Fractal structures in three-body scattering events

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1 Motivation

This short paper describes an attempt at reproducing and improving some of the results of [1] which investigates structures in the initial-value space of a three-body scattering setup.

2 Model setup and theoretical background

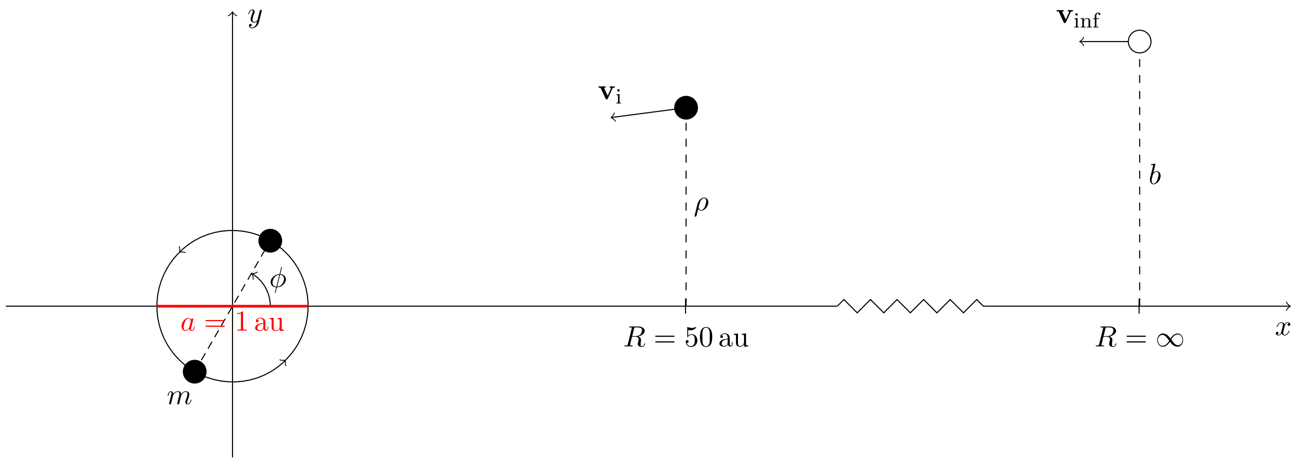

Figure 1: Schematic of the model's initial setup (not to scale).

The model consists of three stars in a two-dimensional plane with equal masses $m = M_\odot$, two of which start off in a circular orbit with semi-major axis $a = 1\,\mathrm{au}$ around each other. The third star approaches this binary system from infinity on an initially hyperbolic orbit with respect to the binary's center of mass with impact parameter b and velocity at infinity v_{inf}. The impact parameter b and the phase of the binary ϕ when its horizontal distance from the third star is $R = 50\,\mathrm{au}$ are the two initial conditions which span the above-mentioned initial-value space. Because the stars have equal masses, one only needs to consider phases between $\phi = 0$ and $\phi = \pi$. The velocity $v_{\mathrm{inf}} = v_{\mathrm{c}}/2$ of the incoming third star is set to half of the critical velocity v_{c} which is derived by comparing the binding energy E_{bind} of the binary to the kinetic energy $E_{\mathrm{kin,3,c}}$ of the incoming star at the critical velocity:

$$
\begin{aligned}
& E_{\mathrm{bind}} \stackrel{!}{=} -E_{\mathrm{kin,3,c}} \\
\Longrightarrow \quad & \frac{-Gm^2}{2a} = -\left(\frac{2m}{3}\right)\frac{v_{\mathrm{c}}^2}{2} \\
\Longrightarrow \quad & v_{\mathrm{c}} = \sqrt{\frac{3Gm}{2a}}\;,
\end{aligned}
\tag{1}
$$

