Definitions and Notes

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Sets

A **Set** is a collection of objects which are called **Elements**.
Note: Let S be a set, let x is an element, then we have either $x \in S$ or $x \notin S$.
Note: Order does not matter in a set. Repeats are not detected in a set.

Let X, Y be sets. X is called a **Subset** of Y, denoted as $X \subseteq Y$, provided that every element of X is an element of Y, that is, if, X is a subset of Y, then for all $x \in X$, we have $x \in Y$. Moreover, if X is a subset of Y, and $X \neq Y$, then we say X is a **Proper Subset** of Y, denoted as $X \subset Y$.

A set with no element in it is called an **Empty Set**, denoted as $\emptyset$.
A set with exactly one element in it is called a **Singleton**.
Note: The empty set is a subset of every set.

Let a, b be elements. $(a, b) \coloneqq \{\{a\}, \{a, b\}\}$ is called an **Ordered Pair**.
Note: By definition, we have $(a, b) = (c, d) \iff a = c \textit{ and } b = d$.

Let X be a set, let A and B be subsets of X:
$A \cup B \coloneqq \{x \in X \mid x \in A \textit{ or } x \in B\}$ is called the **Union** of A and B.
$A \cap B \coloneqq \{x \in X \mid x \in A \textit{ and } x \in B\}$ is called the **Intersection** of A and B.
$A - B = A \setminus B \coloneqq \{x \in A \mid x \notin B\}$ is called the **Difference** between A and B.
$X \setminus A \coloneqq \{x \in X \mid x \notin A\}$ is called the **Complement** of A in X.
$A \times B \coloneqq \{(a, b) \mid a \in A \textit{ and } b \in B\}$ is called the **Cartesian Product** of the sets A and B.

Let A, B be sets, let f be a subset of $A \times B$. f is called a **Function**, denoted as $f : A \to B \quad x \mapsto f(x)$, provided that for all $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$.
Note: The **Domain** of the function $f : A \to B$ is A, and the **Codomain** of f is B.

Let X be a set. $\mathscr{P}(X) \coloneqq \{A \mid A \subseteq X\}$ is called the **Power Set** of X.
Note: The power set of a set X is the set of all subsets of X.

Let S be a set. A function $\diamond : S \times S \to S$ is called a **Binary Operation** on S.
Notation: If $\diamond$ is a binary operation on a set S, for $s_1, s_2 \in S$, we write $s_1 \diamond s_2$ instead of $\diamond(s_1, s_2)$.
Note: If $\diamond$ is a binary operation on a set S, then $\diamond$ is a subset of $(S \times S) \times S$.
Note: Intersection and Union on a set X are binary operations from $\mathscr{P}(X) \times \mathscr{P}(X)$ to $\mathscr{P}(X)$.

Let S be a set, let $f : S \to S$ and $g : S \to S$ be functions, let $Fun(S) \coloneqq \{h : S \to S\}$.
$\circ : Fun(S) \times Fun(S) \to Fun(S) \quad (f, g) \mapsto f \circ g$ is a binary operation called **Function Composition**, and $(f \circ g) : S \to S \quad (f \circ g)(x) \mapsto f(g(x))$ is a function from S to S that sends $s \in S$ to $f(g(s))$.

A binary operation $\diamond$ on a set S is called **Commutative** provided that $\forall a, b \in S$, $a \diamond b = b \diamond a$.
A binary operation $\diamond$ on a set S is called **Associative** provided that $\forall a, b, c \in S$, $(a \diamond b) \diamond c = a \diamond (b \diamond c)$.

Let $\diamond$ be a binary operation on S, the element $e \in S$ is called an **Identity** for $\diamond$ on S, denoted as $\diamond$-identity, provided that for all $s \in S$ we have $e \diamond s = s \diamond e = s$.
Note: The empty set can be an identity element of a binary operation on a set.
Note: 0 is not the identity of subtraction on $\mathbb{R}$.

Let S be a set, let $\diamond$ be a binary operation on S, let e be a $\diamond$-identity. Given $s \in S$, an element $s' \in S$ is called an **Inverse** of s with respect to $\diamond$ on S, denoted as $\diamond$-inverse, provided that $s \diamond s' = s' \diamond s = e$.

Let X be a set. The function $Id_X : X \to X \quad x \mapsto x$ is called the **Identity Function** on X.

Let A and B be sets, let $f : A \to B$ be a function.
f is **Surjective** provided that $\forall b \in B,\ \exists\ a \in A \ s.t.\ f(a) = b$.
f is **Injective** provided that $(f(a) = f(b)) \iff (a = b)$.
f is **Bijective** provided that f is both injective and surjective.
Note: Let f be a bijection from set A to set B, then $\forall b \in B,\ \exists!\ a \in A,\ s.t.\ f(a) = b$.

