Predator Prey Systems[*]

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Abstract. In this report we study the Lotka-Volterra model of simulating predator prey interactions. Given the nature of how this model is defined, we chose to look at two different variations of the model, one with exponential growth and the other with logistic growth. After examining the physical meaning of the system we moved into finding the steady state of each assumption. We also took a look at the stability at those points to get insight on the behavior of each system. Finally, we created an ODE solving algorithm to simulate different scenarios for both systems and went through an analysis of the data. We found out how the parameter affect the system and the limitations of each system

1 Introduction

The Lotka-Volterra model was proposed by mathematician Alfred Lotka in 1910 to describe autocatalytic reactions, but was later used to simulate predator prey relationships in 1920. The model itself is rather simplistic, even though it is a system of nonlinear ODE's; only having a few adjustable parameters and keeping most other variables constant. The goal of this set of model is to represent populations of predators and prey as they develop over time. Our goal is to pick apart the meanings of all these variables and see what their physical meanings are. Given that this is a system of ODE's we can figure out quite a lot about it using some pretty basic linear algebra techniques.

2 The Basic Model

The Lotka-Volterra (L-V) model is a system of ODE's that come as

$$\frac{dx}{dt} = ax - bxy$$
$$\frac{dy}{dt} = -cy + dxy$$

where the parameters a, b, c, d are undetermined, positive, real valued coefficients. For this system x and y correspond to prey and predator, respectively.

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The coefficients are physical and its important to understand what each one means to our system.

$a \rightarrow$ the growth rate of the prey independent of the predator

$b \rightarrow$ the rate at which the predator and prey meet in the system

$c \rightarrow$ the rate at which the predator dies off

$d \rightarrow$ the growth rate of the predator given the amount of prey they consume.

With these four coefficients defined we can see what happens with the system of ODE's. The rate of prey growth is determined by exponential growth in population offset by the interactions between predator and prey. The rate of predator growth is determined by the amount of prey they consume that too offset by the rate at which the population dies off.

3 A More Detailed Model

Now we'll modify our model only slightly, giving our prey a logistic growth which is a much more accurate way of representing population growth. The system of ODE's becomes

$$\frac{dx}{dt} = ax\frac{K-x}{K} - bxy$$
$$\frac{dy}{dt} = -cy + dxy$$

Where all the parameters are the same as before, but now we have this added K term. This term corresponds to the carrying capacity of the prey. Before, the prey had exponential growth which is unbounded and clearly that's non physical; this added term makes the system more accurate.

4 Moving Forward with Stability

Before we can get started on modeling the system we first need to find the stable solution set so that we can have the proper conditions for our system. In order to get a stable solution set we must first figure out the steady state solutions of this system of ordinary differential equations. Clearly there are multiple steady states, with the trivial case being when the initial conditions are $(0, 0)$ (there are no predators or prey). This is not interesting! The interesting case is when the rate of change of both the predator and prey is 0. This implies a system or equilibrium where the predator and prey adjust over time. To solve for this case we say that $x' = y' = 0$ and find relations for both quantities in terms of the undetermined coefficients. First, we'll look at the simpler model, the normal L-V.

$$x' = 0 \rightarrow ax = bxy \rightarrow y = \frac{a}{b}$$
$$y' = 0 \rightarrow cy = dxy \rightarrow x = \frac{c}{d},$$

Since we know the points where its a steady state we can define the jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} a - by & -bx \\ dy & -c + dx \end{pmatrix}$$

which at the steady state is

$$J_{(c/d),(a/b)} = \begin{pmatrix} 0 & -\frac{bc}{d} \\ \frac{da}{b} & 0 \end{pmatrix}$$

From elementary linear algebra we know that the Jacobian matrix has an eigenvalue λ such that $Jx = \lambda x$. From here we can find the eigenvalues by

$$\det\big(J - I\lambda\big) = 0 \rightarrow \begin{vmatrix} -\lambda & -\frac{bc}{d} \\ \frac{da}{b} & -\lambda \end{vmatrix} = \lambda^2 + ac = 0 \rightarrow \lambda = \pm\sqrt{-ac}$$

remember that a, c are both real valued coefficients so that means that at the steady state we'll have periodic solutions. Next, let's go through the same process for the logistic growth model.

$$\begin{aligned} x' &= 0 \rightarrow ax\frac{K - x}{K} = bxy \rightarrow y = \frac{a}{b}\Big(1 - \frac{c}{dK}\Big) \\ y' &= 0 \rightarrow cy = dxy \rightarrow x = \frac{c}{d}, \end{aligned}$$

