Theorem 0.1 (The Generalized Stokes' Theorem)
Let $k > 1$, let M be a compact oriented k-manifold in $\mathbb{R}^n$, with ∂M having the induced orientation if ∂M is not empty, let ω be a $(k-1)$-form defined in an open set of $\mathbb{R}^n$ containing M, then we have the following holds if ∂M is not empty:

$$\int_M d\omega = \int_{\partial M} \omega$$

and we have the following holds if ∂M is empty:

$$\int_{\partial M} \omega = 0$$

Proof. A detailed proof is provided on Munkres Theorem 37.2. Here it suffices to prove the special case where $\operatorname{supp}(\omega) \subseteq V$ with $\alpha : U \to V$ being a coordinate patch on M. Note that $\operatorname{supp}(d\omega) \subseteq \operatorname{supp}(\omega)$, hence we have $\operatorname{supp}(\omega) \subseteq V$. The general case will follow by taking finite sums over regions on M parametrized by coordinate patches.

First we note that we can extend the definition of domain of $d(\alpha^*\omega)$ to $\mathbb{H}^k$

$$\int_M d\omega = \int_U \alpha^* d\omega = \int_U d(\alpha^*\omega) = \int_{\mathbb{H}^k} d(\widetilde{\alpha^*\omega})$$

On the other hand, we can write:

$$\int_{\partial M} \omega = \int_{U \cap \partial \mathbb{H}^k} \alpha^*\omega = \int_{\partial \mathbb{H}^k} \widetilde{\alpha^*\omega}$$

Write $d\widetilde{\alpha^*\omega} = f_1\, dx_2 \wedge dx_3 \wedge \cdots \wedge dx_k + f_2 dx_1 \wedge dx_3 \wedge \cdots \wedge dx_k + \cdots + f_k \wedge dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{k-1}$. Here we have:

$$d\widetilde{\alpha^*\omega} = (D_1 f_1 - D_2 f_2 + \cdots + (-1)^{k-1} D_k f_k) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$$

$$\begin{aligned} \int_{\mathbb{H}^k} d\widetilde{\alpha^*\omega} &= \int_{\mathbb{H}^k} (D_1 f_1 - D_2 f_2 + \cdots + (-1)^{k-1} D_k f_k) \\ &= \int_B (D_1 f_1 - D_2 f_2 + \cdots + (-1)^{k-1} D_k f_k) \end{aligned}$$

where B is a box $a_1 \leq x_1 \leq b_1, \cdots, 0 = a_k \leq x_k \leq b_k$. Here by Fubini's Theorem and Fundamental Theorem of Calculus, we have:

$$\begin{aligned} \int_{\mathbb{H}^k} d\widetilde{\alpha^*\omega} &= \int_B (D_1 f_1 - D_2 f_2 + \cdots + (-1)^{k-1} D_k f_k) \\ &= 0 - 0 + 0 - 0 + \cdots + (-1)^{k-1} \int_{\mathbb{H}^k} D_k f_k \\ &= (-1)^{k-1} \int_{\mathbb{R}^{k-1}} f_k(x_1, x_2, \cdots, x_{k-1}, 0) \\ &= \int_{\partial \mathbb{H}^k} f_k dx_1 \wedge \wedge dx_2 \wedge \cdots \wedge dx_{k-1} \\ &= \int_{\partial \mathbb{H}^k} \alpha^*\omega \end{aligned}$$

The result follows. $\square$