FACULTY
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**BACHELOR THESIS**

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# Electric field of a charge near the wormhole

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Dedication.

Title: Electric field of a charge near
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# Introduction

Problem we're facing is finding field of a point charge, next to a wormhole. Well known approach to problems containing electric fields is direct solution of Poisson equation **?**

$$\Delta\,\Phi = 4\pi\rho,$$

where we consider $\rho$ being charge distribution of point charge at certain position. Solving for $\Phi$ if possible and using correct border conditions we'd probably get an answer. This would not be that interesting, field of a point charge is well known, if it wasn't for the wormhole.

The wormhole essentially doubles the space we care about. Take two $\mathbb{R}^3$ spaces, cut out a disk from both and sew them together through the hole that's left. Now these two spaces together create the space we care about, and into which we'll place our point charge. How do we describe such space? It's simple, enter oblate spheroidal coordinates (OSC). Simply describing $\mathbb{R}^3$ via OSC gives us the wormhole coordinates perfectly. We'll use OSC as defined in *[DLMF] §30.14(i)*:

$$\begin{aligned} x &= \ell\sqrt{1+s^2}\sqrt{1-c^2}\cos\varphi, \\ y &= \ell\sqrt{1+s^2}\sqrt{1-c^2}\sin\varphi, \\ z &= \ell sc, \end{aligned} \tag{1}$$

where $s \in \mathbb{R}^+$, $c \in [-1, 1]$ and $\varphi \in [0, 2\pi]$[1]. Parameter $\ell$ is just a scaling constant - $s, c$ and $\varphi$ are dimensionless. Looking from cartesian coordinates OSC has a disk singularity lying in $xz$ plane with radius $\ell$ and center at $(x, y, z) = (0, 0, 0)$. Everywhere on this this, there is $s = 0$. If we allow $s$ to become negative, we effectively allow for existence of another $\mathbb{R}^3$ space.

[1]Notice that using $s = \sinh\eta, c = \cos\theta$ gives an alternative description as

$$\begin{aligned} x &= \ell\cosh\eta\sin\theta\cos\varphi, \\ y &= \ell\cosh\eta\sin\theta\sin\varphi, \\ z &= \ell\sinh\eta\cos\theta. \end{aligned}$$

# 1. Laplace equation in oblate spheroidal coordinates

## 1.1 Separation of variables

In OSC[1] we can write Laplacian as (*[DLMF] 30.14.6*)

$$\Delta = \frac{1}{\ell^2}\left[\frac{1+s^2}{s^2+c^2}\frac{\partial^2}{\partial s^2} + \frac{1-c^2}{s^2+c^2}\frac{\partial^2}{\partial c^2} + \frac{1}{(1+s^2)(1-c^2)}\frac{\partial^2}{\partial \varphi^2} \right.$$
$$\left. + \frac{2}{s^2+c^2}\left(s\frac{\partial}{\partial s} - c\frac{\partial}{\partial c}\right)\right]. \tag{1.1}$$

We're interested in solution of Laplace equation in OSC

$$\Delta f = 0. \tag{1.2}$$

Let's assume $f = R(s)T(c)U(\varphi)$. Plugging this and (1.3) into (1.2) and after performing some algebraic manipulation we get (omitting arguments for brevity)

$$-\frac{1}{U}\frac{\partial^2 U}{\partial \varphi^2} = \left[\frac{1+s^2}{s^2+c^2}\frac{\partial^2 R}{\partial s^2}T + \frac{1-c^2}{s^2+c^2}\frac{\partial^2 T}{\partial c^2}R + \right.$$
$$\left. + \frac{2}{s^2+c^2}\left(s\frac{\partial R}{\partial s}T - c\frac{\partial T}{\partial c}R\right)\right]\frac{(1+s^2)(1-c^2)}{RT}, \tag{1.3}$$

which after separating using parameter $m^2$ yields

$$(1+s^2)\frac{\partial^2 R}{\partial s^2}T + (1-c^2)\frac{\partial^2 T}{\partial c^2}R + 2\left(s\frac{\partial R}{\partial s}T - c\frac{\partial T}{\partial c}R\right) = \frac{RTm^2(s^2+c^2)}{(1+s^2)(1-c^2)} \tag{1.4}$$

