Row Operations and Determinants

Determinants behave in a predictable manner under row operations.

Proposition #36: Suppose A is a $n \times n$ matrix and r is a nonzero real number. Suppose B is obtained from A by multiplying the k-th row of A by r, for some $k \in \{1, \cdots, n\}$. Then $\det(B) = r\det(A)$.

(Actually, Proposition #36 is valid even when $r = 0$.)

HW #43: Prove Proposition #36.

<u>Proposition</u> #37: Suppose A is a $n \times n$ matrix, $k, \ell \in \{1, \cdots, n\}$ and $k < \ell$. Let B be the matrix obtained by interchanging the k-th and ℓ-th rows of A. Then $\det(B) = -\det(A)$.

<u>Proof</u>. Let $A = [a_{ij}]$ and $B = [b_{ij}]$.

Then

$$b_{ij} = \begin{cases} a_{ij} & \text{if } i \notin \{k, \ell\} \\ a_{\ell j} & \text{if } i = k \\ a_{kj} & \text{if } i = \ell \end{cases} .$$

Now

$$\det(B) = \sum_{\sigma \in S_n} (-1)^\sigma b_{1\sigma(1)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} (-1)^\sigma a_{1\sigma(1)} \cdots a_{\ell\sigma(k)} \cdots a_{k\sigma(\ell)} \cdots a_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} (-1)^\sigma a_{1\sigma(1)} \cdots a_{k\sigma(\ell)} \cdots a_{\ell\sigma(k)} \cdots a_{n\sigma(n)} .$$

In this last sum, row indices increase from 1 to n as we read the factors in each summand from left to right. Thus, the only difficulty lies in the factors $a_{k\sigma(\ell)}$ and $a_{\ell\sigma(k)}$.

To rectify this, let π denote the element of S_n defined by

$$\pi(i) = \begin{cases} i & \text{if } i \in \{1,\cdots,n\} - \{k,\ell\} \\ \ell & \text{if } i = k \\ k & \text{if } i = \ell \end{cases}.$$

Then $\quad a_{k\,\sigma\circ\pi(k)} = a_{k\sigma(\ell)}$

$$a_{\ell\,\sigma\circ\pi(\ell)} = a_{\ell\sigma(k)}$$

and $\quad a_{i\,\sigma\circ\pi(i)} = a_{i\sigma(i)}$ if $i \notin \{k,\ell\}$.

It follows that

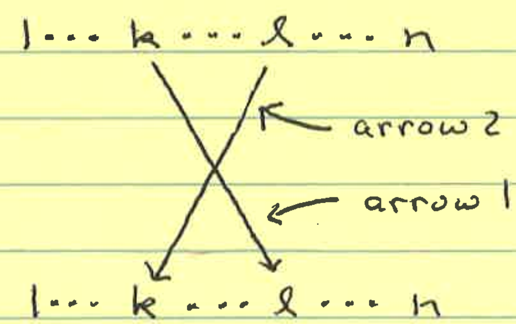
$$\det(B) = \sum_{\sigma \in S_n} (-1)^{\sigma} a_{1\,\sigma\circ\pi(1)} \cdots a_{n\,\sigma\circ\pi(n)} .$$

The delicate point in this argument is relating $(-1)^{\sigma\circ\pi}$ and $(-1)^{\sigma}$.

<u>Claim</u>: $(-1)^{\sigma\circ\pi} = -(-1)^{\sigma}$ for all $\sigma \in S_n$.

To begin, note that all the arrows in the diagram for π are vertical — except for the two arrows depicted below, denoted by arrow 1 and arrow 2.

π:



Now let $i \in \{1, \ldots, n\} - \{k, l\}$. If

$i<k$ or $i>\ell$, the vertical arrow from i to i in the diagram for π misses both arrow 1 and arrow 2. On the other hand, if $k<i<\ell$, then the vertical arrow from i to i intersects arrow 1 in one point and it also intersects arrow 2 in one point. Since there are $\ell-k-1$ integers i such that $k<i<\ell$, and arrow 1 intersects arrow 2 in one point, it follows that the number of intersections of arrows in the diagram for π is

$$2(\ell-k-1)+1.$$

We will now show that π being odd

implies that $\sigma \circ \pi$ and σ have different parities for every $\sigma \in S_n$. This is accomplished in two steps.

