forallχ

Agnes Scott College 2018-2019
Solutions Booklet

P.D. Magnus
University at Albany, State University of New York

Tim Button
University of Cambridge

Jared Millson
Agnes Scott College

This booklet contains model answers to the practice exercises found in `forall`χ`:Cambridge 2014-15`. For several of the questions, there are multiple correct possible answers; in each case, this booklet contains at most one answer. Answers are given in blue.

Typesetting was carried out entirely in LaTeX2ε. The style for typesetting proofs is based on fitch.sty (v0.4) by Peter Selinger, University of Ottawa.

Contents

Arguments 1

Highlight the phrase which expresses the conclusion of each of these arguments:

1. It is sunny. So I should take my sunglasses.
2. It must have been sunny. I did wear my sunglasses, after all.
3. No one but you has had their hands in the cookie-jar. And the scene of the crime is littered with cookie-crumbs. You're the culprit!
4. Miss Scarlett and Professor Plum were in the study at the time of the murder. And Reverend Green had the candlestick in the ballroom, and we know that there is no blood on his hands. Hence Colonel Mustard did it in the kitchen with the lead-piping. Recall, after all, that the gun had not been fired.

Valid arguments 2

A. Which of the following arguments is valid? Which is invalid?

1. Socrates is a man.
2. All men are carrots.

So: Therefore, Socrates is a carrot. Valid

1. Abe Lincoln was either born in Illinois or he was once president.
2. Abe Lincoln was never president.

So: Abe Lincoln was born in Illinois. Valid

1. If I pull the trigger, Abe Lincoln will die.
2. I do not pull the trigger.

So: Abe Lincoln will not die. Invalid
Abe Lincoln might die for some other reason: someone else might pull the trigger; he might die of old age.

1. Abe Lincoln was either from France or from Luxemborg.
2. Abe Lincoln was not from Luxemborg.

So: Abe Lincoln was from France. Valid

1. If the world were to end today, then I would not need to get up tomorrow morning.
2. I will need to get up tomorrow morning.

So: The world will not end today. Valid

1. Joe is now 19 years old.
2. Joe is now 87 years old.

So: Bob is now 20 years old. Valid
An argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. It is impossible for all the premises to be true; so it is certainly impossible that the premises are all true and the conclusion is false.

B. Could there be:

1. A valid argument that has one false premise and one true premise? Yes. Example: the first argument, above.
2. A valid argument that has only false premises? Yes. Example: Socrates is a frog, all frogs are excellent pianists, therefore Socrates is an excellent pianist.

3. A valid argument with only false premises and a false conclusion? Yes. The same example will suffice.
4. A sound argument with a false conclusion? No. By definition, a sound argument has true premises. And a valid argument is one where it is impossible for the premises to be true and the conclusion false. So the conclusion of a sound argument is certainly true.
5. An invalid argument that can be made valid by the addition of a new premise? Yes. Plenty of examples, but let me offer a more general observation. We can *always* make an invalid argument valid, by adding a contradiction into the premises. For an argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. If the premises are contradictory, then it is impossible for them all to be true (and the conclusion false).
6. A valid argument that can be made invalid by the addition of a new premise? No. An argument is valid if and only if it is impossible for all the premises to be true and the conclusion false. Adding another premise will only make it harder for the premises all to be true together.

In each case: if so, give an example; if not, explain why not.

Other logical notions 3

A. For each of the following: Is it necessarily true, necessarily false, or contingent?

1. Caesar crossed the Rubicon. Contingent
2. Someone once crossed the Rubicon. Contingent
3. No one has ever crossed the Rubicon. Contingent
4. If Caesar crossed the Rubicon, then someone has. Necessarily true
5. Even though Caesar crossed the Rubicon, no one has ever crossed the Rubicon. Necessarily false
6. If anyone has ever crossed the Rubicon, it was Caesar. Contingent

B. Look back at the sentences G1–G4 in this section (about giraffes, gorillas and martians in the wild animal park), and consider each of the following:

1. G2, G3, and G4 Jointly consistent
2. G1, G3, and G4 Jointly inconsistent
3. G1, G2, and G4 Jointly consistent
4. G1, G2, and G3 Jointly consistent

Which are jointly consistent? Which are jointly inconsistent?

C. Could there be:

1. A valid argument, the conclusion of which is necessarily false?
   Yes: '$1 + 1 = 3$. So $1 + 2 = 4$.'
2. An invalid argument, the conclusion of which is necessarily true?
   No. If the conclusion is necessarily true, then there is no way to make it false, and hence no way to make it false whilst making all the premises true.
3. Jointly consistent sentences, one of which is necessarily false?
   No. If a sentence is necessarily false, there is no way to make it true, let alone it along with all the other sentences.
4. Jointly inconsistent sentences, one of which is necessarily true?
   Yes. '$1 + 1 = 4$' and '$1 + 1 = 2$'.

In each case: if so, give an example; if not, explain why not.

Connectives 5

A. Using the symbolisation key given, symbolise each English sentence in TFL.

M: Those creatures are men in suits.
C: Those creatures are chimpanzees.
G: Those creatures are gorillas.

1. Those creatures are not men in suits.
   $\neg M$
2. Those creatures are men in suits, or they are not.
   $(M \vee \neg M)$
3. Those creatures are either gorillas or chimpanzees.
   $(G \vee C)$
4. Those creatures are neither gorillas nor chimpanzees.
   $\neg(C \vee G)$
5. If those creatures are chimpanzees, then they are neither gorillas nor men in suits.
   $(C \rightarrow \neg(G \vee M))$
6. Unless those creatures are men in suits, they are either chimpanzees or they are gorillas.
   $(M \vee (C \vee G))$

B. Using the symbolisation key given, symbolise each English sentence in TFL.

A: Mister Ace was murdered.
B: The butler did it.
C: The cook did it.
D: The Duchess is lying.
E: Mister Edge was murdered.
F: The murder weapon was a frying pan.

1. Either Mister Ace or Mister Edge was murdered.
   $(A \vee E)$
2. If Mister Ace was murdered, then the cook did it.
   $(A \rightarrow C)$
3. If Mister Edge was murdered, then the cook did not do it.
   $(E \rightarrow \neg C)$
4. Either the butler did it, or the Duchess is lying.
   $(B \vee D)$
5. The cook did it only if the Duchess is lying.
   $(C \rightarrow D)$

6. If the murder weapon was a frying pan, then the culprit must have been the cook.
   $(F \to C)$
7. If the murder weapon was not a frying pan, then the culprit was either the cook or the butler.
   $(\neg F \to (C \lor B))$
8. Mister Ace was murdered if and only if Mister Edge was not murdered.
   $(A \leftrightarrow \neg E)$
9. The Duchess is lying, unless it was Mister Edge who was murdered.
   $(D \lor E)$
10. If Mister Ace was murdered, he was done in with a frying pan.
    $(A \to F)$
11. Since the cook did it, the butler did not.
    $(C \land \neg B)$
12. Of course the Duchess is lying!
    D

C. Using the symbolisation key given, symbolise each English sentence in TFL.

E_1: Ava is an electrician.
E_2: Harrison is an electrician.
F_1: Ava is a firefighter.
F_2: Harrison is a firefighter.
S_1: Ava is satisfied with her career.
S_2: Harrison is satisfied with his career.

1. Ava and Harrison are both electricians.
   $(E_1 \land E_2)$
2. If Ava is a firefighter, then she is satisfied with her career.
   $(F_1 \to S_1)$
3. Ava is a firefighter, unless she is an electrician.
   $(F_1 \lor E_1)$
4. Harrison is an unsatisfied electrician.
   $(E_2 \land \neg S_2)$
5. Neither Ava nor Harrison is an electrician.
   $\neg(E_1 \lor E_2)$
6. Both Ava and Harrison are electricians, but neither of them find it satisfying.
   $((E_1 \land E_2) \land \neg(S_1 \lor S_2))$
7. Harrison is satisfied only if he is a firefighter.
   $(S_2 \to F_2)$
8. If Ava is not an electrician, then neither is Harrison, but if she is, then he is too.
   $((\neg E_1 \to \neg E_2) \land (E_1 \to E_2))$
9. Ava is satisfied with her career if and only if Harrison is not satisfied with his.
   $(S_1 \leftrightarrow \neg S_2)$
10. If Harrison is both an electrician and a firefighter, then he must be satisfied with his work.
    $((E_2 \land F_2) \to S_2)$

11. It cannot be that Harrison is both an electrician and a firefighter.
$\neg(E_2 \wedge F_2)$
12. Harrison and Ava are both firefighters if and only if neither of them is an electrician.
$((F_2 \wedge F_1) \leftrightarrow \neg(E_2 \vee E_1))$

D. Give a symbolisation key and symbolise the following English sentences in TFL.

A: Alice is a spy.
B: Bob is a spy.
C: The code has been broken.
G: The German embassy will be in an uproar.

1. Alice and Bob are both spies.
$(A \wedge B)$
2. If either Alice or Bob is a spy, then the code has been broken.
$((A \vee B) \rightarrow C)$
3. If neither Alice nor Bob is a spy, then the code remains unbroken.
$(\neg(A \vee B) \rightarrow \neg C)$
4. The German embassy will be in an uproar, unless someone has broken the code.
$(G \vee C)$
5. Either the code has been broken or it has not, but the German embassy will be in an uproar regardless.
$((C \vee \neg C) \wedge G)$
6. Either Alice or Bob is a spy, but not both.
$((A \vee B) \wedge \neg(A \wedge B))$

E. Give a symbolisation key and symbolise the following English sentences in TFL.

F: There is food to be found in the pridelands.
R: Rafiki will talk about squashed bananas.
A: Simba is alive.
K: Scar will remain as king.

1. If there is food to be found in the pridelands, then Rafiki will talk about squashed bananas.
$(F \rightarrow R)$
2. Rafiki will talk about squashed bananas unless Simba is alive.
$(R \vee A)$
3. Rafiki will either talk about squashed bananas or he won't, but there is food to be found in the pridelands regardless.
$((R \vee \neg R) \wedge F)$
4. Scar will remain as king if and only if there is food to be found in the pridelands.
$(K \leftrightarrow F)$
5. If Simba is alive, then Scar will not remain as king.
$(A \rightarrow \neg K)$

F. For each argument, write a symbolisation key and symbolise all of the sentences of the argument in TFL.

1. If Dorothy plays the piano in the morning, then Roger wakes up cranky. Dorothy plays piano in the morning unless she is distracted. So if Roger does not wake up cranky, then Dorothy must be distracted.

   P: Dorothy plays the Piano in the morning.
   C: Roger wakes up cranky.
   D: Dorothy is distracted.

   $(P \to C)$, $(P \lor D)$, $(\neg C \to D)$
2. It will either rain or snow on Tuesday. If it rains, Neville will be sad. If it snows, Neville will be cold. Therefore, Neville will either be sad or cold on Tuesday.

   T_1: It rains on Tuesday
   T_2: It snows on Tuesday
   S: Neville is sad on Tuesday
   C: Neville is cold on Tuesday

   $(T_1 \lor T_2)$, $(T_1 \to S)$, $(T_2 \to C)$, $(S \lor C)$
3. If Zoog remembered to do his chores, then things are clean but not neat. If he forgot, then things are neat but not clean. Therefore, things are either neat or clean; but not both.

   Z: Zoog remembered to do his chores
   C: Things are clean
   N: Things are neat

   $(Z \to (C \land \neg N))$, $(\neg Z \to (N \land \neg C))$, $((N \lor C) \land \neg(N \land C))$.

G. We symbolised an *exclusive or* using '$\lor$', '$\land$', and '$\neg$'. How could you symbolise an *exclusive or* using only two connectives? Is there any way to symbolise an *exclusive or* using only one connective?
For two connectives, we could offer any of the following:

$$\neg(\mathcal{A} \leftrightarrow \mathcal{B})$$
$$(\neg\mathcal{A} \leftrightarrow \mathcal{B})$$
$$(\neg(\neg\mathcal{A} \land \neg\mathcal{B}) \land \neg(\mathcal{A} \land \mathcal{B}))$$

But if we wanted to symbolise it using only one connective, we would have to introduce a new primitive connective.

Sentences of TFL 6

A. For each of the following: (a) Is it a sentence of TFL, strictly speaking? (b) Is it a sentence of TFL, allowing for our relaxed bracketing conventions?

1. (A) — (a) no (b) no
2. $J_{374} \vee \neg J_{374}$ — (a) no (b) yes
3. $\neg\neg\neg\neg F$ — (a) yes (b) yes
4. $\neg \wedge S$ — (a) no (b) no
5. $(G \wedge \neg G)$ — (a) yes (b) yes
6. $(A \to (A \wedge \neg F)) \vee (D \leftrightarrow E)$ — (a) no (b) yes
7. $[(Z \leftrightarrow S) \to W] \wedge [J \vee X]$ — (a) no (b) yes
8. $(F \leftrightarrow \neg D \to J) \vee (C \wedge D)$ — (a) no (b) no

B. Are there any sentences of TFL that contain no atomic sentences? Explain your answer.
No. Atomic sentences contain atomic sentences (trivially). And every more complicated sentence is built up out of less complicated sentences, that were in turn built out of less complicated sentences, . . . , that were ultimately built out of atomic sentences.

C. What is the scope of each connective in the sentence

$$\big[(H \to I) \vee (I \to H)\big] \wedge (J \vee K)$$

The scope of the left-most instance of '$\to$' is '$(H \to I)$'.
The scope of the right-most instance of '$\to$' is '$(I \to H)$'.
The scope of the left-most instance of '$\vee$ is '$\big[(H \to I) \vee (I \to H)\big]$'
The scope of the right-most instance of '$\vee$' is '$(J \vee K)$'
The scope of the conjunction is the entire sentence; so conjunction is the main logical connective of the sentence.

Complete truth tables 10

A. Complete truth tables for each of the following:

1. $A \to A$

A	A	$\to$	A
T	T	**T**	T
F	F	**T**	F

2. $C \to \neg C$

C	C	$\to$	$\neg$	C
T	T	**F**	F	T
F	F	**T**	T	F

3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$

A	B	$(A$	$\leftrightarrow$	$B)$	$\leftrightarrow$	$\neg$	$(A$	$\leftrightarrow$	$\neg$	$B)$
T	T	T	T	T	**T**	T	T	F	F	T
T	F	T	F	F	**T**	F	T	T	T	F
F	T	F	F	T	**T**	F	F	T	F	T
F	F	F	T	F	**T**	T	F	F	T	F

4. $(A \to B) \lor (B \to A)$

A	B	$(A$	$\to$	$B)$	$\lor$	$(B$	$\to$	$A)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	F	T	T
F	T	F	T	T	**T**	T	F	F
F	F	F	T	F	**T**	F	T	F

5. $(A \land B) \to (B \lor A)$

A	B	$(A$	$\land$	$B)$	$\to$	$(B$	$\lor$	$A)$
T	T	T	T	T	**T**	T	T	T
T	F	T	F	F	**T**	F	T	T
F	T	F	F	T	**T**	T	T	F
F	F	F	F	F	**T**	F	F	F

6. $\neg(A \lor B) \leftrightarrow (\neg A \land \neg B)$

A	B	$\neg$	$(A$	$\lor$	$B)$	$\leftrightarrow$	$(\neg$	A	$\land$	$\neg$	$B)$
T	T	F	T	T	T	**T**	F	T	F	F	T
T	F	F	T	T	F	**T**	F	T	F	T	F
F	T	F	F	T	T	**T**	T	F	F	F	T
F	F	T	F	F	F	**T**	T	F	T	T	F

7. $\big[(A \wedge B) \wedge \neg(A \wedge B)\big] \wedge C$

A	B	C	$\big[(A\wedge B)\wedge\neg(A\wedge B)\big] \wedge C$
T	T	T	T T T F F T T T **F** T
T	T	F	T T T F F T T T **F** F
T	F	T	T F F F T T F F **F** T
T	F	F	T F F F T T F F **F** F
F	T	T	F F T F T F F F **F** T
F	T	F	F F T F T F F T **F** F
F	F	T	F F F F T F F T **F** T
F	F	F	F F F F T F F F **F** F

8. $[(A \wedge B) \wedge C] \to B$

A	B	C	$[(A\wedge B)\wedge C] \to B$
T	T	T	T T T T T **T** T
T	T	F	T T T F F **T** T
T	F	T	T F F F T **T** F
T	F	F	T F F F F **T** F
F	T	T	F F T F T **T** T
F	T	F	F F T F F **T** T
F	F	T	F F F F T **T** F
F	F	F	F F F F F **T** F

9. $\neg\big[(C \vee A) \vee B\big]$

A	B	C	$\neg\big[(C\vee A)\vee B\big]$
T	T	T	**F** T T T T T
T	T	F	**F** F T T T T
T	F	T	**F** T T T T F
T	F	F	**F** F T T T F
F	T	T	**F** T T F T T
F	T	F	**F** F F F T T
F	F	T	**F** T T F T F
F	F	F	**T** F F F F F

B. Check all the claims made in introducing the new notational conventions in §10.3, i.e. show that:

1. '$((A \wedge B) \wedge C)$' and '$(A \wedge (B \wedge C))$' have the same truth table

A	B	C	$(A\wedge B) \wedge C$	$A \wedge (B\wedge C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **F** F	T **F** T F F
T	F	T	T F F **F** T	T **F** F F T
T	F	F	T F F **F** F	T **F** F F F
F	T	T	F F T **F** T	F **F** T T T
F	T	F	F F T **F** F	F **F** T F F
F	F	T	F F F **F** T	F **F** F F T
F	F	F	F F F **F** F	F **F** F F F

2. '$((A \vee B) \vee C)$' and '$(A \vee (B \vee C))$' have the same truth table

A	B	C	$(A \vee B) \vee C$	$A \vee (B \vee C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **T** F	T **T** T T F
T	F	T	T T F **T** T	T **T** F T T
T	F	F	T T F **T** F	T **T** F F F
F	T	T	F T T **T** T	F **T** T T T
F	T	F	F T T **T** F	F **T** T T F
F	F	T	F F F **T** T	F **T** F T T
F	F	F	F F F **F** F	F **F** F F F

3. '$((A \vee B) \wedge C)$' and '$(A \vee (B \wedge C))$' do not have the same truth table

A	B	C	$(A \vee B) \wedge C$	$A \vee (B \wedge C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **F** F	T **T** T F F
T	F	T	T T F **T** T	T **T** F F T
T	F	F	T T F **F** F	T **T** F F F
F	T	T	F T T **T** T	F **T** T T T
F	T	F	F T T **F** F	F **F** T F F
F	F	T	F F F **F** T	F **F** F F T
F	F	F	F F F **F** F	F **F** F F F

4. '$((A \rightarrow B) \rightarrow C)$' and '$(A \rightarrow (B \rightarrow C))$' do not have the same truth table

A	B	C	$(A \rightarrow B) \rightarrow C$	$A \rightarrow (B \rightarrow C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **F** F	T **F** T F F
T	F	T	T F F **T** T	T **T** F T T
T	F	F	T F F **T** F	T **T** F T F
F	T	T	F T T **T** T	F **T** T T T
F	T	F	F T T **F** F	F **T** T F F
F	F	T	F T F **T** T	F **T** F T T
F	F	F	F T F **F** F	F **T** F T F

Also, check whether:

5. '$((A \leftrightarrow B) \leftrightarrow C)$' and '$(A \leftrightarrow (B \leftrightarrow C))$' have the same truth table
   Indeed they do:

A	B	C	$(A \leftrightarrow B) \leftrightarrow C$	$A \leftrightarrow (B \leftrightarrow C)$
T	T	T	T T T **T** T	T **T** T T T
T	T	F	T T T **F** F	T **F** T F F
T	F	T	T F F **F** T	T **F** F F T
T	F	F	T F F **T** F	T **T** F T F
F	T	T	F F T **F** T	F **F** T T T
F	T	F	F F T **T** F	F **T** T F F
F	F	T	F T F **T** T	F **T** F F T
F	F	F	F T F **F** F	F **F** F T F

Semantic concepts 11

A. Revisit your answers to §10**A**. Determine which sentences were tautologies, which were contradictions, and which were neither tautologies nor contradictions.

