FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

BACHELOR THESIS

Jan Červeňan

Electric field of a charge near the wormhole

Institute of theoretical physics

Supervisor of the bachelor thesis: prof. RNDr. Pavel Krtouš, Ph.D.
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Dedication.

Title: Electric field of a charge near the wormhole

Author: Jan Červeňan

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Contents

Introduction

Problem we're facing is finding field of a point charge, next to a wormhole. Well known approach to problems containing electric fields is direct solution of Poisson equation **?**

$$\Delta\,\Phi = 4\pi\rho,$$

where we consider ρ being charge distribution of point charge at certain position. Solving for Φ if possible and using correct border conditions we'd probably get an answer. This would not be that interesting, field of a point charge is well known, if it wasn't for the wormhole.

The wormhole essentially doubles the space we care about. Take two $\mathbb{R}^3$ spaces, cut out a disk from both and sew them together through the hole that's left. Now these two spaces together create the space we care about, and into which we'll place our point charge. How do we describe such space? It's simple, enter oblate spheroidal coordinates (OSC). Simply describing $\mathbb{R}^3$ via OSC gives us the wormhole coordinates perfectly. We'll use OSC as defined in *[DLMF] §30.14(i)*:

$$\begin{aligned} x &= \ell\sqrt{1+s^2}\sqrt{1-c^2}\cos\varphi, \\ y &= \ell\sqrt{1+s^2}\sqrt{1-c^2}\sin\varphi, \\ z &= \ell s c, \end{aligned} \tag{1}$$

where $s \in \mathbb{R}^+$, $c \in [-1, 1]$ and $\varphi \in [0, 2\pi]$[1]. Parameter ℓ is just a scaling constant - s, c and φ are dimensionless. Looking from cartesian coordinates OSC has a disk singularity lying in xz plane with radius ℓ and center at $(x, y, z) = (0, 0, 0)$. Everywhere on this this, there is $s = 0$. If we allow s to become negative, we effectively allow for existence of another $\mathbb{R}^3$ space.

[1] Notice that using $s = \sinh\eta, c = \cos\theta$ gives an alternative description as

$$\begin{aligned} x &= \ell\cosh\eta\sin\theta\cos\varphi, \\ y &= \ell\cosh\eta\sin\theta\sin\varphi, \\ z &= \ell\sinh\eta\cos\theta. \end{aligned}$$

1. Laplace equation in oblate spheroidal coordinates

1.1 Separation of variables

In OSC[1] we can write Laplacian as (*[DLMF] 30.14.6*)

$$\Delta = \frac{1}{\ell^2}\left[\frac{1+s^2}{s^2+c^2}\frac{\partial^2}{\partial s^2} + \frac{1-c^2}{s^2+c^2}\frac{\partial^2}{\partial c^2} + \frac{1}{(1+s^2)(1-c^2)}\frac{\partial^2}{\partial \varphi^2} \right.$$
$$\left. + \frac{2}{s^2+c^2}\left(s\frac{\partial}{\partial s} - c\frac{\partial}{\partial c}\right)\right]. \quad (1.1)$$

We're interested in solution of Laplace equation in OSC

$$\Delta f = 0. \quad (1.2)$$

Let's assume $f = R(s)T(c)U(\varphi)$. Plugging this and (1.3) into (1.2) and after performing some algebraic manipulation we get (omitting arguments for brevity)

$$-\frac{1}{U}\frac{\partial^2 U}{\partial \varphi^2} = \left[\frac{1+s^2}{s^2+c^2}\frac{\partial^2 R}{\partial s^2}T + \frac{1-c^2}{s^2+c^2}\frac{\partial^2 T}{\partial c^2}R + \right.$$
$$\left. + \frac{2}{s^2+c^2}\left(s\frac{\partial R}{\partial s}T - c\frac{\partial T}{\partial c}R\right)\right]\frac{(1+s^2)(1-c^2)}{RT}, \quad (1.3)$$

which after separating using parameter m^2 yields

$$(1+s^2)\frac{\partial^2 R}{\partial s^2}T + (1-c^2)\frac{\partial^2 T}{\partial c^2}R + 2\left(s\frac{\partial R}{\partial s}T - c\frac{\partial T}{\partial c}R\right) = \frac{RTm^2(s^2+c^2)}{(1+s^2)(1-c^2)} \quad (1.4)$$