and

$$-\frac{1}{U}\frac{\partial^2 U}{\partial \varphi^2} = m^2 \tag{1.5}$$

Equation (1.5) is solved by

$$U(\varphi) = U_0 \exp(im\varphi). \tag{1.6}$$

Thats one part of $f$. Let's continue with (1.4). By rearanging again[2] and separating using parameter $l(l+1)$ we obtain

$$(1+s^2)\frac{\partial^2 R}{\partial s^2} + 2s\frac{\partial R}{\partial s} + \frac{m^2}{1+s^2}R = l(l+1)R, \tag{1.7}$$

[1]...and naturally in COSC, we just allow for negative s

[2]doing so, one may find useful identity

$$\frac{s^2+c^2}{(1+s^2)(1-c^2)} = \frac{1}{1-c^2} - \frac{1}{1+s^2}$$

$$(1-c^2)\frac{\partial^2 T}{\partial c^2} - 2c\frac{\partial T}{\partial c} - \frac{m^2}{1-c^2}T = -l(l+1)T. \tag{1.8}$$

Both of these formulas resemble Legendre differential equation. Latter of these is in exact same shape, (1.7) requires substitution $s = -i\xi$ to get there (negative sign is unnecessary, but will serve us later to simplify notation. This yields

$$(1-\xi^2)\frac{\partial^2 R}{\partial \xi^2} - 2\xi\frac{\partial R}{\partial \xi} + \left(l(l+1) - \frac{m^2}{1-\xi^2}\right)R = 0, \tag{1.9}$$

$$(1-c^2)\frac{\partial^2 T}{\partial c^2} - 2c\frac{\partial T}{\partial c} + \left(l(l+1) - \frac{m^2}{1-c^2}\right)T = 0. \tag{1.10}$$

Looking into *[DLMF] §14.2(ii)* we can see there is entire family of solutions. We'll need to refine our requirements for $f$ further to choose correct solutions, but first, let's dive into world of solutions to these equations and look around for a bit.

## 1.2 Family of Legendre functions

# 2. Euclidean case

Let's focus on simple euclidean case, in which we know, how the solution looks.

## 2.1 Coulomb's law as is

Electric potential of a point charge is given by well known formula **?**

$$\Phi_c = \frac{1}{4\pi}\frac{A}{|\mathbf{r} - \mathbf{r}'|} \tag{2.1}$$

Plugging (1) into here we get expression for point charge potential directly as

$$\Phi_c = \frac{1}{4\pi\ell}\frac{A}{\sqrt{X + Y\cos(\varphi - \varphi_0)}},$$

where we denote

$$\begin{aligned} X &= -c^2 - 2cc_0ss_0 - c_0^2 + s^2 + s_0^2 + 2, \\ Y &= -2\sqrt{1-c^2}\sqrt{1-c_0^2}\sqrt{s^2+1}\sqrt{s_0^2+1}. \end{aligned} \tag{2.2}$$

Thanks to axial symmetry of coordinates, we can WLOG set $\varphi_0 = 0$ and obtain

$$\Phi_c = \frac{1}{4\pi\ell}\frac{A}{\sqrt{X + Y\cos\varphi}}. \tag{2.3}$$

## 2.2 Euclidean sum shape

Let's fix position of our charge at some constant $s = s_0$, WLOG $s_0 > 0$. This ellipsis divides our space into two spaces without any charge density. Electromagnetic potential there satisfies

$$\Delta^{\pm}\Phi = 0 \tag{2.4}$$

where indexes plus and minus denote our position relative to point charge[1].

[1] If index is + or -, then $s > s_0$ or $s < s_0$ respectively.

# Conclusion

# Bibliography

DLMF. *NIST Digital Library of Mathematical Functions*. `https://dlmf.nist.gov/`, Release 1.1.11 of 2023-09-15. URL `https://dlmf.nist.gov/`. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

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# A. Attachments

## A.1 First Attachment