1. $A \rightarrow A$ — Tautology
2. $C \rightarrow \neg C$ — Neither
3. $(A \leftrightarrow B) \leftrightarrow \neg(A \leftrightarrow \neg B)$ — Tautology
4. $(A \rightarrow B) \vee (B \rightarrow A)$ — Tautology
5. $(A \wedge B) \rightarrow (B \vee A)$ — Tautology
6. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$ — Tautology
7. $\big[(A \wedge B) \wedge \neg(A \wedge B)\big] \wedge C$ — Contradiction
8. $[(A \wedge B) \wedge C] \rightarrow B$ — Tautology
9. $\neg\big[(C \vee A) \vee B\big]$ — Neither

B. Use truth tables to determine whether these sentences are jointly consistent, or jointly inconsistent:

1. $A \rightarrow A$, $\neg A \rightarrow \neg A$, $A \wedge A$, $A \vee A$ — Jointly consistent (see line 1)

A	$A \rightarrow A$	$\neg A \rightarrow \neg A$	$A \wedge A$	$A \vee A$
T	T **T** T	F T **T** F T	T **T** T	T **T** T
F	F **T** F	T F **T** T F	F **F** F	F **F** F

2. $A \vee B$, $A \rightarrow C$, $B \rightarrow C$ — Jointly consistent (see line 1)

A	B	C	$A \vee B$	$A \rightarrow C$	$B \rightarrow C$
T	T	T	T **T** T	T **T** T	T **T** T
T	T	F	T **T** T	T **F** F	T **F** F
T	F	T	T **T** T	T **T** T	F **T** T
T	F	F	T **T** F	T **F** F	F **T** F
F	T	T	F **T** F	F **T** T	T **T** T
F	T	F	F **T** T	F **T** F	T **F** F
F	F	T	F **F** F	F **T** T	F **T** T
F	F	F	F **F** F	F **T** F	F **T** F

3. $B \wedge (C \vee A)$, $A \rightarrow B$, $\neg(B \vee C)$ — Jointly inconsistent

A	B	C	$B \wedge (C \vee A)$	$A \to B$	$\neg (B \vee C)$
T	T	T	T **T** T T T	T **T** T	**F** T T T
T	T	F	T **T** F T T	T **T** T	**F** T T F
T	F	T	F **F** T T T	T **F** F	**F** F T T
T	F	F	F **F** F T T	T **F** F	**T** F F F
F	T	T	T **T** T T F	F **T** T	**F** T T T
F	T	F	T **F** F F F	F **T** T	**F** T T F
F	F	T	F **F** T T F	F **T** F	**F** F T T
F	F	F	F **F** F F F	F **T** F	**T** F F F

4. $A \leftrightarrow (B \vee C)$, $C \to \neg A$, $A \to \neg B$ Jointly consistent (see line 8)

A	B	C	$A \leftrightarrow (B \vee C)$	$C \to \neg A$	$A \to \neg B$
T	T	T	T **T** T T T	T **F** F T	T **F** F T
T	T	F	T **T** T T F	F **T** F T	T **F** F T
T	F	T	T **T** F T T	T **F** F T	T **T** T F
T	F	F	T **F** F F F	F **T** F T	T **T** T F
F	T	T	F **F** T T T	T **T** T F	F **T** F T
F	T	F	F **F** T T F	F **T** T F	F **T** F T
F	F	T	F **F** F T T	T **T** T F	F **T** T F
F	F	F	F **T** F F F	F **T** T F	F **T** T F

C. Use truth tables to determine whether each argument is valid or invalid.

1. $A \to A \therefore A$ Invalid (see line 2)

A	$A \to A$	A
T	T **T** T	T
F	F **T** F	F

2. $A \to (A \wedge \neg A) \therefore \neg A$ Valid

A	$A \to (A \wedge \neg A)$	$\neg A$
T	T **F** T F F T	**F** T
F	F **T** F F T F	**T** F

3. $A \vee (B \to A) \therefore \neg A \to \neg B$ Valid

A	B	$A \vee (B \to A)$	$\neg A \to \neg B$
T	T	T **T** T T T	F T **T** F T
T	F	T **T** F T T	F T **T** T F
F	T	F **F** T F F	T F **F** F T
F	F	F **T** F T F	T F **T** T F

4. $A \vee B, B \vee C, \neg A \therefore B \wedge C$ Invalid (see line 6)

A	B	C	$A \vee B$	$B \vee C$	$\neg A$	$B \wedge C$
T	T	T	T **T** T	T **T** T	**F** T	T **T** T
T	T	F	T **T** T	T **T** F	**F** T	T **F** F
T	F	T	T **T** F	F **T** T	**F** T	F **F** T
T	F	F	T **T** F	F **F** F	**F** T	F **F** F
F	T	T	F **T** T	T **T** T	**T** F	T **T** T
F	T	F	F **T** T	T **T** F	**T** F	T **F** F
F	F	T	F **F** F	F **T** T	**T** F	F **F** T
F	F	F	F **F** F	F **F** F	**T** F	F **F** F

5. $(B \wedge A) \to C, (C \wedge A) \to B \therefore (C \wedge B) \to A$ Invalid (see line 5)

A	B	C	$(B \wedge A) \to C$	$(C \wedge A) \to B$	$(C \wedge B) \to A$
T	T	T	T T T **T** T	T T T **T** T	T T T **T** T
T	T	F	T T T **F** F	F F T **T** T	F F T **T** T
T	F	T	F F T **T** T	T T T **F** F	T F F **T** T
T	F	F	F F T **T** F	F F T **T** F	F F F **T** T
F	T	T	T F F **T** T	T F F **T** T	T T T **F** F
F	T	F	T F F **T** F	F F F **T** T	F F T **T** F
F	F	T	F F F **T** T	T F F **T** F	T F F **T** F
F	F	F	F F F **T** F	F F F **T** F	F F F **T** F

D. Answer each of the questions below and justify your answer.

1. Suppose that ϕ and ψ are tautologically equivalent. What can you say about $\phi \leftrightarrow \psi$?
   ϕ and ψ have the same truth value on every line of a complete truth table, so $\phi \leftrightarrow \psi$ is true on every line. It is a tautology.
2. Suppose that $(\phi \wedge \psi) \to \omega$ is neither a tautology nor a contradiction. What can you say about whether $\phi, \psi \therefore \omega$ is valid?
   Since the sentence $(\phi \wedge \psi) \to \omega$ is not a tautology, there is some line on which it is false. Since it is a conditional, on that line, ϕ and ψ are true and ω is false. So the argument is invalid.
3. Suppose that ϕ, ψ and ω are jointly tautologically inconsistent. What can you say about $(\phi \wedge \psi \wedge \omega)$?
   Since the sentences are jointly tautologically inconsistent, there is no valuation on which they are all true. So their conjunction is false on every valuation. It is a contradiction
4. Suppose that ϕ is a contradiction. What can you say about whether $\phi, \psi \vDash \omega$?
   Since ϕ is false on every line of a complete truth table, there is no line on which ϕ and ψ are true and ω is false. So the entailment holds.
5. Suppose that ω is a tautology. What can you say about whether $\phi, \psi \vDash \omega$?
   Since ω is true on every line of a complete truth table, there is no line on which ϕ and ψ are true and ω is false. So the entailment holds.
6. Suppose that ϕ and ψ are tautologically equivalent. What can you say about $(\phi \vee \psi)$?
   Not much. Since ϕ and ψ are true on exactly the same lines of the truth table, their disjunction is true on exactly the same lines. So, their disjunction is tautologically equivalent to them.
7. Suppose that ϕ and ψ are *not* tautologically equivalent. What can you say about $(\phi \vee \psi)$?
   ϕ and ψ have different truth values on at least one line of a complete truth table, and $(\phi \vee \psi)$ will be true on that line. On other lines, it might be true or false. So $(\phi \vee \psi)$ is either a tautology or it is contingent; it is *not* a contradiction.

E. Consider the following principle:

- Suppose ϕ and ψ are tautologically equivalent. Suppose an argument contains ϕ (either as a premise, or as the conclusion). The validity of the argument would be unaffected, if we replaced ϕ with ψ.

Is this principle correct? Explain your answer.
The principle is correct. Since ϕ and ψ are tautologically equivalent, they have the same truth table. So every valuation that makes ϕ true also makes ψ true, and every valuation that makes ϕ false also makes ψ false. So if no valuation makes all the premises true and the conclusion false, when ϕ was among the premises or the conclusion, then no valuation makes all the premises true and the conclusion false, when we replace ϕ with ψ.

Truth table shortcuts 12

A. Using shortcuts, determine whether each sentence is a tautology, a contradiction, or neither.

1. $\neg B \wedge B$ Contradiction

B	$\neg$	B	$\wedge$	B
T	F		**F**	
F			**F**	

2. $\neg D \vee D$ Tautology

D	$\neg$	D	$\vee$	D
T			**T**	
F	T		**T**	

3. $(A \wedge B) \vee (B \wedge A)$ Neither

A	B	$(A \wedge B)$	$\vee$	$(B \wedge A)$
T	T	T	**T**	
T	F	F	**F**	F
F	T	F	**F**	F
F	F	F	**F**	F

4. $\neg[A \to (B \to A)]$ Contradiction

A	B	$\neg$	$[A$	$\to$	$(B$	$\to$	$A)]$
T	T	**F**		T		T	
T	F	**F**		T		T	
F	T	**F**		T			
F	F	**F**		T			

5. $A \leftrightarrow [A \to (B \wedge \neg B)]$ Contradiction

A	B	A	$\leftrightarrow$	$[A$	$\to$	$(B$	$\wedge$	$\neg$	$B)]$
T	T		**F**		F		F	F	
T	F		**F**		F		F		
F	T		**F**		T				
F	F		**F**		T				

6. $\neg(A \wedge B) \leftrightarrow A$ Neither

A	B	$\neg(A\wedge B)\leftrightarrow A$		
T	T	F	T	**F**
T	F	T	F	**T**
F	T	T	F	**F**
F	F	T	F	**F**

7. $A\rightarrow(B\vee C)$ Neither

A	B	C	$A\rightarrow(B\vee C)$	
T	T	T	**T**	T
T	T	F	**T**	T
T	F	T	**T**	T
T	F	F	**F**	F
F	T	T	**T**	
F	T	F	**T**	
F	F	T	**T**	
F	F	F	**T**	

8. $(A\wedge\neg A)\rightarrow(B\vee C)$ Tautology

A	B	C	$(A\wedge\neg A)\rightarrow(B\vee C)$		
T	T	T	**F**	F	T
T	T	F	**F**	F	T
T	F	T	**F**	F	T
T	F	F	**F**	F	T
F	T	T	**F**		T
F	T	F	**F**		T
F	F	T	**F**		T
F	F	F	**F**		T

9. $(B\wedge D)\leftrightarrow[A\leftrightarrow(A\vee C)]$ Neither

A	B	C	D	$(B\wedge D)\leftrightarrow[A\leftrightarrow(A\vee C)]$			
T	T	T	T	T	**T**	T	T
T	T	T	F	F	**F**	T	T
T	T	F	T	T	**T**	T	T
T	T	F	F	F	**F**	T	T
T	F	T	T	F	**F**	T	T
T	F	T	F	F	**F**	T	T
T	F	F	T	F	**F**	T	T
T	F	F	F	F	**F**	T	T
F	T	T	T	T	**F**	F	T
F	T	T	F	F	**T**	F	T
F	T	F	T	T	**T**	T	F
F	T	F	F	F	**F**	T	F
F	F	T	T	F	**T**	F	T
F	F	T	F	F	**T**	F	T
F	F	F	T	F	**F**	T	F
F	F	F	F	F	**F**	T	F

Partial truth tables 13

A. Use complete or partial truth tables (as appropriate) to determine whether these pairs of sentences are tautologically equivalent:

1. A, $\neg A$ — Not tautologically equivalent

A	A	$\neg A$
T	T	F

2. A, $A \vee A$ — Tautologically equivalent

A	A	$A \vee A$
T	T	T
F	F	F

3. $A \to A$, $A \leftrightarrow A$ — Tautologically equivalent

A	$A \to A$	$A \leftrightarrow A$
T	T	T
F	T	T

4. $A \vee \neg B$, $A \to B$ — Not tautologically equivalent

A	B	$A \vee \neg B$	$A \to B$
T	F	T	F

5. $A \wedge \neg A$, $\neg B \leftrightarrow B$ — Tautologically equivalent

A	B	$A \wedge \neg A$	$\neg B \leftrightarrow B$
T	T	**F** F	F **F**
T	F	**F** F	T **F**
F	T	**F**	F **F**
F	F	**F**	T **F**

6. $\neg(A \wedge B)$, $\neg A \vee \neg B$ — Tautologically equivalent

A	B	$\neg (A \wedge B)$	$\neg A \vee \neg B$
T	T	**F** T	F **F** F
T	F	**T** F	F **T** T
F	T	**T** F	T **T** F
F	F	**T** F	T **T** T

7. $\neg(A \to B)$, $\neg A \to \neg B$ — Not tautologically equivalent

A	B	$\neg (A \to B)$	$\neg A \to \neg B$
T	T	**F** T	F **T** F

8. $(A \to B)$, $(\neg B \to \neg A)$ — Tautologically equivalent

A	B	$(A \to B)$	$(\neg B \to \neg A)$
T	T	T	F **T**
T	F	F	T **F** F
F	T	T	F **T**
F	F	T	T **T** T

B. Use complete or partial truth tables (as appropriate) to determine whether these sentences are jointly tautologically consistent, or jointly tautologically inconsistent:

1. $A \wedge B$, $C \to \neg B$, C — Jointly tautologically inconsistent

A	B	C	$A \wedge B$	$C \to \neg B$	C
T	T	T	T	**F** F	T
T	T	F	T	**T**	F
T	F	T	F	**T** T	T
T	F	F	F	**T**	F
F	T	T	F	**F** F	T
F	T	F	F	**T**	F
F	F	T	F	**T** T	T
F	F	F	F	**T**	F

2. $A \to B$, $B \to C$, A, $\neg C$ — Jointly tautologically inconsistent

A	B	C	$A \to B$	$B \to C$	A	$\neg C$
T	T	T	T	T	T	F
T	T	F	T	F	T	T
T	F	T	F	T	T	F
T	F	F	F	T	T	T
F	T	T	T	T	F	F
F	T	F	T	F	F	T
F	F	T	T	T	F	F
F	F	F	T	T	F	T

3. $A \vee B$, $B \vee C$, $C \to \neg A$ — Jointly tautologically consistent

A	B	C	$A \vee B$	$B \vee C$	$C \to \neg A$
F	T	T	T	T	**T** T

4. A, B, C, $\neg D$, $\neg E$, F — Jointly tautologically consistent

A	B	C	D	E	F	A	B	C	$\neg D$	$\neg E$	F
T	T	T	F	F	T	T	T	T	T	T	T

C. Use complete or partial truth tables (as appropriate) to determine whether each argument is valid or invalid:

1. $A \vee \big[A \to (A \leftrightarrow A)\big] \therefore A$ — Invalid

A	$A \vee \big[A \to (A \leftrightarrow A)\big]$	A
F	**T** T	F

2. $A \leftrightarrow \neg(B \leftrightarrow A) \therefore A$ Invalid

A	B	$A \leftrightarrow \neg(B \leftrightarrow A)$	A
F	F	**T** F T	F

3. $A \to B, B \therefore A$ Invalid

A	B	$A \to B$	B	A
F	T	T	T	F

4. $A \vee B, B \vee C, \neg B \therefore A \wedge C$ Valid

A	B	C	$A \vee B$	$B \vee C$	$\neg B$	$A \wedge C$
T	T	T				T
T	T	F			F	F
T	F	T				T
T	F	F	T	F	T	F
F	T	T			F	F
F	T	F			F	F
F	F	T	F		T	F
F	F	F	F		T	F

5. $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$ Valid

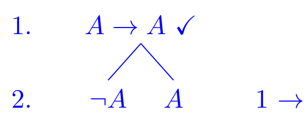
A	B	C	$A \leftrightarrow B$	$B \leftrightarrow C$	$A \leftrightarrow C$
T	T	T			T
T	T	F	T	F	F
T	F	T			T
T	F	F	F		F
F	T	T	F		F
F	T	F			T
F	F	T	T	F	F
F	F	F			T

Truth Trees Rules for TFL 15

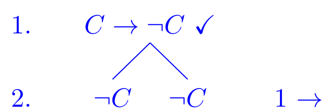
Practice exercises

A. Provide truth trees for each of the following:

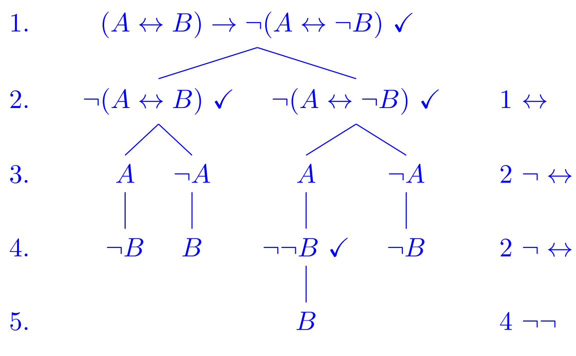
1. $A \to A$



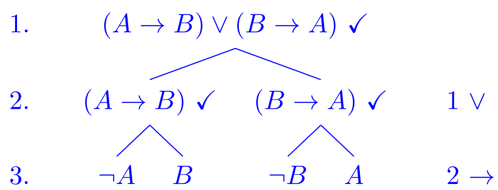
2. $C \to \neg C$



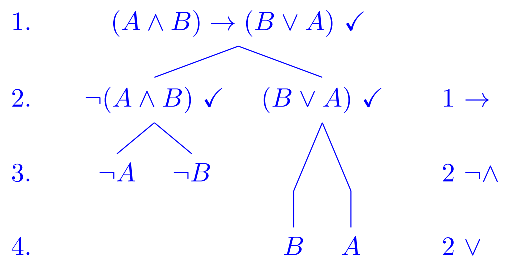
3. $(A \leftrightarrow B) \to \neg(A \leftrightarrow \neg B)$



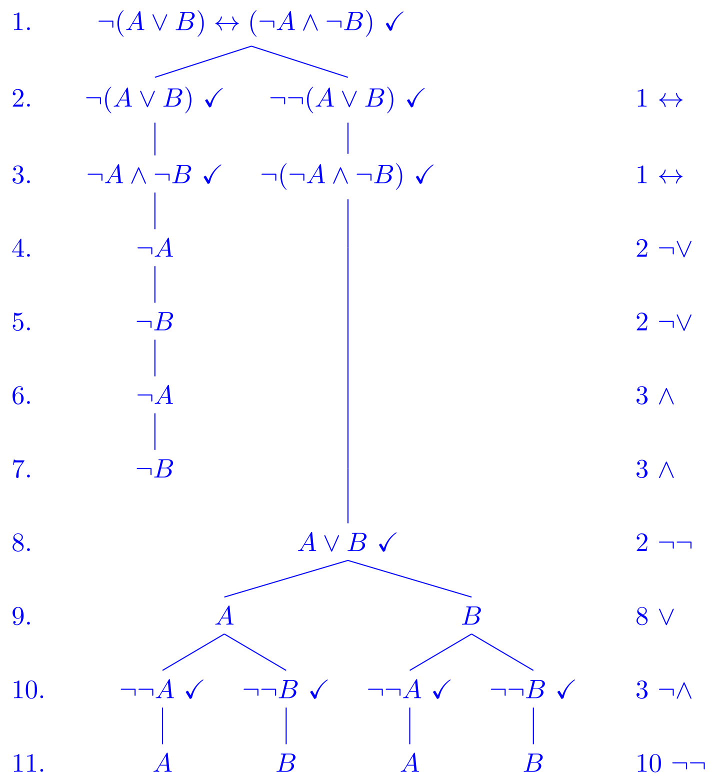
4. $(A \to B) \vee (B \to A)$



5. $(A \wedge B) \rightarrow (B \vee A)$



6. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$



7. $\big[(A \wedge B) \wedge \neg(A \wedge B)\big] \wedge C$

1.	$((A \wedge B) \wedge \neg(A \wedge B)) \wedge C$ ✓		
2.	C		$1 \wedge$
3.	$(A \wedge B) \wedge \neg(A \wedge B)$ ✓		$2 \wedge$
4.	$A \wedge B$ ✓		$3 \wedge$
5.	$\neg(A \wedge B)$ ✓		$2 \wedge$
6.	A		$4 \wedge$
7.	B		$4 \wedge$
8.	$\neg A$	$\neg B$	$5 \wedge$
	$\times$ 6, 8	$\times$ 7, 8	

8. $[(A \wedge B) \wedge C] \rightarrow B$

1.	$\neg((A \wedge B) \wedge C) \rightarrow B$ ✓				
2.	$\neg((A \wedge B) \wedge C)$ ✓			B	$1 \rightarrow$
3.	$\neg(A \wedge B)$ ✓		$\neg C$		$2 \neg\wedge$
4.	$\neg A$	$\neg B$			$3 \neg\wedge$

9. $\neg\big[(C \vee A) \vee B\big]$

1.	$\neg((C \vee A) \vee B)$ ✓	
2.	$\neg(C \vee A)$ ✓	$1 \neg\vee$
3.	$\neg B$	$1 \neg\vee$
4.	$\neg C$	$2 \neg\vee$
5.	$\neg A$	$2 \neg\vee$

Truth Trees and Semantic Properties for TFL 16

A. Use truth trees to determine whether these sentences are jointly consistent, or jointly inconsistent:

1. $A \leftrightarrow (B \vee C)$, $C \rightarrow \neg A$, $A \rightarrow \neg B$
   consistent
2. $A \vee (\neg B \rightarrow \neg C)$, $\neg\big[(C \leftrightarrow B) \vee (A \vee \neg C)\big]$
   inconsistent
3. $A \leftrightarrow (B \wedge C)$, $(A \vee (\neg B \wedge \neg C))$, $\neg(A \leftrightarrow B)$
   inconsistent
4. $A \rightarrow (B \wedge \neg C)$, $\neg C \rightarrow (B \wedge A)$, $\neg(A \leftrightarrow C)$
   consistent
5. $(A \rightarrow B) \rightarrow (C \rightarrow D)$, $\neg\big[(C \rightarrow A) \wedge (D \rightarrow E)\big]$, $\neg E$
   consistent

B. Use truth trees to decide whether the following pairs of statements are equivalent or not.

1. $A \rightarrow (B \rightarrow C)$, $(B \wedge A) \rightarrow C$
   equivalent
2. $A \rightarrow (B \rightarrow A)$, $B \vee \neg(B \rightarrow A)$
   not equivalent
3. $\neg A \vee \neg(B \wedge C)$, $\neg(A \vee B) \wedge (C \vee A)$
   not equivalent
4. $A \vee \neg(\neg B \vee \neg C)$, $(A \vee B) \wedge (A \vee C)$
   equivalent
5. $\neg(A \vee \neg A)$, $(A \vee B) \wedge (\neg B \wedge \neg A)$
   equivalent

C. Use truth trees to determine whether each argument is valid or invalid.

1. $A \vee B$, $B \vee C$, $\neg A \therefore B \wedge C$
   invalid
2. $\neg A \vee \neg(\neg B \wedge C)$, $\neg(A \rightarrow D)$, $D \leftrightarrow B \therefore \neg C \vee \neg B$
   valid
3. $A \leftrightarrow (B \rightarrow D)$, $D \rightarrow (B \vee C) \therefore B \rightarrow (A \rightarrow D)$
   valid
4. $A \rightarrow \big[B \wedge (C \vee D)\big]$, $C \rightarrow \neg\neg D \therefore \neg D \rightarrow \neg C$
   valid

5. $(\neg B \vee \neg C) \rightarrow A$, $C \rightarrow \neg D$ $\therefore$ $A \rightarrow (\neg B \vee \neg D)$
   invalid

Sentences with one quantifier 18

A. Here are the syllogistic figures identified by Aristotle and his successors, along with their medieval names:

- **Barbara.** All G are F. All H are G. So: All H are F
  $\forall x(Gx \rightarrow Fx), \forall x(Hx \rightarrow Gx) \therefore \forall x(Hx \rightarrow Fx)$
- **Celarent.** No G are F. All H are G. So: No H are F
  $\forall x(Gx \rightarrow \neg Fx), \forall x(Hx \rightarrow Gx) \therefore \forall x(Hx \rightarrow \neg Fx)$
- **Ferio.** No G are F. Some H is G. So: Some H is not F
  $\forall x(Gx \rightarrow \neg Fx), \exists x(Hx \wedge Gx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Darii.** All G are H. Some H is G. So: Some H is F.
  $\forall x(Gx \rightarrow Fx), \exists x(Hx \wedge Gx) \therefore \exists x(Hx \wedge Fx)$
- **Camestres.** All F are G. No H are G. So: No H are F.
  $\forall x(Fx \rightarrow Gx), \forall x(Hx \rightarrow \neg Gx) \therefore \forall x(Hx \rightarrow \neg Fx)$
- **Cesare.** No F are G. All H are G. So: No H are F.
  $\forall x(Fx \rightarrow \neg Gx), \forall x(Hx \rightarrow Gx) \therefore \forall x(Hx \rightarrow \neg Fx)$
- **Baroko.** All F are G. Some H is not G. So: Some H is not F.
  $\forall x(Fx \rightarrow Gx), \exists x(Hx \wedge \neg Gx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Festino.** No F are G. Some H are G. So: Some H is not F.
  $\forall x(Fx \rightarrow \neg Gx), \exists x(Hx \wedge Gx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Datisi.** All G are F. Some G is H. So: Some H is F.
  $\forall x(Gx \rightarrow Fx), \exists x(Gx \wedge Hx) \therefore \exists x(Hx \wedge Fx)$
- **Disamis.** Some G is F. All G are H. So: Some H is F.
  $\exists x(Gx \wedge Fx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge Fx)$
- **Ferison.** No G are F. Some G is H. So: Some H is not F.
  $\forall x(Gx \rightarrow \neg Fx), \exists x(Gx \wedge Hx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Bokardo.** Some G is not F. All G are H. So: Some H is not F.
  $\exists x(Gx \wedge \neg Fx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge \neg Fx)$
- **Camenes.** All F are G. No G are H So: No H is F.
  $\forall x(Fx \rightarrow Gx), \forall x(Gx \rightarrow \neg Hx) \therefore \forall x(Hx \rightarrow \neg Fx)$
- **Dimaris.** Some F is G. All G are H. So: Some H is F.
  $\exists x(Fx \wedge Gx), \forall x(Gx \rightarrow Hx) \therefore \exists x(Hx \wedge Fx)$
- **Fresison.** No F are G. Some G is H. So: Some H is not F.
  $\forall x(Fx \rightarrow \neg Gx), \exists x(Gx \wedge Hx) \therefore \exists(Hx \wedge \neg Fx)$

Symbolise each argument in FOL.

