Lesson 12: Independence and Conditioning

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Learning Objectives

1. Identify the formula for joint distributions for independent RVs and conditional distributions (PMFs/PDFs)
2. Find conditional pmf from a joint pmf and check if two RVs are independent.
3. Construct a joint distribution for two independent continuous RVs from their marginal distributions.
4. Calculate conditional probabilities and distributions for continuous RVs.

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Simulations
- Probability axioms
- Probability properties
- Counting
- Rules of probability

Probability for random variables

- Probability functions: pmfs/pdfs
- Cumulative distribution functions (CDFs)
- Important distributions

Two random variables

- Transformations
- Joint distributions
- Independence and conditioning

Expected values

- Means / expected values
- Linear combinations and conditioning
- Variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

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How do we represent conditional pmfs/pdfs?

For events:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

For discrete RVs:

$$p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$p_{Y|X}(y|x) = P(Y = y|X = x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$$

if denominator is greater than 0 ($p_Y(y) > 0$ or $p_X(x) > 0$)

For continuous RVs:

pdf ≠ prob

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

if denominator is greater than 0 ($f_Y(y) > 0$ or $f_X(x) > 0$)

How do we represent independent RVs in a joint pmf/pdf?

What do we know about independence for events?

For events: If $A \perp B$

$$P(A \cap B) = P(A)P(B)$$

$$P(A|B) = P(A)$$

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}}$

$= P(A)$

marginal pmf of x

marginal pmf of y

For discrete RVs: If $X \perp Y$

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

joint CDF → $$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

$$p_{X|Y}(x|y) = p_X(x)$$

$$p_{Y|X}(y|x) = p_Y(y)$$

marginal pdf of x

marginal pdf of y

For continuous RVs: If $X \perp Y$

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

$$f_{X|Y}(x|y) = f_X(x)$$

$$f_{Y|X}(y|x) = f_Y(y)$$

Remember: our probability rules must hold for these!

For discrete RVs

For a valid joint pmf, we need:

- $0 \geq p_{X,Y}(x,y) \leq 1$ for all x, y
- $\sum_{\{all\ x\}} \sum_{\{all\ y\}} p_{X,Y}(x,y) = 1$

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For continuous RVs

For a valid joint pdf, we need:

- $f_{X,Y}(x,y) \geq 0$ for all x, y
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$

For a valid conditional pdf, we need:

- $f_{X|Y}(x|y) \geq 0$ for all x and y
- $\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = 1$

$$\int_{-\infty}^{\infty} f_{Y|X}(y|x)\underline{dy} = 1$$

Extra notes

$P(x_1, x_2, x_3) = P(X_1 = x_1) P(X_2 = x_2) \cdot P(X_3 = x_3)$

- If $X_1, X_2, \ldots, X_n$ are **independent**

$$p_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \prod_{i=1}^{n} p_{X_i}(x_i)$$

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

$$F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n) = \prod_{i=1}^{n} P(X_i \le x_i) = \underbrace{\prod_{i=1}^{n} F_{X_i}(x_i)}_{\text{product of marginal CDF's}}$$

- Don't forget, you can manipulate the conditional density to get the joint:

same goes for discrete RVs!

$$f_{X,Y}(x, y) = f_{X|Y}(x|y) f_Y(y)$$

$$f_Y(y) f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \times f_Y(y)$$

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Last class: joint distribution for two discrete random variables (1/2)

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

7. Find $p_{X|Y}(x|y)$.

joint pmf:

	$Y=1$	$Y=2$	$Y=3$	$P_X(x)$
$X=1$	0	1/6	1/6	1/3
$X=2$	1/6	0	1/6	1/3
$X=3$	1/6	1/6	0	1/3
$P_Y(y)$	1/3	1/3	1/3	

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

ex1) $$P_{X|Y}(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{0}{1/3} = 0$$

ex2) $$P_{X|Y}(X=2 \mid Y=3) = \frac{P(X=2, Y=3)}{P(Y=3)} = \frac{1/6}{1/3} = \frac{1}{2}$$

$$P_{X|Y}(x|y) = \begin{cases} 1/2 & x \neq y \\ 0 & x = y \end{cases}$$

Last class: joint distribution for two discrete random variables (2/2)