and

$$-\frac{1}{U}\frac{\partial^2 U}{\partial \varphi^2} = m^2 \quad (1.5)$$

Equation (1.5) is solved by

$$U(\varphi) = U_0 \exp(im\varphi). \quad (1.6)$$

Thats one part of f. Let's continue with (1.4). By rearanging again[2] and separating using parameter $l(l+1)$ we obtain

$$(1+s^2)\frac{\partial^2 R}{\partial s^2} + 2s\frac{\partial R}{\partial s} + \frac{m^2}{1+s^2}R = l(l+1)R, \quad (1.7)$$

[1]...and naturally in COSC, we just allow for negative s

[2]doing so, one may find useful identity

$$\frac{s^2+c^2}{(1+s^2)(1-c^2)} = \frac{1}{1-c^2} - \frac{1}{1+s^2}$$

$$(1-c^2)\frac{\partial^2 T}{\partial c^2} - 2c\frac{\partial T}{\partial c} - \frac{m^2}{1-c^2}T = -l(l+1)T. \tag{1.8}$$

Both of these formulas resemble *associated Legendre differential equation.* Latter of these is in exact same shape, (1.7) requires substitution $is = \xi$ to get there. This yields

$$(1-\xi^2)\frac{\partial^2 R}{\partial \xi^2} - 2\xi\frac{\partial R}{\partial \xi} + \left(l(l+1) - \frac{m^2}{1-\xi^2}\right)R = 0, \tag{1.9}$$

$$(1-c^2)\frac{\partial^2 T}{\partial c^2} - 2c\frac{\partial T}{\partial c} + \left(l(l+1) - \frac{m^2}{1-c^2}\right)T = 0. \tag{1.10}$$

Looking into *[DLMF] §14.2(ii)* we can see there is entire family of functions that solve these equations. Brief discussion of their definition and properties is in Appendix A.

We'll need to refine our requirements for f further to choose correct ones. What do we want from f, other than it being harmonic? First of all, let's look at the domains at which we operate. Our coordinate system has domains $s \in \mathbb{R}^+, c \in [-1,1], \varphi \in [0, 2\pi]$. Coordinate φ has already been solved and for s and c we require $R(s)$ and $T(c)$ to be defined on respective domains. For T it directly means, we want it to be linear combination of Ferrers function of either first or second kind, but what about R?

1.1.1 Legendre family and its extension

Legendre associated functions satisfy equation

$$\left[(1-z^2)\frac{\partial^2}{\partial z^2} - 2z\frac{\partial}{\partial z} + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right)\right]\mathbb{L}^\mu_\nu = 0. \tag{1.11}$$

Standardized names are "order" for upper index, in this case μ, and "degree" for lower index, in this case ν. We are interested in case $\nu = l \in \mathbb{Z}, \mu = m \in \mathbb{Z}$. Equation is generally solved by Ferrers functions and Associated Legendre functions *[DLMF] §14.3*. A more detailed discussion of these functions is in Appendix A for those, who want to better understand how are they defined.

As we said, exploring properties of $\nu = l \in \mathbb{Z}, \mu = m \in \mathbb{Z}$ is sufficient. For start, let's put $m = 0$. That yields equation

$$\left[(1-z^2)\frac{\partial^2}{\partial z^2} - 2z\frac{\partial}{\partial z} + l(l+1)\right]\mathbb{L}^0_l = 0, \tag{1.12}$$

which is solved by Legendre polynomials

$$\mathsf{P}^0_l(z) = \mathsf{P}_l(z) = \frac{(-1)^l}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}z^l}(1-z^2)^l$$

and Ferrer's functions of second kind

$$\mathsf{Q}_l(z) = \mathsf{P}_l(z)\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) - W_{l-1}(z), \tag{1.13}$$