B. Using the following symbolisation key:

$\mathcal{D} = \{x \mid \text{people}\}$
Kx : _____$_x$ knows the combination to the safe

Sx : ______$_x$ is a spy
Vx : ______$_x$ is a vegetarian
h : Hofthor
i : Ingmar

symbolise the following sentences in FOL:

1. Neither Hofthor nor Ingmar is a vegetarian.
   $\neg Vh \wedge \neg Vi$
2. No spy knows the combination to the safe.
   $\forall x(Sx \to \neg Kx)$
3. No one knows the combination to the safe unless Ingmar does.
   $\forall x \neg Kx \vee Ki$
4. Hofthor is a spy, but no vegetarian is a spy.
   $Sh \wedge \forall x(Vx \to \neg Sx)$

C. Using this symbolisation key:

$\mathcal{D} =$ $\{x \mid$ animals$\}$
Ax : ______$_x$ is an alligator.
Mx : ______$_x$ is a monkey.
Rx : ______$_x$ is a reptile.
Zx : ______$_x$ lives at the zoo.
a : Amos
b : Bouncer
c : Cleo

symbolise each of the following sentences in FOL:

1. Amos, Bouncer, and Cleo all live at the zoo.
   $Za \wedge Zb \wedge Zc$
2. Bouncer is a reptile, but not an alligator.
   $Rb \wedge \neg Ab$
3. Some reptile lives at the zoo.
   $\exists x(Rx \wedge Zx)$
4. Every alligator is a reptile.
   $\forall x(Ax \to Rx)$
5. Any animal that lives at the zoo is either a monkey or an alligator.
   $\forall x(Zx \to (Mx \vee Ax))$
6. There are reptiles which are not alligators.
   $\exists x(Rx \wedge \neg Ax)$
7. If any animal is an reptile, then Amos is.
   $\exists x Rx \to Ra$
8. If any animal is an alligator, then it is a reptile.
   $\forall x(Ax \to Rx)$

D. For each argument, write a symbolisation key and symbolise the argument in FOL.

1. Willard is a logician. All logicians wear funny hats. So Willard wears a funny hat

   $\mathcal{D} =$ $\{x \mid$ people$\}$

Lx : $\_\_\_\_\__x$ is a logician
Hx : $\_\_\_\_\__x$ wears a funny hat
i : Willard

$Li, \forall x(Lx \rightarrow Hx) \therefore Hi$

2. Nothing on my desk escapes my attention. There is a computer on my desk. As such, there is a computer that does not escape my attention.

   $\mathcal{D} =$ $\{x \mid$ physical things$\}$
   Dx : $\_\_\_\_\__x$ is on my desk
   Ex : $\_\_\_\_\__x$ escapes my attention
   Cx : $\_\_\_\_\__x$ is a computer

   $\forall x(Dx \rightarrow \neg Ex), \exists x(Dx \wedge Cx) \therefore \exists x(Cx \wedge \neg Ex)$

3. All my dreams are black and white. Old TV shows are in black and white. Therefore, some of my dreams are old TV shows.

   $\mathcal{D} =$ $\{x \mid$ episodes (psychological and televised)$\}$
   Dx : $\_\_\_\_\__x$ is one of my dreams
   Bx : $\_\_\_\_\__x$ is in black and white
   Ox : $\_\_\_\_\__x$ is an old TV show

   $\forall x(Dx \rightarrow Bx), \forall x(Ox \rightarrow Bx) \therefore \exists x(Dx \wedge Ox)$.
   Comment: generic statements are tricky to deal with. Does the second sentence mean that *all* old TV shows are in black and white; or that most of them are; or that most of the things which are in black and white are old TV shows? I have gone with the former, but it is not clear that FOL deals with these well.

4. Neither Holmes nor Watson has been to Australia. A person could see a kangaroo only if they had been to Australia or to a zoo. Although Watson has not seen a kangaroo, Holmes has. Therefore, Holmes has been to a zoo.

   $\mathcal{D} =$ $\{x \mid$ people$\}$
   Ax : $\_\_\_\_\__x$ has been to Australia
   Kx : $\_\_\_\_\__x$ has seen a kangaroo
   Zx : $\_\_\_\_\__x$ has been to a zoo
   h : Holmes
   a : Watson

   $\neg Ah \wedge \neg Aa, \forall x(Kx \rightarrow (Ax \vee Zx)), \neg Ka \wedge Kh \therefore Zh$

5. No one expects the Spanish Inquisition. No one knows the troubles I've seen. Therefore, anyone who expects the Spanish Inquisition knows the troubles I've seen.

   $\mathcal{D} =$ $\{x \mid$ people$\}$
   Sx : $\_\_\_\_\__x$ expects the Spanish Inquisition
   Tx : $\_\_\_\_\__x$ knows the troubles I've seen
   h : Holmes
   a : Watson

   $\forall x \neg Sx, \forall x \neg Tx \therefore \forall x(Sx \rightarrow Tx)$

6. All babies are illogical. Nobody who is illogical can manage a crocodile. Berthold is a baby. Therefore, Berthold is unable to manage a crocodile.

   $\mathcal{D} =$ $\{x \mid$ people$\}$

$$
\begin{aligned}
Bx &: \underline{\hspace{2em}}_x \text{ is a baby} \\
Ix &: \underline{\hspace{2em}}_x \text{ is illogical} \\
Cx &: \underline{\hspace{2em}}_x \text{ can manage a crocodile} \\
b &: \text{Berthold}
\end{aligned}
$$

$$\forall x(Bx \rightarrow Ix), \forall x(Ix \rightarrow \neg Cx), Bb \therefore \neg Cb$$

Multiple generality 19

A. Using this symbolisation key:

$\mathcal{D} =$ {x | animals}
Ax : $\underline{\quad\quad}_x$ is an alligator.
Mx : $\underline{\quad\quad}_x$ is a monkey.
Rx : $\underline{\quad\quad}_x$ is a reptile.
Zx : $\underline{\quad\quad}_x$ lives at the zoo.
a : Amos
b : Bouncer
c : Cleo

symbolise each of the following sentences in FOL:

1. If Cleo loves Bouncer, then Bouncer is a monkey.
   $Lcb \rightarrow Mb$
2. If both Bouncer and Cleo are alligators, then Amos loves them both.
   $(Ab \wedge Ac) \rightarrow (Lab \wedge Lac)$
3. Cleo loves a reptile.
   $\exists x(Rx \wedge Lcx)$
   Comment: this English expression is ambiguous; in some contexts, it can be read as a generic, along the lines of 'Cleo loves reptiles'. (Compare 'I do love a good pint'.)
4. Bouncer loves all the monkeys that live at the zoo.
   $\forall x((Mx \wedge Zx) \rightarrow Lbx)$
5. All the monkeys that Amos loves love him back.
   $\forall x((Mx \wedge Lax) \rightarrow Lxa)$
6. Every monkey that Cleo loves is also loved by Amos.
   $\forall x((Mx \wedge Lcx) \rightarrow Lax)$
7. There is a monkey that loves Bouncer, but sadly Bouncer does not reciprocate this love.
   $\exists x(Mx \wedge Lxb \wedge \neg Lbx)$

B. Using the following symbolisation key:

$\mathcal{D} =$ {x | animals}
Dx : $\underline{\quad\quad}_x$ is a dog
Sx : $\underline{\quad\quad}_x$ likes samurai movies
Lxy : $\underline{\quad\quad}_x$ is larger than $\underline{\quad\quad}_y$
b : Bertie
e : Emerson
f : Fergis

symbolise the following sentences in FOL:

1. Bertie is a dog who likes samurai movies.
   $Db \land Sb$
2. Bertie, Emerson, and Fergis are all dogs.
   $Db \land De \land Df$
3. Emerson is larger than Bertie, and Fergis is larger than Emerson.
   $Leb \land Lfe$
4. All dogs like samurai movies.
   $\forall x(Dx \to Sx)$
5. Only dogs like samurai movies.
   $\forall x(Sx \to Dx)$
   Comment: the FOL sentence just written does not require that anyone likes samurai movies. The English sentence might suggest that at least some dogs *do* like samurai movies?
6. There is a dog that is larger than Emerson.
   $\exists x(Dx \land Lxe)$
7. If there is a dog larger than Fergis, then there is a dog larger than Emerson.
   $\exists x(Dx \land Lxf) \to \exists x(Dx \land Lxe)$
8. No animal that likes samurai movies is larger than Emerson.
   $\forall x(Sx \to \neg Lxe)$
9. No dog is larger than Fergis.
   $\forall x(Dx \to \neg Lxf)$
10. Any animal that dislikes samurai movies is larger than Bertie.
    $\forall x(\neg Sx \to Lxb)$
    Comment: this is very poor, though! For 'dislikes' does not mean the same as 'does not like'.
11. There is an animal that is between Bertie and Emerson in size.
    $\exists x((Lbx \land Lxe) \lor (Lex \land Lxb))$
12. There is no dog that is between Bertie and Emerson in size.
    $\forall x\big(Dx \to \neg\big[(Lbx \land Lxe) \lor (Lex \land Lxb)\big]\big)$
13. No dog is larger than itself.
    $\forall x(Dx \to \neg Lxx)$
14. Every dog is larger than some dog.
    $\forall x(Dx \to \exists y(Dy \land Lxy))$
    Comment: the English sentence is potentially ambiguous here. I have resolved the ambiguity by assuming it should be paraphrased by 'for every dog, there is a dog smaller than it'.
15. There is an animal that is smaller than every dog.
    $\exists x \forall y(Dy \to Lyx)$
16. If there is an animal that is larger than any dog, then that animal does not like samurai movies.
    $\forall x(\forall y(Dy \to Lxy) \to \neg Sx)$
    Comment: I have assumed that 'larger than any dog' here means 'larger than every dog'.

C. Using the following symbolisation key:

$\mathcal{D} = \{x \mid \text{people and dishes at a potluck}\}$
Rx : _____$_x$ has run out.

Tx : _____$_x$ is on the table.
Fx : _____$_x$ is food.
Px : _____$_x$ is a person.
Lxy : _____$_x$ likes _____$_y$.
e : Eli
f : Francesca
g : the guacamole

symbolise the following English sentences in FOL:

1. All the food is on the table.
   $\forall x(Fx \rightarrow Tx)$
2. If the guacamole has not run out, then it is on the table.
   $\neg Rg \rightarrow Tg$
3. Everyone likes the guacamole.
   $\forall x(Px \rightarrow Lxg)$
4. If anyone likes the guacamole, then Eli does.
   $\exists x(Px \wedge Lxg) \rightarrow Leg$
5. Francesca only likes the dishes that have run out.
   $\forall x\big[(Lfx \wedge Fx) \rightarrow Rx\big]$
6. Francesca likes no one, and no one likes Francesca.
   $\forall x\big[Px \rightarrow (\neg Lfx \wedge \neg Lxf)\big]$
7. Eli likes anyone who likes the guacamole.
   $\forall x((Px \wedge Lxg) \rightarrow Lex)$
8. Eli likes anyone who likes the people that he likes.
   $\forall x\big[\big(Px \wedge \forall y[(Py \wedge Ley) \rightarrow Lxy]\big) \rightarrow Lex\big]$
9. If there is a person on the table already, then all of the food must have run out.
   $\exists x(Px \wedge Tx) \rightarrow \forall x(Fx \rightarrow Rx)$

D. Using the following symbolisation key:

$\mathcal{D} = \{x \mid \text{people}\}$
Dx : _____$_x$ dances ballet.
Fx : _____$_x$ is female.
Mx : _____$_x$ is male.
Cxy : _____$_x$ is a child of _____$_y$.
Sxy : _____$_x$ is a sibling of _____$_y$.
e : Elmer
j : Jane
p : Patrick

symbolise the following arguments in FOL:

1. All of Patrick's children are ballet dancers.
   $\forall x(Cxp \rightarrow Dx)$
2. Jane is Patrick's daughter.
   $Cjp \wedge Fj$
3. Patrick has a daughter.
   $\exists x(Cxp \wedge Fx)$
4. Jane is an only child.
   $\neg\exists x Sxj$

5. All of Patrick's sons dance ballet.
   $\forall x\big[(Cxp \wedge Mx) \rightarrow Dx\big]$
6. Patrick has no sons.
   $\neg\exists x(Cxp \wedge Mx)$
7. Jane is Elmer's niece.
   $\exists x(Sxe \wedge Cjx \wedge Fj)$
8. Patrick is Elmer's brother.
   $Spe \wedge Mp$
9. Patrick's brothers have no children.
   $\forall x\big[(Spx \wedge Mx) \rightarrow \neg\exists y Cyx\big]$
10. Jane is an aunt.
    $Fj \wedge \exists x(Sxj \wedge \exists y Cyx)$
11. Everyone who dances ballet has a brother who also dances ballet.
    $\forall x\big[Dx \rightarrow \exists y(My \wedge Syx \wedge Dy)\big]$
12. Every woman who dances ballet is the child of someone who dances ballet.
    $\forall x\big[(Fx \wedge Dx) \rightarrow \exists y(Cxy \wedge Dy)\big]$

Identity 20

A. Explain why:

- '$\exists x \forall y(Ay \leftrightarrow x = y)$' is a good symbolisation of 'there is exactly one apple'.
  We might naturally read this in English thus:
  - There is something, x, such that, if you choose any object at all, if you chose an apple then you chose x itself, and if you chose x itself then you chose an apple.

  The x in question must therefore be the one and only thing which is an apple.
- '$\exists x \exists y\big[\neg x = y \wedge \forall z(Az \leftrightarrow (x = z \vee y = z)\big]$' is a good symbolisation of 'there are exactly two apples'.
  Similarly to the above, we might naturally read this in English thus:
  - There are two distinct things, x and y, such that if you choose any object at all, if you chose an apple then you either chose x or y, and if you chose either x or y then you chose an apple.

  The x and y in question must therefore be the only things which are apples, and since they are distinct, there are two of them.

Definite descriptions 21

A. Using the following symbolisation key:

$\mathcal{D} =$ {x | people}
Kx : _____$_x$ knows the combination to the safe.
Sx : _____$_x$ is a spy.
Vx : _____$_x$ is a vegetarian.
Txy : _____$_x$ trusts _____$_y$.
h : Hofthor
i : Ingmar

symbolise the following sentences in FOL:

1. Hofthor trusts a vegetarian.
   $\exists x(Vx \wedge Thx)$
2. Everyone who trusts Ingmar trusts a vegetarian.
   $\forall x\big[Txi \rightarrow \exists y(Txy \wedge Vy)\big]$
3. Everyone who trusts Ingmar trusts someone who trusts a vegetarian.
   $\forall x\big[Txi \rightarrow \exists y\big(Txy \wedge \exists z(Tyz \wedge Vz)\big)\big]$
4. Only Ingmar knows the combination to the safe.
   $\forall x(Ki \rightarrow x = i)$
   Comment: does the English claim entail that Ingmar *does* know the combination to the safe? If so, then we should formalise this with a '$\leftrightarrow$'.
5. Ingmar trusts Hofthor, but no one else.
   $\forall x(Tix \leftrightarrow x = h)$
6. The person who knows the combination to the safe is a vegetarian.
   $\exists x\big[Kx \wedge \forall y(Ky \rightarrow x = y) \wedge Vx\big]$
7. The person who knows the combination to the safe is not a spy.
   $\exists x\big[Kx \wedge \forall y(Ky \rightarrow x = y) \wedge \neg Sx\big]$
   Comment: the scope of negation is potentially ambiguous here; I have read it as *inner* negation.

B. Using the following symbolisation key:

$\mathcal{D} =$ {x | cards in a standard deck}
Bx : _____$_x$ is black.
Cx : _____$_x$ is a club.
Dx : _____$_x$ is a deuce.
Jx : _____$_x$ is a jack.
Mx : _____$_x$ is a man with an axe.
Ox : _____$_x$ is one-eyed.
Wx : _____$_x$ is wild.

symbolise each sentence in FOL:

1. All clubs are black cards.
   $\forall x(Cx \to Bx)$
2. There are no wild cards.
   $\neg\exists x Wx$
3. There are at least two clubs.
   $\exists x\exists y(\neg x = y \land Cx \land Cy)$
4. There is more than one one-eyed jack.
   $\exists x\exists y(\neg x = y \land Jx \land Ox \land Jy \land Oy)$
5. There are at most two one-eyed jacks.
   $\forall x\forall y\forall z\big[(Jx \land Ox \land Jy \land Oy \land Jz \land Oz) \to (x = y \lor x = z \lor y = z)\big]$
6. There are two black jacks.
   $\exists x\exists y(\neg x = y \land Bx \land Jx \land By \land Jy)$
   Comment: I am reading this as 'there are *at least* two...'. If the suggestion was that there are *exactly* two, then a different FOL sentence would be required, namely:
   $\exists x\exists y\big(\neg x = y \land Bx \land Jx \land By \land Jy \land \forall z[(Bz \land Jz) \to (x = z \lor y = z)]\big)$
7. There are four deuces.
   $\exists w\exists x\exists y\exists z(\neg w = x \land \neg w = y \land \neg w = z \land \neg x = y \land \neg x = z \land \neg y = z \land Dw \land Dx \land Dy \land Dz)$
   Comment: I am reading this as 'there are *at least* four...'. If the suggestion is that there are *exactly* four, then we should offer instead:
   $\exists w\exists x\exists y\exists z\big(\neg w = x \land \neg w = y \land \neg w = z \land \neg x = y \land \neg x = z \land \neg y = z \land Dw \land Dx \land Dy \land Dz \land \forall v[Dv \to (v = w \lor v = x \lor v = y \lor v = z)]\big)$
8. The deuce of clubs is a black card.
   $\exists x\big[Dx \land Cx \land \forall y\big((Dy \land Cy) \to x = y\big) \land Bx\big]$
9. One-eyed jacks and the man with the axe are wild.
   $\forall x\big[(Jx \land Ox) \to Wx\big] \land \exists x\big[Mx \land \forall y(My \to x = y) \land Wx\big]$
10. If the deuce of clubs is wild, then there is exactly one wild card.
    $\exists x\big(Dx \land Cx \land \forall y\big[(Dy \land Cy) \to x = y\big] \land Wx\big) \to \exists x\big(Wx \land \forall y(Wy \to x = y)\big)$
    Comment: if there is not exactly one deuce of clubs, then the above sentence is true. Maybe that's the wrong verdict. Perhaps the sentence should definitely be taken to imply that there is one and only one deuce of clubs, and then express a conditional about wildness. If so, then we might symbolise it thus:
    $\exists x\big(Dx \land Cx \land \forall y\big[(Dy \land Cy) \to x = y\big] \land \big[Wx \to \forall y(Wy \to x = y)\big]\big)$
11. The man with the axe is not a jack.
    $\exists x\big[Mx \land \forall y(My \to x = y) \land \neg Jx\big]$
12. The deuce of clubs is not the man with the axe.
    $\exists x\exists y\big(Dx \land Cx \land \forall z[(Dz \land Cz) \to x = z] \land My \land \forall z(Mz \to y = z) \land \neg x = y\big)$

C. Using the following symbolisation key:

$\mathcal{D}$: $\{x \mid \text{real animals}\}$
Bx: $\_\_\_\__x$ is in Farmer Brown's field.
Hx: $\_\_\_\__x$ is a horse.
Px: $\_\_\_\__x$ is a Pegasus.
Wx: $\_\_\_\__x$ has wings.

symbolise the following sentences in FOL:

1. There are at least three horses in the world.
$\exists x \exists y \exists z(\neg x = y \land \neg x = z \land \neg y = z \land Hx \land Hy \land Hz)$
2. There are at least three animals in the world.
$\exists x \exists y \exists z(\neg x = y \land \neg x = z \land \neg y = z)$
3. There is more than one horse in Farmer Brown's field.
$\exists x \exists y(\neg x = y \land Hx \land Hy \land Bx \land By)$
4. There are three horses in Farmer Brown's field.
$\exists x \exists y \exists z(\neg x = y \land \neg x = z \land \neg y = z \land Hx \land Hy \land Hz \land Bx \land By \land Bz)$
Comment: I have read this as 'there are *at least* three…'. If the suggestion was that there are *exactly* three, then a different FOL sentence would be required.
5. There is a single winged creature in Farmer Brown's field; any other creatures in the field must be wingless.
$\exists x\big[Wx \land Bx \land \forall y\big((Wy \land By) \to x = y)\big]$
6. The Pegasus is a winged horse.
$\exists x\big[Px \land \forall y(Py \to x = y) \land Wx \land Hx\big]$
7. The animal in Farmer Brown's field is not a horse.
$\exists x\big[Bx \land \forall y(By \to x = y) \land \neg Hx\big]$
Comment: the scope of negation might be ambiguous here; I have read it as *inner* negation.
8. The horse in Farmer Brown's field does not have wings.
$\exists x\big[Hx \land Bx \land \forall y\big((Hy \land By) \to x = y\big) \land \neg Wx\big]$
Comment: the scope of negation might be ambiguous here; I have read it as *inner* negation.

