Class Notes

Physics 351 - Methods of Theoretical Physics
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Complex Numbers

Any time we have periodic behavior, we should look for complex number formulation.
Quantum mechanics is written in terms of complex numbers.

Theorem 1.1 (Euler's Formula)
For $\theta \in \mathbb{R}$, and $i = \sqrt{-1}$, we have the following holds:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

When describing waves, we can describe it using the equation $e^{i(kx-\omega t)}$ where k is the wave number, x is the position, ω is the frequency, and t is time.

Let z be a complex number, that is, we have $z \in \mathbb{C}$, we can write $z = z' + iz''$ for some $z', z'' \in \mathbb{R}$.

Definition 1.1.0.0.1
Let $z = z' + iz'' \in \mathbb{C}$ for $z', z'' \in \mathbb{R}$, the complex conjugate of z is defined to be $z^* = z' - iz''$

For $z_1, z_2 \in \mathbb{C}$, note here the following holds:

1. $(z_1 + z_2)^* = z_1^* + z_2^*$
2. $\left(\frac{1}{z_1}\right)^* = \frac{1}{z_1^*}$
3. $(z_1 z_2)^* = z_1^* z_2^*$

Definition 1.1.0.0.2
Let $z = z' + iz'' \in \mathbb{C}$, the modulus of z is defined to be $|z| = \sqrt{zz^*} = \sqrt{(z')^2 + (z'')^2}$

Here we note that for a complex number z, we have $|z|^2 = zz^*$.

Any complex number $z = z' + iz''$ can be written as $\rho e^{i\theta}$ where $\rho, \theta \in \mathbb{R}$. Here we see that $z = \rho\cos(\theta) + i\rho\sin(\theta)$, and hence $z' = \rho\cos(\theta)$ and $z'' = \rho\sin(\theta)$. Notice that $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{z''}{z'}$. On another note, we have $(z')^2 + (z'')^2 = \rho^2\cos^2(\theta) + \rho^2\sin^2(\theta) = \rho^2 = |z|^2$

Definition 1.1.0.0.3
Let $z = z' + iz'' \in \mathbb{C}$, then the real part of z is $Re\{z\} = z'$, and the complex part of z is $Im\{z\} = z''$.

Notice that, for $z = z' + iz'' \in \mathbb{C}$, we have $Re\{z\}, Im\{z\} \in \mathbb{R}$.
On another note, we have $z + z^* = z' + iz'' + z' - iz'' = 2z' = 2Re\{z\}$.

For $z \in \mathbb{C}$, the equation $z^n = 1$ has n solutions, give by $e^{i2\pi\frac{k}{n}}$ with $k = n - 1$ and $n \in \mathbb{N}$.

Theorem 1.2
For $z, a, b \in \mathbb{R}$, we have the followings hold:

1. $\cos(iz) = \cosh(z) = \frac{e^z + e^{-z}}{2}$ $\qquad \cosh(iz) = \frac{e^{iz} + e^{-iz}}{2} = \cos(z)$
2. $\sin(iz) = \sinh(z) = \frac{e^z - e^{-z}}{2}$ $\qquad \sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} = i\sin(z)$
3. $\tanh(iz) = i\tan(z)$ $\qquad \tanh(z) = -i\tan(iz)$
4. $\cosh(a + ib) = \cosh(a)\cos(b) + i\sinh(a)\sin(b)$
5. $\sinh(a + ib) = \sinh(a)\cos(b) + i\sinh(a)\sin(b)$

When solving a algebraic equation with complex numbers, say $f(z) = a + bi$ for some $z \in \mathbb{C}$, $a, b \in \mathbb{R}$, we want to have $Re(f(z)) = a$ and $Im(f(z)) = b$.

Series

Definition 2.0.0.0.1
Fine structure constant is given by $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}$
The experimental value is $\approx 137.033599206(11)$*, and the theoretical value is around* $\approx 137.035999174(35)$

Definition 2.0.0.0.2
An infinity series is any sum of the form $S = \sum_{n=0}^{\infty} a_n$ *for* $a_n \in \mathbb{R}$*.*

Definition 2.0.0.0.3
Partial sum is defined to be any sum of the form $S_N = \sum_{n=0}^{N} a_n$ *for* $a_n \in \mathbb{R}$ *and* $N \in \mathbb{N}$*.*

Definition 2.0.0.0.4
A series S *is said to be convergent provided that* $\lim_{n\to\infty} S_n$ *exists.* S *is said to be divergent provided that* $\lim_{n\to\infty} S_n$ *does not exist.*

Definition 2.0.0.0.5
A series $S = \sum_{n=0}^{\infty} a_n$ *is said to be absolutely convergent provided that* $\widetilde{S} = \sum_{n=0}^{\infty} |a_n|$ *is convergent.*

For absolutely convergent series, the order of the terms does not affect the value it converges to.

Definition 2.0.0.0.6
A series S *is said to be conditionally convergent provided that* S *is convergent but not absolutely convergent.*

Definition 2.0.0.0.7
For some coefficients a_n *where* $n \in \mathbb{N}$*,* $S(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ *is called a power series.*

Note that each power series has a radius of convergence, or the interval of convergence. In other words, there exists an interval for x over which a power series will converge. For example, consider the series $(1+x)^\alpha = 1 = \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \cdots$, the series converges for $x \in (-1, 1)$.

Notes:

1. Power series can be integrated and differentiated.
2. Power series canbe added, subtracted, multiplied, and divided.
3. The power series of a function, if exists, is unique.

