Abstract Vector Spaces

Although our focus in MA 237 will be restricted to vector spaces associated to $\mathbb{R}^n$, the notion of a vector space applies to a somewhat broader collection of objects - which you may encounter in subsequent courses.

Definition: Suppose V is a set. We say V is a vector space over $\mathbb{R}$ provided V comes equipped with two operations, $\oplus$ and $\odot$, which satisfy ten properties. To be more precise, if $v, w \in V$, there is an associated object, $v \oplus w$, - called the vector sum of v and w and, if $r \in \mathbb{R}$ and $v \in V$, there is an associated object, $r \odot v$, - called the scalar multiple of v by r. Moreover, these operations satisfy the following ten properties:

1) If $v, w \in V$, then $v \oplus w \in V$.

2) If $v, w \in V$, then $v \oplus w = w \oplus v$.

3) If $u, v, w \in V$, then

$$(u \oplus v) \oplus w = u \oplus (v \oplus w).$$

4) There exists an element, $0 \in V$,

such that $v \oplus O = v$ for any $v \in V$.

5) If $v \in V$, there is an element, $-v \in V$, such that $v \oplus (-v) = O$.

6) If $r \in \mathbb{R}$ and $v \in V$, then $r \odot v \in V$.

7) If $r \in \mathbb{R}$ and $v, w \in V$, then $r \odot (v \oplus w) = (r \odot v) \oplus (r \odot w)$.

8) If $r, s \in \mathbb{R}$ and $v \in V$, then $(r+s) \odot v = (r \odot v) \oplus (s \odot v)$.

9) If $r, s \in \mathbb{R}$ and $v \in V$, then $r \odot (s \odot v) = (rs) \odot v$.

10) If $v \in V$, then $1 \odot v = v$.

Note: Technically, we should have written $(\mathbb{V}, \oplus, \odot)$ is a vector space, in the definition above. However, saying $\mathbb{V}$ is a vector space is commonplace - except when confusion might occur.

According to Theorems 1 and 4, both $\mathbb{R}^n$ and $M_{m,n}(\mathbb{R})$ are vector spaces. The following example is useful in Differential Equations.

Let $\mathbb{V}$ denote the set of all functions $f: \mathbb{R} \longrightarrow \mathbb{R}$ such that the n-th derivative of f, $f^{(n)}$, exists for every positive integer n. If $f, g \in V$, define $(f \oplus g)(x) = f(x) + g(x)$. If $r \in \mathbb{R}$ and $f \in V$, define $(r \odot f)(x) = r f(x)$. Then V is a vector space over $\mathbb{R}$.

The following vector space may appear a bit contrived. I include it only so you get a glimpse of just how broad the notion of vector space can be.

Let V denote the set of all positive real numbers. If $x, y \in V$, define $x \oplus y = xy$. If $r \in \mathbb{R}$ and $x \in V$, define $r \odot x = x^r$. Then V is a vector space over $\mathbb{R}$.

HW #12 : Which positive real number acts as 0 in this last example?

This last example suggests that an abstract vector space may have some rather non-obvious properties. On the other hand, the following Proposition suggests that our intuition, drawn from $\mathbb{R}^n$, isn't too misleading.

Proposition 4: Suppose $(V, \oplus, \odot)$ is a vector space. Then

1) if $0_1, 0_2 \in V$ such that

$v \oplus 0_1 = v$ and $v \oplus 0_2 = v$, for all $v \in V$, it follows that $0_1 = 0_2$,

and

2) if $v \in V$ and $(-v_1), (-v_2) \in V$ satisfy $v + (-v_1) = 0 = v + (-v_2)$, it follows that $-v_1 = -v_2$.

Note: In words, Proposition 4 tells us that the additive identity and additive inverses in any vector space is (are) unique.

Proof of Proposition 4):

1) If $v \oplus 0_1 = v$ and $v \oplus 0_2 = v$ for all $v \in V$, then

$$O_1 = O_1 \oplus O_2 = O_2 \oplus O_1 = O_2 .$$

2) $-v_1 = -v_1 \oplus O = -v_1 \oplus (v \oplus (-v_2))$

$$= (-v_1 \oplus v) \oplus (-v_2)$$

$$= (v \oplus (-v_1)) \oplus (-v_2)$$

$$= O \oplus (-v_2)$$

$$= (-v_2) \oplus O$$

$$= -v_2 .$$

Having established that the additive identity and additive inverses are unique in an abstract vector space, the following Proposition tells us how scalar multiplication can be employed to construct these objects.

<u>Proposition 5</u>: Suppose $(V, \oplus, \odot)$ is a vector space and $v \in V$. Then

1) $0 \odot v = 0$, and

2) $(-1) \odot v = -v$.

