If the orbital indices are all different, each entry will occur $N!$ times in $Z_1{}^N$, whereas the entry should occur only once if the particles are identical. Thus, $Z_1{}^N$ overcounts the states by a factor of $N!$, and the correct partition function for N identical particles is

$$\boxed{Z_N = \frac{1}{N!}Z_1{}^N = \frac{1}{N!}(n_Q V)^N} \tag{68}$$

in the classical regime. Here $n_Q = (M\tau/2\pi\hbar^2)^{3/2}$ from (63).

There is a step in the argument where we assume that all N occupied orbitals are always different orbitals. It is no simple matter to evaluate directly the error introduced by this approximation, but later we will confirm by another method the validity of (68) in the classical regime $n \ll n_Q$. The $N!$ factor changes the result for the entropy of the ideal gas. The entropy is an experimentally measurable quantity, and it has been confirmed that the $N!$ factor is correct in this low concentration limit.

Energy. The energy of the ideal gas follows from the N particle partition function by use of (12):

$$U = \tau^2(\partial \log Z_N/\partial\tau) = \tfrac{3}{2}N\tau \ , \tag{69}$$

consistent with (65) for one particle. The free energy is

$$F = -\tau \log Z_N = -\tau \log Z_1{}^N + \tau \log N!. \tag{70}$$

With the earlier result $Z_1 = n_Q V = (M\tau/2\pi\hbar^2)^{3/2}V$ and the Stirling approximation $\log N! \simeq N\log N - N$, we have

$$F = -\tau N \log[(M\tau/2\pi\hbar^2)^{3/2}V] + \tau N \log N - \tau N. \tag{71}$$

From the free energy we can calculate the entropy and the pressure of the ideal gas of N atoms. The pressure follows from (49):

$$p = -(\partial F/\partial V)_\tau = N\tau/V \ , \tag{72}$$

or

$$\boxed{pV = N\tau \ ,} \tag{73}$$