GRAPH THEORY EXERCISES

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Contents

Thoughts and Most Helpful Exercises

This is a personal "journal" of my progress through *Graph Theory by Bondy and Murty.* I am doing this for my own personal growth, as I have always felt a little lacking in discrete mathematics. Note that these are my personal solutions and have not been reviewed so some of my proofs may be wrong! As for the structure of my solutions, I denote questions that I have not yet answered in red and particularly difficult questions with a bold *****. My plan of attack is as follows. One my first pass, I will complete all exercises denoted with a star $\star$ as well as one or two of the challenge exercises in each subsection. These will be marked with a (1) for my first pass. In a second pass of the textbook, I will work through other exercises that interest me (additional challenge problems and any other fun problems).

Below I discuss some topics that I found particularly challenging/interesting and the exercises that best helped me understand the material at a deeper level.

Helpful Questions per Section

- **Section 1:**
- **Section 2:**
- **Section 3:**
- **Section 4:**

Final Thoughts

Chapter 1: Graphs

1.1 Graphs and Their Representation

P.1) (1st pass) *Let G be a simple graph. Show that $m \leq \binom{n}{2}$, and determine when equality holds.*

SOLUTION: A fully-populated graph is one that is complete. This means that we want to count the total number of ways that we can create pairs from n vertices. This is exactly $\binom{n}{2}$. Therefore, equality holds when the graph is complete. If the graph is not complete, $m \leq \binom{n}{2}$.

P.3) (1st pass) *Show that a) every path is bipartite, b) every cycle is bipartite if and only if its length is even*

SOLUTION:

a) Let p be a path $v_1 \dots v_n$ in a graph G. Define $X = \{v_i \mid i \text{ odd}\}$ and $Y = \{v_i \mid i \text{ even}\}$.

   Clearly, X and Y form a bipartition of the path p because for $v \in X$, $v = v_k$ for k odd so $v_{k+1} \in Y$ or $v_{k-1} \in Y$. We say "or" to account for the case where v is the start or end of the path. The same reasoning can be applied to $v \in Y$.

b) ($\leftarrow$): Suppose a cycle $v_0 \dots v_k$ has even length. Therefore, k must be odd since the total number of edges must be even. Apply the same bipartition as problem 1.1.1 so $X = \{v_i \mid v_i \text{ odd}\}$ and $Y = \{v_i \mid v_i \text{ even}\}$.

   For $v \in X$ where $v = v_i$, we know that $v_{(i-1}, v_{i+1} \in Y$ (if $i = k$, let $i + 1 = 0$). We can apply the same logic to $y \in Y$.

   ($\rightarrow$): suppose the length of a cycle is *not* even so suppose the cycle is $v_0 \dots v_k$ for k even. We can construct a bipartition with empty sets X and Y. Start from v_0 without loss of generalization.

   Define the following algorithm to construct a bipartition.

Algorithm 1 Construction of Bipartite Set

```
X := {}
Y := {}
L := [v_0, ..., v_k] for k even and L a queue
▷ use a queue since deg v = 2, we can only move forward or backwards in the cycle. WLOG, move forward
while L not empty do
    v := Pop(L)
    if v adjacent to u ∈ Y or Y empty then
        Add(X, v)                                   ▷ the index of v will be even
    else if v adjacent to u ∈ X or X empty then
        Add(Y, v)                                   ▷ the index of v will be odd
    else
        return error
    end if
end while
```

This algorithm will always return an error because when we get to v_k, v_k is adjacent to v_0. Therefore, there will be an edge $v_k v_0$ with both endpoints in the same partition X.

P.9) (1st pass) *Let $G[X, Y]$ be a bipartite graph.*
a) Show that $\sum_{v \in X} d(v) = \sum_{v \in Y} d(v)$,
b) Deduce that if G is k-regular, with $k \geq 1$, then $|X| = |Y|$.

SOLUTION:

a) Let m be the number of edges. Because $G[X, Y]$ is bipartite, we know that for all $e \in e(G)$, one endpoint $v \in X$ and another $u \in Y$.

   Therefore, we cannot have $\sum_{v \in X} d(v) \neq \sum_{v \in Y} d(v)$, because then that implies that there is an edge with both endpoints in either X or Y.

b) Suppose that G is k-regular with $k \geq 1$. This means that for all $v \in v(G)$, $\deg(v) = k$. Therefore, we get the following...

$$\sum_{v \in X} d(v) = \sum_{v \in Y} d(v)$$
$$\sum_{v \in X} k = \sum_{v \in Y} k$$
$$k(\#\{v \in X\}) = k(\#\{v \in Y\})$$
$$|X| = |Y|$$

P.10) (1st pass) *A k-partite graph is one whose vertex set can be partitioned into k subsets, or parts, in such a way that n edge has both ends in the same part. (Equivalently, one may think of the vertices as being colorable by k colors so that no edge joins two vertices of the same color.) Let G be a simple k-partite graph with parts of sizes $a_1, a_2, \ldots, a_k$. Show that $m \leq \frac{1}{2}\sum_{i=1}^{k} a_i(n - a_i)$.*

SOLUTION: Let's first decipher what $a_i(n-a_i)$ means. Let's say there are k colors. The expression $a_i(n-a_i)$ is equivalent to counting the total number of possible edges between vertices of color i and all other colored vertices since for each vertex of color i, we can have an edge between any of the $n - a_i$ vertices not of color i.