where $(2m/3)$ is the reduced mass of the third star relative to the binary and G is the gravitational constant. It immediately becomes clear that for $|E_{\mathrm{kin,3}}| > |E_{\mathrm{bind}}|$ a fully unbound system with each star flying off into a separate direction is a possible outcome of the scattering event. In fact, a fully bound or even temporarily bound system is impossible in this case. This is different for $v_{\mathrm{inf}} = v_{\mathrm{c}}/2$, as the third star simply does not have enough energy. Instead, the possible remaining outcomes of the scattering event are a simple deflection of the third star that leaves the binary intact, the immediate exchange of a binary partner where the incoming star becomes bound to one of the initial binary members and the other member gets ejected, or so-called *resonance*. In the latter case, the system becomes temporarily bound with complicated and chaotic trajectories until eventually one of the bodies gets ejected.

Because the simulation cannot be started at infinite separation, the alternative "impact parameter" ρ and velocity vector $\mathbf{v}_\mathrm{i}$ (see Figure 1) of the third star at the above-mentioned separation of $R = 50\,\mathrm{au}$ need to be determined. This is achieved analytically by initially treating the system as a two-body system with one "star" (i.e. the binary) of mass $2m$ and the incoming star. As the resulting equations of this approach are not detailed in either [1] or [2], which the former is based on, a quick overview over these equations is given in the following.

By using the fact that the semi-major axis a_2b of a hyperbolic orbit is given by

$$a_\mathrm{2b} = \frac{\mu_\mathrm{2b}}{v_\mathrm{inf}^2} \ , \tag{2}$$

where $\mu_\mathrm{2b} = G(2m + m) = 3Gm$ is the standard gravitational parameter (this simply follows from the vis-viva equation), the shape of the hyperbolic trajectory of the distance vector between both "stars" is simply given by

$$\left(\frac{x}{a_\mathrm{2b}}\right)^2 - \left(\frac{y}{b}\right)^2 = 1 \ . \tag{3}$$

With some geometric considerations it is possible to derive that at $R = 50\,\mathrm{au}$, the new "impact parameter" ρ is

$$\rho = \frac{-B - \sqrt{B^2 - 4AC}}{2A} \ , \tag{4}$$

where

$$A = \left(\frac{\sin\vartheta}{a_\mathrm{2b}}\right)^2 - \left(\frac{\cos\vartheta}{b}\right)^2 \ , \tag{5}$$

$$B = -2\sin\vartheta\left(\frac{R\cos\vartheta + c}{a_\mathrm{2b}^2} + \frac{R\cos\vartheta}{b^2}\right) \ , \tag{6}$$

$$C = \left(\frac{R\cos\vartheta + c}{a_\mathrm{2b}}\right)^2 - \left(\frac{R\sin\vartheta}{b}\right)^2 - 1 \ , \tag{7}$$

and $\vartheta = \arctan(-b/a_\mathrm{2b})$, $c = \sqrt{a_\mathrm{2b}^2 + b^2}$. With the actual distance $d = \sqrt{\rho^2 + R^2}$ between the two "stars" and again using the vis-viva equation, the magnitude of the velocity v_i at $R = 50\,\mathrm{au}$ can be calculated as

$$v_\mathrm{i} = \sqrt{\mu_\mathrm{2b}\left(\frac{2}{d} - \frac{1}{a_\mathrm{2b}}\right)} \ . \tag{8}$$

Finally, the velocity vector $\mathbf{v}_\mathrm{i}$ can be determined through the slope of the hyperbolic trajectory at $R = 50\,\mathrm{au}$:

$$\mathbf{v}_\mathrm{i} = v_\mathrm{i}\begin{pmatrix}\cos(\gamma - \vartheta + \pi)\\ \sin(\gamma - \vartheta + \pi)\end{pmatrix} \ , \tag{9}$$

where

$$\gamma = \arctan\left[\frac{b}{a_\mathrm{2b}^2}\frac{X}{\sqrt{\left(\frac{X}{a_\mathrm{2b}}\right)^2 - 1}}\right] \tag{10}$$

and $X = -\rho\sin\vartheta + R\cos\vartheta + c$. It should be noted that these equations only hold for positive values of b. However, the symmetry of this two-body problem easily allows for their use with negative b values by simply using the absolute value of b and then changing the sign of ρ and the y-direction of $\mathbf{v}_\mathrm{i}$.