D. In this section, I symbolised 'Nick is the traitor' by '$\exists x(Tx \land \forall y(Ty \to x = y) \land x = n)$'. Two equally good symbolisations would be:

- $Tn \land \forall y(Ty \to n = y)$
This sentence requires that Nick is a traitor, and that Nick alone is a traitor. Otherwise put, there is one and only one traitor, namely, Nick. Otherwise put: Nick is the traitor.
- $\forall y(Ty \leftrightarrow y = n)$
This sentence can be understood thus: Take anything you like; now, if you chose a traitor, you chose Nick, and if you chose Nick, you chose a traitor. So there is one and only one traitor, namely, Nick, as required.

Explain why these would be equally good symbolisations.

Sentences of FOL 22

A. Identify which variables are bound and which are free. I shall underline the bound variables, and put free variables in blue.

1. $\exists x L\underline{x}\textcolor{blue}{y} \wedge \forall y L\underline{y}\textcolor{blue}{x}$
2. $\forall x A\underline{x} \wedge B\textcolor{blue}{x}$
3. $\forall x(A\underline{x} \wedge B\underline{x}) \wedge \forall y(C\textcolor{blue}{x} \wedge D\underline{y})$
4. $\forall x \exists y[R\underline{xy} \rightarrow (J\textcolor{blue}{z} \wedge K\underline{x})] \vee R\textcolor{blue}{yx}$
5. $\forall x_1(M\textcolor{blue}{x_2} \leftrightarrow L\textcolor{blue}{x_2}\underline{x_1}) \wedge \exists x_2 L\textcolor{blue}{x_3}\underline{x_2}$

Truth in FOL 24

A. Consider the following interpretation:

- The domain comprises only Corwin and Benedict
- 'Ax' is to be true of both Corwin and Benedict
- 'Bx' is to be true of Benedict only
- 'Nx' is to be true of no one
- 'c' is to refer to Corwin

Determine whether each of the following sentences is true or false in that interpretation:

1. Bc False
2. $Ac \leftrightarrow \neg Nc$ True
3. $Nc \rightarrow (Ac \vee Bc)$ True
4. $\forall x Ax$ True
5. $\forall x \neg Bx$ False
6. $\exists x(Ax \wedge Bx)$ True
7. $\exists x(Ax \rightarrow Nx)$ False
8. $\forall x(Nx \vee \neg Nx)$ True
9. $\exists x Bx \rightarrow \forall x Ax$ True

B. Consider the following interpretation:

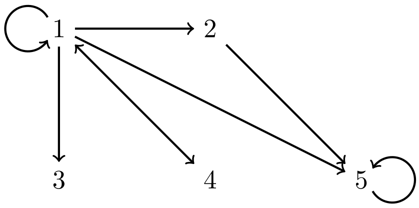
- The domain comprises only Lemmy, Courtney and Eddy
- 'Gx' is to be true of Lemmy, Courtney and Eddy.
- 'Hx' is to be true of and only of Courtney
- 'Mx' is to be true of and only of Lemmy and Eddy
- 'c' is to refer to Courtney
- 'e' is to refer to Eddy

Determine whether each of the following sentences is true or false in that interpretation:

1. Hc True
2. He False
3. $Mc \vee Me$ True
4. $Gc \vee \neg Gc$ True
5. $Mc \rightarrow Gc$ True
6. $\exists x Hx$ True
7. $\forall x Hx$ False
8. $\exists x \neg Mx$ True
9. $\exists x(Hx \wedge Gx)$ True

10. $\exists x(Mx \wedge Gx)$ True
11. $\forall x(Hx \vee Mx)$ True
12. $\exists xHx \wedge \exists xMx$ True
13. $\forall x(Hx \leftrightarrow \neg Mx)$ True
14. $\exists xGx \wedge \exists x\neg Gx$ False
15. $\forall x\exists y(Gx \wedge Hy)$ True

C. Following the diagram conventions introduced at the end of §23, consider the following interpretation:



Determine whether each of the following sentences is true or false in that interpretation:

1. $\exists xRxx$ True
2. $\forall xRxx$ False
3. $\exists x\forall yRxy$ True
4. $\exists x\forall yRyx$ False
5. $\forall x\forall y\forall z((Rxy \wedge Ryz) \rightarrow Rxz)$ False
6. $\forall x\forall y\forall z((Rxy \wedge Rxz) \rightarrow Ryz)$ False
7. $\exists x\forall y\neg Rxy$ True
8. $\forall x(\exists yRxy \rightarrow \exists yRyx)$ True
9. $\exists x\exists y(\neg x = y \wedge Rxy \wedge Ryx)$ True
10. $\exists x\forall y(Rxy \leftrightarrow x = y)$ True
11. $\exists x\forall y(Ryx \leftrightarrow x = y)$ False
12. $\exists x\exists y(\neg x = y \wedge Rxy \wedge \forall z(Rzx \leftrightarrow y = z))$ True

Truth Trees in FOL 28

A. Show that each of the following is not a logical truth: Below are simply *possible* answers. There are infinitely-many alternatives to the countermodels offered.

1. $Da \land Db$
   Countermodel

   $$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(a) &= 1 \\ I(b) &= 2 \\ I(D) &= \{1\} \end{aligned}$$

2. $\exists x Txh$
   Countermodel

   $$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(h) &= 2 \\ I(T) &= \{\langle 1, 1 \rangle\} \end{aligned}$$

3. $Pm \land \neg\forall x Px$
   Countermodel

   $$\begin{aligned} \mathcal{D} &= \{1\} \\ I(m) &= 1 \\ I(P) &= \{\} \end{aligned}$$

4. $\forall z Jz \leftrightarrow \exists y Jy$
   Countermodel

   $$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(J) &= \{1\} \end{aligned}$$

5. $\forall x(Wxmn \lor \exists y Lxy)$
   Countermodel

   $$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(m) &= 1 \\ I(n) &= 2 \\ I(W) &= \{\langle 1,1,1 \rangle\} \\ I(L) &= \{\langle 2,1 \rangle\} \end{aligned}$$

6. $\exists x(Gx \to \forall y My)$
   Countermodel

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(G) &= \{1\} \\ I(M) &= \{1\} \end{aligned}$$

7. $\exists x(x = h \land x = i)$
   Countermodel

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(h) &= 1 \\ I(i) &= 2 \end{aligned}$$

B. Provide a countermodel that shows that the following pairs of sentences are not logically equivalent.

1. Ja, Ka
   Countermodel

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(a) &= 1 \\ I(b) &= 2 \\ I(J) &= \{1\} \\ I(K) &= \{2\} \end{aligned}$$

2. $\exists xJx$, Jm
   Countermodel

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(m) &= 1 \\ I(J) &= \{2\} \end{aligned}$$

3. $\forall xRxx$, $\exists xRxx$
   Countermodel

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(R) &= \{\langle 1, 1\rangle\} \end{aligned}$$

4. $\exists xPx \rightarrow Qc$, $\exists x(Px \rightarrow Qc)$
   Countermodel

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(c) &= 1 \\ I(P) &= \{2\} \\ I(Q) &= \{\} \end{aligned}$$

5. $\forall x(Px \rightarrow \neg Qx)$, $\exists x(Px \land \neg Qx)$
   Countermodel

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(P) &= \{\} \\ I(Q) &= \{1\} \end{aligned}$$

6. $\exists x(Px \land Qx)$, $\exists x(Px \rightarrow Qx)$
   Countermodel

$$\mathcal{D} = \{1\}$$

$$\begin{aligned} I(P) &= \{\} \\ I(Q) &= \{1\} \end{aligned}$$

7. $\forall x(Px \to Qx)$, $\forall x(Px \land Qx)$
   Countermodel

$$\begin{aligned} \mathcal{D} &= \{1\} \\ I(P) &= \{\} \\ I(Q) &= \{1\} \end{aligned}$$

8. $\forall x \exists y Rxy$, $\exists x \forall y Rxy$
   Countermodel

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(R) &= \{\langle 1, 1\rangle, \langle 2, 2\rangle\} \end{aligned}$$

9. $\forall x \exists y Rxy$, $\forall x \exists y Ryx$
   Countermodel

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(R) &= \{\langle 1, 2\rangle, \langle 1, 1\rangle\} \end{aligned}$$

C. Using a truth tree, extract a model showing that each of the following sentences are jointly consistent:

1. $Ma, \neg Na, Pa, \neg Qa$

$$\begin{array}{ll} 1. & Ma \\ & \mid \\ 2. & \neg Na \\ & \mid \\ 3. & Pa \\ & \mid \\ 4. & \neg Qa \end{array}$$

Consistent model

$$\begin{aligned} \mathcal{D} &= \{1\} \\ I(a) &= 1 \\ I(M) &= \{1\} \\ I(N) &= \{\} \\ I(P) &= \{1\} \\ I(Q) &= \{\} \end{aligned}$$

2. $Lee, Leg, \neg Lge, \neg Lgg$

$$\begin{array}{ll} 1. & Lee \\ & \mid \\ 2. & Leg \\ & \mid \\ 3. & \neg Lge \\ & \mid \\ 4. & \neg Lgg \end{array}$$

Consistent model

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(e) &= 1 \\ I(g) &= 2 \\ I(L) &= \{\langle 1, 1\rangle, \langle 1, 2\rangle\} \end{aligned}$$

3. $\neg(Ma \wedge \exists x Ax), Ma \vee Fa, \forall x(Fx \rightarrow Ax)$

1.				$\neg(Ma \wedge \exists x Ax)$ ✓				
2.				$Ma \vee Fa$ ✓				
3.				$\forall x(Fx \to Ax) \setminus a$				
4.				$Fa \to Aa$ ✓				3 $\forall$
5.		$\neg Fa$				Aa		4 $\to$
6.	Ma		Fa × 5, 6	Ma		Fa		2 $\vee$
7.	$\neg Ma$ × 6, 7	$\neg\exists x Ax \setminus a$		$\neg Ma$ × 6, 7	$\neg\exists x Ax \setminus a$	$\neg Ma$	$\neg\exists x Ax \setminus a$	1 $\neg\wedge$
8.		$\neg Aa$			$\neg Aa$ × 5, 8		$\neg Aa$ × 5, 8	7 $\neg\exists$

Consistent model

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(a) &= 1 \\ I(A) &= \{\} \\ I(F) &= \{\} \\ I(M) &= \{1\} \end{aligned}$$

4. $Ma \vee Mb, Ma \to \forall x \neg Mx$

1.		$Ma \vee Mb$ ✓			
2.		$Ma \to \forall x \neg Mx$ ✓			
3.	Ma		Mb		1 $\vee$
4.	$\neg Ma$ × 3, 4	$\forall x \neg Mx \setminus a$	$\neg Ma$	$\forall x \neg Mx \setminus b$	2 $\to$
5.		$\neg Ma$ × 3, 5		$\neg Mb$ × 3, 5	4 $\forall$

Consistent model

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(a) &= 1 \\ I(b) &= 2 \\ I(M) &= \{2\} \end{aligned}$$

5. $\forall y Gy, \forall x(Gx \to Hx), \exists y \neg Iy$

1.	$\forall y Gy \ \backslash a$	
2.	$\forall x(Gx \to Hx) \ \backslash a$	
3.	$\exists y \neg Iy \ \checkmark a$	
4.	$\neg Ia$	$3 \ \exists$
5.	Ga	$1 \ \forall$
6.	$Ga \to Ha \ \checkmark$	$2 \ \forall$
7.	$\neg Ga \quad\quad Ha$	$6 \to$
	$\times$ $5, 7$	

Consistent model

$$
\begin{aligned}
\mathcal{D} &= \{1\} \\
I(a) &= 1 \\
I(G) &= \{1\} \\
I(H) &= \{1\} \\
I(I) &= \{\}
\end{aligned}
$$

6. $\exists x(Bx \vee Ax), \forall x \neg Cx, \forall x\big[(Ax \wedge Bx) \rightarrow Cx\big]$

1.	$\exists x(Bx \vee Ax)$ ✓a				
2.	$\forall x \neg Cx \setminus a$				
3.	$\forall x((Ax \wedge Bx) \rightarrow Cx) \setminus a$				
4.	$Ba \vee Aa$ ✓				1 $\exists$
5.	$\neg Ca$				2 $\forall$
6.	$(Aa \wedge Ba) \rightarrow Ca$ ✓				3 $\forall$
7.	$\neg(Aa \wedge Ba)$ ✓		Ca $\times$ 5, 7		6 $\rightarrow$
8.	$\neg Aa$	$\neg Ba$			7 $\neg\wedge$
9.	Ba	Aa $\times$ 8, 9	Ba $\times$ 8, 9	Aa	4 $\vee$

Consistent model

$$\begin{aligned} \mathcal{D} &= \{1\} \\ I(a) &= 1 \\ I(A) &= \{\} \\ I(B) &= \{1\} \\ I(C) &= \{\} \end{aligned}$$

7. $\exists x Xx, \exists x Yx, \forall x(Xx \leftrightarrow \neg Yx)$

1.	$\exists x Xx$ ✓a			
2.	$\exists x Yx$ ✓a			
3.	$\forall x(Xx \leftrightarrow \neg Yx) \setminus a \setminus b$			
4.	Xa			$1\ \exists$
5.	Yb			$2\ \exists$
6.	$Xa \leftrightarrow \neg Ya$ ✓			$3\ \forall$
7.	$Xb \leftrightarrow \neg Yb$ ✓			$3\ \forall$
8.	Xa		$\neg Xa$	$6 \leftrightarrow$
9.	$\neg Ya$		$\neg\neg Ya$ $\times$ $4, 8$	$6 \leftrightarrow$
10.	Xb	$\neg Xb$		$7 \leftrightarrow$
11.	$\neg Yb$ $\times$ $5, 11$	$\neg\neg Yb$		$7 \leftrightarrow$

Consistent model

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(a) &= 1 \\ I(b) &= 2 \\ I(X) &= \{1\} \\ I(Y) &= \{2\} \end{aligned}$$

8. $\forall x(Px \vee Qx), \exists x \neg(Qx \wedge Px)$

1.	$\forall x(Px \vee Qx) \setminus a$				
2.	$\exists x \neg(Qx \wedge Px)$ ✓a				
3.	$\neg(Qa \wedge Pa)$ ✓				2 $\exists$
4.	$Pa \vee Qa$ ✓				1 $\forall$
5.	Pa		Qa		4 $\vee$
6.	$\neg Qa$	$\neg Pa$	$\neg Qa$	$\neg Pa$	3 $\neg\wedge$
		$\times$ 5, 6	$\times$ 5, 6		

Consistent model

$$\begin{aligned} \mathcal{D} &= \{1\} \\ I(a) &= 1 \\ I(P) &= \{1\} \\ I(Q) &= \{\} \end{aligned}$$

9. $\exists z(Nz \wedge Ozz), \forall x \forall y(Oxy \to Oyx)$

1.	$\exists z(Nz \wedge Ozz)$ ✓a		
2.	$\forall x \forall y(Oxy \to Oyx) \setminus a$		
3.	$Na \wedge Oaa$ ✓		1 $\exists$
4.	Na		3 $\wedge$
5.	Oaa		3 $\wedge$
6.	$\forall y(Oay \to Oya) \setminus a$		2 $\forall$
7.	$Oaa \to Oaa$ ✓		6 $\forall$
8.	$\neg Oaa$	Oaa	7 $\to$
	$\times$ 5, 8		

Consistent model

$$\begin{aligned} \mathcal{D} &= \{1\} \\ I(a) &= 1 \\ I(N) &= \{1\} \\ I(O) &= \{1, 1\} \end{aligned}$$

10. $\neg\exists x\forall y Rxy, \forall x\exists y Rxy$

$$\begin{array}{lcl}
1. & \neg\exists x\forall y Rxy \setminus a & \\
2. & \forall x\exists y Rxy \setminus a & \\
3. & \neg\forall y Ray \checkmark b & 1\ \neg\exists \\
4. & \neg Rab & 3\ \neg\forall \\
5. & \exists y Ray \checkmark c & 2\ \forall \\
6. & Rac & 5\ \exists \\
7. & \infty &
\end{array}$$

Consistent Model

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(a) &= 1 \\ I(b) &= 2 \\ I(R) &= \{\langle 1, 2\rangle, \langle 2, 1\rangle\} \end{aligned}$$

D. Using a truth tree, extract a model showing that each of the following arguments is invalid:

1. $\forall x(Ax \to Bx) \therefore \exists x Bx$

$$\forall x(Ax \to Bx) \therefore \exists x Bx$$

$$\begin{array}{lcl}
1. & \forall x(Ax \to Bx) \setminus a & \text{P} \\
2. & \neg\exists x Bx \setminus a & \neg\text{C} \\
3. & \neg Ba & 2\ \neg\exists \\
4. & Aa \to Ba \checkmark & 2\ \forall \\
5. & \neg Aa \qquad \underset{\underset{3,5}{\times}}{Ba} & 4 \to
\end{array}$$

Countermodel

$$\mathcal{D} = \{1, 2\}$$
$$I(a) = 1$$
$$I(A) = \{2\}$$
$$I(B) = \{2\}$$

2. $\forall x(Rx \to Dx), \forall x(Rx \to Fx) \therefore \exists x(Dx \wedge Fx)$

$\forall x(Rx \to Dx), \forall x(Rx \to Fx) \therefore \exists x(Dx \wedge Fx)$

1.	$\forall x(Rx \to Dx)\ \backslash a$	P
2.	$\forall x(Rx \to Fx)\ \backslash a$	P
3.	$\neg\exists x(Dx \wedge Fx)\ \backslash a$	¬C
4.	$Ra \to Da$ ✓	1 ∀
5.	$Ra \to Fa$ ✓	2 ∀
6.	$\neg(Da \wedge Fa)$ ✓	3 ¬∃
7.	$\neg Da$; $\neg Fa$	6 ¬∧
8.	$\neg Ra$; Da × 7, 8 ; $\neg Ra$; $\neg Da$	4 →
9.	$\neg Ra$; Fa ; $\neg Ra$; Fa × 7, 9 ; $\neg Ra$; Fa × 7, 9	5 →

Countermodel

$$\mathcal{D} = \{1\}$$
$$I(a) = 1$$
$$I(R) = \{\}$$
$$I(D) = \{\}$$
$$I(F) = \{1\}$$

3. $\exists x(Px \to Qx) \therefore \exists x Px$

$$\exists x(Px \to Qx) \therefore \exists x Px$$

$$\begin{array}{lccl}
1. & \multicolumn{2}{c}{\exists x(Px \to Qx) \checkmark a} & \text{P} \\
2. & \multicolumn{2}{c}{\neg\exists x Px \setminus a} & \neg\text{C} \\
3. & \multicolumn{2}{c}{Pa \to Qa \checkmark} & 1\ \exists \\
4. & \neg Pa & Qa & 3 \to \\
5. & \neg Pa & \neg Pa & 2\ \neg\exists
\end{array}$$

Countermodel

$$\begin{aligned}
\mathcal{D} &= \{1\} \\
I(a) &= 1 \\
I(P) &= \{\} \\
I(Q) &= \{1\}
\end{aligned}$$

4. $Na \wedge Nb \wedge Nc \therefore \forall x Nx$

$$Na \wedge Nb \wedge Nc \therefore \forall x Nx$$

$$\begin{array}{lcl}
1. & Na & \text{P} \\
2. & Nb & \text{P} \\
3. & Nc & \text{P} \\
4. & \neg\forall Nx \checkmark d & \neg\text{C} \\
5. & \neg Nd & 4\ \neg\forall
\end{array}$$

Countermodel

$$\begin{aligned}
\mathcal{D} &= \{1, 2\} \\
I(a) &= 1 \\
I(b) &= 1 \\
I(c) &= 1 \\
I(d) &= 2 \\
I(N) &= \{1\}
\end{aligned}$$

5. $Rde, \exists x Rxd \therefore Red$

$Rde, \exists x Rxd \therefore Red$

$$
\begin{array}{lcl}
1. & Rde & \text{P} \\
 & | & \\
2. & \exists x Rxd \checkmark a & \text{P} \\
 & | & \\
3. & \neg Red & \neg\text{C} \\
 & | & \\
4. & Rad & 2\ \exists
\end{array}
$$

Countermodel

$$
\begin{aligned}
\mathcal{D} &= \{1, 2, 3\} \\
I(a) &= 1 \\
I(e) &= 2 \\
I(d) &= 3 \\
I(R) &= \{\langle 1, 2\rangle, \langle 3, 2\rangle\}
\end{aligned}
$$

6. $\exists x(Ex \wedge Fx), \exists x Fx \rightarrow \exists x Gx \therefore \exists x(Ex \wedge Gx)$

$\exists x(Ex \wedge Fx), \exists xFx \to \exists xGx \therefore \exists x(Ex \wedge Gx)$

