Lesson 11: Joint distributions

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2025-10-29

Learning Objectives

1. Define **joint and marginal** distributions for discrete and continuous random variables
2. Calculate or find **joint and marginal** probabilities, pmf's, and CDF's for discrete random variables
3. Calculate or find **joint and marginal** probabilities, pdf's, and CDF's for continuous random variables
4. Extra practice on your own: solve double integrals in a mini lesson

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Simulations
- Probability axioms
- Probability properties
- Counting
- Rules of probability

Probability for random variables

- Probability functions: pmfs/pdfs
- Cumulative distribution functions (CDFs)
- Important distributions

Two random variables

- Transformations
- Joint distributions
- Independence and conditioning

Expected values

- Means / expected values
- Linear combinations and conditioning
- Variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

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What is a joint distribution?

Definition: joint pmf

The **joint pmf** of a pair of discrete RV's X and Y is

$$\begin{aligned} p_{X,Y}(x,y) =& \mathbb{P}(X = x \cap Y = y) \\ =& \mathbb{P}(X = x, Y = y) \end{aligned}$$

Definition: joint pdf

The **joint pdf** for two continuous RVs (X and Y) is $f_{X,Y}(x,y)$, such that we have the following joint probability:

$$\mathbb{P}(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x,y)dydx$$

Important properties of joint distributions

Properties of joint pmf's

- A joint pmf $p_{X,Y}(x,y)$ must satisfy the following properties:
  - $0 \geq p_{X,Y}(x,y) \leq 1$ for all x, y
  - $\sum_{\{all\ x\}} \sum_{\{all\ y\}} p_{X,Y}(x,y) = 1$

Properties of joint pdf's

- A joint pdf $f_{X,Y}(x,y)$ must satisfy the following properties:
  - $f_{X,Y}(x,y) \geq 0$ for all x, y
  - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dxdy = 1$
- Remember that $f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$!!!

Marginal distributions

Marginal pmf's

Suppose X and Y are discrete RV's, with joint pmf $p_{X,Y}(x,y)$. Then the **marginal probability mass functions** are

$$p_X(x) = \sum_{\{all\ y\}} p_{X,Y}(x,y)$$

$$p_Y(y) = \sum_{\{all\ x\}} p_{X,Y}(x,y)$$

Marginal pdf's

Suppose X and Y are continuous RV's, with joint pdf $f_{X,Y}(x,y)$. Then the **marginal probability density functions** are

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

Joint cumulative distribution functions (CDFs)

Joint CDF for discrete RVs

The **joint CDF** of a pair of discrete RV's X and Y is

$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x \; and \; Y \leq y) = \mathbb{P}(X \leq x, Y \leq y)$$

Joint CDF for continuous RVs

The **joint CDF** of continuous random variables X and Y, is the function $F_{X,Y}(x,y)$, such that for all real values of x and y,

$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t) dt ds$$

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Joint distribution for two discrete random variables (1/5)

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

1. Find $p_{X,Y}(x, y)$
2. Find $\mathbb{P}(X + Y = 3)$
3. Find $\mathbb{P}(Y = 1)$
4. Find $\mathbb{P}(Y \leq 2)$
5. Find the joint CDF $F_{X,Y}(x, y)$ for the joint pmf $p_{X,Y}(x, y)$
6. Find the marginal CDFs $F_X(x)$ and $F_Y(y)$

Joint distribution for two discrete random variables (2/5)

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

1. Find $p_{X,Y}(x, y)$
2. Find $\mathbb{P}(X + Y = 3)$

		Y: 1	Y: 2	Y: 3	
X	1				
X	2				
X	3				

Joint distribution for two discrete random variables (3/5)

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

3. Find $\mathbb{P}(Y = 1)$

4. Find $\mathbb{P}(Y \leq 2)$

			Y		
		1	2	3	
	1				
X	2				
	3				

Joint distribution for two discrete random variables (4/5)

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

5. Find the joint CDF $F_{X,Y}(x, y)$ for the joint pmf $p_{X,Y}(x, y)$

		Y			
		1	2	3	
	1				
X	2				
	3				

Joint distribution for two discrete random variables (5/5)