As such, if we sum $a_i(n - a_i)$ across all k colors, this is similar to counting the degree of a complete graph except we want to exclude edges between vertices of the same color.

However, there is a caveat. We want to prevent double-counting because if we continue the procedure above for all colors, we effectively double-count all the edges in a complete graph. This is because, for $a_j(n - a_j)$ with $j \neq i$, we count the total number of possible edges between vertices of color j and all other colored vertices. This is equivalent to counting the total number of possible edges between vertices of color j and all vertices of color i **and** all other colored vertices. We already counted the total number of possible edges between vertices of color j and all vertices of color i via the calculation $a_i(n - a_i)$.

Therefore, $m \leq \frac{1}{2}\sum_{i=1}^{k} a_i(n - a_i)$.

P.11) (1st pass) *A k-partite graph is complete if any two vertices in different parts are adjacent. A simple complete k-partite graph on n vertices whose parts are of equal or almost equal sizes, $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$, is called a Turan graph and denotes $T_{k,n}$.*
a) Show that $T_{k,n}$ has more edges than any other simple complete k-partite graph on n vertices.
b) Determine $e(T_{k,n})$.

a) We can use question 1.1.10's result on k-partite graphs. Specifically, we know that for a complete k-partite graph, $m = \frac{1}{2}\sum_{i=1}^{k} a_i(n-a_i)$. This problem is equivalent to maximizing m subject to the constraint that $\sum_{i=1}^{k} a_i = n$ and $0 < a_i < n$.
$\sum_{i=1}^{k} a_i(n - a_i) = n - 2a_i$

b) Let's consider the simplest case of a_i' of equal sizes to $a_i' = n/k$. Then, $e(T_{k,n}) = m = \frac{1}{2}\sum_{i=1}^{k} a_i(n-a_i) = \frac{1}{2}\sum_{i=1}^{k} \frac{n}{k}(\frac{k-n}{k}) = \frac{n(k-n)}{2k}$.
If a_i' are of almost equal sizes, Then

$$\begin{aligned} e(T_{k,n}) = m &= \frac{1}{2}\sum_{i=1}^{k} a_i(n - a_i) \\ &= \frac{1}{2}\left(\sum_{i=1}^{k}(\lfloor n/k \rfloor)(n - \lfloor n/k \rfloor) + \sum_{i=1}^{k}(\lceil n/k \rceil)(n - \lceil n/k \rceil)\right) \end{aligned}$$

1.2 Isomorphisms and Automorphisms

P.3) (1st pass)

P.5) (1st pass)

P.15) (1st pass)

1.3 Graphs Arising from Other Structures

1.4 Constructing Graphs from Other Graphs

1.5 Directed Graphs

P.2) (1st pass)

P.7) (1st pass)

1.6 Infinite Graphs

P.1)

Chapter 2: Subgraphs

2.1 Subgraphs and Supergraphs

P.1) (1st pass)

P.2) (1st pass)

P.11) (1st pass)

P.17) (1st pass)

P.21) (1st pass)

P.23) (1st pass)

2.2 Spanning and Induced Subgraphs

P.2) (1st pass)

P.11) (1st pass)

P.12) (1st pass)

2.3 Modifying Graphs

P.1) (1st pass)

2.4 Decompositions and Coverings

P.2) (1st pass)

P.6) (1st pass)

2.5 Edge Cuts and Bonds

P.1) (1st pass)

P.2) (1st pass)

P.4) (1st pass)

P.5) (1st pass)

P.6) (1st pass)

P.8) (1st pass)

2.6 Even Subgraphs

P.2) (1st pass)

P.4) (1st pass)

2.7 Graph Reconstruction

P.8) (1st pass)

P.13) (1st pass)

Chapter 3: Connected Graphs

3.1 Walks and Connection

P.1) (1st pass)

P.3) (1st pass)

P.4) (1st pass)

3.2 Cut Edges

P.1) (1st pass)

P.2) (1st pass)

P.3) (1st pass)

3.3 Euler Tours

P.3) (1st pass)

P.4) (1st pass)

3.4 Connection in Digraphs

P.1) (1st pass)