1.	$\exists x(Ex \wedge Fx)$ ✓a				P
2.	$\exists xFx \to \exists xGx$ ✓a				P
3.	$\neg\exists x(Ex \wedge Gx)$ \\$a$ \\$b$				¬C
4.	$Ea \wedge Fa$ ✓				1 ∃
5.	Ea				4 ∧
6.	Fa				4 ∧
7.	$\neg\exists xFx$ \\a	$\exists Gx$ ✓b			2 →
8.	$\neg Fa$ × 6, 8				7 ¬∃
9.		Gb			7 ∃
10.		$\neg(Ea \wedge Ga)$ ✓			3 ¬∃
11.		$\neg Ea$ × 5, 11	$\neg Ga$		10 ¬∧
12.			$\neg(Eb \wedge Gb)$ ✓		3 ¬∃
13.			$\neg Eb$	$\neg Gb$ × 9, 13	12 ¬∧

Countermodel

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(a) &= 1 \\ I(b) &= 2 \\ I(E) &= \{1\} \\ I(F) &= \{1\} \\ I(G) &= \{2\} \end{aligned}$$

7. $\forall xOxc, \forall xOcx \therefore \forall xOxx$

$\forall x Oxc, \forall x Ocx \therefore \forall x Oxx$

1.	$\forall x Oxc \setminus a$	P
2.	$\forall x Ocx \setminus a$	P
3.	$\neg\forall x Oxx$ ✓ a	¬C
4.	$\neg Oaa$	3 ¬∀
5.	Oac	1 ∀
6.	Oca	2 ∀

Countermodel

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(a) &= 1 \\ I(c) &= 2 \\ I(O) &= \{\langle 1, 2\rangle, \langle 2, 1\rangle\} \end{aligned}$$

8. $\exists x(Jx \wedge Kx), \exists x\neg Kx, \exists x\neg Jx \therefore \exists x(\neg Jx \wedge \neg Kx)$

$\exists x(Jx \wedge Kx), \exists x \neg Kx, \exists x \neg Jx \therefore \exists x(\neg Jx \wedge \neg Kx)$

1.	$\exists x(Jx \wedge Kx)$ ✓a						P
2.	$\exists x \neg Kx$ ✓b						P
3.	$\exists x \neg Jx$ ✓c						P
4.	$\neg \exists x(\neg Jx \wedge \neg Kx)$ $\backslash a$ $\backslash b$ $\backslash c$						$\neg$C
5.	$Ja \wedge Ka$ ✓						1 $\exists$
6.	Ja						5 $\wedge$
7.	Ka						5 $\wedge$
8.	$\neg Kb$						2 $\exists$
9.	$\neg Jc$						3 $\exists$
10.	$\neg(\neg Ja \wedge \neg Ka)$ ✓						4 $\neg\exists$
11.	$\neg(\neg Jb \wedge \neg Kb)$ ✓						4 $\neg\exists$
12.	$\neg(\neg Jc \wedge \neg Kc)$ ✓						4 $\neg\exists$
13.	$\neg\neg Ja$ ✓			$\neg\neg Ka$ ✓			10 $\neg\wedge$
14.	$\neg\neg Jb$ ✓		$\neg\neg Kb$ × 8, 14	$\neg\neg Jb$ ✓		$\neg\neg Kb$ × 8, 14	11 $\neg\wedge$
15.	$\neg\neg Jc$ × 9, 15	$\neg\neg Kc$ ✓		$\neg\neg Jc$ × 9, 15	$\neg\neg Kc$ ✓		12 $\neg\wedge$
16.		Kc			Kc		15 $\neg\neg$
17.		Jb			Jb		14 $\neg\neg$
18.		Ja			Ka		13 $\neg\neg$

Countermodel

$$\begin{aligned} \mathcal{D} &= \{1, 2, 3\} \\ I(a) &= 1 \\ I(b) &= 2 \\ I(c) &= 3 \\ I(J) &= \{1, 2\} \end{aligned}$$

$I(K) = \{1, 3\}$

9. $Lab \to \forall x Lxb, \exists x Lxb \therefore Lbb$

$Lab \to \forall x Lxb, \exists x Lxb \therefore Lbb$

1.	$Lab \to \forall x Lxb$ ✓		P
2.	$\exists x Lxb$ ✓ c		P
3.	$\neg Lbb$		¬C
4.	Lcb		2 ∃
5.	$\neg Lab$	$\forall x Lxb \setminus b$	1 →
6.		Lbb	5 ∀
		× 3, 6	

Countermodel

$\mathcal{D} = \{1, 2, 3\}$
$I(a) = 1$
$I(b) = 2$
$I(c) = 3$
$I(L) = \{\langle 3, 2\rangle\}$

E. Using a truth tree, determine whether the following arguments are valid. If they are invalid, provide a countermodel.

1. $\forall x(Ax \lor Bx), \forall y(Ay \to Cy), \exists x \neg Cx \therefore \exists x Bx$
   Valid

$\forall x(Ax \lor Bx), \forall y(Ay \to Cy), \exists x \neg Cx \therefore \exists x Bx$

1.	$\forall x(Ax \lor Bx)\ \backslash a$				P
2.	$\forall y(Ay \to Cy)\ \backslash a$				P
3.	$\exists x \neg Cx$ ✓ a				P
4.	$\neg \exists x Bx\ \backslash a$				¬C
5.	$\neg Ca$				3 ∃
6.	$Aa \lor Ba$ ✓				1 ∀
7.	$Aa \to Ca$ ✓				2 ∀
8.	Aa		Ba		6 ∨
9.	$\neg Aa$	Ca	$\neg Aa$	Ca	7 →
	× 8, 9	× 5, 9		× 5, 9	
10.			$\neg Ba$		4 ¬∃
			× 8, 10		

2. $\exists x(Ax \land Bx) \therefore \exists x Ax \land \exists x Bx$
   Valid

$\exists x(Ax \land Bx) \therefore \exists x Ax \land \exists x Bx$

1.	$\exists x(Ax \land Bx)$ ✓ a		P
2.	$\neg(\exists x Ax \land \exists x Bx)$ ✓		¬C
3.	$Aa \land Ba$ ✓		1 ∃
4.	Aa		3 ∧
5.	Ba		3 ∧
6.	$\neg \exists x Ax\ \backslash a$	$\neg \exists x Bx\ \backslash a$	5 ¬∧
7.	$\neg Aa$	$\neg Ba$	6 ¬∃
	× 4, 7	× 5, 7	

3. $\exists x Ax \land \exists x Bx \therefore \exists x(Ax \land Bx)$
   Invalid

$\exists x Ax \wedge \exists x Bx \therefore \exists x(Ax \wedge Bx)$

1.	$\exists x Ax \wedge \exists x Bx$ ✓			P
2.	$\neg\exists x(Ax \wedge Bx) \setminus a$			¬C
3.	$\exists x Ax$ ✓ a			1 ∧
4.	$\exists x Bx$ ✓ b			2 ∧
5.	Aa			3 ∃
6.	Bb			4 ∃
7.	$\neg(Aa \wedge Ba)$			2 ¬∃
8.	$\neg(Ab \wedge Bb)$			3 ¬∃
9.	$\neg Aa$ × 5, 9	$\neg Ba$		7 ¬∧
10.		$\neg Ab$	$\neg Bb$ × 6, 10	8 ¬∧

Countermodel

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(a) &= 1 \\ I(b) &= 2 \\ I(A) &= \{1\} \\ I(B) &= \{2\} \end{aligned}$$

4. $\forall x(Ax \to Bx), \forall x(Rxb \to Bx) \therefore \forall x((Ax \vee Bx) \to Rxb)$
Invalid

$\forall x(Ax \to Bx), \forall x(Rxb \to Bx) \therefore \forall x((Ax \vee Bx) \to Rxb)$

1.	$\forall x(Ax \to Bx) \setminus a$	P
2.	$\forall x(Rxb \to Bx) \setminus a$	P
3.	$\neg\forall x((Ax \vee Bx) \to Rxb) \checkmark a$	$\neg$C
4.	$\neg((Aa \vee Ba) \to Rab) \checkmark$	3 $\neg\forall$
5.	$Aa \vee Ba \checkmark$	4 $\neg \to$
6.	$\neg Rab$	4 $\neg \to$
7.	$Aa \to Ba \checkmark$	1 $\forall$
8.	$Rab \to Ba \checkmark$	2 $\forall$
9.	Aa ∣ Ba	5 $\vee$
10.	$\neg Aa$ ∣ Ba ∣ $\neg Aa$ ∣ Ba	7 $\to$
11.	$\times$ 9, 10 ∣ $\neg Rab$ ∣ Ba ∣ $\neg Rab$ ∣ Ba ∣ $\neg Rab$ ∣ Ba	8 $\to$

Countermodel

$$\begin{aligned} \mathcal{D} &= \{1, 2\} \\ I(a) &= 1 \\ I(b) &= 2 \\ I(A) &= \{1\} \\ I(B) &= \{1\} \\ I(R) &= \{\} \end{aligned}$$

5. $\forall x(Ax \to Bx), \forall x(\neg Rxb \to \neg Bx) \therefore \forall x((Ax \vee Bx) \to Rxb)$
Valid

$\forall x(Ax \to Bx), \forall x(\neg Rxb \to \neg Bx) \therefore \forall x((Ax \lor Bx) \to Rxb)$

1.	$\forall x(Ax \to Bx) \setminus a$	P
2.	$\forall x(\neg Rxb \to \neg Bx) \setminus a$	P
3.	$\neg\forall x((Ax \lor Bx) \to Rxb)$ ✓a	¬C
4.	$\neg((Aa \lor Ba) \to Rab)$ ✓	3 ¬∀
5.	$Aa \lor Ba$ ✓	4 ¬ →
6.	$\neg Rab$	4 ¬ →
7.	$Aa \to Ba$ ✓	1 ∀
8.	$\neg Rab \to \neg Ba$ ✓	2 ∀
9.	Aa | Ba	5 ∨
10.	$\neg Aa$ × (9, 10) | Ba | $\neg Aa$ | Ba	7 →
11.	$\neg\neg Rab$ × (6, 11) | $\neg Ba$ × (10, 11) | $\neg\neg Rab$ × (6, 11) | $\neg Ba$ × (9, 11) | $\neg\neg Rab$ × (6, 11) | $\neg Ba$ × (9, 11)	8 →

6. $\forall x(Ax \to Bx), \exists y \neg By \to \exists y Ay \therefore \exists x Bx$
Valid

$\forall x(Ax \to Bx), \exists y \neg By \to \exists Ay \therefore \exists x Bx$

1.	$\forall x(Ax \to Bx) \setminus a$		P
2.	$\exists y \neg By \to \exists y Ay$ ✓		P
3.	$\neg \exists x Bx \setminus a$		¬C
4.	$\neg \exists y \neg By \setminus a$	$\exists y Ay$ ✓ a	$2 \to$
5.	$\neg\neg Ba$		$4\ \neg\exists$
6.	$\neg Ba$ × $5, 6$		$3\ \neg\exists$
7.		Aa	$4\ \exists$
8.		$\neg Ba$	$3\ \neg\exists$
9.		$Aa \to Ba$ ✓	$1\ \forall$
10.		$\neg Aa$ × $7, 10$ Ba × $8, 10$	$9 \to$

7. $\forall x \forall y(Lxy \to Lyx), \exists x \forall y Lxy \therefore \forall x \exists y Lxy$
   Valid

$\forall x \forall y(Lxy \to Lyx), \exists x \forall y Lxy \therefore \forall x \exists y Lxy$

1.	$\forall x \forall y(Lxy \to Lyx) \setminus b$	P
2.	$\exists x \forall y Lxy \checkmark b$	P
3.	$\neg \forall x \exists y Lxy \checkmark a$	$\neg$C
4.	$\neg \exists y Lay \setminus b$	3 $\neg\forall$
5.	$\forall y Lby \setminus a$	2 $\exists$
6.	$\neg Lab$	4 $\neg\exists$
7.	Lba	5 $\forall$
8.	$\forall y(Lby \to Lyb) \setminus a$	1 $\forall$
9.	$Lba \to Lab$	8 $\forall$
10.	$\neg Lba$ $\quad$ Lab	9 $\to$
	$\times$ $\quad$ $\times$	
	$7, 10$ $\quad$ $6, 10$	

30 Basic rules for TFL

A. The following two 'proofs' are *incorrect*. Explain the mistakes they make.

1	$\neg L \to (A \wedge L)$	
2	$\quad \neg L$	
3	$\quad A$	$\to$E 1, 2
4	$\quad L$	
5	$\quad \bot$	$\bot$I 4, 2
6	$\quad A$	$\bot$E 5
7	A	TND 2–3, 4–6

$\to$E on line 3 should yield '$A \wedge L$'. 'A' could then be obtained by $\wedge$E.
$\bot$I on line 5 illicitly refers to a line from a closed subproof (line 2).

1	$A \wedge (B \wedge C)$	
2	$(B \vee C) \to D$	
3	B	$\wedge$E 1
4	$B \vee C$	$\vee$I 3
5	D	$\to$E 4, 2

$\wedge$E on line 3 should yield '$B \wedge C$'. 'B' could then be obtained by $\wedge$E again.
The citation for line 5 is the wrong way round: it should be '$\to$E 2, 4'.

B. The following three proofs are missing their citations (rule and line numbers). Add them, to turn them into bona fide proofs. Additionally, write down the argument that corresponds to each proof.

1	$P \wedge S$	
2	$S \to R$	
3	P	$\wedge$E 1
4	S	$\wedge$E 1
5	R	$\to$E 2, 4
6	$R \vee E$	$\vee$I 5

Corresponding argument:
$P \wedge S, S \to R \therefore R \vee E$

1	$A \to D$	
2	$\quad A \wedge B$	
3	$\quad A$	$\wedge$E 2
4	$\quad D$	$\to$E 1, 3
5	$\quad D \vee E$	$\vee$I 4
6	$(A \wedge B) \to (D \vee E)$	$\to$I 2–5

Corresponding argument:
$A \to D \therefore (A \wedge B) \to (D \vee E)$

1	$\neg L \to (J \lor L)$	
2	$\neg L$	
3	$J \lor L$	$\to$E 1, 2
4	$\quad J$	
5	$\quad J \land J$	$\land$I 4, 4
6	$\quad J$	$\land$E 5
7	$\quad L$	
8	$\quad \bot$	$\bot$I 7, 2
9	$\quad J$	$\bot$E 8
10	J	$\lor$E 3, 4–6, 7–9

Corresponding argument:
$\neg L \to (J \lor L), \neg L \therefore J$

C. Give a proof for each of the following arguments:

1. $J \to \neg J \therefore \neg J$

1	$J \to \neg J$	
2	$\quad J$	
3	$\quad \neg J$	$\to$E 1, 2
4	$\quad \bot$	$\bot$I 2, 3
5	$\neg J$	$\neg$I 2–4

2. $Q \to (Q \land \neg Q) \therefore \neg Q$

1	$Q \to (Q \land \neg Q)$	
2	$\quad Q$	
3	$\quad Q \land \neg Q$	$\to$E 1, 2
4	$\quad \neg Q$	$\land$E 3
5	$\quad \bot$	$\bot$I 2, 4
6	$\neg Q$	$\neg$I 2–6

3. $A \to (B \to C) \therefore (A \land B) \to C$

1	$A \to (B \to C)$	
2	$\quad A \land B$	
3	$\quad A$	$\land$E 2
4	$\quad B \to C$	$\to$E 1, 3
5	$\quad B$	$\land$E 2
6	$\quad C$	$\to$E 4, 5
7	$(A \land B) \to C$	$\to$I 2–6

4. $K \wedge L \therefore K \leftrightarrow L$

1	$K \wedge L$	
2	$\quad K$	
3	$\quad L$	$\wedge$E 1
4	$\quad L$	
5	$\quad K$	$\wedge$E 1
6	$K \leftrightarrow L$	$\leftrightarrow$I 2–3, 4–5

5. $(C \wedge D) \vee E \therefore E \vee D$

1	$(C \wedge D) \vee E$	
2	$\quad C \wedge D$	
3	$\quad D$	$\wedge$E 2
4	$\quad E \vee D$	$\vee$I 3
5	$\quad E$	
6	$\quad E \vee D$	$\vee$I 5
7	$E \vee D$	$\vee$E 1, 2–4, 5–6

6. $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$

1	$A \leftrightarrow B$	
2	$B \leftrightarrow C$	
3	$\quad A$	
4	$\quad B$	$\leftrightarrow$E 1, 3
5	$\quad C$	$\leftrightarrow$E 2, 4
6	$\quad C$	
7	$\quad B$	$\leftrightarrow$E 2, 6
8	$\quad A$	$\leftrightarrow$E 1, 7
9	$A \leftrightarrow C$	$\leftrightarrow$I 3–5, 6–8

7. $\neg F \rightarrow G, F \rightarrow H \therefore G \vee H$

1	$\neg F \rightarrow G$	
2	$F \rightarrow H$	
3	$\quad F$	
4	$\quad H$	$\rightarrow$E 2, 3
5	$\quad G \vee H$	$\vee$I 4
6	$\quad \neg F$	
7	$\quad G$	$\rightarrow$E 1, 6
8	$\quad G \vee H$	$\vee$I 7
9	$G \vee H$	TND 3–5, 6–8

8. $(Z \wedge K) \vee (K \wedge M), K \to D \therefore D$

1	$(Z \wedge K) \vee (K \wedge M)$	
2	$K \to D$	
3	$\mid\ Z \wedge K$	
4	$\mid\ K$	$\wedge$E 3
5	$\mid\ K \wedge M$	
6	$\mid\ K$	$\wedge$E 5
7	K	$\vee$E 1, 3–4, 5–6
8	D	$\to$E 2, 7

9. $P \wedge (Q \vee R), P \to \neg R \therefore Q \vee E$

1	$P \wedge (Q \vee R)$	
2	$P \to \neg R$	
3	P	$\wedge$E 1
4	$\neg R$	$\to$E 2, 3
5	$Q \vee R$	$\wedge$E 1
6	$\mid\ Q$	
7	$\mid\ Q \vee E$	$\vee$I 6
8	$\mid\ R$	
9	$\mid\ \bot$	$\bot$I 8, 4
10	$\mid\ Q \vee E$	$\bot$E 9
11	$Q \vee E$	$\vee$E 5, 6–7, 8–10

10. $S \leftrightarrow T \therefore S \leftrightarrow (T \vee S)$

1	$S \leftrightarrow T$	
2	$\mid\ S$	
3	$\mid\ T$	$\leftrightarrow$E 1, 2
4	$\mid\ T \vee S$	$\vee$I 3
5	$\mid\ T \vee S$	
6	$\mid\ \mid\ T$	
7	$\mid\ \mid\ S$	$\leftrightarrow$E 1, 6
8	$\mid\ \mid\ S$	
9	$\mid\ \mid\ S \wedge S$	$\wedge$I 8, 8
10	$\mid\ \mid\ S$	$\wedge$E 9
11	$\mid\ S$	$\vee$E 5, 6–7, 8–10
12	$S \leftrightarrow (T \vee S)$	$\leftrightarrow$I 2–4, 5–11

11. $\neg(P \to Q) \therefore \neg Q$

1	$\neg(P \to Q)$	
2	$\quad Q$	
3	$\quad\quad P$	
4	$\quad\quad Q \wedge Q$	$\wedge$I 2, 2
5	$\quad\quad Q$	$\wedge$E 4
6	$\quad P \to Q$	$\to$I 3–5
7	$\quad \bot$	$\bot$I 6, 1
8	$\neg Q$	$\neg$I 2–7

12. $\neg(P \to Q) \therefore P$

1	$\neg(P \to Q)$	
2	$\quad P$	
3	$\quad P \wedge P$	$\wedge$I 2, 2
4	$\quad P$	$\wedge$E 3
5	$\quad \neg P$	
6	$\quad\quad P$	
7	$\quad\quad \bot$	$\bot$I 6, 5
8	$\quad\quad Q$	$\bot$E 7
9	$\quad P \to Q$	$\to$I 6–8
10	$\quad \bot$	$\bot$I 9, 1
11	$\quad P$	$\bot$E 10
12	P	TND 2–4, 5–11

Additional rules for TFL 31

A. The following proofs are missing their citations (rule and line numbers). Add them wherever they are required:

1	$W \to \neg B$	
2	$A \wedge W$	
3	$B \vee (J \wedge K)$	
4	W	$\wedge$E 2
5	$\neg B$	$\to$E 1, 4
6	$J \wedge K$	DS 3, 5
7	K	$\wedge$E 6

1	$L \leftrightarrow \neg O$	
2	$L \vee \neg O$	
3	$\neg L$	
4	$\neg O$	DS 2, 3
5	L	$\leftrightarrow$E 1, 4
6	$\bot$	$\bot$I 5, 3
7	$\neg\neg L$	$\neg$I 3–6
8	L	DNE 7

1	$Z \to (C \wedge \neg N)$	
2	$\neg Z \to (N \wedge \neg C)$	
3	$\neg(N \vee C)$	
4	$\neg N \wedge \neg C$	DeM 3
5	$\neg N$	$\wedge$E 4
6	$\neg C$	$\wedge$E 4
7	Z	
8	$C \wedge \neg N$	$\to$E 1, 7
9	C	$\wedge$E 8
10	$\bot$	$\bot$I 9, 6
11	$\neg Z$	$\neg$I 7–10
12	$N \wedge \neg C$	$\to$E 2, 11
13	N	$\wedge$E 12
14	$\bot$	$\bot$I 13, 5
15	$\neg\neg(N \vee C)$	$\neg$I 3–14
16	$N \vee C$	DNE 15

B. Give a proof for each of these arguments:

1. $E \vee F$, $F \vee G$, $\neg F \therefore E \wedge G$

1	$E \vee F$	
2	$F \vee G$	
3	$\neg F$	
4	E	DS 1, 3
5	G	DS 2, 3
6	$E \wedge G$	$\wedge$I 4, 5

2. $M \vee (N \to M) \therefore \neg M \to \neg N$

1	$M \vee (N \to M)$	
2	$\quad \neg M$	
3	$\quad N \to M$	DS 1, 2
4	$\quad \neg N$	MT 3, 2
5	$\neg M \to \neg N$	$\to$I 2–4