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

6. Find the marginal CDFs $F_X(x)$ and $F_Y(y)$

Y

		1	2	3	
X	1				
	2				
	3				

Quick remarks on the joint and marginal CDF

- $F_X(x)$: right most columns of the CDF table (where the Y values are largest)
- $F_Y(y)$: bottom row of the table (where X values are largest)
- $F_X(x) = \lim_{y \to \infty} F_{X,Y}(x, y)$
- $F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x, y)$

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Common steps for joint pdfs and CDFs

1. Set up the domain of the pdf with a picture

2. Translate to needed integrands
   - For probability: shade in the area of interest, then translate
   - For expected value: translate domain

3. Set up integral: $dxdy$ or $dydx$?

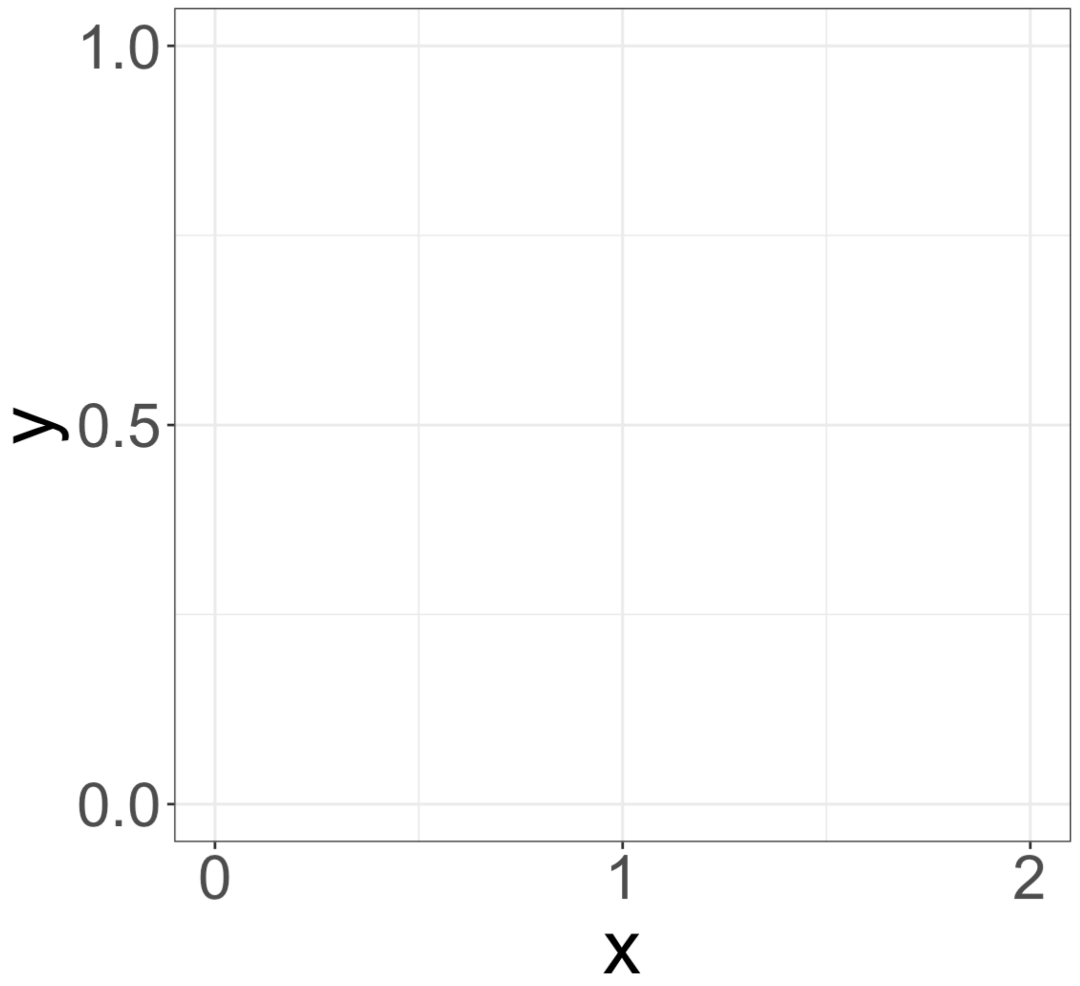
4. Solve integral!