Theorem 2.1 (Taylor's Theorem)
Let $f : \mathbb{R} \to \mathbb{R}$ *be a function. There exists an intercal of convergence* $(-R + x_0, R + x_0)$ *such that* $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$*, such expression is called the Taylor series for* f *around* x_0*.*

Definition 2.1.0.0.1
For some $a, b \in \mathbb{R}$ *and* $N \in \mathbb{N}$*,* $S_N = \sum_{n=0}^{N} (a + b\cdot n) = a(N+1) + b \cdot \sum_{n=1}^{N} n = a(N+1) + b(N+1)\frac{N}{2}$ *is called a Arithmetic Series.*

Definition 2.1.0.0.2
For some $a \in \mathbb{R}$*,* $S = \sum_{n=0}^{\infty} a^n$ *is called a Geometric Series.*

Let $a \in \mathbb{R}$ and $S = \sum_{n=0}^{\infty} a^n$. For some $N \in \mathbb{N}$, the partial sum $S_N = \sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}$. Hence S converges provided that $\lim_{N\to\infty} \frac{1-a^{N+1}}{1-a}$ exists, in which case we have $S = \frac{1}{1-a}$.

Definition 2.1.0.0.3
For $m \in \mathbb{Z}$*,* $p, q \in \mathbb{R}$*,* $(p+q)^m = \sum_{k=0}^{m} p^{m-k} q^k \frac{m!}{(m-k)!k!}$ *is called the Binomial Series.*

Series as approximations

Let $S = a_0 + a_1 x + x_2 x^2 + \cdots$. We might want to truncate the series S to $S_1 = a_0 + a_1 x$. Here we note that, $\frac{S}{S_1} = 1 + \frac{a_2 x^2}{a_0 + a_1 x} + \frac{a_3 x^3}{a_0 + a_1 x} + \cdots$, and hence we see that $\lim_{x \to 0} \frac{S}{S_1} = 1$. In another words, the truncation gives a good approximation when x tends to 0, and we can evaluate the precision of the truncation S_1 by observing the value of $\frac{S}{S_1}$. Here, a_0 is called the leading order term, and for $n \in \mathbb{N}$, a_n is called the n-th correction term.

To explain the idea. Here we suppose we need to solve the equation $x^2 + \epsilon x + 1 = 0$. We note that, for the exactly solution, we have $x = \frac{-\epsilon \pm \sqrt{\epsilon^2 + 4}}{2}$, and we can expand the exactly solution of the equation as Taylor series around $\epsilon = 0$, here we have $x_1(\epsilon) = 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8} + \cdots$, and $x_2(\epsilon) = -1 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \cdots$. While on the other hand, in general, without the quadratic formula, we can find the power series representation of $x(\epsilon)$ using the following way. Fist, we assume that $x(\epsilon)$ can be written as a power series of the following form:

$$x(\epsilon) = a_0 + a_1 \epsilon + a_2 \epsilon^2 + \cdots \tag{1}$$

Here we can plug equation (1) into $x^2 + \epsilon x - 1 = 0$, and get the following:

$$(a_0 + a_1 \epsilon + a_2 \epsilon^2 + \cdots)^2 + \epsilon(a_0 + a_1 \epsilon + a_2 \epsilon^2 + \cdots) - 1 = 0$$

Here we expand and collect terms of equal power of ϵ:

$$\begin{aligned}
\epsilon^0 : &\quad a_0^2 - 1 = 0 \\
\epsilon^1 : &\quad 2a_1 a_0 + a_0 = 0 \\
\epsilon^3 : &\quad 2a_1 a_0 + a_1^2 + a_1 = 0 \\
\cdots &
\end{aligned}$$

Here we can solve for $a_0, a_1, a_2, \cdots$, and get $a_0 = \pm 1$, $a_1 = \frac{-1}{2}$, $a_2 = \pm \frac{1}{8}$, and get the same solution as using the quadratic formula: $x_1(\epsilon) = 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8} + \cdots$, and $x_2(\epsilon) = -1 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \cdots$.

Another example for approximation using series:

[Insert picture here]

We want to find an expression for the potential ϕ in the limit that $r >> a$. For the exact answer, we have the following:

$$\phi = \phi_+ + \phi_- = \frac{q}{4\pi\epsilon_0 R_1} + \frac{-q}{4\pi\epsilon_0 R_2} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

For the approximation, we proceed by the following:

[Insert Picture]

Here we note that: $R_1^2 = r^2 + a^2 - 2ar\cos\theta$

$$\begin{aligned}
\frac{1}{R_1} &= \frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} \\
&= \frac{1}{r\left(1 + \left(\frac{a}{r}\right)^2 - \frac{2a}{r}\cos\theta\right)^{\frac{1}{2}}} \\
&= \frac{1}{r}\left(1 + \left(\frac{a}{r}\right)^2 - 2\frac{a}{r}\cos\theta\right)^{\frac{-1}{2}} \\
&= \frac{1}{r}\left(1 - \frac{1}{2}\left(\left(\frac{a}{r}\right)^2 - \frac{2a}{r}\cos\theta\right) + \frac{-\frac{1}{2}\left(-\frac{1}{2} - 1\right)}{2!}\left(\left(\frac{a}{r}\right)^2 - \frac{2a}{r}\cos\theta\right)^2 + \cdots\right)
\end{aligned}$$

$$\frac{1}{R_2} = \frac{1}{r}\left(1 - \frac{1}{2}\left(\left(\frac{a}{r}\right)^2 + \frac{2a}{r}\cos\theta\right) + \cdots\right)$$

Combining the equations above, we can write the following:

$$\begin{aligned}
\frac{1}{R_1} - \frac{1}{R_2} &= \frac{1}{r}\left(-\frac{1}{2}\left(\left(\frac{a}{r}\right)^2 - \frac{2a}{r}\cos\theta\right) + \frac{1}{2}\left(\left(\frac{a}{r}\right)^2 + \frac{2a}{r}\cos\theta\right)\right) \\
&= \frac{1}{r}\left(\frac{1}{4}\frac{2a}{r}\cos\theta + \text{terms of order } \left(\frac{a}{r}\right)^2 \text{ or higher}\right)
\end{aligned}$$

Here we have $\phi \approx \frac{q}{4\pi\epsilon_0}\frac{1}{r^2} 2a\cos\theta$.

