## APPENDIX F
## COMPARISON OF ROBUST INPUT BOUND

ARMOUR's robust input (given in (52)) is inspired by [35], [51]. We claim that the proposed controller improves on these published controllers because it achieves the same uniform bound with a smaller robust input bound. From [51, Thm. 1, (10)], the robust input given in previous work is

$$v(q_A(t),\Delta_0,[\Delta]) = -\Big(\kappa(t)\,\|w_M(q_A(t),\Delta_0,[\Delta])\| + \phi(t)\Big)r(t), \tag{115}$$

where $\kappa$ and $\phi$ are positive increasing functions with $\kappa_P \geq 1$ and $\phi_P \geq 1$ as their respective minimums, and $w_M(q_A(t),\Delta_0,[\Delta])$ as in (50). With this choice of robust input, [51, Thm. 1] proves that the trajectories of $r$ are ultimately uniformly bounded by $\|r(t)\| \leq \frac{1}{\kappa_P}\sqrt{\frac{\sigma_M}{\sigma_m}} \quad \forall t \geq t_1$, where $t_1$ is some finite time. In the context of Rem. 12, this bound holds for all time if (115) were used in the current framework. This uniform bound can be made identical to (54) by choosing $\kappa_P = \sqrt{\frac{\sigma_M}{2V_M}}$.

To bound the magnitude of the robust input similarly to App. E, note

$$|v(q_A(t),\Delta_0,[\Delta])|_j = \Big(\kappa(t)\,\|w_M(q_A(t),\Delta_0,[\Delta])\| + \phi(t)\Big)\left|r_j(t)\right|. \tag{116}$$

The smallest possible bound on $\left|r_j(t)\right|$ is given by $\|r(t)\| \leq \frac{1}{\kappa_P}\sqrt{\frac{\sigma_M}{\sigma_m}}$, which yields $\left|r_j(t)\right| \leq \|r(t)\| \leq \frac{1}{\kappa_P}\sqrt{\frac{\sigma_M}{\sigma_m}}$. Defining $w_M(*) = w_M(\mathbf{q_A}(\mathbf{T_i};\mathbf{K}),\Delta_0,[\Delta])$ as in (84), we have

$$|v(q_A(t),\Delta_0,[\Delta])|_j \leq \Big(\kappa(t)\,\|w_M(*)\| + \phi(t)\Big)\frac{1}{\kappa_P}\sqrt{\frac{\sigma_M}{\sigma_m}} \tag{117}$$

for all $t \in \mathbf{T_i}$. Next, we show that a lower bound on the right hand side of this equation (i.e., a lower bound on the robust input bound) is larger than the robust input bound (85).

Using the properties of $\kappa$ and $\phi$,

$$\Big(\kappa_P\,\|w_M(*)\|\Big)\frac{1}{\kappa_P}\sqrt{\frac{\sigma_M}{\sigma_m}} \leq \Big(\kappa(t)\,\|w_M(*)\| + \phi(t)\Big)\frac{1}{\kappa_P}\sqrt{\frac{\sigma_M}{\sigma_m}}. \tag{118}$$

Cancelling terms on the left hand side gives

$$\|w_M(*)\|\sqrt{\frac{\sigma_M}{\sigma_m}} \leq \Big(\kappa(t)\,\|w_M(*)\| + \phi(t)\Big)\frac{1}{\kappa_P}\sqrt{\frac{\sigma_M}{\sigma_m}}. \tag{119}$$

The robust input bound for (115) is larger than $\|w_M(*)\|\sqrt{\frac{\sigma_M}{\sigma_m}}$.

We compare this value to ARMOUR's robust input bound in (85). Under what conditions is (85) smaller, i.e.

$$\frac{\alpha_c\varepsilon(\sigma_M-\sigma_m) + \|w_M(*)\| + w_M(*)_j}{2} \overset{?}{\leq} \|w_M(*)\|\sqrt{\frac{\sigma_M}{\sigma_m}}. \tag{120}$$

This relation depends on the user-specified constants $\alpha_c$ and $\varepsilon$ and $\sigma_M$ and $\sigma_m$, but generally the following are true. First, $\sigma_M \geq \sigma_m$ by definition, and usually $\sigma_M$ is much larger than $\sigma_m$. Second, $\varepsilon$ is the user-specified uniform bound and a small constant is desired to minimize tracking error. Third, the $j^{\text{th}}$ component of the worst case disturbance $w_M(*)_j$ is smaller than $\|w_M(*)\|$. Therefore, ignoring the term involving $\varepsilon$ (which is small), ARMOUR's robust input bound (85) is smaller than (117) by at least a factor of $\sqrt{\frac{\sigma_M}{\sigma_m}}$.

To give an idea of the differences, consider the Kinova Gen3 robot as reported in Sec. IX-A7.: $\alpha_c = 1$, $\varepsilon = 0.049089$, $\sigma_M = 18.2726$, and $\sigma_m = 8.2993$. Plugging in yields

$$\frac{\alpha_c\varepsilon(\sigma_M-\sigma_m) + \|w_M(*)\| + w_M(*)_j}{2} \leq 0.4895 + \|w_M(*)\|, \tag{121}$$

and

$$\|w_M(*)\|\sqrt{\frac{\sigma_M}{\sigma_m}} = 1.48380\,\|w_M(*)\|. \tag{122}$$