Example 29.3

Evaluate $\ln \mathcal{Z}$ for a gas of (i) fermions and (ii) bosons.

Solution:

(i) For fermions, each state can either be empty or singly occupied, so that $\{n_i\} = \{0, 1\}$, and hence eqn 29.17 becomes

$$\mathcal{Z} = \prod_i 1 + e^{\beta(\mu - E_i)}. \tag{29.18}$$

Hence

$$\ln \mathcal{Z} = \sum_i \ln(1 + e^{\beta(\mu - E_i)}). \tag{29.19}$$

(ii) For bosons, each state can contain any integer number of particles, so that $\{n_i\} = \{0, 1, 2, 3, \ldots\}$, and hence eqn 29.17 becomes

$$\mathcal{Z} = \prod_i 1 + e^{\beta(\mu - E_i)} + e^{2\beta(\mu - E_i)} + \cdots \tag{29.20}$$

and therefore, by summing this geometric series, we have that

$$\mathcal{Z} = \prod_i \frac{1}{1 - e^{\beta(\mu - E_i)}}, \tag{29.21}$$

and hence

$$\ln \mathcal{Z} = -\sum_i \ln(1 - e^{\beta(\mu - E_i)}). \tag{29.22}$$

Summarizing the results of the previous example, we can write

$$\boxed{\ln \mathcal{Z} = \pm \sum_i \ln(1 \pm e^{\beta(\mu - E_i)}),} \tag{29.23}$$

where the $\pm$ sign means $+$ for fermions and $-$ for bosons.

The number of particles in each energy level is given by

$$\langle n_i \rangle = -\frac{1}{\beta}\left(\frac{\partial \ln \mathcal{Z}}{\partial E_i}\right) = \frac{e^{\beta(\mu - E_i)}}{1 \pm e^{\beta(\mu - E_i)}}, \tag{29.24}$$

and hence dividing top and bottom by $e^{\beta(\mu - E_i)}$ gives

$$\boxed{\langle n_i \rangle = \frac{1}{e^{\beta(E_i - \mu)} \pm 1},} \tag{29.25}$$

where, again, the $\pm$ sign means $+$ for fermions and $-$ for bosons.

If μ and T are fixed for a particular system, eqn 29.25 shows that the mean occupation of the i^{th} state, $\langle n_i \rangle$, is a function only of the energy E_i. It is therefore convenient to consider the **distribution function** $f(E)$ for fermions and bosons, which is defined to be the mean occupation of

$f(E) \to f(E)$

↓ BANAKETA-FUNTZIOAREN AZTERKETA:

(μ, T) FINKOAK

egia esateko; T FINKO

(gainerakoak aldatu egiten, μ FINKO)

$P, V, \ldots$