Let E be the total energy, T be the kinetic energy, and $U(x)$ be the potential energy. We know that $E = T + U(x)$. We have the following diagram:
[Insert a picture]
For Harmonic oscillator, we can write $E = \frac{1}{2}mv^2 + \frac{1}{2}k(x - x_0)^2$ where m is the mass of the oscillator and v is the speed of the oscillator, and k is some constant. We have the following diagram:
[Insert a picture]
Notice that in both diagrams, the curve have a minimum, and tends to be quadratic around the minimum. We can expand the $U(x)$ using Taylor series around the minimum point x_0:

$$U(x) = U(x_0) + \frac{dU}{dx} |_{x_0} (x - x_0) + \frac{1}{2}\frac{d^2U}{dx^2} |_{x_0} (x - x_0)^2 + \cdots$$

Here we note that $U(x_0)$ is a constant, and since we are expanding around x_0, we have $\frac{dU}{dx} |_{x_0} = 0$. Hence we can write the following:

$$U(x) = U(x_0) + \frac{1}{2}U''(x - x_0)^2 + \cdots = U(x_0) + \frac{1}{2}t(x - x_0)^2 + \cdots \qquad \text{for some } t \in \mathbb{R}$$

Hence we say every potential function $U(x)$ looks like a harmonic oscillation, which is quadratic, near its minimum. If the system is originally at rest, and we perturb the system by adding a small energy to it, then the perturbation makes the system oscillate, and if the perturbation is small enough, then such oscillation will tend to be quadratic.
[Insert a picture]

On the other hand, we can derive this from Newton's Law. By Newton's Law, we have $m\frac{d^2x}{dt^2} = -f(x)$ for some force of the form $-f(x)$. Say there exists a position x_0 such that $f(x_0) = 0$. We can expand f around x_0, by the following:

$$f(x) = f(x_0) + \frac{df}{dx}(x_0)(x - x_0) + \cdots = k(x - x_0) + \cdots \qquad \text{for some } k \in \mathbb{R}$$

If we keep $f(x) = k(x - x_0) = -m\frac{d^2x}{dt^2}$, we can solve for x and get $x = x_0 + A\cos(\omega t + \phi)$, where $\omega^2 = \frac{k}{m}$, and has a period $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$.

[Insert a picture for energy diagram of planet going around the sun]

Vectors

For Newton's Law in vector notation, we can write:

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}$$

where $\vec{r}$ is the position of the particle, $\vec{F}$ is the sum of all external forces acting on that particle.

Definition 4.0.0.0.1
A vector is a mathematical object with a magnitude with a direction.

Definition 4.0.0.0.2
Given a vector $\vec{r}$.
The magnitude of $\vec{r}$ is denoted as $|\vec{r}| = r$, which is a scalar. The direction of $\vec{r}$ is denoted as $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$.

EX. The center of mass of two objects, one has mass m_1 and position $\vec{r}_1$, one has mass m_2 and position $\vec{r}_2$, is given by the following:

$$\vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \qquad \text{Let } \alpha = \frac{m_1}{m_1 + m_2}, \text{ note that } 0 \le \alpha \le 1$$

$$\text{We have } \vec{R}_{cm} = \alpha\vec{r}_1 + (1-\alpha)\vec{r}_2 = \alpha(\vec{r}_1 - \vec{r}_2) + \vec{r}_2$$

Definition 4.0.0.0.3
Given vector $\vec{a}$ and $\vec{b}$, the dot product of $\vec{a}, \vec{b}$ is given by $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$, where θ is the angle between the two vectors.

Here we have some properties for the dot product. Let $\vec{a}, \vec{b}, \vec{c}$ be vectors.

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
2. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
3. $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

EX. Let $\vec{a}, \vec{b}$ be vectors such that $\vec{a} \neq \vec{b}$, with $|\vec{a}|, |\vec{b}| \in (0, \infty)$. If we have $\vec{a} \cdot \vec{b} = 0$, then we know that $\cos(\theta) = 0 \Rightarrow \theta = \frac{\pi}{2}$, where θ is the angle between the two vectors. Here we say the two vectors $\vec{a}, \vec{b}$ are orthogonal to each other.

EX. Let $\vec{a}, \vec{R}, \vec{r}$ be vectors such that $\vec{a} + \vec{R} = \vec{r}$, let θ denote the angle between $\vec{a}$ and $\vec{r}$.

$$\begin{aligned} |\vec{R}|^2 &= \vec{R} \cdot \vec{R} = (\vec{r} - \vec{a}) \cdot (\vec{r} - \vec{a}) \\ &= \vec{r} \cdot \vec{r} - \vec{r} \cdot \vec{a} - \vec{a} \cdot \vec{r} + \vec{a} \cdot \vec{r} \\ &= |\vec{r}|^2 - 2|r||a|\cos(\theta) + |\vec{a}|^2 \end{aligned}$$

Here we derived the Law of Cosine.

Definition 4.0.0.0.4
Give two vectors $\vec{a}$ and $\vec{b}$, we define a new vector $\vec{c} = \vec{a} \times \vec{b}$ which has the properties:

1. *$\vec{c}$ is orthogonal to both $\vec{a}$ and $\vec{b}$*
2. *$|\vec{c}| = |\vec{a}||\vec{b}|\sin(\theta)$*
3. *The direction $\vec{c}$ is set by the Right Hand Rule*

Here we have some properties for the cross product. Let $\vec{a}, \vec{b}, \vec{c}$ be vectors.

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
3. $(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$

Here we note that, for vectors $\vec{a}$ and $\vec{b}$:
If $\vec{a} \times \vec{b} = \vec{0}$, then $\vec{a}$ and $\vec{b}$ are parallel.
If $\vec{a} \cdot \vec{b} = \vec{0}$, then $\vec{a}$ and $\vec{b}$ are orthogonal.

For vector $\vec{a}, \vec{b}, \vec{c}$, $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ gives the volume of the Parallelepiped form by the three vectors.

