

at energy  $U_0$ . The higher order terms in the expansion vanish in the limit of an infinitely large reservoir.\*

Therefore  $\Delta\sigma_R$  defined by (4) becomes

$$\Delta\sigma_R = -(\varepsilon_1 - \varepsilon_2)/\tau. \quad (8)$$

The final result of (5) and (8) is

$$\frac{P(\varepsilon_1)}{P(\varepsilon_2)} = \frac{\exp(-\varepsilon_1/\tau)}{\exp(-\varepsilon_2/\tau)}. \quad (9)$$

A term of the form  $\exp(-\varepsilon/\tau)$  is known as a **Boltzmann factor**. This result is of vast utility. It gives the ratio of the probability of finding the system in a single quantum state 1 to the probability of finding the system in a single quantum state 2.

### Partition Function

It is helpful to consider the function

$$Z(\tau) = \sum_s \exp(-\varepsilon_s/\tau), \quad (10)$$

called the **partition function**. The summation is over the Boltzmann factor  $\exp(-\varepsilon_s/\tau)$  for all states  $s$  of the system. The partition function is the proportionality factor between the probability  $P(\varepsilon_s)$  and the Boltzmann factor  $\exp(-\varepsilon_s/\tau)$ :

$$P(\varepsilon_s) = \frac{\exp(-\varepsilon_s/\tau)}{Z}. \quad (11)$$

We see that  $\sum P(\varepsilon_s) = Z/Z = 1$ : the sum of all probabilities is unity.

The result (11) is one of the most useful results of statistical physics. The average energy of the system is  $U = \langle \varepsilon \rangle = \sum \varepsilon_s P(\varepsilon_s)$ , or

$$U = \frac{\sum \varepsilon_s \exp(-\varepsilon_s/\tau)}{Z} = \tau^2 (\partial \log Z / \partial \tau). \quad (12)$$

\* We expand  $\sigma(U_0 - \varepsilon)$  and not  $g(U_0 - \varepsilon)$  because the expansion of the latter quantity immediately gives convergence difficulties.