One problem with all these types of formula, such as eqns 30.7, 30.12 and 30.13, is that to simplify them any further, you have to do a difficult integral. Fortunately, we can show that these integrals are related to the **polylogarithm** function $\mathrm{Li}_n(x)$ (see Appendix C.5), so that

$$\int_0^\infty \frac{E^{n-1}\,\mathrm{d}E}{z^{-1}\mathrm{e}^{\beta E} \pm 1} = (k_\mathrm{B}T)^n \Gamma(n)[\mp \mathrm{Li}_n(\mp z)], \tag{30.14}$$

where $\Gamma(n)$ is a gamma function. This result is proved in the appendix (eqn C.36). The crucial thing to realize is that $\mathrm{Li}_n(z)$ is just a numerical function of z, i.e. of the temperature and the chemical potential. This integral then allows us to establish, after a small amount of algebra, that the number N of particles is given by

$$N = \frac{(2S+1)V}{\lambda_\mathrm{th}^3}[\mp \mathrm{Li}_{3/2}(\mp z)], \tag{30.15}$$

and the internal energy U is given by

$$\begin{aligned} U &= \frac{3}{2}k_\mathrm{B}T\frac{(2S+1)V}{\lambda_\mathrm{th}^3}[\mp \mathrm{Li}_{5/2}(\mp z)] \\ &= \frac{3}{2}Nk_\mathrm{B}T\frac{\mathrm{Li}_{5/2}(\mp z)}{\mathrm{Li}_{3/2}(\mp z)}. \end{aligned} \tag{30.16}$$

We will use these equations in subsequent sections. Note also that we have from eqns 30.7 and 30.13 that

$$\boxed{\Phi_\mathrm{G} = -\frac{2}{3}U.} \tag{30.17}$$

$z = e^{\frac{\mu}{k_B T}}$

Γ GAMMA FUNTZIOA
(C ERANSKINA)

$\Gamma(z+1) = z\Gamma(z)$

$n! = \int_0^\infty x^n e^{-x}\,dx$

$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$

$\Gamma(n) = (n-1)!$

$\Gamma(3/2) = \frac{\sqrt{\pi}}{2}$

$\Gamma(5/2) = \frac{3}{2}\Gamma(3/2) = \frac{3}{4}\sqrt{\pi}$

Example 30.2

Evaluate N, U and Φ_G (from eqns 30.15, 30.16 and 30.17) in the high-temperature limit.

Solution:

In the high–temperature limit, namely $\beta\mu \ll 1$, we can use the fact that $\mathrm{Li}_n(z) \approx z$ when $|z| \ll 1$. Hence

$$N \approx \frac{(2S+1)V}{\lambda_\mathrm{th}^3}, \tag{30.18}$$

$$U \approx \frac{3}{2}Nk_\mathrm{B}T, \tag{30.19}$$

$$\Phi_\mathrm{G} \approx -Nk_\mathrm{B}T. \tag{30.20}$$

$\frac{N}{V} \approx \frac{2S+1}{\lambda_T^3}$

These three equations are reassuringly familiar. The equation for N shows that the number density of particles N/V is such that, on average, $2S+1$ particles (one for each spin state) occupy a volume λ_th^3. The equation for U asserts that the energy per particle is the familiar equipartition result $\frac{3}{2}k_\mathrm{B}T$. The equation for Φ_G, together with $\Phi_\mathrm{G} = -pV$ (from eqn 22.49) yields the ideal gas law $pV = Nk_\mathrm{B}T$.