Proof:

1) First note that

$$0 \odot v = (0+0) \odot v = (0 \odot v) \oplus (0 \odot v).$$

It follows that

$$\begin{aligned} 0 &= (0 \odot v) \oplus (-0 \odot v) \\ &= \big((0 \odot v) \oplus (0 \odot v)\big) \oplus (-0 \odot v) \\ &= (0 \odot v) \oplus \big((0 \odot v) \oplus (-0 \odot v)\big) \\ &= (0 \odot v) \oplus 0 \\ &= 0 \odot v. \end{aligned}$$

2) $v \oplus ((-1) \odot v) = (1 \odot v) \oplus ((-1) \odot v)$

$= (1 + (-1)) \odot v$

$= 0 \odot v$

$= 0$, by 1).

So $(-1) \odot v$ is the additive inverse of v, i.e. $(-1) \odot v = -v$.

HW #13: Use Proposition 5 to check your answer for HW #12.

Abstract Subspaces

Definition: Suppose $(\mathbb{V}, \oplus, \odot)$ is a vector space and $\mathbb{W}$ is a subset of $\mathbb{V}$. We say that $\mathbb{W}$ is a subspace of $\mathbb{V}$ provided $\mathbb{W}$ satisfies the following three properties:

1) $\mathbb{W} \neq \emptyset$,

2) if $w_1, w_2 \in \mathbb{W}$, then $w_1 \oplus w_2 \in \mathbb{W}$,

   and

3) if $r \in \mathbb{R}$ and $w \in \mathbb{W}$, then $r \odot w \in \mathbb{W}$.

Terminology: Property 2) is referred to as "$\mathbb{W}$ is closed under vector addition. Similarly, Property 3) is referred to as "W is closed under scalar multiplication".

The term "subspace" is due to the following Theorem.

<u>Theorem 7</u>: Suppose $(V, \oplus, \odot)$ is a vector space and W is a subspace of V. Then W, endowed with the vector addition and scalar multiplication defined on V, is, itself, a vector space.

<u>Proof</u>: First note that properties 1) and 6) for a vector space are satisfied by W – due to W being closed under

vector addition and scalar multiplication, respectively.

That W satisfies properties 4) and 5) for a vector space follows from Proposition 5, $W \neq \phi$ and W is closed under scalar multiplication.

The remaining properties for a vector space are satisfied by W - since they are valid for any scalars and any vectors in V - and W is a subset of V.

<u>HW #14</u>: Suppose $(V, \oplus, \odot)$ is a vector space, m is a positive integer

and $\overline{W}_1, \cdots, \overline{W}_m$ denote m subspaces of $\overline{V}$. Give an induction proof that $\overline{W}_1 \cap \overline{W}_2 \cap \cdots \cap \overline{W}_m$ is also a subspace of $\overline{V}$.

Subspaces of $\mathbb{R}^n$

The subspaces of $\mathbb{R}^n$ and their basic properties are two of the primary focal points of this course. Here, we provide two extremely important methods for constructing subspaces of $\mathbb{R}^n$. We'll also discuss how subspaces of $\mathbb{R}^n$ can be viewed as geometric objects in $\mathbb{R}^n$.

Method 1: Spans

Suppose m is a positive integer

and $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$. Set $S = \{\vec{v}_1, \cdots, \vec{v}_m\}$. Then the span of S, denoted by $\text{span}(S)$ or $\text{span}(\{\vec{v}_1, \cdots, \vec{v}_m\})$, is defined to be:

$$\text{span}(S) = \left\{ \sum_{i=1}^{m} c_i \vec{v}_i \,\middle|\, c_1, \cdots, c_m \in \mathbb{R} \right\}.$$

In words, $\text{span}(S)$ is the set of all the vectors in $\mathbb{R}^n$ which can be written as a linear combination of the elements in S.

<u>Ex</u>: Suppose $S = \{\vec{v}\}$, where

where $\vec{v}$ is a nonzero vector in $\mathbb{R}^n$. Then $\text{span}(S) = \{cv \mid c \in \mathbb{R}\}$.

Thus, in this case, $\text{span}(S)$ is just the set of all scalar multiples of $\vec{v}$. Recall that the Claim on page 11 tells us that the set of all such scalar multiples determines a line in $\mathbb{R}^n$ – which passes through the origin and contains the position vector which represents $\vec{v}$.

<u>Ex</u>: We would expect that 3 non-colinear points in $\mathbb{R}^n$ would determine a plane in $\mathbb{R}^n$. Let O, P and Q denote 3 non-colinear points in $\mathbb{R}^n$ – where O denotes

the origin, $P=(p_1,\cdots,p_n)$ and $Q=(q_1,\cdots,q_n)$. We'll describe a method for constructing the points in the plane determined by O, P and Q.