Let's also define the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} a - \frac{2ax}{K} - by & -bx \\ dy & -c + dx \end{pmatrix}$$

where we evaluate it at the steady state

$$J_{(c/d),((a/b)(1-c/dK))} = \begin{pmatrix} -\frac{ac}{dK} & -\frac{bc}{d} \\ \frac{da}{b}\Big(1 - \frac{c}{dK}\Big) & 0 \end{pmatrix}$$

and then find the determinate like before

$$\det\big(J - I\lambda\big) = 0 \rightarrow \begin{vmatrix} -\frac{ac}{dK} - \lambda & -\frac{bc}{d} \\ \frac{da}{b}\Big(1 - \frac{c}{dK}\Big) & -\lambda \end{vmatrix} = \lambda^2 + \frac{ac}{dK}\lambda + ac\Big(1 - \frac{c}{dK}\Big) = 0$$

To find the eigenvalue, we need to use the quadratic equation

$$\lambda = \frac{1}{2}\Bigg(-\frac{ac}{dK} \pm \sqrt{\Big(\frac{ac}{dK}\Big)^2 - 4ac\Big(1 - \frac{c}{dK}\Big)}\Bigg).$$

First, lets look at the term in the square root

$$\Big(\frac{ac}{dK}\Big)^2 - 4ac\Big(1 - \frac{c}{dK}\Big).$$

We need to find conditions on which this term is negative and positive. If the sum is positive then we have all real roots and if its negative then we have complex roots. We have the condition that if $1 < c/dK$ then the term inside the square root is positive. That means that

$$\sqrt{\Big(\frac{ac}{dK}\Big)^2 - 4ac\Big(1 - \frac{c}{dK}\Big)} > \frac{ac}{dK}$$

so we'll get that the real value of the eigenvalues take both positive and negative values so it is not stable at that condition.

Now let's take the condition that $1 > c/dK$. It needs to be noted that there are two outcomes here, either the value under the square root is negative or its positive and the magnitude is less than ac/dK. The reason why this is important is because either way we'll get stable solutions because for both cases we'll have $RE(\lambda) < 0$. First, let's look at the case where we want the sum to be negative.

$$\Big(\frac{ac}{dK}\Big)^2 - 4ac\Big(1 - \frac{c}{dK}\Big) < a^2 - 4ac \rightarrow a < 4c$$

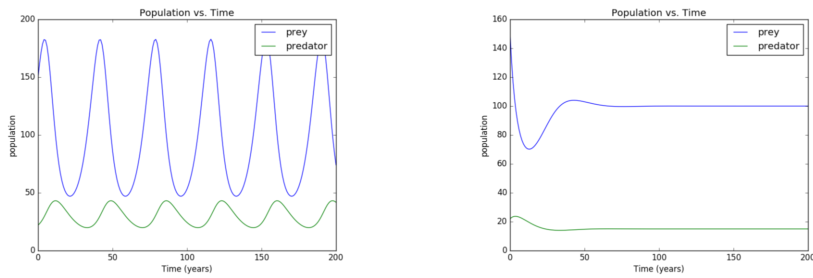
and if we want it to where its positive the resulting inequality is just the opposite of what we have, i.e. $a > 4c$. The previous will contain a complex part while the latter has only real values. In order to have stable solutions then we must have

$$1 > \frac{c}{dk}$$

Comparing the two systems we can see that adding in the logistic growth factor changes the behavior of the system. For the normal L-V, we saw that we only had solutions of the periodic form. In contrast, the logistic growth version has stable solutions, where the prey and predator population will reach a point of neither growth nor decay. That being said for certain parameter values, $a < 4c$, we will see some periodic behavior because those eigenvalues will contain complex parts, but will still be stable, i.e. zero out to a constant.