For the standard unit vectors $\hat{\imath} = (1,0,0), \hat{\jmath} = (0,1,0), \hat{k} = (0,0,1)$ in $\mathbb{R}^3$, we have the followings:

1. $\hat{\imath} \cdot \hat{\jmath} = \hat{\jmath} \cdot \hat{k} = \hat{\imath} \cdot \hat{k} = 0$
2. $\hat{\imath} \times \hat{\jmath} = \hat{k}, \ \hat{\jmath} \times \hat{k} = \hat{\imath}, \ \hat{k} \times \hat{\imath} = \hat{\jmath}$

For $a, b, c, d, e, f \in \mathbb{R}$, we have the followings:

1. $(a\hat{\imath} + b\hat{\jmath} + c\hat{k}) \cdot (d\hat{\imath} + f\hat{\jmath} + g\hat{k}) = ad + bf + cg$
2. $(a\hat{\imath} + b\hat{\jmath} + c\hat{k}) \times (d\hat{\imath} + f\hat{\jmath} + g\hat{k}) = (bf - ce)\hat{\imath} + (cd - af)\hat{\jmath} + (ae - bd)\hat{k}$

For vectors $\vec{a}, \vec{b}, \vec{c}$, we have the followings:

1. $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$
2. $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$
3. $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

To distinguish two planes in the three dimensional Euclidean space, we need a normal vectors of the planes, and two points, one on each plane. As a result, a plane can be denoted by $P_{\hat{n},\vec{r}_0}$, where $\hat{n}$ is the normal vector of the plane, and $\vec{r}_0$ is any point on that plane. Here we have $\hat{n} \cdot (\vec{r} - \vec{r}_0) = 0$ for all $\vec{r} \in P_{\hat{n},\vec{r}_0}$.

In $\mathbb{R}^3$, suppose we have $\vec{r}(t)$ as the position vector of an object. Then we can write the components of $\vec{r}(t)$ as the followings:

$$x(t) = \hat{\imath} \cdot \vec{r}(t) \qquad y(t) = \hat{\jmath} \cdot \vec{r}(t) \qquad z(t) = \hat{k} \cdot \vec{r}(t)$$

Now suppose an object is going in a circular path on a plane parallel to the $x - y$ plane in $\mathbb{R}^3$, then the path of the object can be described as the following:

$$\vec{r}(t) = \vec{r}_0 + R(\cos(\omega t)\hat{\imath} + \sin(\omega t)\hat{\jmath})$$

where R is the radius of the circular path, ω is the angular velocity of the object, and $\vec{r}_0$ is the initial position of the object.

Given a position vector of an object in $\mathbb{R}^3$ as $\vec{r}(t) = (x(t), y(t), z(t))$, the velocity of the object can be defined by the following:

$$\begin{aligned}
\vec{v}(t) &= \lim_{h\to 0} \frac{(x(t+h) - x(t))\hat{\imath} + (y(t+h) - y(t))\hat{\jmath} + (z(t+h) - z(t))\hat{k}}{h} \\
&= \lim_{h\to 0} \left[\frac{(x(t+h) - x(t))\hat{\imath}}{h}\right] + \lim_{h\to 0} \left[\frac{(y(t+h) - y(t))\hat{\jmath}}{h}\right] + \lim_{h\to 0} \left[\frac{(z(t+h) - z(t))\hat{k}}{h}\right] \\
&= x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}
\end{aligned}$$

In Polar Coordinate system, we describe the position of some object as $\vec{r}(t) = \rho(t)\hat{\rho}$, where $\rho(t)$ is the length of the vector and $\hat{\rho}$ has some direction pointing from the origin outwards, not necessarily to be constant. As a result, if an object is in circular motion, then in Polar Coordinate system, the path of the object can be described as $\vec{r}(t) = R\hat{\rho}$, where R is the radius of the circular path. Here, using Polar Coordinate system, given the path of an object $\vec{r}(t) = r(t)\hat{\rho}$ we can find the velocity of the object, given by the following:

$$\vec{v} = \frac{d}{dt}(r\hat{\rho}) = \frac{dr}{dt}\hat{\rho} + r\frac{d\theta}{dt}\hat{\theta}$$

And acceleration:

$$\begin{aligned}
\vec{a} &= \frac{d^2\vec{r}}{dt^2} \\
&= \frac{d}{dt}\left(\frac{dr}{dt}\hat{\rho}\right) + \frac{d}{dt}\left(r\frac{d\theta}{dt}\hat{\theta}\right) \\
&= \left(\frac{d^2r}{dt^2} - r\frac{d^2\theta}{dt^2}\right)\hat{\rho} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\hat{\theta}
\end{aligned}$$

For a pendulum hanging from the origin, making an angle θ with the vertical. The pendulum is experiencing a force $-mg\sin(\theta)$ in the direction of $\hat{\theta}$, let L be the length of the pendulum, and m be the mass of the pendulum, we can write the following:

$$-mg\sin(\theta) = mL\frac{d^2\theta}{dt^2}$$

Similarly, if the tension is T, in the direction of $\hat{\rho}$ we can write the following:

$$-T + mg\cos(\theta) = -L\left(\frac{d\theta}{dt}\right)^2 m$$

For time dependent vector $\vec{v} = (v_1, v_2, v_3)$, we can write the following:

$$\vec{v} = \sum_{i=1}^{k} v_i \hat{e}_i \qquad \text{where } \hat{e}_i \text{ are orthonormal basis}$$

Then we can write the following:

$$\frac{d\vec{v}}{dt} = \sum_{i=1}^{k} \left[\frac{dv_i}{dt} \hat{e}_i + v_i \frac{d\hat{e}_i}{dt} \right]$$

In Cartesian coordinate system, we have $\frac{d\hat{e}_i}{dt} = 0$, while in Polar coordinate, we have $\frac{d\hat{e}_i}{dt} \neq 0$.

Matrix

Say we have two coordinate systems, one with orthonormal basis $\hat{\imath}, \hat{\jmath}$, called the S coordinate system, and the other one with orthonormal basis $\hat{\imath}', \hat{\jmath}'$, called the S' coordinate system. Suppose further S' is obtained by rotating S at an angle θ counterclockwise. Then for $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ in S, we can write:

$$\vec{r} \cdot \hat{\imath}' = x' \qquad \vec{r} \cdot \hat{\jmath}' = y' \qquad \text{and we get } \vec{r} = x'\hat{\imath}' + y'\hat{\jmath} \text{ in } S'$$

Definition 5.0.0.0.1
A matrix is an array of elements arranged in rows and columns.
An $m \times n$ matrix has m rows and n columns.