3. $(M \vee N) \wedge (O \vee P)$, $N \to P$, $\neg P \therefore M \wedge O$

1	$(M \vee N) \wedge (O \vee P)$	
2	$N \to P$	
3	$\neg P$	
4	$\neg N$	MT 2, 3
5	$M \vee N$	$\wedge$E 1
6	M	DS 5, 4
7	$O \vee P$	$\wedge$E 1
8	O	DS 7, 3
9	$M \wedge O$	$\wedge$I 6, 8

4. $(X \wedge Y) \vee (X \wedge Z)$, $\neg(X \wedge D)$, $D \vee M \therefore M$

1	$(X \wedge Y) \vee (X \wedge Z)$	
2	$\neg(X \wedge D)$	
3	$D \vee M$	
4	$\quad X \wedge Y$	
5	$\quad X$	$\wedge$E 4
6	$\quad X \wedge Z$	
7	$\quad X$	$\wedge$E 6
8	X	$\vee$E 1, 4–5, 6–7
9	$\quad D$	
10	$\quad X \wedge D$	$\wedge$I 8, 9
11	$\quad \bot$	$\bot$I 10, 2
12	$\neg D$	$\neg$I 9–11
13	M	DS 3, 12

Proof-theoretic concepts 32

A. Show that each of the following sentences is a theorem:

1. $O \to O$

1	$\quad\mid\ \underline{O}$	
2	$\quad\mid\ O$	R 1
3	$O \to O$	$\to$I 1–2

2. $N \lor \neg N$

1	$\quad\mid\ \underline{N}$	
2	$\quad\mid\ N \lor \neg N$	$\lor$I 1
3	$\quad\mid\ \underline{\neg N}$	
4	$\quad\mid\ N \lor \neg N$	$\lor$I 3
5	$N \lor \neg N$	TND 1–2, 3–4

3. $J \leftrightarrow [J \lor (L \land \neg L)]$

1	$\quad\mid\ \underline{J}$	
2	$\quad\mid\ J \lor (L \land \neg L)$	$\lor$I 1
3	$\quad\mid\ \underline{J \lor (L \land \neg L)}$	
4	$\quad\mid\quad\mid\ \underline{L \land \neg L}$	
5	$\quad\mid\quad\mid\ L$	$\land$E 4
6	$\quad\mid\quad\mid\ \neg L$	$\land$E 4
7	$\quad\mid\quad\mid\ \bot$	$\bot$I 5, 6
8	$\quad\mid\ \neg(L \land \neg L)$	$\neg$I 4–7
9	$\quad\mid\ J$	DS 3, 8
10	$J \leftrightarrow [J \lor (L \land \neg L)]$	$\leftrightarrow$I 1–2, 3–9

4. $((A \to B) \to A) \to A$

1	$(A \to B) \to A$	
2	$\neg A$	
3	$\neg(A \to B)$	MT 1, 2
4	A	
5	$\bot$	$\bot$I 4, 2
6	B	$\bot$E 5
7	$A \to B$	$\to$I 4–6
8	$\bot$	$\bot$I 7, 3
9	$\neg\neg A$	$\neg$I 2
10	A	DNE 9
11	$((A \to B) \to A) \to A$	$\to$I 1–10

B. Provide proofs to show each of the following:

1. $C \to (E \wedge G), \neg C \to G \vdash G$

1	$C \to (E \wedge G)$	
2	$\neg C \to G$	
3	C	
4	$E \wedge G$	$\to$E 1, 3
5	G	$\wedge$E 4
6	$\neg C$	
7	G	$\to$E 2, 6
8	G	TND 3–5, 6–7

2. $M \wedge (\neg N \to \neg M) \vdash (N \wedge M) \vee \neg M$

1	$M \wedge (\neg N \to \neg M)$	
2	M	$\wedge$E 1
3	$\neg N \to \neg M$	$\wedge$E 1
4	$\neg N$	
5	$\neg M$	$\to$E 3, 4
6	$\bot$	$\bot$I 2, 5
7	$\neg\neg N$	$\neg$I 4–6
8	N	DNE 7
9	$N \wedge M$	$\wedge$I 8, 2
10	$(N \wedge M) \vee \neg M$	$\vee$I 9

3. $(Z \wedge K) \leftrightarrow (Y \wedge M), D \wedge (D \to M) \vdash Y \to Z$

1	$(Z \wedge K) \leftrightarrow (Y \wedge M)$	
2	$D \wedge (D \to M)$	
3	D	$\wedge$E 2
4	$D \to M$	$\wedge$E 2
5	M	$\to$E 4, 3
6	$\quad Y$	
7	$\quad Y \wedge M$	$\wedge$I 6, 5
8	$\quad Z \wedge K$	$\leftrightarrow$E 1, 7
9	$\quad Z$	$\wedge$E 8
10	$Y \to Z$	$\to$I 6–9

4. $(W \vee X) \vee (Y \vee Z), X \to Y, \neg Z \vdash W \vee Y$

1	$(W \vee X) \vee (Y \vee Z)$	
2	$X \to Y$	
3	$\neg Z$	
4	$\quad W \vee X$	
5	$\quad\quad W$	
6	$\quad\quad W \vee Y$	$\vee$I 5
7	$\quad\quad X$	
8	$\quad\quad Y$	$\to$E 2, 7
9	$\quad\quad W \vee Y$	$\vee$I 8
10	$\quad W \vee Y$	$\vee$E 4, 5–6, 7–9
11	$\quad Y \vee Z$	
12	$\quad Y$	DS 11, 3
13	$\quad W \vee Y$	$\vee$I 12
14	$W \vee Y$	$\vee$E 1, 4–10, 11–13

C. Show that each of the following pairs of sentences are provably equivalent:

1. $R \leftrightarrow E$, $E \leftrightarrow R$

1	$R \leftrightarrow E$	
2	$\quad E$	
3	$\quad R$	$\leftrightarrow$E 1, 2
4	$\quad R$	
5	$\quad E$	$\leftrightarrow$E 1, 4
6	$E \leftrightarrow R$	$\leftrightarrow$I 2–3, 4–5

1	$E \leftrightarrow R$	
2	$\quad E$	
3	$\quad R$	$\leftrightarrow$E 1, 2
4	$\quad R$	
5	$\quad E$	$\leftrightarrow$E 1, 4
6	$R \leftrightarrow E$	$\leftrightarrow$I 4–5, 2–3

2. G, $\neg\neg\neg\neg G$

1	G	
2	$\quad \neg\neg\neg G$	
3	$\quad \neg G$	DNE 2
4	$\quad \bot$	$\bot$I 1, 3
5	$\neg\neg\neg\neg G$	$\neg$I 2–4

1	$\neg\neg\neg\neg G$	
2	$\neg\neg G$	DNE 1
3	G	DNE 2

3. $T \to S$, $\neg S \to \neg T$

1	$T \to S$	
2	$\quad \neg S$	
3	$\quad \neg T$	MT 1, 2
4	$\neg S \to \neg T$	$\to$I 2–3

1	$\neg S \to \neg T$	
2	$\quad T$	
3	$\quad\quad \neg S$	
4	$\quad\quad \neg T$	$\to$E 1, 3
5	$\quad\quad \bot$	$\bot$I 2, 4
6	$\quad \neg\neg S$	$\neg$I 3–5
7	$\quad S$	DNE 6
8	$T \to S$	$\to$I 2–7

4. $U \to I$, $\neg(U \wedge \neg I)$

1	$U \to I$	
2	$\quad U \wedge \neg I$	
3	$\quad U$	$\wedge$E 2
4	$\quad \neg I$	$\wedge$E 2
5	$\quad I$	$\to$E 1, 3
6	$\quad \bot$	$\bot$I 5, 4
7	$\neg(U \wedge \neg I)$	$\neg$I 2–6

1	$\neg(U \wedge \neg I)$	
2	$\quad U$	
3	$\quad\quad \neg I$	
4	$\quad\quad U \wedge \neg I$	$\wedge$I 2, 3
5	$\quad\quad \bot$	$\bot$I 4, 1
6	$\quad \neg\neg I$	$\neg$I 3–5
7	$\quad I$	DNE 6
8	$U \to I$	$\to$I 2–7

5. $\neg(C \to D), C \land \neg D$

1	$C \land \neg D$	
2	C	$\land$E 1
3	$\neg D$	$\land$E 1
4	$\vert\ C \to D$	
5	$\vert\ D$	$\to$E 4, 2
6	$\vert\ \bot$	$\bot$I 5, 3
7	$\neg(C \to D)$	$\neg$I 4–6

1	$\neg(C \to D)$	
2	$\vert\ D$	
3	$\vert\ \vert\ C$	
4	$\vert\ \vert\ D$	R 2
5	$\vert\ C \to D$	$\to$I 3–4
6	$\vert\ \bot$	$\bot$I 5, 1
7	$\neg D$	$\neg$I 2–6
8	$\vert\ \neg C$	
9	$\vert\ \vert\ C$	
10	$\vert\ \vert\ \bot$	$\bot$I 9, 8
11	$\vert\ \vert\ D$	$\bot$E 10
12	$\vert\ C \to D$	$\to$I 9–11
13	$\vert\ \bot$	$\bot$I 12, 1
14	$\neg\neg C$	$\neg$I 8–13
15	C	DNE 14
16	$C \land \neg D$	$\land$I 15, 7

6. $\neg G \leftrightarrow H, \neg(G \leftrightarrow H)$

1	$\neg G \leftrightarrow H$	
2	$\vert\ G \leftrightarrow H$	
3	$\vert\ \vert\ G$	
4	$\vert\ \vert\ H$	$\leftrightarrow$E 2, 3
5	$\vert\ \vert\ \neg G$	$\leftrightarrow$E 1, 4
6	$\vert\ \vert\ \bot$	$\bot$I 3, 5
7	$\vert\ \vert\ \neg G$	
8	$\vert\ \vert\ H$	$\leftrightarrow$E 1, 7
9	$\vert\ \vert\ G$	$\leftrightarrow$E 2, 8
10	$\vert\ \vert\ \bot$	$\bot$I 9, 7
11	$\vert\ \bot$	TND 3–6, 7–10
12	$\neg(G \leftrightarrow H)$	$\neg$I 2–11

1	$\neg(G \leftrightarrow H)$	
2	$\quad \neg G$	
3	$\quad\quad \neg H$	
4	$\quad\quad\quad G$	
5	$\quad\quad\quad \bot$	$\bot$I 4, 2
6	$\quad\quad\quad H$	$\bot$E 5
7	$\quad\quad\quad H$	
8	$\quad\quad\quad \bot$	$\bot$I 7, 3
9	$\quad\quad\quad G$	$\bot$E 8
10	$\quad\quad G \leftrightarrow H$	$\leftrightarrow$I 4–6, 7–9
11	$\quad\quad \bot$	$\bot$I 10, 1
12	$\quad \neg\neg H$	$\neg$I 3–11
13	$\quad H$	DNE 12
14	$\quad H$	
15	$\quad\quad G$	
16	$\quad\quad\quad G$	
17	$\quad\quad\quad H$	R 14
18	$\quad\quad\quad H$	
19	$\quad\quad\quad G$	R 15
20	$\quad\quad G \leftrightarrow H$	$\leftrightarrow$I 16–17, 18–19
21	$\quad\quad \bot$	$\bot$I 20, 1
22	$\quad \neg G$	$\neg$I 15–21
23	$\neg G \leftrightarrow H$	$\leftrightarrow$I 2–13, 14–22

D. If you know that $\mathcal{A} \vdash \mathcal{B}$, what can you say about $(\mathcal{A} \wedge \mathcal{C}) \vdash \mathcal{B}$? What about $(\mathcal{A} \vee \mathcal{C}) \vdash \mathcal{B}$? Explain your answers.
If $\mathcal{A} \vdash \mathcal{B}$, then $(\mathcal{A} \wedge \mathcal{C}) \vdash \mathcal{B}$. After all, if $\mathcal{A} \vdash \mathcal{B}$, then there is some proof with assumption $\mathcal{A}$ that ends with $\mathcal{B}$, and no undischarged assumptions other than $\mathcal{A}$. Now, if we start a proof with assumption $(\mathcal{A} \wedge \mathcal{C})$, we can obtain $\mathcal{A}$ by $\wedge$E. We can now copy and paste the original proof of $\mathcal{B}$ from $\mathcal{A}$, adding 1 to every line number and line number citation. The result will be a proof of $\mathcal{B}$ from assumption $\mathcal{A}$.

However, we cannot prove much from $(\mathcal{A} \vee \mathcal{C})$. After all, it might be impossible to prove $\mathcal{B}$ from $\mathcal{C}$.

E. In this section, I claimed that it is just as hard to show that two sentences are not provably equivalent, as it is to show that a sentence is not a theorem. Why did I claim this? (*Hint*: think of a sentence that would be a theorem iff $\mathcal{A}$ and $\mathcal{B}$ were provably equivalent.)
Consider the sentence $\mathcal{A} \leftrightarrow \mathcal{B}$. Suppose we can show that this is a theorem.

So we can prove it, with no assumptions, in m lines, say. Then if we assume $\mathcal{A}$ and copy and paste the proof of $\mathcal{A} \leftrightarrow \mathcal{B}$ (changing the line numbering), we will have a deduction of this shape:

1	$\mathcal{A}$	
$m+1$	$\mathcal{A} \leftrightarrow \mathcal{B}$	
$m+2$	$\mathcal{B}$	$\leftrightarrow$E $m+1$, 1

This will show that $\mathcal{A} \vdash \mathcal{B}$. In exactly the same way, we can show that $\mathcal{B} \vdash \mathcal{A}$. So if we can show that $\mathcal{A} \leftrightarrow \mathcal{B}$ is a theorem, we can show that $\mathcal{A}$ and $\mathcal{B}$ are provably equivalent.

Conversely, suppose we can show that $\mathcal{A}$ and $\mathcal{B}$ are provably equivalent. Then we can prove $\mathcal{B}$ from the assumption of $\mathcal{A}$ in m lines, say, and prove $\mathcal{A}$ from the assumption of $\mathcal{B}$ in n lines, say. Copying and pasting these proofs together (changing the line numbering where appropriate), we obtain:

1		$\mathcal{A}$	
m		$\mathcal{B}$	
$m+1$		$\mathcal{B}$	
$m+n$		$\mathcal{A}$	
$m+n+1$	$\mathcal{A} \leftrightarrow \mathcal{B}$		$\leftrightarrow$I 1–m, $m+1$–$m+n$

Thus showing that $\mathcal{A} \leftrightarrow \mathcal{B}$ is a theorem.

There was nothing special about $\mathcal{A}$ and $\mathcal{B}$ in this. So what this shows is that the problem of showing that two sentences are provably equivalent is, essentially, the same problem as showing that a certain kind of sentence (a biconditional) is a theorem.

Derived rules 34

A. Provide proof schemes that justify the addition of the third and fourth De Morgan rules as derived rules.

Third rule:

m	$\neg\mathcal{A} \wedge \neg\mathcal{B}$	
k	$\neg\mathcal{A}$	$\wedge$E m
$k+1$	$\neg\mathcal{B}$	$\wedge$E m
$k+2$	$\mid$ $\mathcal{A} \vee \mathcal{B}$	
$k+3$	$\mid\ \mid$ $\mathcal{A}$	
$k+4$	$\mid\ \mid$ $\bot$	$\bot$I $k+3$, k
$k+5$	$\mid\ \mid$ $\mathcal{B}$	
$k+6$	$\mid\ \mid$ $\bot$	$\bot$I $k+5$, $k+1$
$k+7$	$\mid$ $\bot$	$\vee$E $k+2$, $k+3$–$k+4$, $k+5$–$k+6$
$k+8$	$\neg(\mathcal{A} \vee \mathcal{B})$	$\neg$I $k+2$–$k+7$

Fourth rule:

m	$\neg(\mathcal{A} \vee \mathcal{B})$	
k	$\mid$ $\mathcal{A}$	
$k+1$	$\mid$ $\mathcal{A} \vee \mathcal{B}$	$\vee$I k
$k+2$	$\mid$ $\bot$	$\bot$I $k+1$, m
$k+3$	$\neg\mathcal{A}$	$\neg$I k–$k+2$
$k+4$	$\mid$ $\mathcal{B}$	
$k+5$	$\mid$ $\mathcal{A} \vee \mathcal{B}$	$\vee$I $k+4$
$k+6$	$\mid$ $\bot$	$\bot$I $k+5$, m
$k+7$	$\neg\mathcal{B}$	$\neg$I $k+4$–$k+6$
$k+8$	$\neg\mathcal{A} \wedge \neg\mathcal{B}$	$\wedge$I $k+3$, $k+7$

Basic rules for FOL 35

A. The following two 'proofs' are *incorrect*. Explain why both are incorrect. Also, provide interpretations which would invalidate the fallacious argument forms the 'proofs' enshrine:

1	$\forall x Rxx$	
2	Raa	$\forall$E 1
3	$\forall y Ray$	$\forall$I 2
4	$\forall x \forall y Rxy$	$\forall$I 3

When using $\forall$I, you must replace *all* names with the new variable. So line 3 is bogus. As a counterinterpretation, consider the following:

1	$\forall x \exists y Rxy$	
2	$\exists y Ray$	$\forall$E 1
3	Raa	
4	$\exists x Rxx$	$\exists$I 3
5	$\exists x Rxx$	$\exists$E 2, 3–4

The instantiating constant, 'a', occurs in the line (line 2) to which $\exists$E is to be applied on line 5. So the use of $\exists$E on line 5 is bogus. As a counterinterpretation, consider the following:

$$1 \longleftrightarrow 2$$

B. The following three proofs are missing their citations (rule and line numbers). Add them, to turn them into bona fide proofs.

1	$\forall x \exists y (Rxy \lor Ryx)$	
2	$\forall x \neg Rmx$	
3	$\exists y (Rmy \lor Rym)$	$\forall$E 1
4	$Rma \lor Ram$	
5	$\neg Rma$	$\forall$E 2
6	Ram	DS 4, 5
7	$\exists x Rxm$	$\exists$I 6
8	$\exists x Rxm$	$\exists$E 3, 4–7

1	$\forall x (\exists y Lxy \to \forall z Lzx)$	
2	Lab	
3	$\exists y Lay \to \forall z Lza$	$\forall$E 1
4	$\exists y Lay$	$\exists$I 2
5	$\forall z Lza$	$\to$E 3, 4
6	Lca	$\forall$E 5
7	$\exists y Lcy \to \forall z Lzc$	$\forall$E 1
8	$\exists y Lcy$	$\exists$I 6
9	$\forall z Lzc$	$\to$E 7, 8
10	Lcc	$\forall$E 9
11	$\forall x Lxx$	$\forall$I 10

1	$\forall x(Jx \rightarrow Kx)$	
2	$\exists x \forall y Lxy$	
3	$\forall x Jx$	
4	$\quad \forall y Lay$	
5	$\quad Laa$	$\forall$E 4
6	$\quad Ja$	$\forall$E 3
7	$\quad Ja \rightarrow Ka$	$\forall$E 1
8	$\quad Ka$	$\rightarrow$E 7, 6
9	$\quad Ka \wedge Laa$	$\wedge$I 8, 5
10	$\quad \exists x(Kx \wedge Lxx)$	$\exists$I 9
11	$\exists x(Kx \wedge Lxx)$	$\exists$E 2, 4–10

C. In §18 problem part A, we considered fifteen syllogistic figures of Aristotelian logic. Provide proofs for each of the argument forms. NB: You will find it *much* easier if you symbolise (for example) 'No F is G' as '$\forall x(Fx \rightarrow \neg Gx)$'.
I shall prove the four Figure I syllogisms; the rest are *extremely* similar.

Barbara

1	$\forall x(Gx \rightarrow Fx)$	
2	$\forall x(Hx \rightarrow Gx)$	
3	$Ga \rightarrow Fa$	$\forall$E 1
4	$Ha \rightarrow Ga$	$\forall$E 2
5	$\quad Ha$	
6	$\quad Ga$	$\rightarrow$E 4, 5
7	$\quad Fa$	$\rightarrow$E 3, 6
8	$Ha \rightarrow Fa$	$\rightarrow$I 5–7
9	$\forall x(Hx \rightarrow Fx)$	$\forall$I 8

Celerant is exactly as Barbara, replacing 'F' with '$\neg F$' throughout.

Ferio

1	$\forall x(Gx \rightarrow \neg Fx)$	
2	$\exists x(Hx \wedge Gx)$	
3	$\quad Ha \wedge Ga$	
4	$\quad Ha$	$\wedge$E 3
5	$\quad Ga$	$\wedge$E 3
6	$\quad Ga \rightarrow \neg Fa$	$\forall$E 1
7	$\quad \neg Fa$	$\rightarrow$E 6, 5
8	$\quad Ha \wedge \neg Fa$	$\wedge$I 4, 7
9	$\quad \exists x(Hx \wedge \neg Fx)$	$\exists$I 8
10	$\exists x(Hx \wedge \neg Fx)$	$\exists$E 2, 3–9

Darii is exactly as Ferio, replacing '$\neg F$' with 'F' throughout.

D. Aristotle and his successors identified other syllogistic forms which depended upon 'existential import'. Symbolise each of the following argument forms in FOL and offer proofs.