5 Using The Models

Finally we can get to solving these systems to simulate a basic predator prey model. Lets first just compare the two models with the same parameter values, except for K since its not in the original L-V system.



exponential growth with parameter values $a = .3, b = .01, c = .1, d = .001$

logistic growth with parameter values $a = .3, b = .01, c = .1, d = .001, K = 200$

Fig. 2: Exponential and logistic growth of the predator-prey model

So from our stability analysis we can see that our solutions match up. The figure on the left has the periodic solutions that we derived and also have seen before while the figure on the right has both the predators and prey converge to a constant value. Let's look at the exponential growth model where we'll tweak the parameters slightly to see how the system changes. We'll treat the system we already generated as the control group. As you can see in fig. 3, We chose

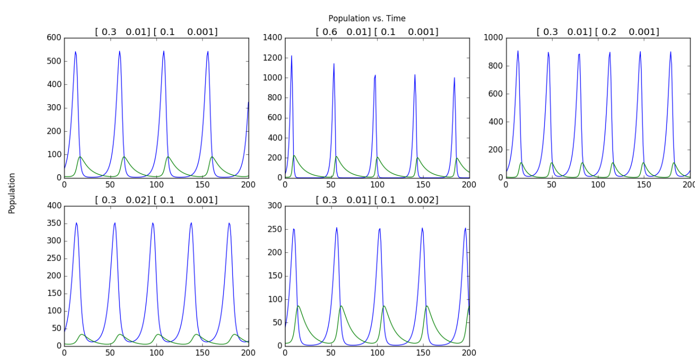


Fig. 3: Exponential growth model of various parameter values. Note that for this analysis prey are represented with a blue curves while the predators will be green curves. Also the titles show the parameter values of the form $[a, b][c, d]$

to double each rate individually to see the specific change each one has on the system. The information we get out of it is

Increase a : Prey and predator population double, period increases slightly.
Increase b : Prey and predator populations drop by about half.
Increase c : Prey and predator population roughly double, period increases noticeably.
Increase d : Prey population drops drastically, predator population stays the same.

Going through each one, when we increase a its pretty clear that we should see an increase in both predator and prey populations since there will be more prey for the predators to consume; since we left encounter rates the same, more prey will be around. When we double b we again see pretty obvious behavior. The two species will interact more which will lead to more prey being consumed, but since we never increased the prey birth rate there will be less prey for the predators. c increasing is interesting, because I would've assumed that we may have lost some predator population, given that they die off at a faster rate. The reasoning for the contrary seems to be that since there is more prey around the term dxy in the predator growth equation will be a lot larger, making the population higher. Finally, d increasing really destroyed the stability of the system. This makes sense since if we have to many predators they will have to die out at a rate of $-cy$. It interesting to see how damaging this increase is to the system. From these plots we see that each parameter really has its own impact on the system. a and c saw great increase in population levels, while b and d saw pretty damaging affects to the population.

Before we move on to the logistic growth model, we should increase both a and c to see if we can notice any change in the period. Fig. 4 shows us what we

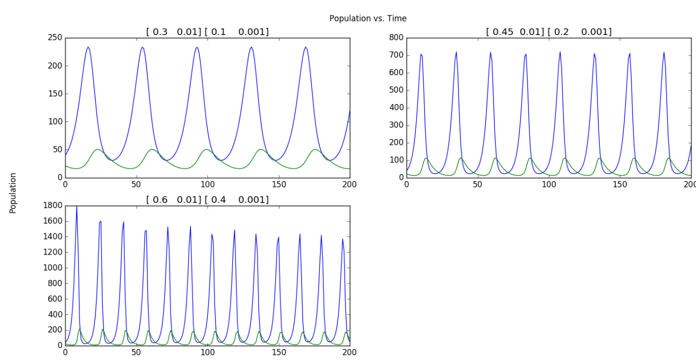


Fig. 4: Varying increases in the parameters a and c in the exponential growth model

expected from the stability analysis, as we increase a and c, we see increases in the period of the system.

Now we'll look at the logistic growth model under the same parameters from before but now with an added figure, of $K = 200$. For clarity I removed the titles of the plots, but they follow the same parameters from figure 3. Now this output