P.2) (1st pass)

P.3) (1st pass)

P.6) (1st pass)

P.8) (1st pass)

P.11) (1st pass)

P.12) (1st pass)

3.5 Cycle Double Covers

P.3) (1st pass)

Chapter 4: Trees

4.1 Forests and Trees

P.2) (1st pass)

P.6) (1st pass)

P.20) (1st pass)

4.2 Spanning Trees

P.1) (1st pass)

P.4) (1st pass)

P.8) (1st pass)

P.9) (1st pass)

4.3 Fundamental Cycles and Bonds

P.10) (1st pass)

Chapter 5: Nonseparable Graphs

5.1 Cut Vertices

P.1) (1st pass)

P.2) (1st pass)

5.2 Separations and Bonds

P.1) (1st pass)

P.2) (1st pass)

P.8) (1st pass)

P.9) (1st pass)

P.11) (1st pass)

5.3 Ear Decompositions

P.1) (1st pass)

P.2) (1st pass)

P.9) (1st pass)

P.12) (1st pass)

5.4 Directed Ear Decompositions

P.1) (1st pass)

Chapter 6: Tree-Search Algorithms

6.1 Tree-Search

P.1) `(1st pass)`

P.6) `(1st pass)`

P.10) `(1st pass)`

P.12) `(1st pass)`

6.2 Minimum-Weight Spanning Trees

P.1) `(1st pass)`

6.3 Branching Search

P.2) `(1st pass)`

P.3) `(1st pass)`

P.4) `(1st pass)`

P.9) `(1st pass)`

P.11) `(1st pass)`

P.12) `(1st pass)`

Chapter 7: Flows in Networks

7.1 Transportation Networks

P.1) (1st pass)

P.2) (1st pass)

P.3) (1st pass)

7.2 The Max-Flow Min-Cut Theorem

P.1) (1st pass)

P.3) (1st pass)

7.3 Arc-Disjoint Directed Paths

P.1) (1st pass)

P.2) (1st pass)

P.4) (1st pass)

P.5) (1st pass)

Chapter 8: Complexity of Algorithms

8.1 Computation Complexity

P.1) (1st pass)

P.4) (1st pass)

8.2 Polynomial Reductions

P.1) (1st pass)

8.3 $\mathcal{NP}$-Complete Problems

P.1) (1st pass)

P.2) (1st pass)

P.3) (1st pass)

P.4) (1st pass)

P.5) (1st pass)

P.9) (1st pass)

8.4 Approximation Algorithms

P.1) (1st pass)

P.4) (1st pass)

8.5 Greedy Heuristics

P.1) (1st pass)

P.2) (1st pass)

P.3) (1st pass)

8.6 Linear and Integer Programming

P.2) (1st pass)

P.7) (1st pass)

P.9) (1st pass)

P.10) (1st pass)

Chapter 9: Connectivity

9.1 Vertex Connectivity

P.2) (1st pass)

P.6) (1st pass)

9.2 The Fan Lemma

P.1) (1st pass)

P.3) (1st pass)

9.3 Edge Connectivity

P.2) (1st pass)

P.5) (1st pass)

P.8) (1st pass)

P.9) (1st pass)

P.13) (1st pass)

9.4 Three-Connected Graphs

P.3) (1st pass)

P.10) (1st pass)

9.5 Submodularity

P.4) (1st pass)

P.5) (1st pass)

9.6 Gomory-Hu Trees

P.1) (1st pass)

9.7 Chordal Graphs

P.1) (1st pass)

P.2) (1st pass)

P.3) (1st pass)

P.4) (1st pass)

Chapter 10: Planar Graphs

10.1 Plane and Planar Graphs

P.1) (1st pass)

P.2) (1st pass)

P.3) (1st pass)

P.4) (1st pass)

P.5) (1st pass)

10.2 Duality

P.2) (1st pass)

P.4) (1st pass)

P.5) (1st pass)

P.9) (1st pass)

P.13) (1st pass)

10.3 Euler's Formula

P.1) (1st pass)

10.4 Bridges

P.1) (1st pass)

10.5 Kuratowski's Theorem

P.3) (1st pass)

10.6 Surface Embeddings of Graphs

P.3) (1st pass)

Chapter 11: The Four-Color Problem

11.2 The Five-Color Problem

P.1) (1st pass)

P.2) (1st pass)

P.9) (1st pass)

Chapter 12: Stable Sets and Cliques

12.1 Stable Sets

P.2) (1st pass)

P.10) (1st pass)

12.2 Turan's Theorem

P.1)

12.3 Ramsey's Theorem

P.1)

12.4 The Regularity Lemma

P.1)

Chapter 13: The Probabilistic Method

13.1 Random Graphs

P.1) (1st pass)

P.2) (1st pass)