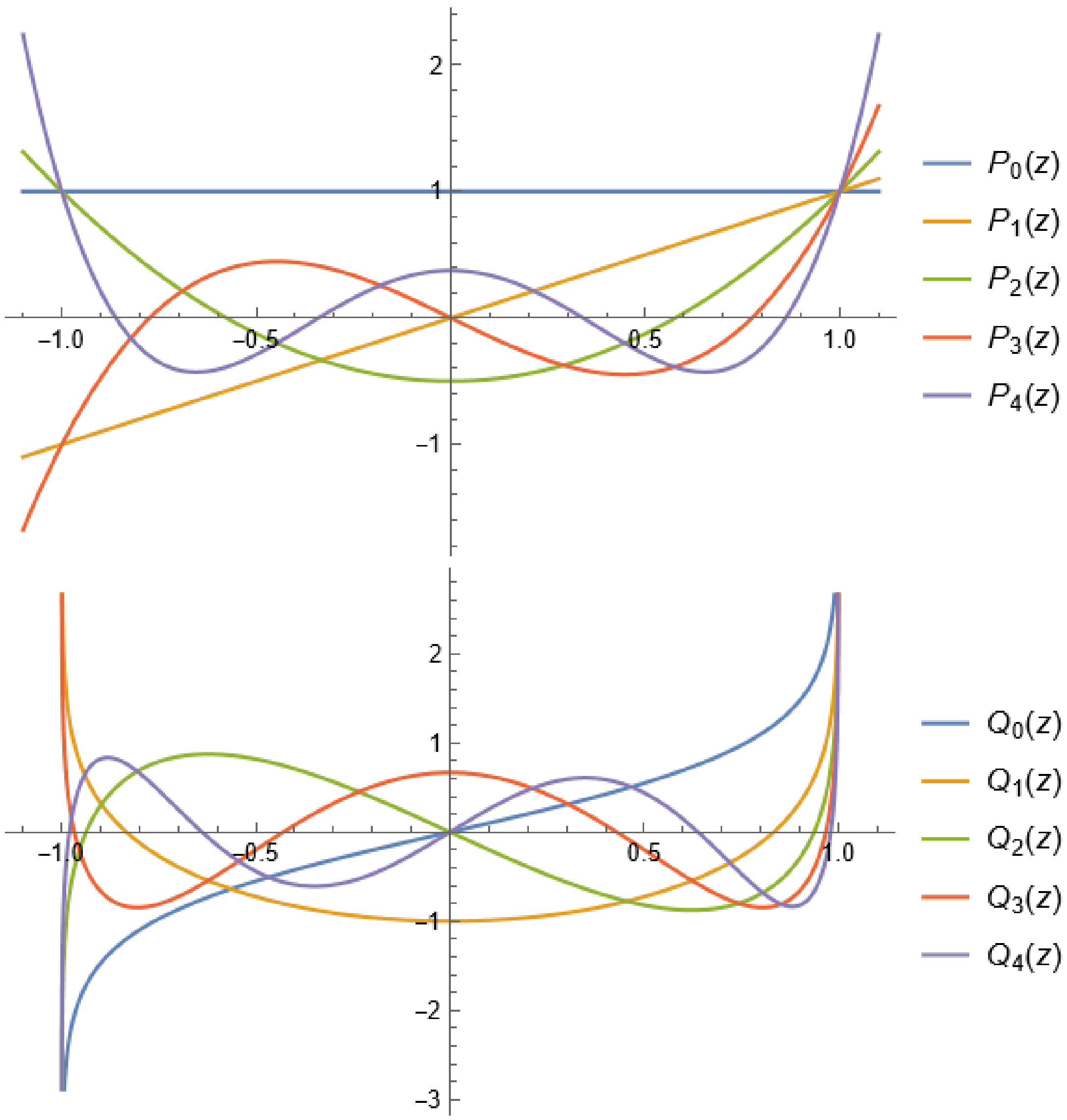

Figure 1.1: Graphics of P and Q functions for first five degrees.

where

$$W_{l-1}(z) = \sum_{k=1}^{l} \frac{1}{k} \mathsf{P}_{k-1}(z) \mathsf{P}_{l-k}(z).$$

Notice we're omitting order when it's zero. Visuals of these functions are at figure 1.1. To read more about them and their properties go to *[DLMF] §14.7*.