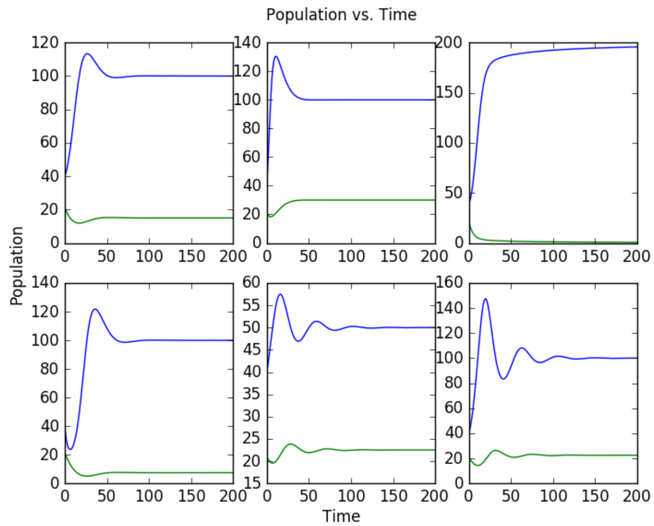


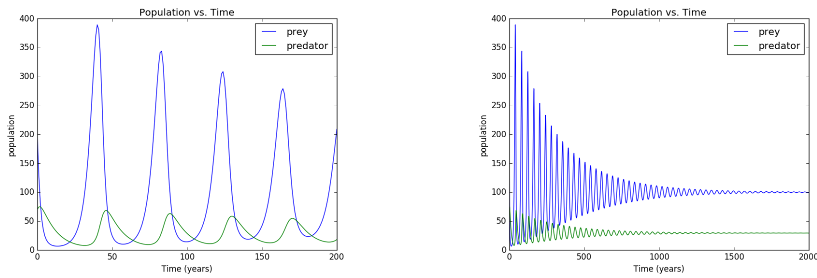
Fig. 5: Logistic growth model for various parameter schemes

is very interesting right off the bat because we can see our first instance of a non stable system. The parameter values for the top right sub-figure in figure 5 are $a = .3, b = .01, c = .2, d = .001, K = 200$. The reason why this is important is because if we refer to out stability analysis we see that

$$\frac{c}{dK} = 1 \rightarrow \frac{c}{dK} \not< 1$$

which means we have to get an unstable system. The data shows that with this set up the predators will die and the carrying capacity of the prey will be their upper population bound. Another important feature of this system is that for almost all the plots, the prey converge to a population of 100 while the predator population varies between 5 to 30. The only time this is different (except for the unstable case), this happens is when we doubled the parameter d, which lead to a bound of 50 for the prey. This is clear from the definition of the steady state solution of this system. No growth in prey population happens when $x = c/d$, which is 50 for this specific set of parameters. Not too important, but its nice to see that we have consistency throughout the entire analysis process.

Going off that condition above, if we increase either the parameter d or K, we should see this system start to resemble the older model with exponential growth because the periodic term in the eigenvalues will get bigger. The only difference is that it is still a stable system so it will converge to a bound. Also, the parameter that should be changed here is K, because if we increase d then like said above we will see the bound of the prey decrease more and more until it reaches 0. As we thought the periodic nature shows up for large K, but is still



exponential growth with parameter values $a = .3, b = .01, c = .1, d = .001, K = 5000$

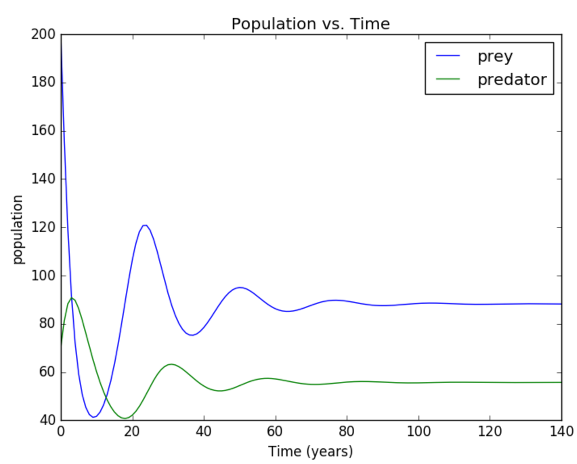
logistic growth with the same parameters ran for 2000 years to show convergence is still upheld

Fig. 7

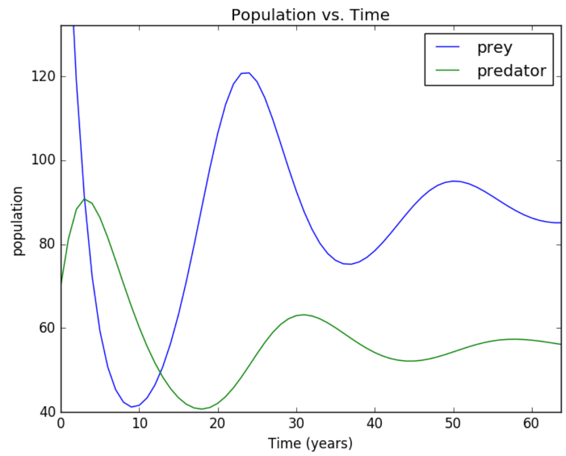
bounded.