Initially, we depict $\sigma \circ \pi$ by placing the arrow diagram for π on top of the arrow diagram for σ.

$$\begin{array}{cc} & 1 \;\cdots\; n \\ \pi: & \\ & 1 \;\cdots\; n \\ \sigma: & \\ & 1 \;\cdots\; n \end{array}$$

Observe that, as depicted above, the total number of crossings of arrows equals the number of crossings of arrows in the arrow diagram for σ plus an odd number.

Of course, the picture of $\sigma\circ\pi$ drawn above is not the arrow diagram for $\sigma\circ\pi$ — and the arrow diagram for $\sigma\circ\pi$ is the picture used to determine the parity of $\sigma\circ\pi$. We must determine how the number of crossings in the picture, above, relates to the number of crossings in the arrow diagram for $\sigma\circ\pi$.

To this end, let $i, j \in \{1, \cdots, n\}$ where $i < j$. There are four basic cases which we need to consider — as depicted below:

Case 1:

$$
\begin{tikzcd}[column sep=small]
 & 1 & \cdots & i \arrow[dd] & \cdots & j \arrow[dd] & \cdots & n \\
\pi: & & & & & & & \\
 & 1 & \cdots & \pi(i) \arrow[dd] & \cdots & \pi(j) \arrow[dd] & \cdots & n \\
\sigma: & & & & & & & \\
 & 1 & \cdots & \sigma\circ\pi(i) & \cdots & \sigma\circ\pi(j) & \cdots & n
\end{tikzcd}
$$

Case 2:

$$
\begin{tikzcd}[column sep=small]
 & 1 & \cdots & i \arrow[ddrr] & \cdots & j \arrow[ddll] & \cdots & n \\
\pi: & & & & & & & \\
 & 1 & \cdots & \pi(j) \arrow[dd] & \cdots & \pi(i) \arrow[dd] & \cdots & n \\
\sigma: & & & & & & & \\
 & 1 & \cdots & \sigma\circ\pi(j) & \cdots & \sigma\circ\pi(i) & \cdots & n
\end{tikzcd}
$$

<u>Case 3:</u>

$$\begin{tikzcd}
 & 1 & \cdots & i \arrow[d] & \cdots & j \arrow[d] & \cdots & n \\
\pi: & 1 & \cdots & \pi(i) \arrow[drr] & \cdots & \pi(j) \arrow[dll] & \cdots & n \\
\sigma: & 1 & \cdots & \sigma\circ\pi(j) & \cdots & \sigma\circ\pi(i) & \cdots & n
\end{tikzcd}$$

<u>Case 4:</u>

$$\begin{tikzcd}
 & 1 & \cdots & i \arrow[drr] & \cdots & j \arrow[dll] & \cdots & n \\
\pi: & 1 & \cdots & \pi(j) \arrow[drr] & \cdots & \pi(i) \arrow[dll] & \cdots & n \\
\sigma: & 1 & \cdots & \sigma\circ\pi(i) & \cdots & \sigma\circ\pi(j) & \cdots & n
\end{tikzcd}$$

Now note that if we were to draw arrows directly from i to $\sigma \circ \pi(i)$ and from j to $\sigma \circ \pi(j)$ – as we would do in the arrow diagram for $\sigma \circ \pi$ – the number of intersections of these arrows is the same as the number of intersections of the arrows depicted in Cases 1, 2 and 3 – but two fewer than the number of intersections of arrows as seen in Case 4. We conclude that the number of crossings in the arrow diagram for $\sigma \circ \pi$ differs from the number of crossings of arrows in the stacked picture of $\sigma \circ \pi$, given above, by an even number.