The Claim on page 11 implies that

$$L_1=\{(tp_1,\cdots,tp_n)\mid t\in\mathbb{R}\}$$

is the line through O containing P and

$$L_2=\{(sq_1,\cdots,sq_n)\mid s\in\mathbb{R}\}$$

is the line through O containing Q. Certainly, L_1 and L_2 should be contained in the plane determined by O, P and Q.

Of course, the points lying only on L_1 and L_2 will not form a plane. However, we can interpret L_1 and L_2 as being rather like the coordinate axes of this plane. As such, we would expect the plane to consist of all the points which lie on the translates of L_1 along L_2.

Now, we know that the typical point on L_2 is of the form

$(sq_1, \cdots, sq_n)$ where $s \in \mathbb{R}$. Let $f_s : \mathbb{R}^n \to \mathbb{R}^n$ denote the translation

$$f_s(x_1, \cdots, x_n) = (x_1 + sq_1, \cdots, x_n + sq_n).$$

Note that $f_s(0) = (sq_1, \cdots, sq_n)$.

Next, observe that the typical point in L_1 is of the form $(tp_1, \cdots, tp_n)$ where $t \in \mathbb{R}$. Noting that

$$f_s(tp_1, \cdots, tp_n) = (tp_1 + sq_1, \cdots, tp_n + sq_n),$$

it follows that $(tp_1 + sq_1, \cdots, tp_n + sq_n)$, $t \in \mathbb{R}$, is the typical point lying on the translate of L_1 to $(sq_1, \cdots, sq_n)$ so that 0 maps to $(sq_1, \cdots, sq_n)$.

Consequently, we would expect that the plane determined by O, P and Q in $\mathbb{R}^n$ is just the set of all points of the form $(tp_1+sq_1, \cdots, tp_n+sq_n)$ - where t and s vary over the real numbers.

If we set $\vec{p} = [p_1, \cdots, p_n]$ and $\vec{q} = [q_1, \cdots, q_n]$, then $t\vec{p}+s\vec{q} = [tp_1+sq_1, \cdots, tp_n+sq_n]$. Thus, the plane determined by O, P and Q is just the set of points in $\mathbb{R}^n$ associated to the set of vectors $\{t\vec{p}+s\vec{q} \mid t, s \in \mathbb{R}\}$.

Finally, note that

$$\operatorname{span}(\{\vec{p},\vec{q}\}) = \{t\vec{p}+s\vec{q} \mid t,s\in\mathbb{R}\}.$$

We conclude that if $\vec{v}$ and $\vec{w}$ are two non-parallel vectors in $\mathbb{R}^n$, then $\operatorname{span}(\{\vec{v},\vec{w}\})$ can be viewed as a plane in $\mathbb{R}^n$ which passes through the origin.

The second example is very suggestive. For instance, it suggests how one could construct a copy of space in $\mathbb{R}^n$ — from four non-coplanar points in $\mathbb{R}^n$, O, P, Q and R, where O

denotes the origin. One simply constructs the plane determined by O, P and Q and then translates this plane along the line determined by O and R.

The second example also suggests how different coordinate systems might be imposed on $\mathbb{R}^n$. (Use your imagination.) This too will be discussed later in the course.

<u>Proposition 6</u>: Suppose m is a positive integer and $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$. Set $S = \{\vec{v}_1, \cdots, \vec{v}_m\}$. Then span$(S)$

is a subspace of $\mathbb{R}^n$.

<u>Proof</u>: We must verify that span(S) is nonempty and closed under both vector addition and scalar multiplication on $\mathbb{R}^n$.

That span$(S) \neq \phi$ follows from our requiring that m be a positive integer – so $S \neq \phi$. From this it follows that $\vec{0} = \sum_{i=1}^{m} 0\vec{v}_i \in$ span(S). Thus, span$(S) \neq \phi$.

Suppose $\vec{w}_1, \vec{w}_2 \in$ span(S). Since span(S) is the set of all linear combinations of $\vec{v}_1, \cdots, \vec{v}_m$, it follows that $\vec{w}_1 = \sum_{i=1}^{m} c_i \vec{v}_i$ and

$\vec{w}_2 = \sum_{i=1}^{m} d_i \vec{v}_i$ for some

$c_1, \cdots, c_n, d_1, \cdots, d_n \in \mathbb{R}$. Then

$$\vec{w}_1 + \vec{w}_2 = \sum_{i=1}^{m} c_i \vec{v}_i + \sum_{i=1}^{m} d_i \vec{v}_i$$

$$= \sum_{i=1}^{m} (c_i + d_i) \vec{v}_i \in \text{span}(S).$$

Thus, span(S) is closed under vector addition.