For $n, m, l \in \mathbb{N}$. A $n \times m$ matrix P can left multiply an $m \times l$ matrix Q, and get an $n \times l$ matrix L. That is, we have $PQ = L$. Note that in general, we have $PQ \neq QP$.

Definition 5.0.0.0.2
The transpose of a matrix M is written as M^T, the rows of M becomes the columns of M^T.

For a matrix M, the (i, j)-entry of M is the element in M in the i-row and j-column.

For matrix P and matrix Q, suppose matrix PQ exists. Let p_i denote the rows of P, and q_j denote the columns of Q, then the (i, j)-entry of PQ is given by $p_i \cdot q_j$.
Let A, B be $n \times n$ matrices and $c \in \mathbb{R}$, then we have the followings:

1. $\det(AB) = \det(A)\det(B)$
2. $\det(A^T) = \det(A)$
3. $\det(cA) = c^n \det(A)$

Let A' be obtained by switching two rows of an $n \times n$ matrix A, then we can write the following:

$$\det(A) = -\det(A')$$

Let A'' be obtained by adding a scalar multiple of one row of an $n \times n$ matrix A to another row of A, then we can write the following:

$$\det(A) = \det(A'')$$

Let A''' be obtained by multiplying a row of an $n \times n$ matrix A with a scalar λ, then we can write the following:

$$\lambda \det(A) = \det(A''')$$

Let A be an $n \times n$ matrix. If A is invertible, then the (j, i)-entry of A^{-1} is given by the following:

$$(A^{-1})_{ji} = (-1)^{i+j} \frac{\det(C^{ij})}{\det(A)}$$

where C^{ij} is obtained by deleting the i-th row and j-th column of A.

Linear Least Squares
When applying linear regression, say we have a set of N data, dependent variable $\{y_i\}$ and independent variable $\{x_i\}$, and suppose we have a prediction linear model $\{\widetilde{y}_i(x_i) = mx_i + b\}$, then we can make use of the Error Function to generate the best value for m and b:

$$E(m, b) := \sum_{i=1}^{N} (y_i - \widetilde{y}_i(x_i))^2 = \sum_{i=1}^{N} (y_i - (mx_i + b))^2$$

Notice here $E(m, b)$ is a function of m and b, then we can differentiate $E(m, b)$ to find the minimum value of E.

$$\frac{\partial E}{\partial m} = \frac{\partial}{\partial m}\left[\sum_{i=1}^{N} (y_i - (mx_i + b))^2\right] = 0 \qquad \Rightarrow \qquad \sum_{i=1}^{N} (y_i - mx_i - b)(x_i) = 0$$

$$\sum_{i=1}^{N} x_i y_i = \sum_{i=1}^{N} mx_i^2 + \sum_{i=1}^{N} bx_i = m\sum_{i=1}^{N} x_i^2 + b\sum_{i=1}^{N} x_i$$

Let $\sum_{i=1}^{N} x_i y_i = S_{xy}$ and let $\sum_{i=1}^{N} x_i^2 = S_{xx}$, $\sum_{i=1}^{N} x_i = S_x$. Then we can write:

$$S_{xx}m + S_x b = S_{xy}$$

We can similarly find that the following holds:

$$\frac{\partial E}{\partial b} = 0 \qquad \Rightarrow \qquad S_x m + Nb = S_y$$

where $S_y = \sum_{i=1}^{N} y_i$. Here we can solve the following system to minimize m and b:

$$\begin{bmatrix} S_x x & S_x \\ S_x & N \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} S_{xy} \\ S_y \end{bmatrix}$$

Hence we can write the following:

$$\begin{bmatrix} m \\ b \end{bmatrix} = \frac{1}{NS_{xx} - S_x^2} \begin{bmatrix} N & -S_x \\ -S_x & S_{xx} \end{bmatrix} \begin{bmatrix} S_{xy} \\ S_y \end{bmatrix}$$

In other words, we can write the following:

$$\widetilde{Y} := \begin{bmatrix} \widetilde{y}_1 \\ \widetilde{y}_2 \\ \vdots \\ \widetilde{y}_N \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix}$$

We define the following:

$$X = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix} \qquad \vec{p} = \begin{bmatrix} m \\ b \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

Then we can write:

$$\begin{aligned} &E(m,b) = \sum_{i=1}^{N} (y_i - \widetilde{y}_i(x_i))^2 \\ &\Rightarrow E = (Y - \widetilde{Y})^T (Y - \widetilde{Y}) = Y^T Y - 2\widetilde{Y}^T Y + \widetilde{Y}^T \widetilde{Y} = Y^T Y - 2(X\vec{p})^T Y + (X\vec{p})^T (X\vec{p}) \\ &\Rightarrow E = Y^T Y - 2\vec{p}^T X^T Y + \vec{p}^T X^T X \vec{p} \end{aligned}$$

Here we can take the derivative of E:

$$\frac{\partial E}{\partial m} = 0 - 2\left(\frac{\partial}{\partial m}\vec{p}^T\right) X^T Y + \frac{\partial \vec{p}^T}{\partial m} X^T X \vec{p} + \vec{p}^T X^T X \frac{\partial \vec{p}}{\partial m}$$

where we have:

$$\frac{\partial}{\partial^p}\vec{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \frac{\partial}{\partial^p}\vec{p}^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Let $\vec{u} = \sum_k u_k \vec{e}_k$ for some scalars u_k and standard basis vector $\vec{e}_k$, similarly let $\vec{v} = \sum_l v_l \vec{e}_l$. Then we can write the following, with a matrix M whose entries is $M_{i,j} = \langle \vec{e}_i, \vec{e}_j \rangle$:

$$\langle \vec{u}, \vec{v} \rangle = \left\langle \sum_k u_k \vec{e}_k, \sum_l v_l \vec{e}_l \right\rangle = \sum_l v_l \left\langle \sum_k u_k \vec{e}_k, \vec{e}_l \right\rangle = \sum_k \bar{u}_k \sum_l v_l \langle \vec{e}_l, \vec{e}_k \rangle := \sum_k \bar{u}_k (M\vec{v})_k$$

Let M be a matrix, $\overline{M}$ denotes the matrix whose entries are complex conjugate of the corresponding entries in M. Hermitian conjugate of a matrix M is defined to be $M^+ := \overline{M^T}$. A matrix is said to be Hermitian provided that $M^+ = M$.