3 Numerical simulations and results

For each initial condition pair (ϕ, b) the system is simulated with the fourth-order Runge-Kutta method (RK4) and an adaptive time step based on the maximum acceleration $a_{\max}$ of a particle after each integration step. The criterion for the adaptive time step was set to $1\,\mathrm{au\,yr^{-1}}$ so that the next time step was always calculated as $\Delta t = 1\,\mathrm{au\,yr^{-1}}/a_{\max}$. Simulations using a smaller time step criterion of $0.1\,\mathrm{au\,yr^{-1}}$ were tested for lower resolution versions of Figure 2 and showed very slight deviations of the features as well as a small shift in positive ϕ-direction but otherwise almost identical results. The simulations were run with a maximum time of $t_{\max} = 3200\,\mathrm{yr}$ which is equal to approximately 4500 orbits of the undisturbed binary star ($T = \pi\sqrt{2a^3/(Gm)} \approx 0.707\,\mathrm{yr}$). Additionally, in order to not waste too much time on simulations in which very small time steps start occurring due to extremely close encounters (i.e. high accelerations) between two bodies, the simulations were terminated after reaching a certain threshold of iterations which was varied between the images and will therefore be mentioned in the respective image captions. The pixel values of the images represent the escape angle θ_{esc} (see the caption of Figure 2 for further explanation), which is the angle of the ejected or deflected star's final position (in this case amplified by adding the 10 000-fold difference vector between the final two positions to the final position) with the positive x-axis. Technically this is of course not entirely correct, as the scattering does not happen at $(0, 0)$ but one can easily see that with increasing distances from the center (as achieved by amplifying the distance vector) this error approaches zero.

The first image shown in Figure 2 shows the simulation results for a 301×301 grid of $b \in [-4.5, 7.5]$ and $\phi \in [0, \pi]$. One can clearly see smooth regions in which all scattering events lead to the exchange of a binary member (red) and smooth regions, where the incoming star gets deflected (blue). In between there are regions with seemingly random pixel colors and values which are the result of *resonance* behaviour. The overall structures look very similar to the corresponding image in [1] although there are a few significant differences. Primarily, apart from a very slight distortion, most of the structure seems to be shifted by about -0.6 in ϕ-direction compared to [1]. Additionally, the upper exchange region with the *Starfleet logo* shape is much more extended in ϕ-direction in this version. Although the cause of this is not quite clear, one could speculate that the different integration method and tricks used in [1] provide more accurate solutions than the RK4 with variable "time steps" of $1\,\mathrm{au\,yr^{-1}}$ used in this work. As already mentioned above, smaller time steps (and therefore more accurate simulations) did in fact lead to a small shift in positive ϕ-direction.

Although in a slightly different position, I was also able to find the small scale structure (Figure 3) that initially drew my interest to this project, equally located in b-range $[-3.760, -3.755]$ but in this case in the ϕ-interval $[2.747, 3.009]$. This region is located in the bottom resonance region of Figure 2 and clearly shows that there are smooth sections in the resonance area which are separated by sections that even at this magnification still appear "random". As one would expect, the extremely small b-interval indicates that inside this resonance region very small changes in the initial conditions can lead to very different outcomes, clearly showing that the three-body problem is of chaotic and fractal nature. Because of the higher resolution, the image in this work does not only show the large bands (in red) which correspond to scattering events with two passes of the distance between the furthest star and the center of mass through a minimum ($N_{\min} = 2$) before ejection but also the thinner bands (in blue) with $N_{\min} = 3$ which are mentioned in [1].