- **Barbari.** Something is H. All G are F. All H are G. So: Some H is F
  $\exists x Hx, \forall x(Gx \to Fx), \forall x(Hx \to Gx) \therefore \exists x(Hx \land Fx)$

1	$\exists x Hx$	
2	$\forall x(Gx \to Fx)$	
3	$\forall x(Hx \to Gx)$	
4	$\quad Ha$	
5	$\quad Ha \to Ga$	$\forall$E 3
6	$\quad Ga$	$\to$E 5, 4
7	$\quad Ga \to Fa$	$\forall$E 2
8	$\quad Fa$	$\to$E 7, 6
9	$\quad Ha \land Fa$	$\land$I 4, 8
10	$\quad \exists x(Hx \land Fx)$	$\exists$I 9
11	$\exists x(Hx \land Fx)$	$\exists$E 1, 4–10

- **Celaront.** Something is H. No G are F. All H are G. So: Some H is not F
  $\exists x Hx, \forall x(Gx \to \neg Fx), \forall x(Hx \to Gx) \therefore \exists x(Hx \land \neg Fx)$
  Proof is exactly as for Barbari, replacing 'F' with '$\neg F$' throughout.
- **Cesaro.** Something is H. No F are G. All H are G. So: Some H is not F.
  $\exists x Hx, \forall x(Fx \to \neg Gx), \forall x(Hx \to Gx) \therefore \exists x(Hx \land \neg Fx)$

1	$\exists x Hx$	
2	$\forall x(Fx \to \neg Gx)$	
3	$\forall x(Hx \to Gx)$	
4	$\quad Ha$	
5	$\quad Ha \to Ga$	$\forall$E 3
6	$\quad Ga$	$\to$E 5, 4
7	$\quad Fa \to \neg Ga$	$\forall$E 2
8	$\qquad Fa$	
9	$\qquad \neg Ga$	$\to$E 7, 8
10	$\qquad \bot$	$\bot$I 6, 9
11	$\quad \neg Fa$	$\neg$I 8–10
12	$\quad Ha \land \neg Fa$	$\land$I 4, 11
13	$\quad \exists x(Hx \land \neg Fx)$	$\exists$I 12
14	$\exists x(Hx \land \neg Fx)$	$\exists$E 1, 4–13

- **Camestros.** Something is H. All F are G. No H are G. So: Some H is not F.
  $\exists x Hx, \forall x(Fx \to Gx), \forall x(Hx \to \neg Gx) \therefore \exists x(Hx \land \neg Fx)$

1	$\exists x Hx$	
2	$\forall x(Fx \to Gx)$	
3	$\forall x(Hx \to \neg Gx)$	
4	$\quad Ha$	
5	$\quad Ha \to \neg Ga$	$\forall$E 3
6	$\quad \neg Ga$	$\to$E 5, 4
7	$\quad Fa \to Ga$	$\forall$E 2
8	$\quad \neg Fa$	MT 7, 6
9	$\quad Ha \land \neg Fa$	$\land$I 4, 8
10	$\quad \exists x(Hx \land \neg Fx)$	$\exists$I 9
11	$\exists x(Hx \land \neg Fx)$	$\exists$E 1, 4–10

- **Felapton.** Something is G. No G are F. All G are H. So: Some H is not F.
  $\exists x Gx, \forall x(Gx \to \neg Fx), \forall x(Gx \to Hx) \therefore \exists x(Hx \land \neg Fx)$

1	$\exists x Gx$	
2	$\forall x(Gx \to \neg Fx)$	
3	$\forall x(Gx \to Hx)$	
4	$\quad Ga$	
5	$\quad Ga \to Ha$	$\forall$E 3
6	$\quad Ha$	$\to$E 5, 4
7	$\quad Ga \to \neg Fa$	$\forall$E 2
8	$\quad \neg Fa$	$\to$E 7, 4
9	$\quad Ha \land \neg Fa$	$\land$I 6, 8
10	$\quad \exists x(Hx \land \neg Fx)$	$\exists$I 9
11	$\exists x(Hx \land Fx)$	$\exists$E 1, 4–10

- **Darapti.** Something is G. All G are F. All G are H. So: Some H is F.
  $\exists x Gx, \forall x(Gx \to Fx), \forall x(Gx \to Hx) \therefore \exists x(Hx \land Fx)$
  Proof is exactly as for Felapton, replacing '$\neg F$' with 'F' throughout.

- **Calemos.** Something is H. All F are G. No G are H. So: Some H is not F.
  $\exists x Hx, \forall x(Fx \to Gx), \forall x(Gx \to \neg Hx) \therefore \exists x(Hx \land \neg Fx)$

1	$\exists x Hx$	
2	$\forall x(Fx \to Gx)$	
3	$\forall x(Gx \to \neg Hx)$	
4	$\quad Ha$	
5	$\quad Ga \to \neg Ha$	$\forall$E 3
6	$\quad\quad Ga$	
7	$\quad\quad \neg Ha$	$\to$E 5, 6
8	$\quad\quad \bot$	$\bot$I 4, 7
9	$\quad \neg Ga$	$\neg$I 6–8
10	$\quad Fa \to Ga$	$\forall$E 2
11	$\quad \neg Fa$	MT 10, 9
12	$\quad Ha \land \neg Fa$	$\land$I 4, 11
13	$\quad \exists x(Hx \land Fx)$	$\exists$I 12
14	$\exists x(Hx \land Fx)$	$\exists$E 1, 4–13

- **Fesapo.** Something is G. No F is G. All G are H. So: Some H is not F.
  $\exists x Gx, \forall x(Fx \to \neg Gx), \forall x(Gx \to Hx) \therefore \exists x(Hx \land \neg Fx)$

1	$\exists x Gx$	
2	$\forall x(Fx \to \neg Gx)$	
3	$\forall x(Gx \to Hx)$	
4	$\quad Ga$	
5	$\quad Ga \to Ha$	$\forall$E 3
6	$\quad Ha$	$\to$E 5, 4
7	$\quad Fa \to \neg Ga$	$\forall$E 2
8	$\quad\quad Fa$	
9	$\quad\quad \neg Ga$	$\to$E 7, 8
10	$\quad\quad \bot$	$\bot$I 4, 9
11	$\quad \neg Fa$	$\neg$I 8–10
12	$\quad Ha \land \neg Fa$	$\land$I 6, 11
13	$\quad \exists x(Hx \land Fx)$	$\exists$I 12
14	$\exists x(Hx \land Fx)$	$\exists$E 1, 4–13

- **Bamalip.** Something is F. All F are G. All G are H. So: Some H are F.
  $\exists x Fx, \forall x(Fx \to Gx), \forall x(Gx \to Hx) \therefore \exists x(Hx \land Fx)$

1	$\exists x Fx$	
2	$\forall x(Fx \to Gx)$	
3	$\forall x(Gx \to Hx)$	
4	$\quad Fa$	
5	$\quad Fa \to Ga$	$\forall$E 2
6	$\quad Ga$	$\to$E 5, 4
7	$\quad Ga \to Ha$	$\forall$E 3
8	$\quad Ha$	$\to$E 7, 6
9	$\quad Ha \land Fa$	$\land$I 8, 4
10	$\quad \exists x(Hx \land Fx)$	$\exists$I 9
11	$\exists x(Hx \land Fx)$	$\exists$E 1, 4–10

E. Provide a proof of each claim.

1. $\vdash \forall x Fx \lor \neg\forall x Fx$

1	$\quad \forall x Fx$	
2	$\quad \forall x Fx \lor \neg\forall x Fx$	$\lor$I 1
3	$\quad \neg\forall x Fx$	
4	$\quad \forall x Fx \lor \neg\forall x Fx$	$\lor$I 3
5	$\forall x Fx \lor \neg\forall x Fx$	TND 1–2, 3–4

2. $\vdash \forall z(Pz \lor \neg Pz)$

1	$\quad Pa$	
2	$\quad Pa \lor \neg Pa$	$\lor$I 1
3	$\quad \neg Pa$	
4	$\quad Pa \lor \neg Pa$	$\lor$I 3
5	$Pa \lor \neg Pa$	TND 1–2, 3–4
6	$\forall x(Px \lor \neg Px)$	$\forall$I 5

3. $\forall x(Ax \to Bx), \exists x Ax \vdash \exists x Bx$

1	$\forall x(Ax \to Bx)$	
2	$\exists x Ax$	
3	$\quad Aa$	
4	$\quad Aa \to Ba$	$\forall$E 1
5	$\quad Ba$	$\to$E 4, 3
6	$\quad \exists x Bx$	$\exists$I 5
7	$\exists x Bx$	$\exists$E 2, 3–6

4. $\forall x(Mx \leftrightarrow Nx), Ma \wedge \exists xRxa \vdash \exists xNx$

$$\begin{array}{rll}
1 & \mid \forall x(Mx \leftrightarrow Nx) & \\
2 & \mid \underline{Ma \wedge \exists xRxa} & \\
3 & \mid Ma & \wedge\text{E } 2 \\
4 & \mid Ma \leftrightarrow Na & \forall\text{E } 1 \\
5 & \mid Na & \leftrightarrow\text{E } 4, 3 \\
6 & \mid \exists xNx & \exists\text{I } 5
\end{array}$$

5. $\forall x\forall yGxy \vdash \exists xGxx$

$$\begin{array}{rll}
1 & \mid \underline{\forall x\forall yGxy} & \\
2 & \mid \forall yGay & \forall\text{E } 1 \\
3 & \mid Gaa & \forall\text{E } 2 \\
4 & \mid \exists xGxx & \exists\text{I } 3
\end{array}$$

6. $\vdash \forall xRxx \to \exists x\exists yRxy$

$$\begin{array}{rll}
1 & \mid \quad \mid \underline{\forall xRxx} & \\
2 & \mid \quad \mid Raa & \forall\text{E } 1 \\
3 & \mid \quad \mid \exists yRay & \exists\text{I } 2 \\
4 & \mid \quad \mid \exists x\exists yRxy & \exists\text{I } 3 \\
5 & \mid \forall xRxx \to \exists x\exists yRxy & \to\text{I } 1\text{–}4
\end{array}$$

7. $\vdash \forall y\exists x(Qy \to Qx)$

$$\begin{array}{rll}
1 & \mid \quad \mid \underline{Qa} & \\
2 & \mid \quad \mid Qa & \text{R } 1 \\
3 & \mid Qa \to Qa & \to\text{I } 1\text{–}2 \\
4 & \mid \exists x(Qa \to Qx) & \exists\text{I } 3 \\
5 & \mid \forall y\exists x(Qy \to Qx) & \forall\text{I } 4
\end{array}$$

8. $Na \to \forall x(Mx \leftrightarrow Ma), Ma, \neg Mb \vdash \neg Na$

$$\begin{array}{rll}
1 & \mid Na \to \forall x(Mx \leftrightarrow Ma) & \\
2 & \mid Ma & \\
3 & \mid \underline{\neg Mb} & \\
4 & \mid \quad \mid \underline{Na} & \\
5 & \mid \quad \mid \forall x(Mx \leftrightarrow Ma) & \to\text{E } 1, 4 \\
6 & \mid \quad \mid Mb \leftrightarrow Ma & \forall\text{E } 5 \\
7 & \mid \quad \mid Mb & \leftrightarrow\text{E } 6, 2 \\
8 & \mid \quad \mid \bot & \bot\text{I } 7, 3 \\
9 & \mid \neg Na & \neg\text{I } 4\text{–}8
\end{array}$$

9. $\forall x \forall y(Gxy \to Gyx) \vdash \forall x \forall y(Gxy \leftrightarrow Gyx)$

1	$\forall x \forall y(Gxy \to Gyx)$	
2	$\quad \vert\ Gab$	
3	$\quad \vert\ \forall y(Gay \to Gya)$	$\forall$E 1
4	$\quad \vert\ Gab \to Gba$	$\forall$E 3
5	$\quad \vert\ Gba$	$\to$E 4, 2
6	$\quad \vert\ Gba$	
7	$\quad \vert\ \forall y(Gby \to Gyb)$	$\forall$E 1
8	$\quad \vert\ Gba \to Gab$	$\forall$E 7
9	$\quad \vert\ Gab$	$\to$E 8, 6
10	$Gab \leftrightarrow Gba$	$\leftrightarrow$I 2–5, 6–9
11	$\forall y(Gay \leftrightarrow Gya)$	$\forall$I 10
12	$\forall x \forall y(Gxy \leftrightarrow Gyx)$	$\forall$I 11

10. $\forall x(\neg Mx \lor Ljx), \forall x(Bx \to Ljx), \forall x(Mx \lor Bx) \vdash \forall x Ljx$

1	$\forall x(\neg Mx \lor Ljx)$	
2	$\forall x(Bx \to Ljx)$	
3	$\forall x(Mx \lor Bx)$	
4	$\neg Ma \lor Ljx$	$\forall$E 1
5	$Ba \to Lja$	$\forall$E 2
6	$Ma \lor Ba$	$\forall$E 3
7	$\quad \vert\ \neg Ma$	
8	$\quad \vert\ Ba$	DS 6, 7
9	$\quad \vert\ Lja$	$\to$E 5, 8
10	$\quad \vert\ Lja$	
11	$\quad \vert\ Lja$	R 10
12	Lja	$\lor$E 4, 7–9, 10–11
13	$\forall x Ljx$	$\forall$I 12

F. Write a symbolisation key for the following argument, symbolise it, and prove it:

> There is someone who likes everyone who likes everyone that she likes. Therefore, there is someone who likes herself.

Symbolisation key:

$\mathcal{D} = \{x \mid \text{people}\}$
Lxy : _____$_x$ likes _____$_y$

$\exists x \forall y(\forall z(Lxz \to Lyz) \to Lxy) \therefore \exists x Lxx$

1	$\exists x \forall y(\forall z(Lxz \to Lyz) \to Lxy)$	
2	$\quad \forall y(\forall z(Laz \to Lyz) \to Lay)$	
3	$\quad \forall z(Laz \to Laz) \to Laa$	$\forall$E 2
4	$\quad\quad Lac$	
5	$\quad\quad Lac$	R 4
6	$\quad Lac \to Lac$	$\to$I 4–5
7	$\quad \forall z(Laz \to Laz)$	$\forall$I 6
8	$\quad Laa$	$\to$E 3, 7
9	$\quad \exists x Lxx$	$\exists$I 8
10	$\exists x Lxx$	$\exists$E 1, 2—9

G. For each of the following pairs of sentences: If they are provably equivalent, give proofs to show this. If they are not, construct an interpretation to show that they are not logically equivalent.

1. $\forall x Px \to Qc, \forall x(Px \to Qc)$ Not logically equivalent
   Countermodel

   $\mathcal{D} = \{1, 2\}$
   $I(c) = 1$
   $I(P) = \{1\}$
   $I(Q) = \{1\}$

2. $\forall x \forall y \forall z Bxyz, \forall x Bxxx$ Not logically equivalent
   Countermodel

   $\mathcal{D} = \{1, 2\}$
   $I(B) = \{\langle 1, 1, 1\rangle, \langle 2, 2, 2\rangle\}$

3. $\forall x \forall y Dxy, \forall y \forall x Dxy$ Provably equivalent

1	$\forall x \forall y Dxy$	
2	$\forall y Day$	$\forall$E 1
3	Dab	$\forall$E 2
4	$\forall x Dxb$	$\forall$I 3
5	$\forall y \forall x Dxy$	$\forall$I 4

1	$\forall y \forall x Dxy$	
2	$\forall x Dxa$	$\forall$E 1
3	Dba	$\forall$E 2
4	$\forall y Dby$	$\forall$I 3
5	$\forall x \forall y Dxy$	$\forall$I 4

4. $\exists x \forall y Dxy, \forall y \exists x Dxy$ Not logically equivalent
   Countermodel

   $\mathcal{D} = \{1, 2\}$
   $I(B) = \{\langle 1, 2\rangle, \langle 2, 1\rangle\}$

5. $\forall x(Rca \leftrightarrow Rxa), Rca \leftrightarrow \forall x Rxa$ Not logically equivalent Countermodel

   $\mathcal{D} = \{1, 2\}$
   $I(a) = 1$

$I(c) = 2$
$I(R) = \{\langle 1, 1\rangle\}$

H. For each of the following arguments: If it is valid in FOL, give a proof. If it is invalid, construct an interpretation to show that it is invalid.

1. $\exists y \forall x Rxy \therefore \forall x \exists y Rxy$ — Valid

1	$\exists y \forall x Rxy$	
2	$\quad \forall x Rxa$	
3	$\quad Rba$	$\forall$E 2
4	$\quad \exists y Rby$	$\exists$I 3
5	$\exists y Rby$	$\exists$E 1, 2–4
6	$\forall x \exists y Rxy$	$\forall$I 5

2. $\exists x(Px \wedge \neg Qx) \therefore \forall x(Px \to \neg Qx)$ — Not valid

Countermodel

$\mathcal{D} = \{1, 2\}$
$I(P) = \{1, 2\}$
$I(Q) = \{2\}$

3. $\forall x(Sx \to Ta), Sd \therefore Ta$ — Valid

1	$\forall x(Sx \to Ta)$	
2	Sd	
3	$Sd \to Ta$	$\forall$E 1
4	Ta	$\to$E 3, 2

4. $\forall x(Ax \to Bx), \forall x(Bx \to Cx) \therefore \forall x(Ax \to Cx)$ — Valid

1	$\forall x(Ax \to Bx)$	
2	$\forall x(Bx \to Cx)$	
3	$Aa \to Ba$	$\forall$E 1
4	$Ba \to Ca$	$\forall$E 2
5	$\quad Aa$	
6	$\quad Ba$	$\to$E 3, 5
7	$\quad Ca$	$\to$E 4, 6
8	$Aa \to Ca$	$\to$I 5–7
9	$\forall x(Ax \to Cx)$	$\forall$I 8

5. $\exists x(Dx \vee Ex), \forall x(Dx \to Fx) \therefore \exists x(Dx \wedge Fx)$ — Invalid Countermodel

$\mathcal{D} = \{1\}$
$I(D) = \{\}$
$I(E) = \{1\}$
$I(F) = \{1\}$

6. $\forall x \forall y(Rxy \vee Ryx) \therefore Rjj$ Valid

1	$\forall x \forall y(Rxy \vee Ryx)$	
2	$\forall y(Rjy \vee Ryj)$	$\forall$E 1
3	$Rjj \vee Rjj$	$\forall$E 2
4	$\quad Rjj$	
5	$\quad Rjj$	R 4
6	$\quad Rjj$	
7	$\quad Rjj$	R 6
8	Rjj	$\vee$E 3, 4–5, 6–7

7. $\exists x \exists y(Rxy \vee Ryx) \therefore Rjj$ Invalid Countermodel

$$\mathcal{D} = \{1, 2\}$$
$$I(j) = 2$$
$$I(R) = \{\langle 1, 1\rangle\}$$

8. $\forall x Px \rightarrow \forall x Qx, \exists x \neg Px \therefore \exists x \neg Qx$ Invalid Countermodel

$$\mathcal{D} = \{1\}$$
$$I(P) = \{\}$$
$$I(Q) = \{1\}$$

Conversion of quantifiers 36

A. Show that the following are jointly contrary:

1. $Sa \to Tm, Tm \to Sa, Tm \land \neg Sa$

1	$Sa \to Tm$	
2	$Tm \to Sa$	
3	$Tm \land \neg Sa$	
4	Tm	$\land$E 3
5	$\neg Sa$	$\land$E 3
6	Sa	$\to$E 2, 4
7	$\bot$	$\bot$I 5, 6

2. $\neg\exists x Rxa, \forall x \forall y Ryx$

1	$\neg\exists x Rxa$	
2	$\forall x \forall y Ryx$	
3	$\forall x \neg Rxa$	CQ 1
4	$\neg Rba$	$\forall$E 3
5	$\forall y Rya$	$\forall$E 2
6	Rba	$\forall$E 5
7	$\bot$	$\bot$I 6, 4

3. $\neg\exists x \exists y Lxy, Laa$

1	$\neg\exists x \exists y Lxy$	
2	Laa	
3	$\forall x \neg\exists y Lxy$	CQ 1
4	$\neg\exists y Lay$	$\forall$E 3
5	$\forall y \neg Lay$	CQ 4
6	$\neg Laa$	$\forall$E 5
7	$\bot$	$\bot$I 2, 6

4. $\forall x(Px \to Qx), \forall z(Pz \to Rz), \forall y Py, \neg Qa \land \neg Rb$

1	$\forall x(Px \to Qx)$	
2	$\forall z(Pz \to Rz)$	
3	$\forall y Py$	
4	$\neg Qa \land \neg Rb$	
5	$\neg Qa$	$\land$E 4
6	$Pa \to Qa$	$\forall$E 1
7	$\neg Pa$	MT 6, 5
8	Pa	$\forall$E 3
9	$\bot$	$\bot$I 8, 7

B. Show that each pair of sentences is provably equivalent:

1. $\forall x(Ax \to \neg Bx), \neg\exists x(Ax \land Bx)$

1	$\forall x(Ax \to \neg Bx)$	
2	$\exists x(Ax \land Bx)$	
3	$Aa \land Ba$	
4	Aa	$\land$E 3
5	Ba	$\land$E 3
6	$Aa \to \neg Ba$	$\forall$E 1
7	$\neg Ba$	$\to$E 6, 4
8	$\bot$	$\bot$I 5, 7
9	$\bot$	$\exists$E 2, 3–8
10	$\neg\exists x(Ax \land Bx)$	$\neg$I 2–9

1	$\neg\exists x(Ax \land Bx)$	
2	$\forall x \neg(Ax \land Bx)$	CQ 1
3	$\neg(Aa \land Ba)$	$\forall$E 2
4	Aa	
5	Ba	
6	$Aa \land Ba$	$\land$I 4, 5
7	$\bot$	$\bot$I 6, 3
8	$\neg Ba$	$\neg$I 5–7
9	$Aa \to \neg Ba$	$\to$I 4–8
10	$\forall x(Ax \to \neg Bx)$	$\forall$I 9

2. $\forall x(\neg Ax \to Bd), \forall x Ax \lor Bd$

1	$\forall x(\neg Ax \to Bd)$	
2	$\neg Aa \to Bd$	$\forall$E 1
3	Bd	
4	$\forall x Ax \lor Bd$	$\lor$I 6
5	$\neg Bd$	
6	$\neg\neg Aa$	MT 2, 5
7	Aa	DNE 6
8	$\forall x Ax$	$\forall$E 7
9	$\forall x Ax \lor Bd$	$\lor$I 8
10	$\forall x Ax \lor Bd$	TND 3–4, 5–9

1	$\forall x Ax \lor Bd$	
2	$\neg Aa$	
3	$\forall x Ax$	
4	Aa	$\forall$E 3
5	$\bot$	$\bot$I 4, 2
6	$\neg\forall x Ax$	$\neg$I 3–5
7	Bd	DS 1, 6
8	$\neg Aa \to Bd$	$\to$I 2–7
9	$\forall x(Ax \to Bd)$	$\forall$I 8

C. In §18, I considered what happens when we move quantifiers 'across' various logical operators. Show that each pair of sentences is provably equivalent:

1. $\forall x(Fx \wedge Ga), \forall x Fx \wedge Ga$

1	$\forall x(Fx \wedge Ga)$	
2	$Fb \wedge Ga$	$\forall$E 1
3	Fb	$\wedge$E 2
4	Ga	$\wedge$E 6
5	$\forall x Fx$	$\forall$I 3
6	$\forall x Fx \wedge Ga$	$\wedge$I 5, 4