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

8. Are X and Y independent? Why or why not?

Remark:

- To show that X and Y are *not* independent, we just need to find one counter example
- However, to show that they *are* independent, we need to verify this for all possible pairs of x and y

X \ Y	1	2	3	$P_X(x)$
1	0	1/6	1/6	1/3
2	1/6	0	1/6	1/3
3	1/6	1/6	0	1/3
$P_Y(y)$	1/3	1/3	1/3	

If $X \perp Y$:

$P(X=x, Y=y) = P(X=x)P(Y=y)$

$x=1, y=1.$

$P(X=x, Y=y) = 0$

$P(X=1) = 1/3$

$P(Y=1) = 1/3$

$P(X=1, Y=1) = 0$

$\neq \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

$\Rightarrow$ X & Y are not independent

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Constructing a joint pdf from two independent, continuous RVs

$f_{X,Y}(x,y) = f_X(x) f_Y(y)$ ← when $X \perp Y$

$= \frac{1}{2} \cdot 3y^2 = \frac{3}{2} y^2$

$f_{X,Y}(x,y) = \frac{3}{2} y^2$ for $0 \le x \le 2$, $0 \le y \le 1$

Example 1.1

Let X and Y be independent r.v.'s with $f_X(x) = \frac{1}{2}$, for $0 \leq x \leq 2$ and $f_Y(y) = 3y^2$, for $0 \leq y \leq 1$.

1. Find $f_{X,Y}(x,y)$

ex) domain w/ x dep on y

$0 \le x \le y/2$

immediately know $X \not\perp Y$

Probability from joint pdf from two independent, continuous RVs

Example 1.2

Let X and Y be independent r.v.'s with $f_X(x) = \frac{1}{2}$, for $0 \leq x \leq 2$ and $f_Y(y) = 3y^2$, for $0 \leq y \leq 1$.

2. Find $\mathbb{P}(0 \leq X \leq 1, 0 \leq Y \leq \frac{1}{2})$

$$f_{X,Y}(x,y) = \frac{3}{2}y^2$$

OPT 1: $P(0 \le X \le 1, 0 \le Y \le 1/2) = \int_0^1 \int_0^{1/2} \frac{3}{2} y^2 \, dy \, dx$

OPT 2: $P(0 \le X \le 1, 0 \le Y \le 1/2) =$

$$F(X=1, Y=\tfrac{1}{2}) - \underbrace{F(X=0, Y=0)}_{F=0}$$

$P(X \le 0, Y \le 0) = 0$

$$= F(X=1, Y=\tfrac{1}{2})$$

$P(X \le 1)$

$$= F_X(X=1) F_Y(Y=\tfrac{1}{2})$$

$$= \left[\int_0^1 \underbrace{\tfrac{1}{2}}_{f_X(x)} dx\right]\left[\int_0^{1/2} \underbrace{3y^2}_{f_Y(y)} dy\right]$$

$$= \left(\tfrac{1}{2}x \Big|_{x=0}^{x=1}\right)\left[y^3 \Big|_{y=0}^{y=1/2}\right] = \left(\tfrac{1}{2} - 0\right)\left(\left(\tfrac{1}{2}\right)^3 - 0\right) = \frac{1}{16}$$

