Heat Capacity of Electron Gas

We derive a quantitative expression for the heat capacity of a degenerate Fermi gas of electrons in three dimensions. The calculation is perhaps the most impressive accomplishment of the theory of the degenerate Fermi gas. For an ideal monatomic gas the heat capacity is $\frac{3}{2}N$, but for electrons in a metal very much lower values are found. The calculation that follows gives excellent agreement with the experimental results. The increase in the total energy of a system of N electrons when heated from 0 to τ is denoted by $\Delta U \equiv U(\tau) - U(0)$, whence

$$\Delta U = \int_0^\infty d\varepsilon\, \varepsilon \mathfrak{D}(\varepsilon) f(\varepsilon) - \int_0^{\varepsilon_F} d\varepsilon\, \varepsilon \mathfrak{D}(\varepsilon). \tag{24}$$

Here $f(\varepsilon)$ is the Fermi-Dirac function, and $\mathfrak{D}(\varepsilon)$ is the number of orbitals per unit energy range. We multiply the identity

$$N = \int_0^\infty d\varepsilon\, f(\varepsilon)\mathfrak{D}(\varepsilon) = \int_0^{\varepsilon_F} d\varepsilon\, \mathfrak{D}(\varepsilon) \tag{25}$$

by ε_F to obtain

$$\left(\int_0^{\varepsilon_F} + \int_{\varepsilon_F}^\infty\right) d\varepsilon\, \varepsilon_F f(\varepsilon)\mathfrak{D}(\varepsilon) = \int_0^{\varepsilon_F} d\varepsilon\, \varepsilon_F \mathfrak{D}(\varepsilon). \tag{26}$$

We use (26) to rewrite (24) as

$$\boxed{\Delta U = \int_{\varepsilon_F}^\infty d\varepsilon\, (\varepsilon - \varepsilon_F) f(\varepsilon)\mathfrak{D}(\varepsilon) + \int_0^{\varepsilon_F} d\varepsilon\, (\varepsilon_F - \varepsilon)[1 - f(\varepsilon)]\,\mathfrak{D}(\varepsilon).} \tag{27}$$

The first integral on the right-hand side of (27) gives the energy needed to take electrons from ε_F to the orbitals of energy $\varepsilon > \varepsilon_F$, and the second integral gives the energy needed to bring the electrons to ε_F from orbitals below ε_F. Both contributions to the energy are positive. The product $f(\varepsilon)\mathfrak{D}(\varepsilon)d\varepsilon$ in the first integral is the number of electrons elevated to orbitals in the energy range $d\varepsilon$ at an energy ε. The factor $[1 - f(\varepsilon)]$ in the second integral is the probability that an electron has been removed from an orbital ε. The function ΔU is plotted in Figure 7.4. In Figure 7.5 we plot the Fermi-Dirac distribution function versus ε, for six values of the temperature. The electron concentration of the Fermi gas was taken such that $\varepsilon_F/k_B = 50\,000\,K$, characteristic of the conduction electrons in a metal.

The heat capacity of the electron gas is found on differentiating ΔU with respect to τ. The only temperature-dependent term in (27) is $f(\varepsilon)$, whence we