Lesson 5: Equally Likely Outcomes and Counting

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Learning Objectives

1. Define permutations and combinations
2. Characterize difference between sampling with and without replacement
3. Characterize difference between sampling when order matters and when order does not matter
4. Calculate the probability of sampling any combination of the following: *with or without replacement* and *order does or does not matter*

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Simulations
- Probability axioms
- Probability properties
- Counting
- Rules of probability

Probability for random variables

- Probability functions: pmfs/pdfs
- Cumulative distribution functions (CDFs)
- Important distributions

Two random variables

- Transformations
- Joint distributions
- Independence and conditioning

Expected values

- Means / expected values
- Linear combinations and conditioning
- Variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

Birthday example

ThePudding

Welcome to the party! Today you will participate in an **interactive experiment** about the birthday paradox. We will use you, the reader, as part of our data, to help explain what it is, why it is cool, and how it works.

By **Russell Samora**

Let's do this!

Basic Counting Examples

Basic Counting Examples (1/3)

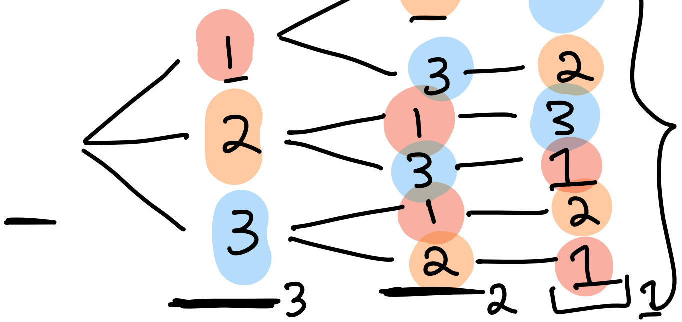
Example 1

Suppose we have 10 (distinguishable) subjects for study.

1. How many possible ways are there to order them?
2. How many ways to order them if we can reuse the same subject and
   - need 10 total?
   - need 6 total?
3. How many ways to order them *without replacement* and only need 6?
4. How many ways to choose 6 subjects without replacement if the order doesn't matter?

Basic Counting Examples (2/3)

Suppose we have 10 (distinguishable) subjects for study.



6 total combos

$3!$

Example 1.1

How many possible ways are there to order them?

order matters
w/out replacement

$\underline{10} \times \underline{9} \times \underline{8} \times \underline{7} \times \underline{6} \times \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1}$

$10!$

Example 1.2

How many ways to order them if we can reuse the same subject and

order
replacement

1. • need 10 total?
2. • need 6 total?

① $\underline{10} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} = 10^{10}$

② $\underline{10} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{10} = 10^{6}$

Basic Counting Examples (3/3)

Suppose we have 10 (distinguishable) subjects for study.

Example 1.3

How many ways to order them without replacement and only need 6?

$$10 \times 9 \times 8 \times 7 \times 6 \times 5 = \frac{10!}{4!}$$

$$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{4 \cdot 3 \cdot 2 \cdot 1}}$$

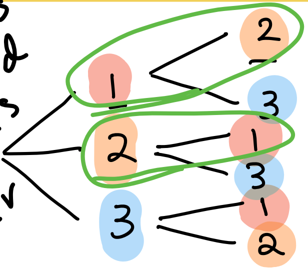
Example 1.4

How many ways to choose 6 subjects without replacement if the order doesn't matter?

$$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{\left(\frac{10!}{4!}\right)}{6!} = \frac{10!}{4!\,6!}$$

How many ways to order the 6 picked subjects?

3 options
2 picked
order does ~~NOT~~ matter



times duplicated

How many ways are there to order 2 picks?