★ $F_{X,Y}(x,y) = F_X(x) F_Y(y)$

$$P(0 \le X \le 1, 0 \le Y \le 1/2) = \frac{1}{16}$$

Showing independence from joint pdf

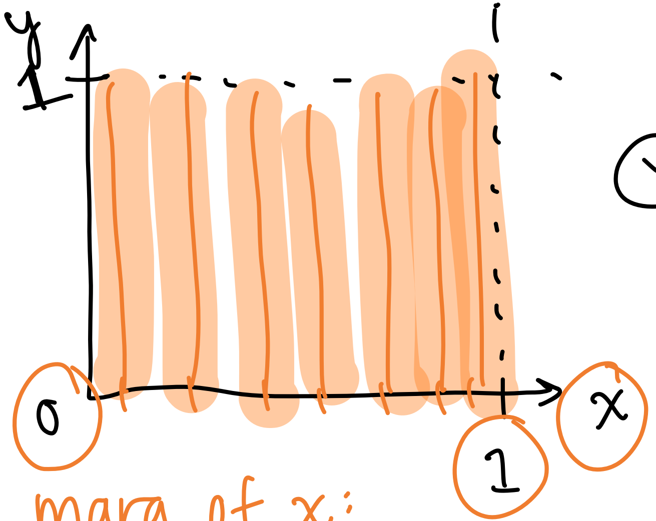
1. find marginals
2. mult them together
3. look if equal to joint

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

Example 2.1

Let $f_{X,Y}(x,y) = 18x^2y^5$, for $0 \le x \le 1$, $0 \le y \le 1$.

1. Are X and Y independent?



marg of x: integrate over y

$\boxed{X}$ $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{y=0}^{y=1} 18x^2y^5\,dy$$

$$= 3x^2y^6 \Big|_{y=0}^{y=1} = 3x^2(1)^6 - 3x(0)^6$$

$$f_X(x) = 3x^2 \text{ for } 0 \le x \le 1$$

(Y) $$f_Y(y) = \int_{x=0}^{x=1} 18x^2y^5\,dx = 6x^3y^5 \Big|_{x=0}^{x=1}$$

$$= 6y^5 \text{ for } 0 \le y \le 1$$

$$f_X(x) f_Y(y) = [3x^2][6y^5] = 18x^2y^5$$

$$f_{X,Y}(x,y) = 18x^2y^5 \overset{?}{=} f_X(x) f_Y(y)$$

for $0 \le x \le 1$, $0 \le y \le 1 \Rightarrow X \perp Y$

Finding CDF from two independent RVs

Example 2.2

Let $f_{X,Y}(x,y) = 18x^2y^5$, for $0 \le x \le 1,\ 0 \le y \le 1$.

2. Find $F_{X,Y}(x,y)$.

STEPS

1. given joint pdf
2. get marg pdf
3. int marg pdfs to get CDFs
4. mult marg CDFs

③ $F_X(x) = P(X \le x) = \int_0^x f_X(s)\,ds = \int_0^x 3s^2\,ds$

$= s^3 \Big|_{s=0}^{s=x} = x^3$ for $0 \le x \le 1$

$F_Y(y) = P(Y \le y) = \int_0^y f_Y(t)\,dt = \int_0^y 6t^5\,dt$

$= \dots = y^6$ for $0 \le y \le 1$

④ $F_{X,Y}(x,y) = F_X(x)F_Y(y)$

$$= \begin{bmatrix} 0 & x<0 \\ x^3 & 0 \le x \le 1 \\ 1 & x>1 \end{bmatrix} \begin{bmatrix} 0 & y<0 \\ y^6 & 0 \le y \le 1 \\ 1 & y>1 \end{bmatrix}$$

$$F_{X,Y}(x,y) = \begin{cases} 0 & x<0 \text{ OR } y<0 \\ x^3y^6 & 0 \le x \le 1 \ \&\ 0 \le y \le 1 \\ x^3 & 0 \le x \le 1 \ \&\ y>1 \\ y^6 & 0 \le y \le 1 \ \&\ x>1 \\ 1 & x>1 \ \&\ y>1 \end{cases}$$

Showing independence from joint pdf

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 3

Let $f_{X,Y}(x,y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$. Are X and Y independent?

Final statement on independence

1. If $f_{X,Y}(x,y) = g(x)h(y)$, where $g(x)$ and $h(y)$ are pdf's, then X and Y are independent.
   - The domain of the joint pdf needs to be independent as well!!

transformations also independent

2. If $F_{X,Y}(x,y) = G(x)H(y)$, where $G(x)$ and $H(y)$ are cdf's, then X and Y are independent.
   - The domain of the joint CDF needs to be independent as well!!

