Ex: If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

then

$$A_{32} = \begin{bmatrix} 1 & \cancel{2} & 3 \\ 4 & \cancel{5} & 6 \\ \cancel{7} & \cancel{8} & \cancel{9} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}.$$

$$M_{32} = \det\left(\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}\right) = 6 - 12 = -6$$

$$C_{32} = (-1)^{3+2} M_{32} = (-1)(-6) = 6.$$

In Calculus, you probably used row expansion along the 1-st row to compute determinants of 3×3 matrices. The following Theorem justifies such computations (and generalizes such computations to rows, in general, and columns as well).

Theorem #15: Let $A = [a_{ij}]$ denote a $n \times n$ matrix, M_{rs} denote the (r,s)-minor of A and C_{rs} denote the (r,s)-cofactor of A. Then the following formulae are valid:

1) If $k \in \{1, \cdots, n\}$, then

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det(A_{kj})$$
$$= \sum_{j=1}^{n} (-1)^{k+j} a_{kj} M_{kj}$$
$$= \sum_{j=1}^{n} a_{kj} C_{kj} .$$

2) If $l \in \{1, \cdots, n\}$, then

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+l} a_{il} \det(A_{il})$$
$$= \sum_{i=1}^{n} (-1)^{i+l} a_{il} M_{il}$$
$$= \sum_{i=1}^{n} a_{il} C_{il} .$$

Formula 1) is frequently referred to as the cofactor expansion along the k-th row of A, while formula 2) is called cofactor expansion along the l-th column of A.

Ex: Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

We'll compute $\det(A)$ by expanding along the third row, then by expanding along the second column.

The third row expansion:

$$A_{31} = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}, \quad M_{31} = 12 - 15 = -3, \quad C_{31} = -3$$

$$A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}, \quad M_{32} = 6 - 12 = -6, \quad C_{32} = 6$$

$$A_{33} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \quad M_{33} = 5 - 8 = -3, \quad C_{33} = -3$$

$$\det(A) = a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33}$$

$$= 7(-3) + 8(6) + 9(-3) = 0$$

The second column expansion:

$A_{12} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$, $M_{12} = 36 - 42 = -6$, $C_{12} = 6$

$A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$, $M_{22} = 9 - 21 = -12$, $C_{22} = -12$

$A_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$, $M_{32} = 6 - 12 = -6$, $C_{32} = 6$

$$\det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

$$= 2(6) + 5(-12) + 8(6) = 0.$$

HW #48: Suppose $A = [a_{ij}]$ is a $n \times n$ upper triangular matrix, i.e. $a_{ij} = 0$ if $i > j$. Use Theorem #15 and induction to show $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

Now let $A = [a_{ij}]$ denote a $n \times n$ matrix and let C_{ij} denote the (i,j)-cofactor of A. The matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the cofactor matrix of A.

The adjoint of A, denoted by $\text{adj}(A)$, is the transpose of the cofactor matrix of A, i.e.

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

The following formula is a consequence of cofactor expansion along rows and the determinant of a matrix having two equal rows is zero.

$$A\,(\mathrm{adj}(A)) = \det(A)\, I_n$$

The following Theorem follows immediately from this formula.

<u>Theorem #16</u>: Suppose A is a $n \times n$ matrix and $\mathrm{adj}(A)$ denotes the adjoint of A. If $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A).$$

Ex: Theorem #16 provides a frequently employed method for constructing A^{-1} when A is an invertible 2×2 matrix.

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $\det(A) = ad-bc \neq 0$. Note that

$A_{11} = [d]$, $M_{11} = d$, $C_{11} = d$

$A_{12} = [c]$, $M_{12} = c$, $C_{12} = -c$

$A_{21} = [b]$, $M_{21} = b$, $C_{21} = -b$

$A_{22} = [a]$, $M_{22} = a$, $C_{22} = a$.

So the cofactor matrix for A is

$$\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}.$$

Then

$$\operatorname{adj}(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^t = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

So

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$$= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

To check this ...

$$\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= I_2 \checkmark$$

We now consider the system of linear equations

$$\mathcal{S}: \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Letting $A = [a_{ij}]_{\substack{i=1,\cdots,n \\ j=1,\cdots,n}}$ and

$$\mathbb{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbb{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

this system becomes $A\mathbb{X} = \mathbb{B}$. Note that if A is invertible, then

$$\mathbb{X} = A^{-1}\mathbb{B}.$$

Therefore, if $\det(A) \neq 0$, $\mathcal{S}$ has the unique solution

$$X^t = (A^{-1}B)^t.$$

Using Theorem #16, the following formula for computing this unique solution for $\mathcal{S}$ is obtained.

<u>Cramer's Rule</u>: Let $AX = B$ denote the system of linear equation, $\mathcal{S}$, as above. For each $i \in \{1, \cdots, n\}$, let A_i denote the $n \times n$ matrix obtained by replacing the i-th column of A with B. If $\det(A) \neq 0$, then the unique solution

to $\mathcal{S}$ is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \ \ldots, \ x_n = \frac{\det(A_n)}{\det(A)}.$$

Ex: Consider the system $\mathcal{S}$ below.

$$x_1 + 2x_2 = 3$$
$$4x_1 + 5x_2 = 6$$

or

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

So $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

It follows that

$$A_1 = \begin{bmatrix} 3 & 2 \\ 6 & 5 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}.$$

Note that

$$\det(A) = 5-8 = -3$$

$$\det(A_1) = 15-12 = 3$$

$$\det(A_2) = 6-12 = -6 .$$

Cramer's Rule tells us that the unique solution to $\mathcal{S}$ is:

$$x_1 = \frac{3}{-3} = -1$$

$$x_2 = \frac{-6}{-3} = 2 .$$