whence we have for the heat capacity of an electron gas, when $\tau \ll \tau_F$,

$$\boxed{C_{el} = \tfrac{1}{3}\pi^2 \mathcal{D}(\varepsilon_F)\tau.} \tag{34}$$

In conventional units,

$$C_{el} = \tfrac{1}{3}\pi^2 \mathcal{D}(\varepsilon_F) k_B{}^2 T. \tag{35}$$

We found that the density of orbitals at the Fermi energy is

$$\mathcal{D}(\varepsilon_F) = 3N/2\varepsilon_F = 3N/2\tau_F \tag{36}$$

for a free electron gas, with $\tau_F \equiv \varepsilon_F$. Do not be deceived by the notation τ_F: it is *not* the temperature of the Fermi gas, but only a convenient reference point. For $\tau \ll \tau_F$ the gas is degenerate; for $\tau \gg \tau_F$ the gas is in the classical regime. Thus (34) becomes

$$\boxed{C_{el} = \tfrac{1}{2}\pi^2 N\tau/\tau_F.} \tag{37}$$

In conventional units there is an extra factor k_B, so that

$$\boxed{C_{el} = \tfrac{1}{2}\pi^2 N k_B T/T_F\ ,} \tag{38}$$

where $k_B T_F \equiv \varepsilon_F$. Again, T_F is not an actual temperature, but only a reference point.

We can give a physical explanation of the form of the result (37). When the specimen is heated from absolute zero, chiefly those electrons in states within an energy range τ of the Fermi level are excited thermally, because the FD distribution function is affected over a region of the order of τ in width, illustrated by Figures 7.3 and 7.5. Thus the number of excited electrons is of the order of $N\tau/\varepsilon_F$, and each of these has its energy increased approximately by τ. The total electronic thermal energy is therefore of the order of $U_{el} \approx N\tau^2/\varepsilon_F$. Thus the electronic contribution to the heat capacity is given by

$\mathcal{D}(\varepsilon_F) \sim \dfrac{N}{\tau_F}$ ε_F

$$C_{el} = dU_{el}/d\tau \approx N\tau/\varepsilon_F \approx N\tau/\tau_F\ , \tag{39}$$

which is directly proportional to τ, in agreement with the exact result (34) and with the experimental results.