Lemma 5.0.1
For $\vec{u}, \vec{v}$ and matrix A, we have $\langle u, A\vec{v} \rangle = \langle A^+\vec{u}, \vec{v} \rangle$, where $\langle \cdot \rangle$ denotes a well-defined inner product.

Proof. $\langle \vec{u}, A\vec{v} \rangle = (\vec{u}^+ A)\vec{v} = (A^+\vec{u})^+\vec{v} = \langle A^+\vec{u}, \vec{v} \rangle$, with $(A^+\vec{u})^+ = u^+(A^+)^+ = u^+A$ □

Theorem 5.1
In an well-defined inner product space, the eigenvectors of a Hermitian matrix with different eigenvalues are orthogonal to each other. Furthermore, the eigenvalues are real.

Proof. Let v_1 and v_2 be eigenvectors of a Hermitian matrix M, corresponds to different eigenvectors λ_1 and λ_2. We see that $\langle v_1, Mv_1 \rangle = \langle v_1, \lambda_1 v_1 \rangle = \lambda_1 \langle v_1, v_1 \rangle$. By the Lemma, we can write $\langle v_1, Mv_1 \rangle = \langle M^+ v_1, v_1 \rangle = \langle Mv_1, v_1 \rangle = \langle \lambda_1 v_1, v_1 \rangle = \bar{\lambda}_1 \langle v_1, v_1 \rangle = \lambda_1 \langle v_1, v_1 \rangle \Rightarrow \bar{\lambda}_1 = \lambda_1$. Moreover, $\langle v_1, Mv_2 \rangle = \langle v_1, \lambda_2 v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle = \langle M^+ v_1, v_2 \rangle = \langle Mv_1, v_2 \rangle = \langle \lambda_1 v_1, v_2 \rangle = \lambda_1 \langle v_1, v_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle \Rightarrow \langle v_1, v_2 \rangle = 0$. The result follows. □

Partial Derivative

Definition 5.1.0.0.1
The partial derivative of a function f is a derivative in which all variables except one are held fixed. Given $f(x_1, x_2, \cdots, x_n)$, we write $\frac{\partial f}{\partial x_k}$ being the partial derivative of f in the x_k direction.

Vector Field
For vector $\vec{r}$ in $\mathbb{R}^3$:

$$\vec{F}(\vec{r}) = F_x(\vec{r})\hat{e}_x + F_y(\vec{r})\hat{e}_y + F_z(\vec{r})\hat{e}_z = F_\rho(\vec{r})\hat{e}_\rho + F_\theta(\vec{r})\hat{e}_\theta + F_z(\vec{r})\hat{e}_z$$

describes a vector field, with $(\hat{e}_x, \hat{e}_y, \hat{e}_z)$ being the standard Cartesian $\mathbb{R}^3$ basis, and $(\hat{e}_\rho, \hat{e}_\theta, \hat{e}_z)$ being the standard cylindrical $\mathbb{R}^3$ basis.

The divergence of a vector field is given by $\nabla \cdot \vec{F}$ where

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Physically, the divergence of a vector field describes the net flow through an infinitesimal volume at a given point in the vector field.

The curl of a vector field is given by $\nabla \times \vec{F} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (F_x, F_y, F_z) = (\frac{\partial}{\partial \rho}, \frac{1}{\rho}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}) \times (F_\rho, \rho F_\theta, F_z)$. Physically, the curl of a vector field is equal to the rotation produced at a given infinitesimal point.

Fourier Series

Theorem 6.1
Given $f : [a, b] \to \mathbb{R}$, there exists (c_m) such that the following holds:

$$f(x) = \sum_{m=-\infty}^{\infty} c_m e^{ik_m x} \qquad\qquad \textit{with } k_m = \frac{2\pi m}{b-a}$$

In the following discussion, we will find a way to construct (c_m) for given $f : [a, b] \to \mathbb{R}$.

Denote $I_{mn} \coloneqq \int_a^b e^{ik_m x} e^{-ik_n x}\, dx$. First we notice that, if $m \neq n$, we can write

$$\begin{aligned}
I_{mn} &= \int_a^b e^{ik_m x} e^{-ik_n x}\, dx \\
&= \int_a^b \exp\left(\frac{2\pi i(m-n)}{b-a} x\right) dx \\
&= \frac{b-a}{2\pi i(m-n)} \left[\exp\left(\frac{2\pi i(m-n)x}{b-a}\right)\right]_a^b \\
&= \frac{b-a}{2\pi i(m-n)} \left[\exp\left(\frac{2\pi i(m-n)b}{b-a}\right) - \exp\left(\frac{2\pi i(m-n)a}{b-a}\right)\right] \\
&= \frac{b-a}{2\pi i(m-n)} \exp\left(\frac{2\pi i(m-n)a}{b-a}\right) \left[\exp\left(\frac{2\pi i(m-n)(b-a)}{b-a}\right) - 1\right] \\
&= \frac{b-a}{2\pi i(m-n)} \exp\left(\frac{2\pi i(m-n)a}{b-a}\right) [\cos(2\pi(m-n)) + i\sin(2\pi(m-n)) - 1] \\
&= \frac{b-a}{2\pi i(m-n)} \exp\left(\frac{2\pi i(m-n)a}{b-a}\right) [0] \\
&= 0
\end{aligned}$$

