

$$z\phi(T) = \frac{N}{N+1} \simeq 1 - \frac{1}{N} \quad (N \gg 1)$$

$$1 - z\phi(T) = \frac{1}{N+1} \simeq \frac{1}{N} \Rightarrow z_s \cong \frac{1}{\phi(T)}$$

$$\frac{U}{N} = k_B T^2 \frac{\phi'(T)}{\phi(T)}$$

$$\frac{A}{N} = -k_B T \ln \phi(T) + O\left(\frac{\ln N}{N}\right),$$

$$\frac{S}{Nk} = \ln \phi(T) + T \frac{\phi'(T)}{\phi(T)} O\left(\frac{\ln N}{N}\right)$$

$$\phi(T) = \left[2 \sinh \left(\frac{\hbar\omega}{2k_B T} \right) \right]^{-1}$$

$$\phi(T) = \frac{k_B T}{\hbar\omega}$$