AVERAGE VALUES

The average value, or mean value, of a function $f(s)$ taken over a probability distribution function $P(s)$ is defined as

$$\langle f \rangle = \sum_s f(s)\,P(s)\,, \tag{39}$$

provided that the distribution function is normalized to unity:

$$\sum_s P(s) = 1. \tag{40}$$

The binomial distribution (15) has the property (17) that

$$\sum_s g(N,s) = 2^N\,, \tag{41}$$

and is not normalized to unity. If all states are equally probable, then $P(s) = g(N,s)/2^N$, and we have $\sum P(s) = 1$. The average of $f(s)$ over this distribution will be

$$\langle f \rangle = \sum_s f(s)\,P(N,s). \tag{42}$$

Consider the function $f(s) = s^2$. In the approximation that led to (35) and (36), we replace in (42) the sum $\sum$ over s by an integral $\int \cdots ds$ between $-\infty$ and $+\infty$. Then

$$\begin{aligned}\langle s^2 \rangle &= \frac{(2/\pi N)^{1/2}\,2^N \int ds\, s^2 \exp(-2s^2/N)}{2^N}\,,\\ &= (2/\pi N)^{1/2}\,(N/2)^{3/2} \int_{-\infty}^{\infty} dx\, x^2 e^{-x^2}\\ &= (2/\pi N)^{1/2}\,(N/2)^{3/2}\,(\pi/4)^{1/2}\,,\end{aligned}$$

whence

$$\langle s^2 \rangle = \tfrac{1}{4}N; \qquad \langle (2s)^2 \rangle = N. \tag{43}$$

The quantity $\langle (2s)^2 \rangle$ is the mean square spin excess. The root mean square spin excess is

$$\langle (2s)^2 \rangle^{1/2} = \sqrt{N}\,, \tag{44}$$