4 Conclusion

I was able to reproduce the results of [1] and generate images with a higher resolution as well as use colors to better visualize the behaviour of the model. As already theorized above, using a more sophisticated algorithm and/or more strictly following the integration approach in [1] could possibly eliminate the obvious differences between both versions and allow for the generation of even higher resolution images that are closer to the original work in an even shorter time.

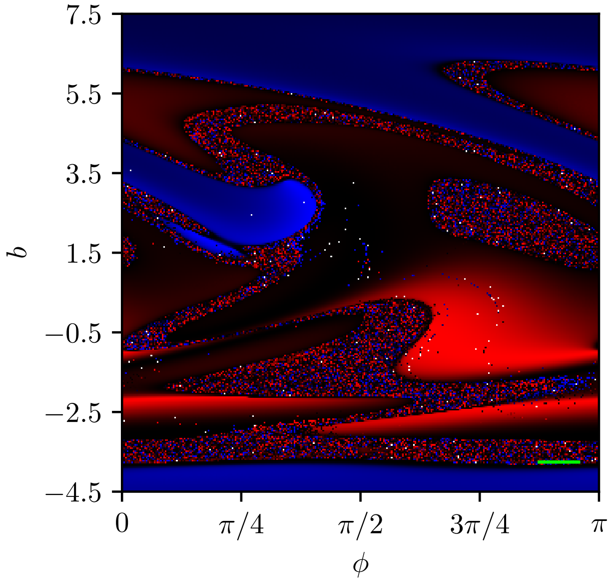

Figure 2: Escape angles for 301×301 simulations with initial conditions of (ϕ, b). The pixel values V (between 0 and 1) for the red and blue channels are calculated via $V = (1 + \cos\theta_{\mathrm{esc}})/2$. Pixels in red represent simulations that end with the ejection of one of the binary members while pixels in blue correspond to outcomes where the incoming star gets deflected/ejected. White pixels are simulations that were terminated due to reaching the maximum iteration threshold of 200 000 000. The green rectangle in the bottom right visualizes the small region displayed in Figure 3 (note that the region is actually much thinner in b-direction).

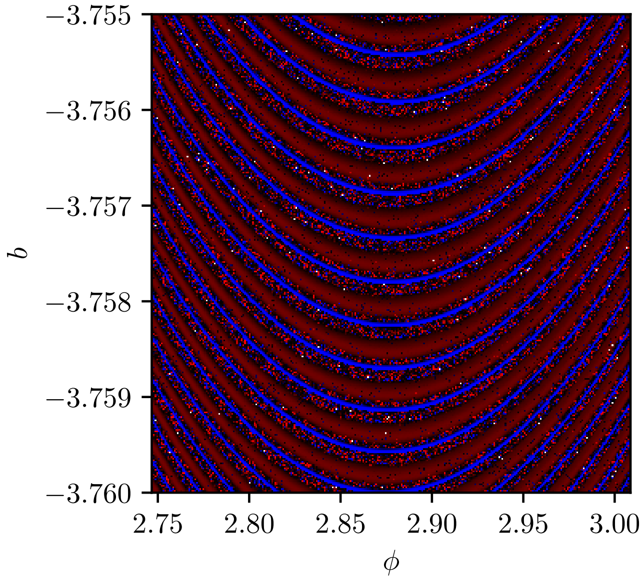

Figure 3: Escape angles for 301×301 simulations with initial conditions in the small region of $b \in [-3.760, -3.755]$, $\phi \in [2.747, 3.009]$ and with a maximum iteration threshold of 500 000 000. The coloring scheme is identical to Figure 3.

References

[1] P. T. Boyd and S. L. W. McMillan. Initial-value space structure in irregular gravitational scattering. *Physical Review A*, 46(10), 1992.

[2] P. Hut and J. N. Bahcall. Binary-single star scattering. I - Numerical experiments for equal masses. *The Astrophysical Journal*, 268:319–341, 1983.