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Coupled two-photonic components system

Consider a coupled two-photonic components system with a coupling constant $C > 0$, each has energy E one with gain $\gamma > 0$ and the other one with loss $-\gamma$. The Hamiltonian of this system is given by:

$$\mathscr{H} = \begin{bmatrix} E + i\gamma & C \\ C & E - i\gamma \end{bmatrix}$$

By simple computation, the eigen-energies of this system is given by the followings:

$$\lambda_1 = E + \sqrt{C^2 - \gamma^2} \qquad\qquad \lambda_2 = E - \sqrt{C^2 - \gamma^2}$$

Here we see that:

$$\begin{cases} C > \gamma & \mathscr{H} \text{ is in unbroken PT-symmetric phase} \\ C = \gamma & \text{Exceptional point occurs} \\ C < \gamma & \mathscr{H} \text{ is in broken PT-symmetric phase} \end{cases}$$

We can also compute the analytic eigensolutions of the system described by $\mathscr{H}$.

For λ_1, the eigensolution is given by the following:

$$|\psi\rangle_1 = \begin{bmatrix} m \\ \left(\frac{\sqrt{C^2-\gamma^2}}{C} - \frac{\gamma}{C}\, i\right) m \end{bmatrix}$$

For λ_2, the eigensolution is given by the following:

$$|\psi\rangle_2 = \begin{bmatrix} m \\ \left(-\frac{\sqrt{C^2-\gamma^2}}{C} - \frac{\gamma}{C}\, i\right) m \end{bmatrix}$$

where $m \in \mathbb{C}$ is a constant that is used to normalize the eigensolution such that $\langle\psi|\psi\rangle = 1$.

We claim that, for system like the 2×2 Hamiltonian $\mathscr{H}$ described above, if the system has nondegenerate spectrum, the eigenvalues are real if and only if the eigensolutions $|\psi\rangle$ are invariant under the PT operator up to a phase factor λ, that is, we have:

$$PT\,|\psi\rangle = \lambda\,|\psi\rangle \qquad \text{with } |\lambda| = 1 \qquad\qquad \text{(in unbroken PT-symmetric phase)}$$

Mathematically, this statement can be generalized by Theorem 1.1 on next page. Many former studies on PT-symmetric systems, such as *PT-symmetry in optics, by Zyablovsky et al (2014)*, and *PT-symmetric quantum mechanics, by Bender et al (1998)*, have also demonstrated this result, but here we will look further on how this result can be analytically applied to our two-photonic components system.

For the two-photonic components system, the time operator T is taking the complex conjugate, and the parity operator P is given by the following matrix:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hence we can see that:

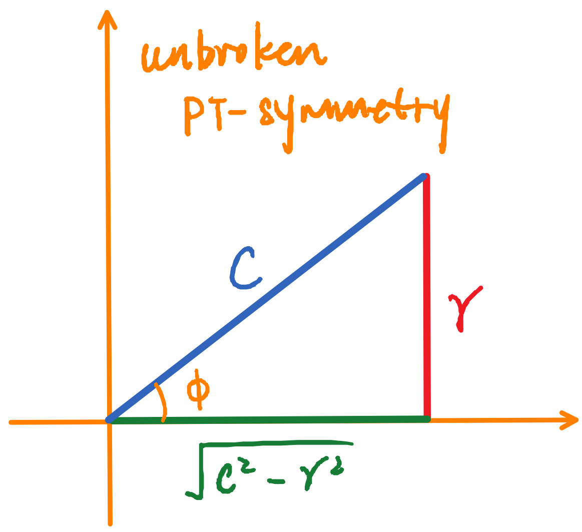
$$PT\,|\psi\rangle_1 = \begin{bmatrix} \left(\frac{\sqrt{C^2-\gamma^2}}{C} + \frac{\gamma}{C} i\right) m \\ m \end{bmatrix} = \left(\frac{\sqrt{C^2-\gamma^2}}{C} + \frac{\gamma}{C} i\right) |\psi\rangle_1$$

When $\gamma < C$, that is, the system is in unbroken PT-symmetry state, then we have:

$$\left|\frac{\sqrt{C^2-\gamma^2}}{C} + \frac{\gamma}{C} i\right| = 1 \qquad \text{when } \gamma < C$$

it follows that, when we have $\gamma > C$, the system is in broken PT-symmetry state, and we have:

$$\left|\frac{\sqrt{C^2-\gamma^2}}{C} + \frac{\gamma}{C} i\right| \neq 1 \qquad \text{when } \gamma > C$$



Without lost of generality, we suppose that the P and T operators are isometries, that is, they preserve the length of the eigensolutions of the system.

Theorem 1.1
Let $n \in \mathbb{N}$, let $\mathscr{H} \in Mat_{n\times n}(\mathbb{C})$ be commute with PT where P is a linear isometry and T is an antilinear isometry. Consider the system $\mathscr{H}\psi = E\psi$ for some $E \in \mathbb{C}$ and $\psi \in \mathbb{C}^n$, with $\dim(\{\phi \mid \mathscr{H}\phi = E\phi\}) = 1$. We have $E \in \mathbb{R}$ if and only if $PT\psi = \lambda\psi$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.

Proof. First we will show the $\Rightarrow$ direction holds. Suppose that we have $E \in \mathbb{R}$, here we denote $PT\psi = \mu$, then by the linearity of P and the antilinearity of T, we have:

$$PT\mathscr{H}\psi = PTE\psi = E^*PT\psi = EPT\psi = E\mu \tag{1}$$

Since PT commutes with $\mathscr{H}$, then we also have:

$$PT\mathscr{H}\psi = \mathscr{H}PT\psi = \mathscr{H}\mu \tag{2}$$

Hence combining (1) and (2) we get:

$$E\mu = \mathscr{H}\mu \tag{3}$$

Since PT is an isometry, then we get:

$$||\psi|| = ||\mu|| \tag{4}$$

From (3), and since $\dim(\{\phi \mid \mathscr{H}\phi = E\phi\}) = 1$, we get:

$$PT\psi = \mu = \lambda\psi \qquad \text{for some } \lambda \in \mathbb{C} \tag{5}$$

Combining (4) and (5), we get:

$$||\psi|| = ||\mu|| = ||\lambda\psi|| = |\lambda| \cdot ||\psi|| \qquad \Rightarrow \qquad |\lambda| = 1$$

This completes the proof of the $\Rightarrow$ direction.
For the $\Leftarrow$ direction, we suppose that $PT\psi = \lambda\psi$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. By linearity of P and antilinearity of T, we can write:

$$\mathscr{H}PT\psi = PT\mathscr{H}\psi = PTE\psi = E^*PT\psi = E^*\lambda\psi \tag{6}$$

Since $\dim(\{\phi \mid \mathscr{H}\phi = E\phi\}) = 1$ and $\lambda\psi \in \{\phi \mid \mathscr{H}\phi = E\phi\}$, equation (6) forces $E^* = E$, and hence $E \in \mathbb{R}$. This completes the proof of the $\Leftarrow$ direction, the result of this theorem follows. $\square$