Example 2: Joint pdf (1/2)

Example 2.1

Let $f_{X,Y}(x,y) = \frac{3}{2}y^2$, for $0 \le x \le 2,\ 0 \le y \le 1.$

1. Find $\mathbb{P}(0 \le X \le 1, 0 \le Y \le \frac{1}{2})$

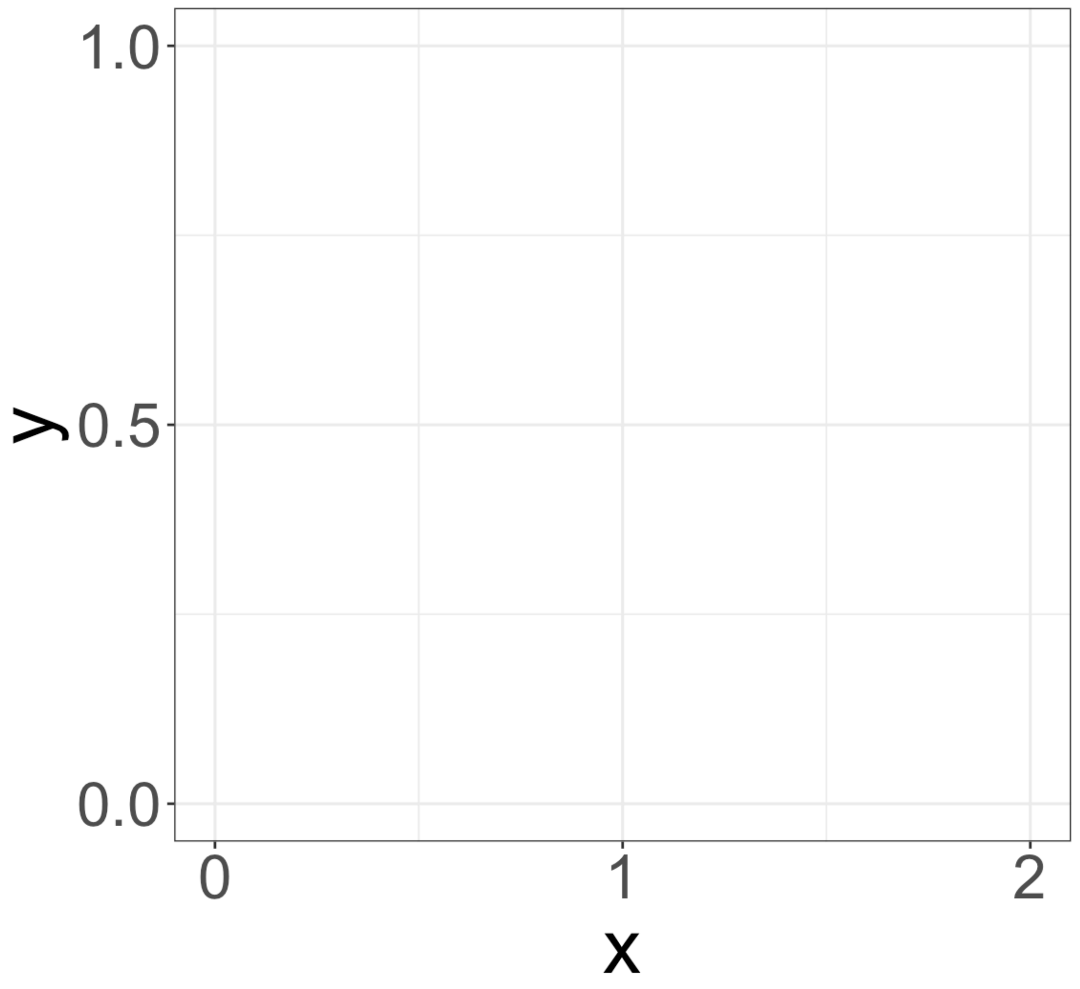


Example 2: Joint pdf (2/2)

Example 2.2

Let $f_{X,Y}(x,y) = \frac{3}{2}y^2$, for $0 \leq x \leq 2,\ 0 \leq y \leq 1.$

2. Find $f_X(x)$ and $f_Y(y)$.



Example with more complicated pdf (1/2)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Example 3.1

Let $f_{X,Y}(x,y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$.