Now let's try to set up some systems that incorporate all the parameters. Now we'll morph the control group to see the changes they produce

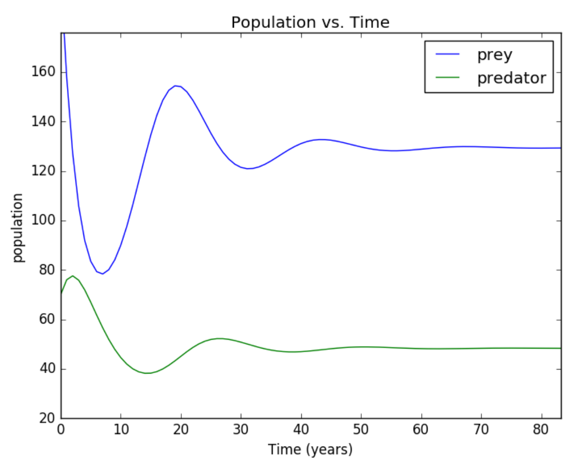
$$a : .3 \to .5,\ \ b : .01 \to .007,\ \ c : .1 \to .15,\ \ d : .001 \to .0017,\ \ K : 200 \to 400$$



If we reference the top left plot in figure 5 we can see the periodic nature is increased as well as the bound of the predator population. This is to be expected since we kind of already know how these parameters affect the system from the last part. The interesting behavior comes as the system is trying to reach equilibrium so lets get a close look.

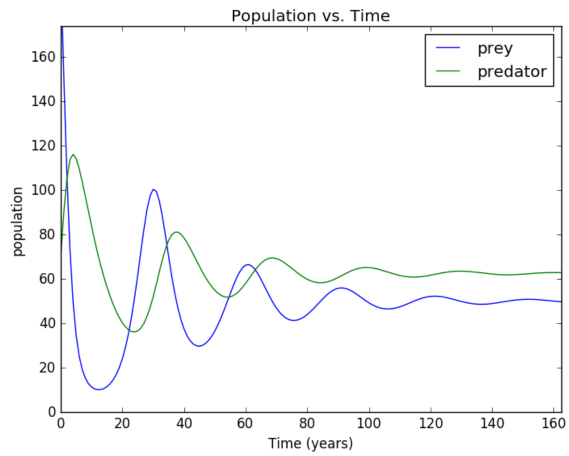


This system has a lot more prey than before with a much higher carrying capacity. This means that its a much better system for the predators who will have a higher population count, than the original system, once they reach their population bound. With the doubling of the carrying capacity we see the system obtain that periodic nature as well as helping the predator population grow larger. From here if we take the rate at which the predators die off and increase it ($c = .15 \rightarrow .22$), we get the following



So the rate at which takes to stabilize is about the same, but behavior is a little less periodic and the population bound of the prey has increased considerably.

Further increase will see the prey population increase while the predator population decrease. Considering the stability condition, c can only be so large before the system becomes unstable. Something of importance is when we decrease c or increase d too much, we'll see a system in which the bound for the predator is larger than that of the prey.



This is a shift from $c = .22 \to .1, d = .0017 \to .002$ which lines up with the physical interpretation, we effectively decrease the rate of death and increased the birth rate for the predator. The problem is that there is no way this is physically possible, but it does adhere to the stability we derived earlier; this must be a limitation of the system.

6 Conclusion

Starting from the Lotka-Volterra model with exponential growth, we were able to figure out what each of the parameters a, b, c, d meant to the system as well as their physically meaning. From there we concluded that the steady state of that system was when $(x, y) = (c/d, a/b)$ through steady state analysis. After generating a series of plots we were able to see the nature of the system, periodic, as well as the sensitively of each parameter and how they each individually affect the system.

Using the same ideas on the logistic growth Lotka-Volterra model, we concluded the parameters meant the same except for K meaning the carrying capacity for the prey. From there we do the standard steady state analysis to find no change in population at $(x, y) = (c/d, (a/b)(1 - c/dK))$. After a pretty long derivation we found the stability condition of this system to be $1 > c/dK$ given that we will have Eigen values that has real parts always negative. Like the former model, we were able to generate plots given certain parameter values and see how each one affects the system. Given the definition of the stability, we say that the factors that play a part in the periodic behavior of the system are the terms c, d, K. If we decrease c or increase d, K, we will say the model take on strong periodic shape. As shown earlier, if we go to far with changing these values we end up getting non physical systems.

References

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