Here we get $I_{mn} = 0$ if $m \neq n$, and $I_{mn} = b - a$ if $m = n$.
Now we can write the following:

$$\begin{aligned}
\int_a^b f(x) e^{-ik_n x}\, dx &= \int_a^b \left(\sum_{m=-\infty}^{\infty} c_m e^{ik_m x}\right) e^{-ik_n x}\, dx \\
&= \sum_{m=-\infty}^{\infty} c_m \int_a^b e^{i(k_m x - k_n x)}\, dx \\
&= \sum_{m=-\infty}^{\infty} c_m I_{mn} \\
&= (b-a) c_n
\end{aligned}$$

Hence we get the following:

$$c_n = \frac{\int_a^b f(x) e^{-ik_n x}\, dx}{b-a}$$

Moreover, for real-valued function $f(x)$, we can write the following:

$$f(x) = \sum_{m=-\infty}^{\infty} c_m e^{imx} = c_0 + \sum_{m=1}^{\infty} (c_m e^{imx} + c_{-m} e^{-imx}) \qquad\qquad \text{with } c_{-m} = \overline{c_m}$$

Rewriting with $c_m = c'_m + ic''_m$,

$$\begin{aligned}
f(x) &= c_0 + \sum_{m=1}^{\infty} 2\, Re(c_m e^{imx}) \\
&= c_0 + \sum_{m=1}^{\infty} 2(c'_m \cos(mx) - c''_m \sin(mx)) \\
&= a_0 + \sum_{m=1}^{\infty} (a_m \cos(mx) + b_m \sin(mx))
\end{aligned}$$

where we define $a_0 = c_0$, $a_m = 2\, Re(c_m)$, and $b_m = -2\, Im(c_m)$.
Or conversely, we define $c_0 = a_0$, $c_m = \frac{a_m}{2} - \frac{b_m}{2} i$.

Here we have an observation, for periodic functions $f(x) = f(x+h)$ defined on $\mathbb{R}$, once we calculate the Fourier Series for $f(x)$ on $[0, h]$, the series is automatically periodic on $\mathbb{R}$ when defined.

For real Fourier Series for function f defined on $[a, b]$, one can write the following:

$$f(x) = a_0 + \sum_{m=1}^{\infty} (a_m \cos(k_m x) + b_m \sin(k_m x))$$

where we have:

$$k_m = \frac{2\pi m}{b-a} \qquad a_0 = \frac{1}{b-a}\int_a^b f(x)dx$$

$$a_m = \frac{2}{b-a}\int_a^b \cos(k_m x) f(x)\, dx \qquad b_m = \frac{2}{b-a}\int_a^b \sin(k_m x) f(x)\, dx$$

Let f be a function defined on the interval $[0, \frac{L}{2}]$. Suppose we want a Fourier Series with a basic period of $L/2$ that agrees with $f(x)$ on $[0, \frac{L}{2}]$, then we can simply let the Fourier Series be defined with coefficients:

$$c_m = \frac{1}{L/2}\int_0^{L/2} f(x) \exp(-ik_m x)\, dx \tag{C}$$

One might extend the period of the Fourier Series to be L by extending the definition of f on $[-\frac{L}{2}, 0]$. If we let $f(x) = f(-x)$, we obtain the following for such Fourier Series that agrees with $f(x)$ on on $[-L/2, L/2]$:

$$S(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L} x\right) \tag{E}$$

where (a_n) is determined by (c_m) in equation (C). If instead we let $f(x) = -f(-x)$, then one would obtain the following for the corresponding Fourier Series that agrees with $f(x)$ on $[-L/2, L/2]$:

$$S(x) = \sum_{n-1}^{\infty} b_n \sin\left(\frac{2\pi n}{L} x\right) \tag{O}$$

where (b_n) is determined by (c_m) in equation (C). Conversely, if $f(x)$ is odd or even being defined originally, the Fourier Series obtained correspondingly will also be given in the form in equation (E) and equation (O), respectively.

Linear Partial Differential Equations

Wave Equation Consider the wave equation given by the following:

$$\nabla^2\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t}$$

For Cartesian Coordinate, we have $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. Note that the principle of superposition holds for solutions of the wave equation.

For one dimensional space, we get the following wave equation:

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} \tag{1}$$

For two dimensional space, we get the wave equation in polar coordinates:

$$\frac{\partial^2\Psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial\Psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2\Psi}{\partial \theta} = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} \tag{2}$$

Consider a string that is held at the two ends at $x = 0$ and $x = L$. The string is perturb at small vertical distance, and we can model the motion of the string through a function $\Psi(x, t)$, with $\Psi(0, t) = 0$ and $\Psi(L, t) = 0$. First we set some condition where we have $\Psi(x, 0) = h(x)$, and $\frac{\partial\Psi}{\partial t}(x, 0) = v(x)$. Assume that we can write the following:

$$\Phi(x, t) = w(x)T(t)$$

Then we can write:

$$\frac{\partial^2}{\partial x^2}(wT) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(wT)$$
$$\left(\frac{d^2}{dx^2}w\right)\frac{1}{w} = \left(\frac{1}{c^2}\frac{d^2}{dt^2}T\right)\frac{1}{T}$$

By the property of w and T, we can introduce a separation constant α:

$$\left(\frac{d^2}{dx^2}w\right)\frac{1}{w} = \left(\frac{1}{c^2}\frac{d^2}{dt^2}T\right)\frac{1}{T} = \alpha$$

Hence we obtain the following:

$$\frac{w''}{w} = \alpha \qquad\qquad \frac{T''}{T} = \alpha c^2$$

Solving w and T we get the followings, for $\alpha \neq 0$:

$$w = ae^{\sqrt{\alpha}x} + be^{-\sqrt{\alpha}x} \qquad\qquad T = Ae^{\sqrt{\alpha}ct} + Be^{-\sqrt{\alpha}ct}$$

with $a, b, A, B \in \mathbb{R}$ being constant. Now we apply the boundary condition where we have $\Psi(0, t) = 0 = w(0)T(t) \Rightarrow w(0) = 0$. $\Psi(L, t) = 0 = w(L)T(t) \Rightarrow w(L) = 0$. Here we can write the following, for $\alpha \neq 0$:

$$w(0) = a + b = 0 \qquad\qquad w(L) = ae^{\sqrt{\alpha}L} + be^{-\sqrt{\alpha}L} = 0$$
$$a\left(e^{\sqrt{\alpha}L} - e^{-\sqrt{\alpha}L}\right) = 0 \qquad \Rightarrow \qquad e^{2\sqrt{\alpha}L} = 1$$