Finally, suppose $r \in \mathbb{R}$ and $\vec{w} \in \text{span}(S)$. Then $\vec{w} = \sum_{i=1}^{m} c_i \vec{v}_i$ for some $c_1, \cdots, c_n \in \mathbb{R}$. Then

$$r\vec{v} = r \sum_{i=1}^{m} c_i \vec{v}_i = \sum_{i=1}^{m} (rc_i) \vec{v}_i \in \text{span}(S).$$

This implies that span(S) is closed

under scalar multiplication - which completes the proof of Proposition 6.

HW #15: Suppose $S = \{\vec{v}_1, \cdots, \vec{v}_m\}$, as above, and $k \in \{1, \cdots, m\}$. Show that $\vec{v}_k \in \text{span}(S)$.

Method 2: Solution Sets of Homogeneous, Linear Equations.

Let $\vec{v} \in \mathbb{R}^n$. We'll denote the set of all the vectors in $\mathbb{R}^n$ which are orthogonal to $\vec{v}$ by $\vec{v}^{\perp}$ - read $\vec{v}$-perp. Note that if $\vec{v} = \vec{0}$, then $\vec{v}^{\perp} = \mathbb{R}^n$. If $\vec{v} \neq \vec{0}$, the situation is similar to the construction of planes in space – $\vec{v}^{\perp}$ will be a copy of $\mathbb{R}^{n-1}$ in $\mathbb{R}^n$, which contains the origin. Of course, this copy of $\mathbb{R}^{n-1}$ may appear to be rotated – much like planes in space need not look like the x,y-plane.

<u>Proposition 7</u>: If $\vec{v} \in \mathbb{R}^n$, then $\vec{v}^{\perp}$ is a subspace of $\mathbb{R}^n$.

<u>Proof</u>: That $\vec{v}^{\perp} \neq \emptyset$ follows from $\vec{v} \cdot \vec{0} = 0$ – so $\vec{0} \in \vec{v}^{\perp}$.

If $\vec{w}_1, \vec{w}_2 \in \vec{v}^{\perp}$, then $\vec{v} \cdot \vec{w}_1 = 0$ and $\vec{v} \cdot \vec{w}_2 = 0$. Thus,

$$\vec{v} \cdot (\vec{w}_1 + \vec{w}_2) = \vec{v} \cdot \vec{w}_1 + \vec{v} \cdot \vec{w}_2 = 0 + 0 = 0,$$ so

$\vec{w}_1 + \vec{w}_2 \in \vec{v}^{\perp}$. As such, $\vec{v}^{\perp}$ is closed under vector addition.

Finally, if $r \in \mathbb{R}$ and $\vec{w} \in \vec{v}^{\perp}$, then $\vec{v} \cdot \vec{w} = 0$ — so

$$\vec{v} \cdot (r\vec{w}) = r(\vec{v} \cdot \vec{w}) = r(0) = 0.$$

We conclude that $\vec{v}^{\perp}$ is closed under scalar multiplication. This completes the proof of Proposition 7.

Now note that, if $\vec{v} = [v_1, \ldots, v_n]$ and $\vec{x} = [x_1, \ldots, x_n]$, then

$$\vec{v} \cdot \vec{x} = v_1 x_1 + \cdots + v_n x_n.$$

Therefore, $\vec{v}^{\perp}$ is just the set of all $\vec{x} \in \mathbb{R}^n$ such that

$$v_1 x_1 + \cdots + v_n x_n = 0.$$

The equation, $v_1 x_1 + \cdots + v_n x_n = 0$, is a homogeneous, linear equation. The term "homogeneous" is employed due to $v_1 x_1 + \cdots + v_n x_n$ being equal to 0 as opposed to being equal to some nonzero constant. For instance,

$$2x + 3y = 0$$

is homogeneous, while

$$2x + 3y = 2$$

is not homogeneous.

Let $a_1, \cdots, a_n \in \mathbb{R}$. Since the set of vectors, $\vec{x} = [x_1, \cdots, x_n]$, such that

$ax_1 + \cdots + a_n x_n = 0$ coincides with $\vec{a}^\perp$, where $\vec{a} = [a_1, \cdots, a_n]$, the following Proposition is just another way of phrasing Proposition 7.

<u>Proposition 8</u>: The set of vectors which form the solution set of a homogeneous, linear equation in n variables is a subspace of $\mathbb{R}^n$.

We are frequently interested in solution sets of systems of linear equations. The following is a generic system of m homogeneous, linear equations in n variables.

$$\mathcal{S}: \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0. \end{aligned}$$

Recall that the solution set for the system $\mathcal{S}$ consists of all those vectors $\vec{x} = [x_1, \cdots, x_n]$ such that

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = 0$$

for $k = 1, 2, \cdots, m$.

<u>HW #16</u>: Show that the solution set for the system $\mathcal{S}$, as above, is

a subspace of $\mathbb{R}^n$

Hint: A simple and enlightening approach would be to use Proposition 8 and HW #14.