We'll be defining altered Ferrer's functions of second kind

$$^{\sigma}\mathsf{Q}_l^0(z) = {}^{\sigma}\mathsf{Q}_l(z).$$

Definition of this altered Q function requires us to introduce something one might call generalized argument of hyperbolic tangent:

$$
{}^{\infty}\mathrm{ath}(z) = \begin{cases} \operatorname{arctanh}(z) - \dfrac{i\pi}{2}\operatorname{sign}\left(\operatorname{Im}(z)\right) & \text{if } |z| < 1 \\ \operatorname{arccoth}(z) & \text{if } |z| > 1 \end{cases}, \tag{1.14}
$$

$$
{}^{+}\mathrm{ath}(z) = {}^{\infty}\mathrm{ath}(z) - \begin{cases} \dfrac{i\pi}{2} & \text{if } \operatorname{Im}(z) < 1 \\ 0 & \text{if } \operatorname{Im}(z) \geq 1 \end{cases}, \tag{1.15}
$$

$$
{}^{-}\mathrm{ath}(z) = {}^{\infty}\mathrm{ath}(z) + \begin{cases} \dfrac{i\pi}{2} & \text{if } \operatorname{Im}(z) > 1 \\ 0 & \text{if } \operatorname{Im}(z) \leq 1 \end{cases}, \tag{1.16}
$$

$$
{}^{0}\mathrm{ath}(z) = {}^{\infty}\mathrm{ath}(z) + \frac{i\pi}{2}\operatorname{sign}\left(\operatorname{Im}(z)\right). \tag{1.17}
$$

These functions are all the same as ${}^{\infty}\mathrm{ath}$ on the real axis, as you can deduce from all the subcases being dependent on imaginary part of z, it's graph can be seen on figure 1.2. Pre-indices will make sense the moment we look at imaginary axis, which is depicted at fig. 1.3. The σ denotes, where in imaginary axis does the ${}^{\sigma}\mathrm{ath}$ values zero. That means

$$
\begin{aligned}
{}^{\infty}\mathrm{ath}(iz) &\xrightarrow{z\to\pm\infty} 0 \\
{}^{+}\mathrm{ath}(iz) &\xrightarrow{z\to+\infty} 0 \\
{}^{-}\mathrm{ath}(iz) &\xrightarrow{z\to-\infty} 0 \\
{}^{0}\mathrm{ath}(0) &= 0
\end{aligned}
$$

Similarly to (1.13) we define

$$
{}^{\sigma}\mathsf{Q}_l(z) = \mathsf{P}_l(z)\,{}^{\sigma}\mathrm{ath}(z) - W_{l-1}(z) \tag{1.18}
$$

Remember substitution $is = \xi$ we used earlier, when discussing separation of variables? Now it comes in handy... You may have noticed the P and Q functions are not defined on the real axis for any point lying outside the interval (-1,1). On the other hand, the ξ parameter lies purely on the imaginary axis! Thus defining

$$
{}^{\sigma}\mathrm{at}(s) = -i\mathrm{ath}(is) \tag{1.19}
$$

$$
\mathsf{p}_l(s) = (-i)^l\mathsf{P}_l(is) \tag{1.20}
$$

$$
{}^{\sigma}\mathsf{q}_l(s) = (-i)^{l+1\sigma}\mathsf{Q}_l(is) = (-i)^{l+1}\mathsf{P}_l(is)\,{}^{\sigma}\mathrm{ath}(is) - (-i)^{l+1}W_{l-1}(is) \tag{1.21}
$$

gives us real valued functions ${}^{\sigma}\mathsf{q}_l$ defined for $s \in \mathbb{R}$. Now using identities

$$
\mathsf{P}_l^m(z) = (-1)^m(1-z^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\mathsf{P}_l(z) \tag{1.22}
$$

$$
\mathsf{Q}_l^m(z) = (-1)^m(1-z^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\mathsf{Q}_l(z) \tag{1.23}
$$

$$
\tag{1.24}
$$

from *[DLMF] §14.7(ii)* we easily obtain

$$
\mathsf{p}_l^m(s) = (-i)^{l+m}\mathsf{P}_l^m(is) \tag{1.25}
$$

$$
{}^{\sigma}\mathsf{q}_l^m(s) = (-i)^{l+m+1\sigma}\mathsf{Q}_l^m(is) \tag{1.26}
$$

That is exactly what we wanted. The R function from (1.7) and (1.9) is then solved by linear combinations of ${}^{\sigma}\mathsf{q}_l^m$ and p_l^m functions.