We get that $2\sqrt{\alpha}L = (2n\pi)i$ for $n \in \mathbb{Z} \setminus \{0\}$. In this case, for $\alpha \neq 0$, arranging we get the following:

$$a\left(e^{i\frac{m\pi x}{L}} - e^{-\frac{im\pi x}{L}}\right) = a\sin\left(\frac{m\pi x}{L}\right)$$

Notice that for $\alpha = 0$, one can show that we get the trivial solution $w(t) = 0$. Notice here, for the equation $T = Ae^{\sqrt{\alpha}ct} + Be^{-\sqrt{\alpha}ct}$, the LHS is real, so the RHS must also be real, this implies A and B must be chosen such that the RHS is real. Here with the same $\alpha = i\frac{\pi m}{L}$ for some $m \in \mathbb{Z} \setminus \{0\}$, here we can write the following:

$$\Psi_m(x, t) = a_m \sin\left(\frac{m\pi x}{L}\right)\left(A_m \sin\left(\frac{c\pi mt}{L}\right) + B_m \cos\left(\frac{c\pi mt}{L}\right)\right)$$

Then the general solution is given by the following:

$$\Psi(x, t) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi x}{L}\right)\left(A_m \sin\left(\frac{c\pi mt}{L}\right) + B_m \cos\left(\frac{c\pi mt}{L}\right)\right) \tag{3}$$

where A_m and B_m are constants depending on the initial conditions of the problem. Here we can write the following:

$$\Psi(x,0) = h(x) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{L}x\right) B_m$$

which can be viewed as a Fourier Series with length period $2L$, defined on $[-L, L]$, with $\Psi(x,0) = -\Psi(-x,0)$ on the interval $[-L, 0]$. Now we can write the following:

$$\int_{-L}^{L} h(x)\sin\left(\frac{\pi n}{L}x\right) = \sum_{m=1}^{\infty} B_m \int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right) = B_n L$$

Hence we can write the following:

$$B_n = \frac{1}{L}\int_{-L}^{L} h(x)\sin\left(\frac{n\pi}{L}x\right) = \frac{2}{L}\int_{0}^{L} h(x)\sin\left(\frac{n\pi}{L}x\right)\,dx$$

By equation (3), we can write the following:

$$\frac{\partial\Psi}{\partial t}(x,t) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{L}x\right)\left(-\frac{c\pi m}{L}B_m \sin\left(\frac{c\pi m t}{L}\right) + \frac{c\pi m}{L}A_m\cos\left(\frac{c\pi m t}{L}\right)\right)$$

With the initial condition that $\frac{\partial\Phi}{\partial t}(x,0) = v(x)$, one can solve for A_n.

Heat equation is given by the following:

$$\nabla^2 T = \frac{\partial T}{\partial t}$$

with $T(r,\theta,\phi) = T(r,\theta) = f(r)g(\theta)$ in Spherical Coordinate. Then we can write the following:

$$\frac{\nabla^2 T}{fg} = \frac{1}{f}\frac{\partial}{\partial r}\left(r^2\frac{df}{dr}\right) + \frac{1}{g\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{dg}{d\theta}\right) = 0$$

Hence we must have the following holds:

$$\frac{1}{f}\frac{d}{dr}\left(r^2 f'\right) = \alpha \qquad\qquad -\frac{1}{\sin(\theta)}\frac{1}{g}\frac{d}{d\theta}(\sin(\theta)g') = \alpha$$

for some arbitrary α. Now we need to solve the following ODE:

$$-\frac{1}{\sin(\theta)}\frac{1}{g}\left(\cos(\theta)g' + \sin(\theta)g''\right) = \alpha \tag{4}$$

let $z = \cos(\theta)$, and $y(z) = g(\theta)$, then we get:

$$\frac{\partial g}{\partial\theta} = \frac{\partial}{\partial\theta}y(z) = \frac{dy}{dz}\frac{dz}{d\theta} = -\sin(\theta)\frac{dy}{dz} = -(1-z^2)^{1/2}y'$$

Here we can rewrite ODE (4) as the following:

$$(1-z^2)y'' - 2zy + \alpha y = 0 \tag{5}$$

Such ODE (5) has solution given by the following:

$$y = \sum_{m=0}^{\infty} a_m z^{m+s} \qquad\qquad a_{m+1} = a_m\frac{m(m+1) - n(n+1)}{(m+2)(m+1)}$$

If $\alpha \neq n(n+1)$ with n being an integer, the resulting infinite series diverges at $z = \pm 1$. To keep our solutions free of singularities, we demand that $\alpha = n(n+1)$. Here we conclude that $g(\theta)$ is given by the following, with P_n being the Legendre Polynomial:

$$g(\theta) = P_n(\cos(\theta))$$

On the other hand, we need to solve for f with the following ODE:

$$r^2 f'' + 2rf' - \alpha f = 0$$

Here we get the following:

$$f(r) = Ar^{\lambda} \quad \text{with } \lambda = \frac{1+\sqrt{1+4\alpha}}{2},\ A \in \mathbb{R}$$

Combining the results, we can write:

$$T_n(r,\theta) = A_n \cdot P_n(\cos(\theta)) \cdot r^{\lambda_n} \quad \text{with } \lambda_n = \frac{1+\sqrt{1+4n(n+1)}}{2},\ A_n \in \mathbb{R}$$

Here the general solution to the Heat equation is given by the following:

$$T(r,\theta) = \sum_{n=0}^{\infty} A_n \cdot P_n(\cos(\theta)) \cdot r^{\lambda_n}$$