1. Find $f_X(x)$ and $f_Y(y)$.

Example with more complicated pdf (2/2)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Example 3.2

Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$.

2. Find $\mathbb{P}(Y < 3)$.

Recall: Finding the pdf of a transformation

- Let M be a transformation of X and Y: $M = g(X, Y)$
- When we have a transformation of X and Y, M, we need to follow the **CDF method** to find the pdf of M

We follow **CDF method**:

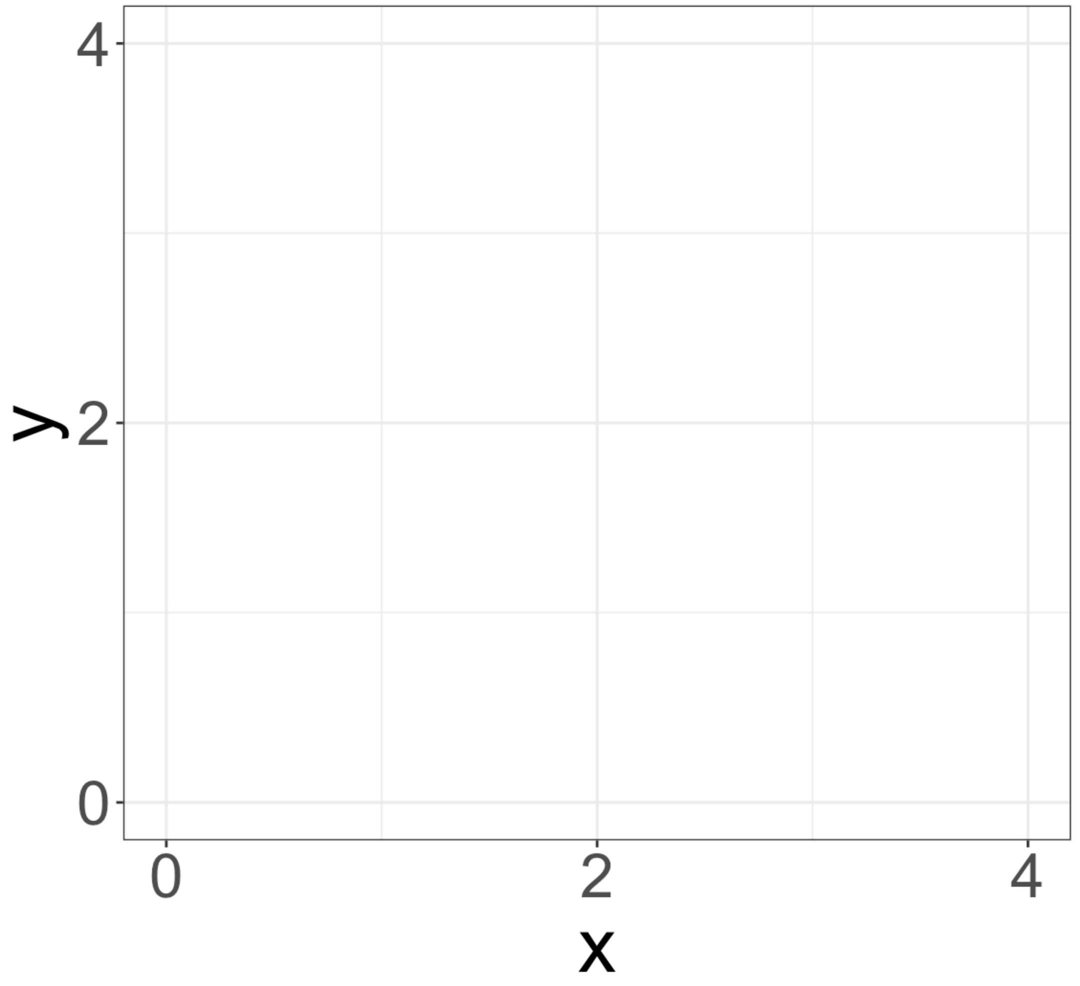
1. Start with the joint pdf for X and Y
   - aka $f_{X,Y}(x, y)$
2. Translate the domain of X and Y to M: find possible values of M
3. Find the CDF of M
   - aka $F_M(m) = P(M \leq m) = P(g(X, Y) \leq m)$
4. Take the derivative of the CDF of M with respect to m to find the pdf of M
   - aka $f_M(m) = \frac{d}{dm} F_M(m)$