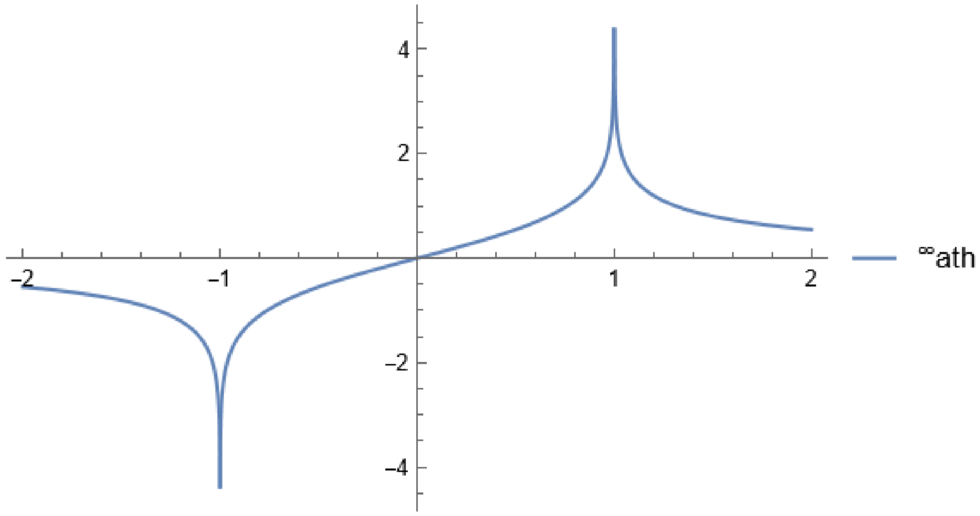

Figure 1.2: Graph of $^{\infty}$ath on real axis.

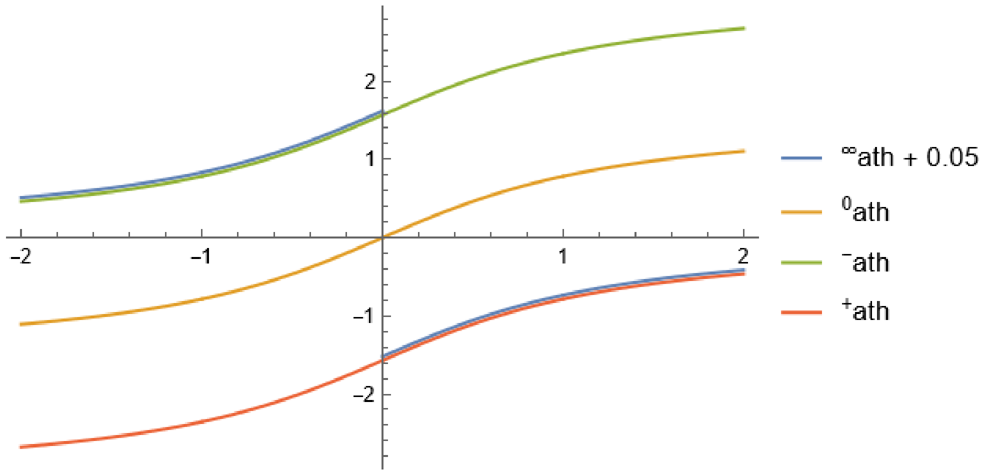

Figure 1.3: Graph of $^{\sigma}$ath on imaginary axis. Specifically $^{\infty}$ath was shifted by 0.05 upwards to be visible.