Permutations and Combinations

Permutations and Combinations

Definition: Permutations

Permutations are the number of ways to **arrange in order** r distinct objects when there are n total.

$$nPr = \frac{n!}{(n-r)!}$$

Definition: Combinations

Combinations are the number of ways to choose (**order doesn't matter**) r objects from n without replacement.

$$nCr = \text{"n choose r"} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

See this a lot in Binomial distribution

Some combinations properties

Property	Proof
$\binom{n}{r} = \binom{n}{n-r}$	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}$ n-r is r from slide above
$\binom{n}{1} = n$	$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n \cdot \cancel{(n-1)} \cdot (n-2) \cdots 1}{1! \cdot \cancel{(n-1)} \cdot (n-2) \cdots 1} = \frac{n \cdot (n-1)!}{1 \cdot (n-1)!} = \frac{n}{1} = n$
$\binom{n}{0} = 1$	$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = 1$ $0! = 1$

More Examples: order matters vs. not

Table of different cases

- n = total number of objects
- r = number objects needed

	with replacement	**without replacement**
order matters	n^r	$nPr = \dfrac{n!}{(n-r)!}$ permutations
order doesn't matter	$\dbinom{n+r-1}{r}$	$nCr = \dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$ combinations

Enumerating Events and Sample Space

- Recall, $P(A) = \frac{|A|}{|S|}$
  - Within combinatorics, we can use the previous equations to help enumerate the event and sample space
  - But A might be a combination of enumerations

- For example in the following example drawing 2 spades when order does not matter, we actually need to enumerate the other cards that are NOT spades. So the event is choosing 2 spades out of 13 AND choosing 0 other cards of 39 cards (13 hearts + 13 clubs + 13 diamonds).
- Thus the probability is actually:

13C2

39 C 0 = 1

$$P(\text{two spades}) = \frac{\binom{13}{2}\binom{39}{0}}{\binom{52}{2}}$$

= 52

= 2

- Note that $13 + 39 = 52$ and $2 + 0 = 2$. So the numerator's n's add up to the denominator's n and the numerator's r's add up to the denominator's r's

Another example: order matters vs. not (1/2)

Example 2

Suppose we draw 2 cards from a standard deck without replacement. What is the probability that both are spades when

1. order matters?
2. order doesn't matter?

$n = 13$
$r = 2$

All other cards $n = 39$
$r = 0$

$n = 52$
$r = 2$

Let
A = both cards are spades
S = picking 2 cards

① $|S|: 52 \cdot 51 = {}_{52}P_2 = \frac{52!}{(52-2)!} = \frac{52!}{50!}$

$|A|: 13 \cdot 12 = \frac{13!}{11!}$

$P(1st\ spade)\ P(2nd\ sp \mid 1st\ sp)$

$P(A) = \frac{|A|}{|S|} = \frac{13 \cdot 12}{52 \cdot 51} = \left(\frac{13}{52}\right) \cdot \left(\frac{12}{51}\right)$

$= 0.05882$

② $|S|: {}_{52}C_2 = \frac{52!}{5!(52-2)!}$

$|A|: {}_{13}C_2 = \frac{13!}{2!(13-2)!}$

$P(A) = \frac{|A|}{|S|} = \frac{\binom{13}{2}}{\binom{52}{2}} = \frac{\frac{13!}{\cancel{2}!(11!)}}{\frac{52!}{\cancel{2}!(50!)}} = \frac{13 \cdot 12}{52 \cdot 51}$

(same as above!)

Another example: order matters vs. not (2/2)

Example 2

Suppose we draw 2 cards from a standard deck without replacement. What is the probability that both are spades when

1. order matters?
2. order doesn't matter?

We can do a simulation!

```r
set.seed(1234)
n_sim <- 1000000
cards = c(rep("S", 13),
          rep("H", 13),
          rep("C", 13),
          rep("D", 13))
draws <- replicate(n_sim,
                   sample(cards, 2, replace = FALSE))
draws[, 1:10]
```

```
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
[1,] "C"  "H"  "D"  "S"  "C"  "S"  "C"  "H"  "H"  "D"
[2,] "H"  "C"  "D"  "S"  "H"  "C"  "H"  "S"  "H"  "H"
```

```r
spades_2 = sum( draws[1, ] == "S" & draws[2, ] == "S" )
spades_2 / n_sim
```

```
[1] 0.058727
```