Example of a joint pdf with a transformation (1/2)

Example 4.1

Let X and Y have constant density on the square $0 \leq X \leq 4, 0 \leq Y \leq 4.$

1. Find $\mathbb{P}(|X - Y| < 2)$

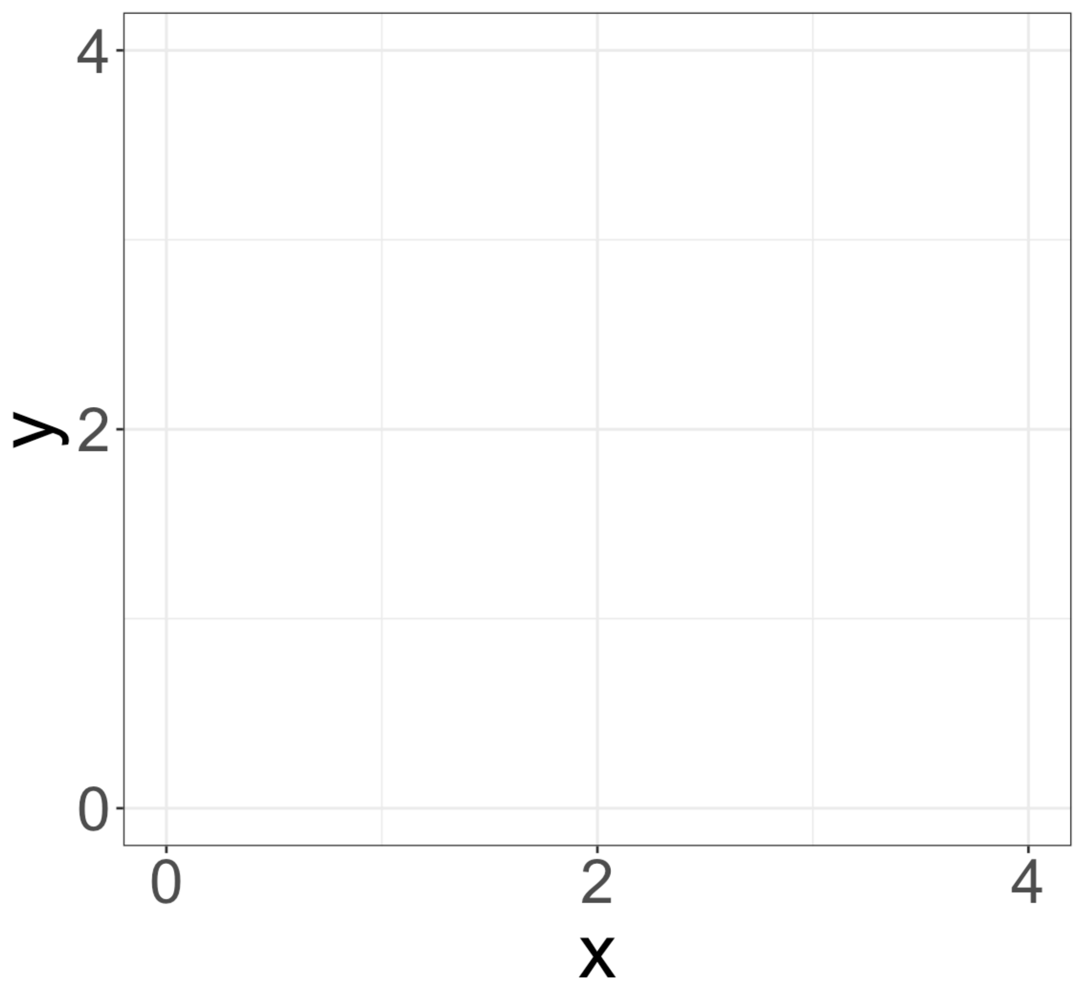


Example of a joint pdf with a transformation (1/2)