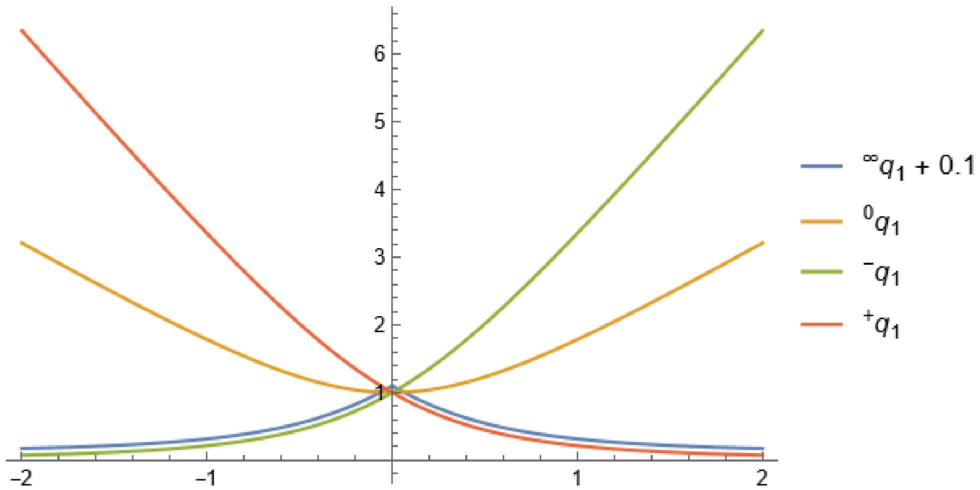

Figure 1.4: Graph of $^{\sigma}\mathsf{q}_1$ on real axis. $^{\infty}\mathsf{q}$ was shifted 0.1 upwards to be visible. We can see that in cases $^{\sigma} = \infty, +, -$ the $^{\sigma}\mathsf{q}$ functions has also asymptotic behaviour as $^{\sigma}$ath

2. Euclidean case

Let's focus on simple euclidean case, in which we know, how the solution looks.

2.1 Coulomb's law as is

Electric potential of a point charge is given by well known formula **?**

$$\Phi_c = \frac{1}{4\pi}\frac{A}{|\mathbf{r}-\mathbf{r}'|} \tag{2.1}$$

Plugging (1) into here we get expression for point charge potential directly as

$$\Phi_c = \frac{1}{4\pi\ell}\frac{A}{\sqrt{X + Y\cos(\varphi-\varphi_0)}},$$

where we denote

$$\begin{aligned} X &= -c^2 - 2cc_0ss_0 - c_0^2 + s^2 + s_0^2 + 2, \\ Y &= -2\sqrt{1-c^2}\sqrt{1-c_0^2}\sqrt{s^2+1}\sqrt{s_0^2+1}. \end{aligned} \tag{2.2}$$

Thanks to axial symmetry of coordinates, we can WLOG set $\varphi_0 = 0$ and obtain

$$\Phi_c = \frac{1}{4\pi\ell}\frac{A}{\sqrt{X + Y\cos\varphi}}. \tag{2.3}$$

2.2 Euclidean sum shape

Let's fix position of our charge at some constant $s = s_0$, WLOG $s_0 > 0$. This ellipsis divides our space into two spaces without any charge density. Electromagnetic potential there satisfies

$$\Delta^{\pm}\Phi = 0 \tag{2.4}$$

where indexes plus and minus denote our position relative to point charge[1].

[1] If index is + or -, then $s > s_0$ or $s < s_0$ respectively.

Conclusion

A. Family of Legendre functions

Legendre associated functions satisfy equation

$$\left[(1-z^2)\frac{\partial^2}{\partial z^2} - 2z\frac{\partial}{\partial z} + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right)\right]\mathbb{L}^{\mu}_{\nu} = 0. \tag{A.1}$$

We're interested just in certain solutions, however It's instructive to provide at least some general overview of the topic.

A.1 Hypergeometric function

In following definitions it is convenient to have basic knowledge of what so called *Hypergeometric function* is and how it works. For $z \in \mathbb{C}, a, b, c \in \mathbb{C}\backslash\mathbb{Z}_0^-$ we write

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!} z^k,$$

where $(n)_i$ is *Pochhammer symbol* defined as

$$(n)_k = n(n+1)\cdots(n+k-1) = \frac{\Gamma(n+k)}{\Gamma(n)} \xrightarrow{n\in\mathbb{N}} \frac{(n+k-1)!}{(n-1)!}.$$

This can be generalized in following manner[1]

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^i}{k!}.$$

Obviously, our case is $F = {}_2F_1$ but it is so common, that we omit indexes. We say that ${}_pF_q$ has p upper arguments and q lower arguments. Clearly:

