Writing $\psi(E)$ as a power series in x as

$$\psi(E) = \sum_{s=0}^{\infty} \frac{x^s}{s!} \left(\frac{\mathrm{d}^s\psi}{\mathrm{d}x^s}\right)_{x=0}, \tag{30.35}$$

we can express I as a power series of integrals as follows:

$$I = \sum_{s=0}^{\infty} \frac{1}{s!} \left(\frac{\mathrm{d}^s\psi}{\mathrm{d}x^s}\right)_{x=0} \int_{-E_\mathrm{F}/k_\mathrm{B}T}^{\infty} \frac{x^s \mathrm{e}^x\,\mathrm{d}x}{(\mathrm{e}^x+1)^2}. \tag{30.36}$$

The integral part of this can be simplified by replacing[6] the lower limit by $-\infty$. It vanishes for odd s, but for even s

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^s \mathrm{e}^x\,\mathrm{d}x}{(\mathrm{e}^x+1)^2} &= 2\int_0^{\infty} \frac{x^s \mathrm{e}^x\,\mathrm{d}x}{(\mathrm{e}^x+1)^2} \\
&= 2\int_0^{\infty} \mathrm{d}x \sum_{n=0}^{\infty} \mathrm{e}^x x^s \times \left[(n+1)(-1)^{n+1}\mathrm{e}^{-nx}\right] \\
&= 2\sum_{n=1}^{\infty}(-1)^{n+1} n \int_0^{\infty} x^s \mathrm{e}^{-nx}\,\mathrm{d}x \\
&= 2(s!)\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \\
&= 2(s!)(1-2^{1-s})\zeta(s),
\end{aligned} \tag{30.37}$$

where $\zeta(s)$ is the Riemann zeta function.

Thus the integral is

$$\begin{aligned}
I &= \sum_{s=0,s\ \text{even}}^{\infty} 2\left(\frac{\mathrm{d}^s\psi}{\mathrm{d}x^s}\right)_{x=0} (1-2^{1-s})\zeta(s) \\
&= \psi + \frac{\pi^2}{6}\left(\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}\right)_{x=0} + \frac{7\pi^4}{360}\left(\frac{\mathrm{d}^4\psi}{\mathrm{d}x^4}\right)_{x=0} + \cdots \\
&= \int_{-\infty}^{\mu} \phi(E)\,\mathrm{d}E + \frac{\pi^2}{6}(k_\mathrm{B}T)^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}E}\right)_{E=\mu} \\
&\quad + \frac{7\pi^4}{360}(k_\mathrm{B}T)^4\left(\frac{\mathrm{d}^3\phi}{\mathrm{d}E^3}\right)_{E=\mu} + \cdots
\end{aligned} \tag{30.38}$$

This expression is known as the **Sommerfeld formula**.

Having derived the Sommerfeld formula, we can now evaluate N and U quite easily. Let us choose $S = \frac{1}{2}$, just to make the equations a little less cumbersome. Then

$$\begin{aligned}
N &= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^{\infty} E^{1/2} f(E)\,\mathrm{d}E \\
&= \frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \mu^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{k_\mathrm{B}T}{\mu}\right)^2 + \ldots\right],
\end{aligned} \tag{30.39}$$

$\int_0^{\infty} \phi(E)\cdot f(E)\,\mathrm{d}E$
$\phi = E^{1/2}$

[6]This approximation is valid when $k_\mathrm{B}T \gg E_\mathrm{F}$.