Example 4.2

Let X and Y have constant density on the square $0 \le X \le 4, 0 \le Y \le 4.$

2. Let $M = \max(X, Y)$. Find the pdf for M, that is $f_M(m)$



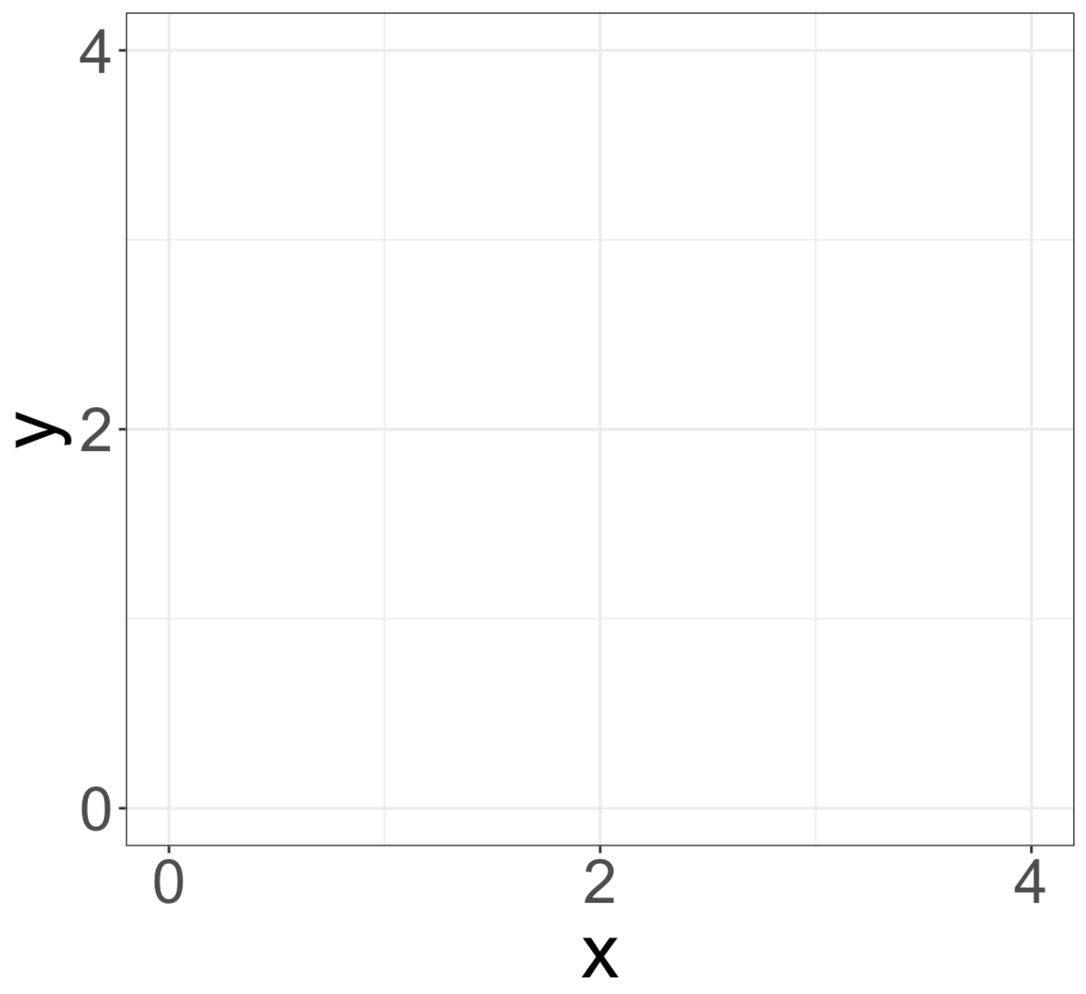
Example of a joint pdf with a transformation (1/2)

Example 4.3

Let X and Y have constant density on the square $0 \leq X \leq 4, 0 \leq Y \leq 4.$

