2x2 PT-symmetric Hamiltonian

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H is PT-symmetric, PT is isometry

$PTH = HPT$

$H\varphi = E\varphi$ for some $E \in \mathbb{C}$

Show that:

if $E \in \mathbb{R}$, and $\dim(\{\varphi \mid H\varphi = E\varphi\}) = 1$

then $PT\varphi = \lambda\varphi$ for some $\lambda \in \mathbb{C}$ with $\lambda^*\lambda = 1$

Proof:

Denote $PT\varphi = \mu$

$PTH\varphi = PTE\varphi = EPT\varphi = E\mu$

$PTH\varphi = HPT\varphi = H\mu$

$\Rightarrow E\mu = H\mu \quad (*)$

Here both P and T are isometry

then $\|\varphi\| = \|\mu\| \quad (**)$

from $(*)$, we get $\varphi = \lambda\mu$ for some $\lambda \in \mathbb{C}$

because $\dim(\{\varphi \mid H\varphi = E\varphi\}) = 1$

from $(**)$ $\|\varphi\| = \|\lambda\mu\| = |\lambda| \cdot \|\mu\| = \|\mu\|$

$\Rightarrow |\lambda| = 1 \Rightarrow \lambda^*\lambda = 1$

$\square$

Consequences

order parameter

$$\eta = \sum_k \left| \varphi_k^* \varphi_k - PT\varphi_k^* \varphi_k \right|$$

$$PT\varphi_k = \lambda \varphi_k$$

$$(\lambda\varphi)^* \varphi = \lambda^* \varphi^* \varphi = \lambda^*$$

Application

$$H = \begin{bmatrix} E+i\gamma & c \\ c & E-i\gamma \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$T: \mathbb{C}^2 \to \mathbb{C}^2 \quad \vec{v} \mapsto \vec{v}^*$$

$$\det(H-\lambda I) = (E+i\gamma-\lambda)(E-i\gamma-\lambda) - c^2$$
$$= E^2 - 2\lambda E + \gamma^2 + \lambda^2 - c^2$$

$$\Rightarrow 0 = \lambda^2 - 2E\lambda + (E^2+\gamma^2-c^2)$$

$$\lambda = \frac{2E \pm \sqrt{4E^2-4E^2-4\gamma^2+4c^2}}{2}$$
$$= E \pm \sqrt{c^2-\gamma^2}$$

$H\varphi = E\varphi \Rightarrow$ let $\varphi = \begin{bmatrix} m \\ n \end{bmatrix}$ for $\lambda = E + \sqrt{c^2-\gamma^2}$

$$\begin{bmatrix} m(E+i\gamma)+cn \\ cm+(E-i\gamma)n \end{bmatrix} = \begin{bmatrix} (E+\sqrt{c^2-\gamma^2})m \\ (E+\sqrt{c^2-\gamma^2})n \end{bmatrix}$$

$$\cancel{mE} + i\gamma m + cn = \cancel{mE} + \sqrt{c^2-\gamma^2}\, m$$

$$i\gamma m + cn = \sqrt{c^2-\gamma^2}\, m$$

$$(i\gamma - \sqrt{c^2-\gamma^2})m = -cn$$

$$n = -\frac{i\gamma - \sqrt{c^2-\gamma^2}}{c} m$$

$$v = \begin{bmatrix} m \\ -\frac{i\gamma - \sqrt{c^2-\gamma^2}}{c} m \end{bmatrix}$$

when λ is real

$$\lambda = E + \sqrt{c^2-\gamma^2} \Rightarrow c^2 \geq \gamma^2$$

$$v^* = \begin{bmatrix} m \\ \frac{i\gamma + \sqrt{c^2-\gamma^2}}{c} m \end{bmatrix}$$

$$PT\, v = \begin{bmatrix} \frac{i\gamma + \sqrt{c^2-\gamma^2}}{c} m \\ m \end{bmatrix}$$

$$PT\, v = \left(\frac{i\gamma + \sqrt{c^2-\gamma^2}}{c}\right) v$$

$$\underline{\left(\frac{i\gamma + \sqrt{c^2-\gamma^2}}{c}\right)}\left(-\frac{i\gamma - \sqrt{c^2-\gamma^2}}{c}\right)$$

$$= -\frac{-\gamma^2 - c^2 + \gamma^2}{c^2} = 1$$

$H\varphi = E\varphi \Rightarrow$ let $\varphi = \begin{bmatrix} m \\ n \end{bmatrix}$ for $\lambda = E - \sqrt{c^2-\gamma^2}$

$$\begin{bmatrix} m(E+i\gamma)+cn \\ cm+(E-i\gamma)n \end{bmatrix} = \begin{bmatrix} (E-\sqrt{c^2-\gamma^2})m \\ (E-\sqrt{c^2-\gamma^2})n \end{bmatrix}$$

$$\cancel{mE} + i\gamma m + cn = \cancel{mE} - \sqrt{c^2-\gamma^2}\, m$$

$$i\gamma m + cn = -\sqrt{c^2-\gamma^2}\, m$$

$$(i\gamma + \sqrt{c^2-\gamma^2})m = -cn$$

$$n = -\frac{i\gamma + \sqrt{c^2-\gamma^2}}{c} m$$