1	$\forall x Fx \wedge Ga$	
2	$\forall x Fx$	$\wedge$E 1
3	Ga	$\wedge$E 1
4	Fb	$\forall$E 2
5	$Fb \wedge Ga$	$\wedge$I 4, 3
6	$\forall x(Fx \wedge Ga)$	$\forall$I 5

2. $\exists x(Fx \vee Ga), \exists x Fx \vee Ga$

1	$\exists x(Fx \vee Ga)$	
2	$\quad Fb \vee Ga$	
3	$\quad\quad Fb$	
4	$\quad\quad \exists x Fx$	$\exists$I 3
5	$\quad\quad \exists x Fx \vee Ga$	$\vee$I 4
6	$\quad\quad Ga$	
7	$\quad\quad \exists x Fx \vee Ga$	$\vee$I 6
8	$\quad \exists x Fx \vee Ga$	$\vee$E 2, 3–5, 6–7
9	$\exists x Fx \vee Ga$	$\exists$E 1, 2–8

1	$\exists x Fx \vee Ga$	
2	$\quad \exists x Fx$	
3	$\quad\quad Fb$	
4	$\quad\quad Fb \vee Ga$	$\vee$I 3
5	$\quad\quad \exists x(Fx \vee Ga)$	$\exists$I 4
6	$\quad \exists x(Fx \vee Ga)$	$\exists$E 2, 3–5
7	$\quad Ga$	
8	$\quad Fb \vee Ga$	$\vee$I 7
9	$\quad \exists x(Fx \vee Ga)$	$\exists$I 8
10	$\exists x(Fx \vee Ga)$	$\vee$E 1, 2–6, 7–9

3. $\forall x(Ga \to Fx), Ga \to \forall x Fx$

1	$\forall x(Ga \to Fx)$	
2	$Ga \to Fb$	$\forall$E 1
3	$\quad Ga$	
4	$\quad Fb$	$\to$E 2, 3
5	$\quad \forall x Fx$	$\forall$I 4
6	$Ga \to \forall x Fx$	$\to$I 3–5

1	$Ga \to \forall x Fx$	
2	$\quad Ga$	
3	$\quad \forall x Fx$	$\to$E 1, 2
4	$\quad Fb$	$\forall$E 3
5	$Ga \to Fb$	$\to$I 2–4
6	$\forall x(Ga \to Fx)$	$\forall$I 5

4. $\forall x(Fx \to Ga), \exists x Fx \to Ga$

1	$\forall x(Fx \to Ga)$	
2	$\quad \exists x Fx$	
3	$\quad\quad Fb$	
4	$\quad\quad Fb \to Ga$	$\forall$E 1
5	$\quad\quad Ga$	$\to$E 4, 3
6	$\quad Ga$	$\exists$E 2, 3–5
7	$\exists x Fx \to Ga$	$\to$I 2–6

1	$\exists x Fx \to Ga$	
2	$\quad Fb$	
3	$\quad \exists x Fx$	$\exists$I 2
4	$\quad Ga$	$\to$E 1, 3
5	$Fb \to Ga$	$\to$I 2–4
6	$\forall x(Fx \to Ga)$	$\forall$I 5

5. $\exists x(Ga \to Fx), Ga \to \exists x Fx$

1	$\exists x(Ga \to Fx)$	
2	$\quad Ga$	
3	$\quad\quad Ga \to Fb$	
4	$\quad\quad Fb$	$\to$E 3, 2
5	$\quad\quad \exists x Fx$	$\exists$I 4
6	$\quad \exists x Fx$	$\exists$E 1, 3–5
7	$Ga \to \exists x Fx$	$\to$I 2–6

1	$Ga \to \exists x Fx$	
2	$\quad Ga$	
3	$\quad \exists x Fx$	
4	$\quad\quad Fb$	
5	$\quad\quad\quad Ga$	
6	$\quad\quad\quad Fb$	R 4
7	$\quad\quad Ga \to Fb$	$\to$I 5–6
8	$\quad\quad \exists x(Ga \to Fx)$	$\exists$I 7
9	$\quad \exists x(Ga \to Fx)$	$\exists$E 3, 4–8
10	$\quad \neg Ga$	
11	$\quad\quad Ga$	
12	$\quad\quad \bot$	$\bot$I 11, 10
13	$\quad\quad Fb$	$\bot$E 12
14	$\quad Ga \to Fb$	$\to$E 11–13
15	$\quad \exists x(Ga \to Fx)$	$\exists$I 14
16	$\exists x(Ga \to Fx)$	TND 2–9, 10–15

6. $\exists x(Fx \to Ga), \forall x Fx \to Ga$

1	$\mid\ \exists x(Fx \to Ga)$	
2	$\mid\ \mid\ \forall x Fx$	
3	$\mid\ \mid\ \mid\ Fb \to Ga$	
4	$\mid\ \mid\ \mid\ Fb$	$\forall$E 2
5	$\mid\ \mid\ \mid\ Ga$	$\to$E 3, 4
6	$\mid\ \mid\ Ga$	$\exists$E 1, 3–5
7	$\mid\ \forall x Fx \to Ga$	$\to$I 2–6

1	$\mid\ \forall x Fx \to Ga$	
2	$\mid\ \mid\ \forall x Fx$	
3	$\mid\ \mid\ Ga$	$\to$E 1, 2
4	$\mid\ \mid\ \mid\ Fb$	
5	$\mid\ \mid\ \mid\ Ga$	R 3
6	$\mid\ \mid\ Fb \to Ga$	$\to$I 4–5
7	$\mid\ \mid\ \exists x(Fx \to Ga)$	$\exists$I 6
8	$\mid\ \mid\ \neg\forall x Fx$	
9	$\mid\ \mid\ \exists x \neg Fx$	CQ 8
10	$\mid\ \mid\ \mid\ \neg Fb$	
11	$\mid\ \mid\ \mid\ \mid\ Fb$	
12	$\mid\ \mid\ \mid\ \mid\ \bot$	$\bot$I 11, 10
13	$\mid\ \mid\ \mid\ \mid\ Ga$	$\bot$E 12
14	$\mid\ \mid\ \mid\ Fb \to Ga$	$\to$I 11–13
15	$\mid\ \mid\ \mid\ \exists x(Fx \to Ga)$	$\exists$I 14
16	$\mid\ \mid\ \exists x(Fx \to Ga)$	$\exists$E 9, 10–15
17	$\mid\ \exists x(Fx \to Ga)$	TND 2–7, 8–16

NB: the variable 'x' does not occur in 'Ga'.

When all the quantifiers occur at the beginning of a sentence, that sentence is said to be in *prenex normal form*. Together with the CQ rules, these equivalences are sometimes called *prenexing rules*, since they give us a means for putting any sentence into prenex normal form.

Rules for identity 37

A. Provide a proof of each claim.

1. $Pa \vee Qb, Qb \to b = c, \neg Pa \vdash Qc$

1	$Pa \vee Qb$	
2	$Qb \to b = c$	
3	$\neg Pa$	
4	Qb	DS 1, 3
5	$b = c$	$\to$E 2, 4
6	Qc	=E 5, 4

2. $m = n \vee n = o, An \vdash Am \vee Ao$

1	$m = n \vee n = o$	
2	An	
3	$m = n$	
4	Am	=E 3, 2
5	$Am \vee Ao$	$\vee$I 4
6	$n = o$	
7	Ao	=E 6, 7
8	$Am \vee Ao$	$\vee$I 7
9	$Am \vee Ao$	$\vee$E 1, 3–5, 6–8

3. $\forall x\ x = m, Rma \vdash \exists x Rxx$

1	$\forall x\ x = m$	
2	Rma	
3	$a = m$	$\forall$E 1
4	Raa	=E 3, 2
5	$\exists x Rxx$	$\exists$I 4

4. $\forall x \forall y(Rxy \to x = y) \vdash Rab \to Rba$

1	$\forall x \forall y(Rxy \to x = y)$	
2	$\quad\mid\ Rab$	
3	$\quad\mid\ \forall y(Ray \to a = y)$	$\forall$E 1
4	$\quad\mid\ Rab \to a = b$	$\forall$E 3
5	$\quad\mid\ a = b$	$\to$E 4, 2
6	$\quad\mid\ Raa$	$=$E 5, 2
7	$\quad\mid\ Rba$	$=$E 5, 6
8	$Rab \to Rba$	$\to$I 2–7

5. $\neg\exists x \neg x = m \vdash \forall x \forall y(Px \to Py)$

1	$\neg\exists x \neg x = m$	
2	$\forall x \neg\neg x = m$	CQ 1
3	$\neg\neg a = m$	$\forall$E 2
4	$a = m$	DNE 3
5	$\neg\neg b = m$	$\forall$E 2
6	$b = m$	DNE 5
7	$\quad\mid\ Pa$	
8	$\quad\mid\ Pm$	$=$E 3, 7
9	$\quad\mid\ Pb$	$=$E 5, 8
10	$Pa \to Pb$	$\to$I 7–9
11	$\forall y(Pa \to Py)$	$\forall$I 10
12	$\forall x \forall y(Px \to Py)$	$\forall$I 11

6. $\exists x Jx, \exists x \neg Jx \vdash \exists x \exists y\ \neg x = y$

1	$\exists x Jx$	
2	$\exists x \neg Jx$	
3	$\quad\mid\ Ja$	
4	$\quad\mid\quad\mid\ \neg Jb$	
5	$\quad\mid\quad\mid\quad\mid\ a = b$	
6	$\quad\mid\quad\mid\quad\mid\ Jb$	$=$E 5, 3
7	$\quad\mid\quad\mid\quad\mid\ \bot$	$\bot$I 6, 4
8	$\quad\mid\quad\mid\ \neg a = b$	$\neg$I 5–7
9	$\quad\mid\quad\mid\ \exists y \neg a = y$	$\exists$I 8
10	$\quad\mid\quad\mid\ \exists x \exists y \neg x = y$	$\exists$I 9
11	$\quad\mid\ \exists x \exists y \neg x = y$	$\exists$E 2, 4–10
12	$\exists x \exists y \neg x = y$	$\exists$E 1, 3–11

7. $\forall x(x = n \leftrightarrow Mx), \forall x(Ox \vee \neg Mx) \vdash On$

1	$\forall x(x = n \leftrightarrow Mx)$	
2	$\forall x(Ox \vee \neg Mx)$	
3	$n = n \leftrightarrow Mn$	$\forall$E 1
4	$n = n$	=I
5	Mn	$\leftrightarrow$E 3, 4
6	$On \vee \neg Mn$	$\forall$E 2
7	$\quad \neg On$	
8	$\quad \neg Mn$	DS 6, 7
9	$\quad \bot$	$\bot$I 5, 8
10	$\neg\neg On$	$\neg$I 7–9
11	On	DNE 10

8. $\exists x Dx, \forall x(x = p \leftrightarrow Dx) \vdash Dp$

1	$\exists x Dx$	
2	$\forall x(x = p \leftrightarrow Dx)$	
3	$\quad Dc$	
4	$\quad c = p \leftrightarrow Dc$	$\forall$E 2
5	$\quad c = p$	$\leftrightarrow$E 4, 3
6	$\quad Dp$	=E 5, 3
7	Dp	$\exists$E 1, 3–6

9. $\exists x\big[(Kx \wedge \forall y(Ky \to x = y)) \wedge Bx\big], Kd \vdash Bd$

1	$\exists x\big[(Kx \wedge \forall y(Ky \to x = y) \wedge Bx\big]$	
2	Kd	
3	$\quad (Ka \wedge \forall y(Ky \to a = y)) \wedge Ba$	
4	$\quad Ka \wedge \forall y(Ky \to a = y)$	$\wedge$E 3
5	$\quad Ka$	$\wedge$E 4
6	$\quad \forall y(Ky \to a = y)$	$\wedge$E 4
7	$\quad Kd \to a = d$	$\forall$E 6
8	$\quad a = d$	$\to$E 7, 2
9	$\quad Ba$	$\wedge$E 3
10	$\quad Bd$	=E 8, 9
11	Bd	$\exists$E 1, 3–10

10. $\vdash Pa \to \forall x(Px \lor \neg x = a)$

1	Pa	
2	$b = a$	
3	Pb	=E 2, 1
4	$Pb \lor \neg b = a$	∨I 3
5	$\neg b = a$	
6	$Pb \lor \neg b = a$	∨I 5
7	$Pb \lor \neg b = a$	TND 2–4, 5–6
8	$\forall x(Px \lor \neg x = a)$	∀I 7
9	$Pa \to \forall x(Px \lor \neg x = a)$	→I 1–8

B. Show that the following are provably equivalent:

- $\exists x\big([Fx \land \forall y(Fy \to x = y)] \land x = n\big)$
- $Fn \land \forall y(Fy \to n = y)$

And hence that both have a decent claim to symbolise the English sentence 'Nick is the F'.

In one direction:

1	$\exists x\big([Fx \land \forall y(Fy \to x = y)] \land x = n\big)$	
2	$[Fa \land \forall y(Fy \to a = y)] \land a = n$	
3	$a = n$	∧E 2
4	$Fa \land \forall y(Fy \to a = y)$	∧E 2
5	Fa	∧E 4
6	Fn	=E 3, 5
7	$\forall y(Fy \to a = y)$	∧E 4
8	$\forall y(Fy \to n = y)$	=E 3, 7
9	$Fn \land \forall y(Fy \to n = y)$	∧I 6, 8
10	$Fn \land \forall y(Fy \to n = y)$	∃E 1, 2–9

And now in the other:

1	$Fn \land \forall y(Fy \to n = y)$	
2	$n = n$	=I
3	$[Fn \land \forall y(Fy \to n = y)] \land n = n$	∧I 1, 2
4	$\exists x\big([Fx \land \forall y(Fy \to x = y)] \land x = n\big)$	∃I 3

C. In §20, I claimed that the following are logically equivalent symbolisations of the English sentence 'there is exactly one F':

- $\exists x Fx \land \forall x\forall y\big[(Fx \land Fy) \to x = y\big]$
- $\exists x\big[Fx \land \forall y(Fy \to x = y)\big]$
- $\exists x\forall y(Fy \leftrightarrow x = y)$

Show that they are all provably equivalent. (*Hint*: to show that three claims are provably equivalent, it suffices to show that the first proves the second, the second proves the third and the third proves the first; think about why.)

It suffices to show that the first proves the second, the second proves the third and the third proves the first, for we can then show that any of them prove any others, just by chaining the proofs together (numbering lines, where necessary. Armed with this, we start on the first proof:

1	$\exists x Fx \wedge \forall x \forall y\big[(Fx \wedge Fy) \rightarrow x = y\big]$	
2	$\exists x Fx$	$\wedge$E 1
3	$\forall x \forall y\big[(Fx \wedge Fy) \rightarrow x = y\big]$	$\wedge$E 1
4	$\quad Fa$	
5	$\quad \forall y\big[(Fa \wedge Fy) \rightarrow a = y\big]$	$\forall$E 3
6	$\quad (Fa \wedge Fb) \rightarrow a = b$	$\forall$E 5
7	$\quad\quad Fb$	
8	$\quad\quad Fa \wedge Fb$	$\wedge$I 4, 7
9	$\quad\quad a = b$	$\rightarrow$E 6, 8
10	$\quad Fb \rightarrow a = b$	$\rightarrow$I 7–9
11	$\quad \forall y(Fy \rightarrow a = y)$	$\forall$I 10
12	$\quad Fa \wedge \forall y(Fy \rightarrow a = y))$	$\wedge$I 4, 11
13	$\quad \exists x\big[Fx \wedge \forall y(Fy \rightarrow x = y)\big]$	$\exists$I 12
14	$\exists x\big[Fx \wedge \forall y(Fy \rightarrow x = y)\big]$	$\exists$E 2, 4–13

Now for the second proof:

1	$\exists x\big[Fx \wedge \forall y(Fy \rightarrow x = y)\big]$	
2	$\quad Fa \wedge \forall y(Fy \rightarrow a = y)$	
3	$\quad Fa$	$\wedge$E 2
4	$\quad \forall y(Fy \rightarrow a = y)$	$\wedge$E 2
5	$\quad\quad Fb$	
6	$\quad\quad Fb \rightarrow a = b$	$\forall$E 4
7	$\quad\quad a = b$	$\rightarrow$E 6, 5
8	$\quad\quad a = b$	
9	$\quad\quad Fb$	$=$E 8, 3
10	$\quad Fb \leftrightarrow a = b$	$\leftrightarrow$I 5–7, 8–9
11	$\quad \forall y(Fy \leftrightarrow a = y)$	$\forall$I 10
12	$\quad \exists x \forall y(Fy \leftrightarrow x = y)$	$\exists$I 11
13	$\exists x \forall y(Fy \leftrightarrow x = y)$	$\exists$E 1, 2–12

And finally, the third proof:

1	$\exists x \forall y(Fy \leftrightarrow x = y)$	
2	$\mid\ \forall y(Fy \leftrightarrow a = y)$	
3	$\mid\ Fa \leftrightarrow a = a$	$\forall$E 2
4	$\mid\ a = a$	=I
5	$\mid\ Fa$	$\leftrightarrow$E 3, 4
6	$\mid\ \exists x Fx$	$\exists$I 5
7	$\mid\ \mid\ Fb \wedge Fc$	
8	$\mid\ \mid\ Fb$	$\wedge$E 7
9	$\mid\ \mid\ Fb \leftrightarrow a = b$	$\forall$E 2
10	$\mid\ \mid\ a = b$	$\leftrightarrow$E 9, 8
11	$\mid\ \mid\ Fc$	$\wedge$E 7
12	$\mid\ \mid\ Fc \leftrightarrow a = c$	$\forall$E 2
13	$\mid\ \mid\ a = c$	$\leftrightarrow$E 12, 11
14	$\mid\ \mid\ b = c$	=E 10, 13
15	$\mid\ (Fb \wedge Fc) \rightarrow b = c$	$\rightarrow$I 8–14
16	$\mid\ \forall y\big[(Fb \wedge Fy) \rightarrow b = y\big]$	$\forall$I 15
17	$\mid\ \forall x \forall y\big[(Fx \wedge Fy) \rightarrow x = y\big]$	$\forall$I 16
18	$\mid\ \exists x Fx \wedge \forall x \forall y\big[(Fx \wedge Fy) \rightarrow x = y\big]$	$\wedge$I 6, 17
19	$\exists x Fx \wedge \forall x \forall y\big[(Fx \wedge Fy) \rightarrow x = y\big]$	$\exists$E 1, 2–18

D. Symbolise the following argument

> There is exactly one F. There is exactly one G. Nothing is both F and G. So: there are exactly two things that are either F or G.

And offer a proof of it.

Here's the symbolisation, the proof will come over the page:

$\exists x\big[Fx \wedge \forall y(Fy \rightarrow x = y)\big]$,
$\exists x\big[Gx \wedge \forall y(Gy \rightarrow x = y)\big]$,
$\forall x(\neg Fx \vee \neg Gx) \therefore$
$\exists x \exists y\big[\neg x = y \wedge \forall z((Fz \vee Gz) \rightarrow (x = z \vee y = z))\big]$

1	$\exists x\big[Fx \wedge \forall y(Fy \to x = y)\big]$	
2	$\exists x\big[Gx \wedge \forall y(Gy \to x = y)\big]$	
3	$\forall x(\neg Fx \vee \neg Gx)$	
4	$Fa \wedge \forall y(Fy \to a = y)$	
5	Fa	$\wedge$E 4
6	$\forall y(Fy \to a = y)$	$\wedge$E 4
7	$\neg Fa \vee \neg Ga$	$\forall$E 3
8	$\neg Ga$	DS 7, 5
9	$Gb \wedge \forall y(Gy \to b = y)$	
10	Gb	$\wedge$E 9
11	$\forall y(Gy \to b = y)$	$\wedge$E 9
12	$a = b$	
13	Ga	=E 12, 10
14	$\bot$	$\bot$I 13, 8
15	$\neg a = b$	$\neg$I 12–14
16	$Fc \vee Gc$	
17	Fc	
18	$Fc \to a = c$	$\forall$E 6
19	$a = c$	$\to$E 18, 17
20	$a = c \vee b = c$	$\vee$I 19
21	Gc	
22	$Gc \to b = c$	$\forall$E 11
23	$b = c$	$\to$E 22, 21
24	$a = c \vee b = c$	$\vee$I 23
25	$a = c \vee b = c$	$\vee$E 16, 17–20, 21–24
26	$(Fc \vee Gc) \to (a = c \vee b = c)$	$\to$I 16–25
27	$\forall z((Fz \vee Gz) \to (a = z \vee b = z))$	$\forall$I 26
28	$\neg a = b \wedge \forall z((Fz \vee Gz) \to (a = z \vee b = z))$	$\wedge$I 15, 27
29	$\exists y\big[\neg a = y \wedge \forall z((Fz \vee Gz) \to (a = z \vee y = z))\big]$	$\exists$I 28
30	$\exists x \exists y\big[\neg x = y \wedge \forall z((Fz \vee Gz) \to (x = z \vee y = z))\big]$	$\exists$I 29
31	$\exists x \exists y\big[\neg x = y \wedge \forall z((Fz \vee Gz) \to (x = z \vee y = z))\big]$	$\exists$E 2, 9–30
32	$\exists x \exists y\big[\neg x = y \wedge \forall z((Fz \vee Gz) \to (x = z \vee y = z))\big]$	$\exists$E 1, 4–31

Derived rules 38

A. Offer proofs which justify the addition of the third and fourth CQ rules as derived rules.

Justification for the third rule:

1	$\neg\exists x Ax$	
2	$\quad Aa$	
3	$\quad \exists x Ax$	$\exists$I 2
4	$\quad \bot$	$\bot$I 3, 1
5	$\neg Aa$	$\neg$I 2–4
6	$\forall x \neg Ax$	$\forall$I 5

Justification for the fourth rule:

1	$\forall x \neg Ax$	
2	$\quad \exists x Ax$	
3	$\quad\quad Aa$	
4	$\quad\quad \neg Aa$	$\forall$E 1
5	$\quad\quad \bot$	$\bot$I 3, 4
6	$\quad \bot$	$\exists$E 2, 3–5
7	$\neg\exists x Ax$	$\neg$I 2–6