3. Let $Z = \min(X, Y)$. Find the pdf for Z, that is $f_Z(z)$.

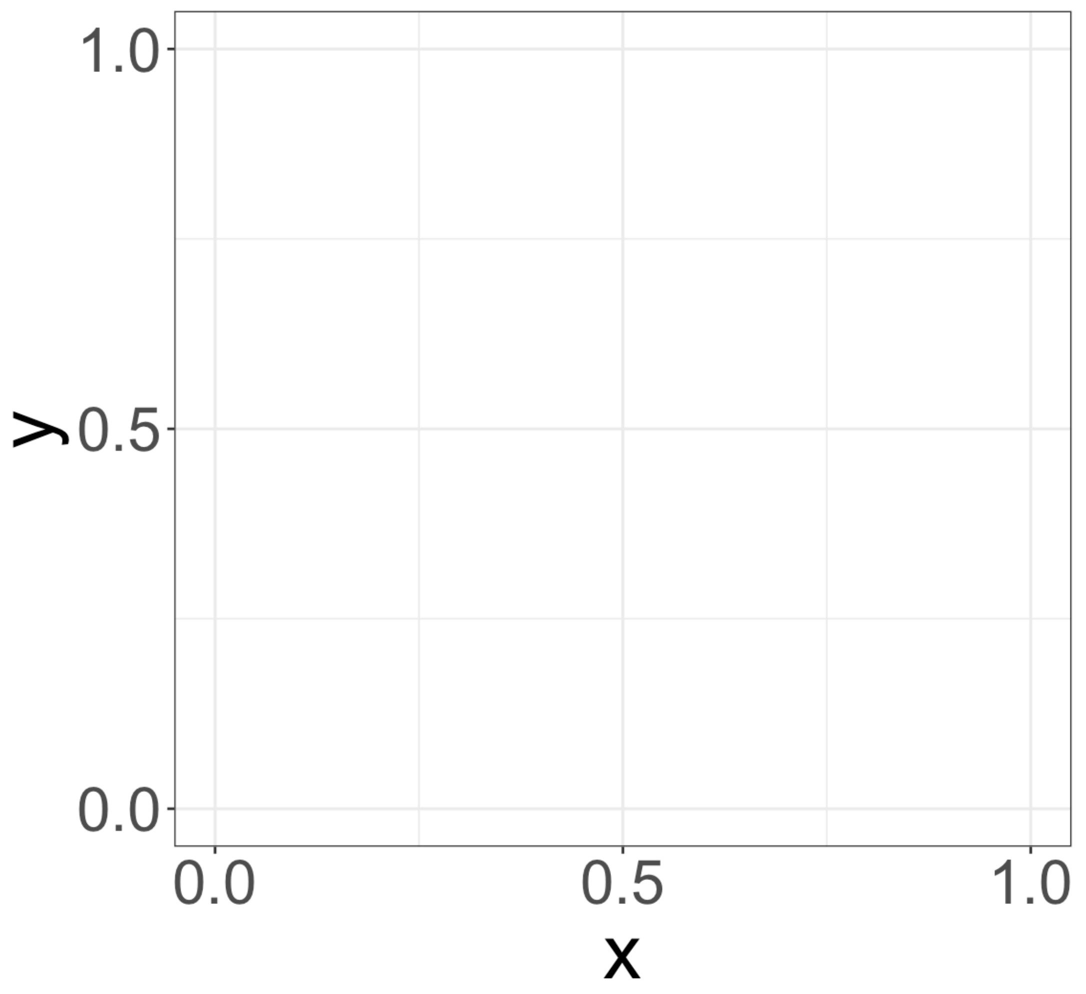
Do this problem at home for extra practice. I'll add the solution to the annotated notes!



Last example *for home*: more complicated transformation

Example 5

Let X and Y have joint density $f_{X,Y}(x,y) = \frac{8}{5}(x+y)$ in the region $0 < x < 1,\ \frac{1}{2} < y < 1$. Find the pdf of the RV Z, where $Z = XY$.



Learning Objectives

1. Solve double integrals in our mini lesson!
2. Calculate probabilities for a pair of continuous random variables
3. Calculate a *joint and marginal* probability density function (pdf)
4. Calculate a *joint and marginal* cumulative distribution function (CDF) from a pdf

Double Integrals Mini Lesson (1/3)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Mini Lesson Example 1

Solve the following integral:

$$\int_2^3 \int_0^1 xy\,dy\,dx$$

Double Integrals Mini Lesson (2/3)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Mini Lesson Example 2

Solve the following integral:

$$\int_2^3 \int_0^1 (x+y)dydx$$

Double Integrals Mini Lesson (3/3)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Mini Lesson Example 3

Solve the following integral:

$$\int_2^3 \int_0^1 e^{x+y} dydx$$