30.2 The Fermi gas

What we have done so far is to consider bosons and fermions on an equal footing. Let us now restrict our attention to a gas of fermions (known as a **Fermi gas**) and to get a feel for what is going on, let us also consider $T = 0$. Fermions will occupy the lowest-energy states, but we can only put one fermion in each state, and thus only $2S+1$ in each energy level. The fermions will fill up the energy levels until they get to an energy E_{F}, known as the **Fermi energy**, which is the energy of the highest occupied state at a temperature of absolute zero.[4] Thus we define

$$E_{\mathrm{F}} = \mu(T=0). \tag{30.21}$$

This makes sense because $\mu(T=0) = (\partial E/\partial N)_0$ which gives $\mu(T=0) = E(N) - E(N-1) = E_{\mathrm{F}}$. At absolute zero, we have that $\beta \to \infty$, and hence the occupation n_k is given by

$$n_k = \frac{1}{e^{\beta(E_k-\mu)}+1} = \theta(\mu - E_{\mathrm{F}}), \tag{30.22}$$

where $\theta(x)$ is a Heaviside step function.[5] At absolute zero, therefore, the number of states is given by

$$N = \int_0^{k_{\mathrm{F}}} g(k)\,\mathrm{d}^3k, \tag{30.23}$$

where k_{F} is the **Fermi wave vector**, defined by

$$E_{\mathrm{F}} = \frac{\hbar^2 k_{\mathrm{F}}^2}{2m}. \tag{30.24}$$

Hence the number of fermions N is given by

$$N = \frac{(2S+1)V}{2\pi^2}\frac{k_{\mathrm{F}}^3}{3}, \tag{30.25}$$

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so that writing $n = N/V$, we have

$$k_{\mathrm{F}} = \left[\frac{6\pi^2 n}{2S+1}\right]^{1/3}, \tag{30.26}$$

and hence

$$E_{\mathrm{F}} = \frac{\hbar^2}{2m}\left[\frac{6\pi^2 n}{2S+1}\right]^{2/3}. \tag{30.27}$$

Example 30.3

Evaluate k_{F} and E_{F} for spin-$\frac{1}{2}$ particles.

Solution:

When $S = \frac{1}{2}$, $2S+1 = 2$ and hence eqns 30.26 and 30.27 become

$$k_{\mathrm{F}} = \left[3\pi^2 n\right]^{1/3}, \tag{30.28}$$

and

$$E_{\mathrm{F}} = \frac{\hbar^2}{2m}\left[3\pi^2 n\right]^{2/3}. \tag{30.29}$$

[4] The highest filled energy level at $T = 0$ is known as the **Fermi level**, though this can be a misleading term as, for example in semiconductors, there may not be any states at the chemical potential (which lies somewhere in the energy gap).

[5] The Heaviside step function $\theta(x)$ is defined by

$$\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

It is plotted in Fig. 30.1.

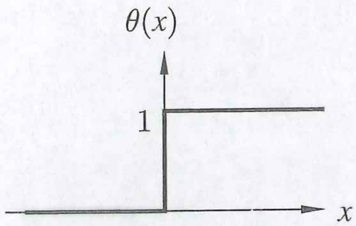

Fig. 30.1 The Heaviside step function.