Entropy as a Logarithm

Several useful properties follow from the definition of the entropy as the logarithm of the number of accessible states, instead of as the number of accessible states itself. First, the entropy of two independent systems is the sum of the separate entropies.

Second, the entropy is entirely insensitive—for all practical purposes—to the precision δU with which the energy of a closed system is defined. We have never meant to imply that the system energy is known exactly, a circumstance that for a discrete spectrum of energy eigenvalues would make the number of accessible states depend erratically on the energy. We have simply not paid much attention to the precision, whether it be determined by the uncertainty principle $\delta U\,\delta(\text{time}) \sim \hbar$, or determined otherwise. Define $\mathfrak{D}(U)$ as the number of accessible states per unit energy range; $\mathfrak{D}(U)$ can be a suitable smoothed average centered at U. Then $g(U) = \mathfrak{D}(U)\,\delta U$ is the number of accessible states in the range δU at U. The entropy is

$$\sigma(U) = \log \mathfrak{D}(U)\,\delta U = \log \mathfrak{D}(U) + \log \delta U. \tag{37}$$

Typically, as for the system of N spins, the total number of states will be of the order of 2^N. If the total energy is of the order of N times some average one-particle energy Δ, then $\mathfrak{D}(U) \sim 2^N/N\Delta$. Thus

$$\sigma(U) = N \log 2 - \log N\Delta + \log \delta U. \tag{38}$$

Let $N = 10^{20}$; $\Delta = 10^{-14}$ erg; and $\delta U = 10^{-1}$ erg.

$$\sigma(U) = 0.69 \times 10^{20} - 13.82 - 2.3. \tag{39}$$

We see from this example that the value of the entropy is dominated overwhelmingly by the value of N; the precision δU is without perceptible effect on the result. In the problem of N free particles in a box, the number of states is proportional to something like $U^N \delta U$, whence $\sigma \sim N \log U + \log \delta U$. Again the term in N is dominant, a conclusion independent of even the system of units used for the energy.

Example: Perpetual motion of the second kind. Early in our study of physics we came to understand the impossibility of a perpetual motion machine, a machine that will give forth more energy than it absorbs.