This implies that

$$(-1)^{\sigma\circ\pi} = (-1)^{\sigma}(-1)^{\pi} = -(-1)^{\sigma}.$$

It follows that

$$\det(B) = \sum_{\sigma \in S_n} (-1)^{\sigma} a_{1\,\sigma\circ\pi(1)} \cdots a_{n\,\sigma\circ\pi(n)}$$

$$= \sum_{\sigma \in S_n} -(-1)^{\sigma\circ\pi} a_{1\,\sigma\circ\pi(1)} \cdots a_{n\,\sigma\circ\pi(n)}$$

$$= -\sum_{\sigma \in S_n} (-1)^{\sigma\circ\pi} a_{1\,\sigma\circ\pi(1)} \cdots a_{n\,\sigma\circ\pi(n)}.$$

By an argument similar to the one given when showing $\det(A^t) = \det(A)$,

$$\sum_{\sigma \in S_n} (-1)^{\sigma\circ\pi} a_{1\,\sigma\circ\pi(1)} \cdots a_{n\,\sigma\circ\pi(n)}$$

$$= \sum_{\sigma\circ\pi \in S_n} (-1)^{\sigma\circ\pi} a_{1\,\sigma\circ\pi(1)} \cdots a_{n\,\sigma\circ\pi(n)}.$$

Finally, letting $\tau = \sigma\circ\pi$ we obtain

$$\det(B) = -\sum_{\sigma\circ\pi \in S_n} (-1)^{\sigma\circ\pi} a_{1\,\sigma\circ\pi(1)} \cdots a_{n\,\sigma\circ\pi(n)}$$

$$= -\sum_{\tau \in S_n} (-1)^{\tau} a_{1\tau(1)} \cdots a_{n\tau(n)}$$

$$= -\det(A).$$

Proposition #37 has a useful

<u>Corollary</u>: Suppose A is an $n\times n$ matrix and there exist $k, \ell \in \{1, \cdots, n\}$ such that $k<\ell$ and the k-th row of A equals the ℓ-th row of A. Then $\det(A) = 0$.

<u>Proof</u>: Let B be obtained by interchanging

rows k and l in A. Then

$\det(B) = -\det(A)$, by Proposition #37.

On the other hand, the k-th row of A equals the l-th row of A - so $B = A$.

Thus $\det(A) = -\det(A)$ - so $\det(A) = 0$.

<u>Proposition #38</u>: Suppose A is a $n \times n$ matrix, $c \in \mathbb{R}$ and l and k are distinct elements of $\{1, \dots, n\}$. Let B be obtained by adding c times the l-th row of A to the k-th row of A. Then

$\det(B) = \det(A)$.

<u>Proof</u>: Let $A = [a_{ij}]$ and $B = [b_{ij}]$.

Then

$$b_{ij} = \begin{cases} a_{ij} & \text{if } i \neq k \\ a_{kj} + c a_{\ell j} & \text{if } i = k \,. \end{cases}$$

It follows that

$$\det(B) = \sum_{\sigma \in S_n} (-1)^\sigma b_{1\sigma(1)} \cdots b_{k\sigma(k)} \cdots b_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} (-1)^\sigma a_{1\sigma(1)} \cdots \left(a_{k\sigma(k)} + c\, a_{\ell\sigma(k)} \right) \cdots a_{n\sigma(n)}$$

$$= \sum_{\sigma \in S_n} (-1)^\sigma a_{1\sigma(1)} \cdots a_{k\sigma(k)} \cdots a_{n\sigma(n)}$$

$$+ c \sum_{\sigma \in S_n} (-1)^\sigma a_{1\sigma(1)} \cdots a_{\ell\sigma(k)} \cdots a_{n\sigma(n)}$$

$$= \det(A) + c \sum_{\sigma \in S_n} (-1)^\sigma a_{1\sigma(1)} \cdots a_{\ell\sigma(k)} \cdots a_{n\sigma(n)} \,.$$

Observe now that $\sum_{\sigma \in S_n} (-1)^\sigma a_{1\sigma(1)} \cdots a_{\ell\sigma(k)} \cdots a_{n\sigma(n)}$

corresponds to the determinant of the matrix obtained by replacing the k-th row of A with a copy of the l-th row of A. Since the l-th and k-th rows of this matrix are equal

$$\sum_{\sigma \in S_n} (-1)^{\sigma} a_{1\sigma(1)} \cdots a_{l\sigma(k)} \cdots a_{n\sigma(n)} = 0$$

- by the Corollary to Proposition #37. We conclude that $\det(B) = \det(A)$.

HW #44: Compute

$$\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right).$$

HW #45: Let E denote an elementary $n \times n$ matrix and let A be a $n \times n$

matrix. Prove that

$$\det(EA) = \det(E)\det(A).$$

<u>Hint</u>: Recall that EA can be interpreted as a matrix obtained by performing an elementary row operation on A and consider the three row operations separately.