1. $(1)_k = \Gamma(k+1) = k!$
2. $(n)_0 = 1$
3. ${}_pF_q$ is symmetric in all upper arguments and all lower arguments.
4. If any $a_i = b_j$, they can be cancelled out. For example we have

$${}_3F_2(a_1,c,a_3;c,b_2;z) = {}_2F_1(a_1,a_3;b_2;z)$$

In following, we'll use "normalized" version known as *Olver's function.* We'll denote it $\mathbf{F}$ and it's defined by

$$\mathbf{F}(a,b;c;z) = \frac{F(a,b;c;z)}{\Gamma(c)}\ .$$

More about Hypergeometric functions in *[DLMF] §15* or Gradshteyn and Ryzhik [2007] chapter 9.1.

[1] Other, often seen, possible syntax convention

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) = F\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};z\right)$$

will not be used here.

A.2 Definitions

Now for the solutions. It is convenient to divide them into two main categories based on their definition domain - Ferrer's functins and Associated Legendre functions, both containing so called "first" and "second" kind of functions. Following are going to be real valued solutions, where we take $\mu, \nu \in \mathbb{R}$.

A.2.1 Ferrer's functions

On definition domain $x \in (-1,1)$ we consider solutions of first kind in shape

$$\mathsf{P}_\nu^\mu(x) = \left(\frac{1+x}{1-x}\right)^{\mu/2} \mathbf{F}\left(\nu+1, -\nu; 1-\mu; \tfrac{1}{2} - \tfrac{1}{2}x\right). \tag{A.2}$$

Second kind is then

$$\begin{aligned}\mathsf{Q}_\nu^\mu(x) = \frac{\pi}{2\sin(\mu\pi)} \Bigg(&\cos(\mu\pi)\left(\frac{1+x}{1-x}\right)^{\mu/2} \mathbf{F}\left(\nu+1, -\nu; 1-\mu; \tfrac{1}{2} - \tfrac{1}{2}x\right) - \\ &- \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\left(\frac{1-x}{1+x}\right)^{\mu/2} \mathbf{F}\left(\nu+1, -\nu; 1+\mu; \tfrac{1}{2} - \tfrac{1}{2}x\right)\Bigg).\end{aligned} \tag{A.3}$$

Q_ν^μ is undefined when $\nu + \mu \in \mathbb{Z}^-$. P_ν^μ exists for all μ, ν.

A.2.2 Associated Legendre functions

On definition domain $x \in (1, \infty)$ we consider solutions of first kind in shape

$$P_\nu^\mu(x) = \left(\frac{x+1}{x-1}\right)^{\mu/2} \mathbf{F}\left(\nu+1, -\nu; 1-\mu; \tfrac{1}{2} - \tfrac{1}{2}x\right),$$

and of second kind

$$Q_\nu^\mu(x) = \\ e^{\mu\pi i}\frac{\pi^{1/2}\Gamma(\nu+\mu+1)(x^2-1)^{\mu/2}}{2^{\nu+1}x^{\nu+\mu+1}} \mathbf{F}\left(\tfrac{1}{2}\nu + \tfrac{1}{2}\mu + 1, \tfrac{1}{2}\nu + \tfrac{1}{2}\mu + \tfrac{1}{2}; \nu + \tfrac{3}{2}; \frac{1}{x^2}\right),$$

which also isn't defined for $\nu + \mu \in \mathbb{Z}^-$ and P_ν^μ exists for all μ, ν.

A.3 Extension to complex argument

Looking at the definitions it is obvious that functions van be defined on the complex plane in exactly the same manner. We just need to introduce cuts along real axes.

By introducing cut along interval $(-\infty, -1] \cup [1, \infty)$ we get $\mathsf{P}^\mu?\nu$ and Q_ν^μ defined easily as direct exchange of $x \in \mathbb{R} \to z \in \mathbb{C}\backslash(-\infty, -1] \cup [1, \infty)$. For Associated Legendre functions we can get the same, while introducing cut along $(-\infty, 1]$, thus getting definition domain $\mathbb{C}\backslash(-\infty, 1]$.

Visualisation of definition domains on

A.4 Relations summary

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A. Attachments

A.1 First Attachment