- Make sure that:
  - X domain does NOT depend on Y
  - Y domain does NOT depend on X

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Example starting from a joint pdf: first try!

$P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Example 1.1

Let $f_{X,Y}(x,y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$.

1. Find $\mathbb{P}(2 < X < 10|Y = 4)$

WAY NOT TO DO THIS:

$$P(2 < X < 10 \mid Y = 4) = \frac{P(2 < X < 10,\ Y = 4)}{P(Y=4)}$$

$$\downarrow$$

$P(Y=4) = 0$ for a cont RV

RIGHT WAY: first find $f_Y(y)$

then $f_{X|Y}(x|y) \longrightarrow \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

then we can consider intervals & probability

Example starting from a joint pdf: second try! (1/2)

$e^{x+y} = e^x e^y$

Example 1.1

Let $f_{X,Y}(x,y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$.

1. Find $\mathbb{P}(2 < X < 10 | Y = 4)$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{\cancel{5}e^{-x}e^{-3y}}{\cancel{5}e^{-5y}} \quad \begin{matrix} 0<y<\frac{x}{2} \\ y>0 \end{matrix}$$

$$= e^{-x}e^{-3y-(-5y)}$$

$$f_{X|Y}(x|y) = e^{-x}e^{2y} \quad \text{for } 0<y<\frac{x}{2}$$

$$f_{X|Y}(x|y=4) = \underbrace{e^{-x}e^{8}}_{\text{red plane}} \qquad 4<\frac{x}{2}, \quad x>8$$

to get green plane: $P(2<X<10 | Y=4)$

$$= \int_{\cancel{2}\,8}^{10} e^8 e^{-x}\,dx = \ldots = 1-e^{-2}$$

when $y=4$, x must be >8

$$f_Y(y) = \int_{2y}^{\infty} 5e^{-x-3y}\,dx$$

bounds of x:

$y < \frac{x}{2}$

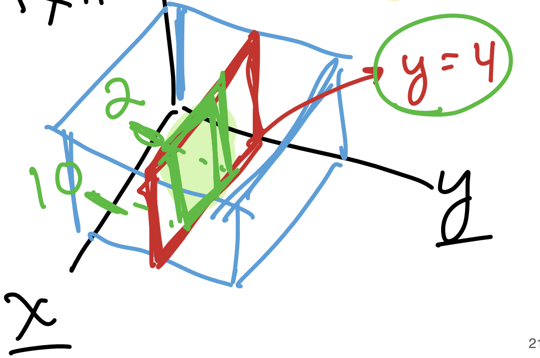
$2y < x < \infty$

$$= \int_{2y}^{\infty} 5e^{-x}\underbrace{e^{-3y}}_{\text{constants}}dx$$

$$= 5e^{-3y}\int_{2y}^{\infty} e^{-x}\,dx$$

$$= \ldots = 5e^{-5y} \quad y>0$$

$f_{X|Y}(x|y)$



Example starting from a joint pdf: second try! (2/2)

why not $F_{X,Y}(x=10,\ y=4) - F_{X,Y}(x=2,\ y=4)$

$$= \int_{8}^{10} \int_{4}^{4} f_{X,Y}(x,y)\,dy\,dx$$

↳ ends up @ 0 again

when $P(\underline{\qquad\quad} \mid X = x)$, must find conditional pdf $(f_{Y|X}(y|x))$ first!

Example starting from a joint pdf

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 1.2

Let $f_{X,Y}(x,y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$.

2. Find $\mathbb{P}(X > 20 | Y = 5)$

Finding probability with conditional domain and pdf

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 2

Randomly choose a point X from the interval $[0, 1]$, and given $X = x$, randomly choose a point Y from $[0, x]$. Find $\mathbb{P}(0 < Y < \frac{1}{4})$.