P.3) (1st pass)

P.4) (1st pass)

13.2 Expectation

P.1) (1st pass)

P.2) (1st pass)

P.3) (1st pass)

P.11) (1st pass)

13.3 Variance

P.1) (1st pass)

P.2) (1st pass)

P.4) (1st pass)

13.4 Evolution of Random Graphs

P.1) (1st pass)

13.5 The Local Lemma

P.1) (1st pass)

Chapter 14: Vertex Colorings

14.1 Chromatic Number

P.2) (1st pass)

P.3) (1st pass)

P.15) (1st pass)

14.2 Critical Graphs

P.6) (1st pass)

14.3 Girth and Chromatic Number

P.1) (1st pass)

14.4 Perfect Graphs

P.5) (1st pass)

14.5 List Colorings

P.1) (1st pass)

P.5) (1st pass)

P.10) (1st pass)

14.6 The Adjacency Polynomial

P.1) (1st pass)

14.7 The Chromatic Polynomial

P.1) (1st pass)

P.11) (1st pass)

P.12) (1st pass)

Chapter 15: Colorings of Maps

15.1 Chromatic Numbers of Surfaces

P.1) (1st pass)

15.2 The Four-Color Theorem

P.1) (1st pass)

P.3) (1st pass)

P.4) (1st pass)

15.3 List Colorings of Planar Graphs

P.1) (1st pass)

15.4 Hadwiger's Conjecture

P.1) (1st pass)

Chapter 16: Matchings

16.1 Maximum Matchings

P.5) (1st pass)

P.7) (1st pass)

P.15) (1st pass)

16.2 Matchings in Bipartite Graphs

P.5) (1st pass)

P.7) (1st pass)

P.8) (1st pass)

P.11) (1st pass)

16.3 Matchings in Arbitrary Graphs

P.1) (1st pass)

P.2) (1st pass)

P.3) (1st pass)

P.4) (1st pass)

P.5) (1st pass)

P.6) (1st pass)

P.7) (1st pass)

16.4 Perfect Matchings and Factors

P.2) (1st pass)

P.16) (1st pass)

P.22) (1st pass)

16.5 Matching Algorithms

P.3) (1st pass)

P.8) (1st pass)

P.9) (1st pass)

P.10) (1st pass)

Chapter 17: Edge Colorings

17.1 Edge Chromatic Number

P.3) (1st pass)

P.9) (1st pass)

P.16) (1st pass)

17.2 Vizing's Theorem

P.1) (1st pass)

17.3 Snarks

P.4) (1st pass)

17.4 Coverings by Perfect Matchings

P.6) (1st pass)

17.5 List Edge Colorings

P.1) (1st pass)

Chapter 18: Hamilton Cycles

18.1 Hamiltonian and Nonhamiltonian Cycles

P.1) (1st pass)

18.2 Nonhamiltonian Planar Graphs

P.1) (1st pass)

18.3 Path and Cycle Exchanges

P.1) (1st pass)

18.4 Path Exchanges and Parity

P.1) (1st pass)

18.5 Hamilton Cycles in Random Graphs

P.1) (1st pass)

Chapter 19: Coverings and Packings in Directed Graphs

19.1 Coverings and Packings in Hypergraphs

P.1) (1st pass)

19.2 Coverings by Directed Paths

P.1) (1st pass)

19.3 Coverings by Directed Cycles

P.2) (1st pass)

19.4 Packings of Branchings

P.4) (1st pass)

19.5 Packings of Directed Cycles and Directed Bonds

P.4) (1st pass)

P.5) (1st pass)

P.6) (1st pass)

P.7) (1st pass)

Chapter 20: Electrical Networks

20.1 Circulations and Tensions

P.1) (1st pass)

P.2) (1st pass)

P.3) (1st pass)

P.5) (1st pass)

P.6) (1st pass)

20.2 Basis Matrices

P.1)

20.3 Feasible Circulations and Tensions

P.1) (1st pass)

20.4 The Matrix-Tree Theorem

P.1) (1st pass)

P.2) (1st pass)

P.3) (1st pass)

P.4) (1st pass)

20.5 Resistive Electrical Networks

P.1)

20.6 Perfect Squares

P.1)

20.7 Random Walks on Graphs

P.1)

Chapter 21: Integer Flows and Coverings

21.1 Circulations and Colorings

P.1) (1st pass)

21.2 Integer Flows

P.1) (1st pass)

21.3 Tutte's Flow Conjectures

P.1) (1st pass)

21.4 Edge-Disjoint Spanning Trees

P.1) (1st pass)

21.5 The Four-Flow and Eight-Flow Theorems

P.1) (1st pass)

21.6 The Six-Flow Theorem

P.1) (1st pass)

21.7 The Tutte Polynomial

P.1) (1st pass)