Let A and B be sets. a function $f : A \to B$ is said to be **Invertible** provided that f^{-1} is a function.
If f is invertible, the function $f^{-1} : B \to A \quad f^{-1}(b) = a \iff f(a) = b$ is called the **Inverse** of f.
Note: Given sets A, B and $f : A \to B$, $f^{-1} = \{(b, a) \in B \times A \mid (a, b) \in f\}$ is a subset of $B \times A$.

Let A and B be sets, and let $f : A \to B$ be a function. The set $f(A) := \{f(a) \in B \mid a \in A\}$ is called the **Image** of f.
Note: If $f(A)$ is the image of the function f from set A to set B, then $f(A)$ is a subset of B.

Let A, B be sets. The sets A and B are said to have the same **Cardinality** provided that exists a bijection from the set A to the set B.

Let S be a set. A **Relation** on S is a subset of $S \times S$.
Let R be a relation on the set S, and let $x, y \in S$. We write xRy whenever $(x, y) \in R$.

Let S be a set, let R be a relation on S. R is called an **Equivalence Relation** on S provided that the followings hold:

1. R is **Reflexive**, that is, $\forall s \in S$, we have sRs.
2. R is **Symmetric**, that is, $\forall s, t \in S$, if sRt, then tRs.
3. R is **Transitive**, that is, $\forall s, t, u \in S$, if sRt and tRu, then sRu.

If R is an equivalence relation on S, then for $s, t \in S$, we write $s \sim t$ whenever $(s, t) \in R$.

Let S be a set, let $\sim$ be an equivalence relation on S, let $x \in S$. $C(x) := \{y \in S \mid y \sim x\}$ is called the **Class** of x with respect to $\sim$, or the **Equivalence Class** of x with respect to $\sim$.
Notation: If $C(x)$ is a class, then $C(x)$ can also be denoted as $[(x)]$.

Let S be a set, and let $\sim$ be an equivalence relation on S. The set $S/\sim := \{C(x) \mid x \in X\}$ is called the **Quotient** of X with respect to $\sim$, or the **Factor Set** of X with respect to $\sim$.

Fields

Let F be a set, let $*$ and $+$ be commutative and associative binary operations on F.
$(F, *, +)$ is called a **Field** provided that the followings hold:

1. There exists a $+$-identity in F, denoted as 0_F.
2. There exists a $*$-identity in F, denoted as 1_F.
3. For all $f \in F$, there exists a $+$-inverse of f in F, denoted as $-f$.
4. For all $f \in (F \setminus \{0_F\})$, there exists a $*$-inverse of f in F, denoted as f^{-1}.
5. For all $a, b, c \in F$, we have $a * (b + c) = a * b + a * c$.
6. $0_F \neq 1_F$.

Note: If $(F, *, +)$ is a field, then F is not empty.

A field $(F, +, *)$ is said to have an **Order Structure** provided that there exists a subset P of the set F such that the followings hold:

1. P is closed with respect to the binary operations $+$ and $*$.
2. For $a \in F$, **Trichotomy** holds: • $a \in P$ • $-a \in P$ • $a = 0$

If P satisfies the requirements listed above, then we say the pair (F, P) has an ordered structure.

A field $(F, +, *)$ is said to be **Ordered** provided that there exists a subset P of F such that (F, P) has an ordered structure, and we called the pair (F, P) an **Ordered Field**.
Note: Given (F, P) is an ordered field, P is always not empty.
Note: In general, a field can have more than one ordered structure.

Let (F, P) be an ordered field, let $a, b \in F$.
We say a is **Greater than** b, denoted as $a > b$, iff $(a - b) \in P$.
We say a is **Less than** b, denoted as $a < b$, iff $-(a - b) \in P$.
We say a is **Greater than or Equal to** b, denoted as $a \geq b$, iff $(a - b) \in P \cup \{0_F\}$.
We say a is **Less than or Equal to** b, denoted as $a \leq b$, iff $-(a - b) \in P \cup \{0_F\}$.

Let (F, P) be an ordered field.

The function $|\ | : F \to F \quad x \mapsto \begin{cases} x & x \in P \\ -x & -x \in P \\ 0_F & x = 0_F \end{cases}$ is called the **Absolute Value Function** on F

Let (F, P) be an ordered field, let $a \in F$, $|a|$ is called the **Absolute Value** of a.

Consider using the ordered field (F, P) from now on.

Let A be a subset of F, and let u be an element in F. u is called an **Upper Bound** for A provided that for all $a \in A$, we have $u \geq a$. If u is an upper bound for the set A, then we say the set A is **Bounded Above** by u, and A is bounded above in F.

Let A be a bounded above subset of F, and let α be an element in F. α is called a **Least Upper Bound** or **Supremum** for the set A provided that the followings hold:

1. α is an upper bound for A.
2. α is the least such, that is, if $u \in F$ is an upper bound for A, then $\alpha \leq u$.

Note: Not every set has a least upper bound.
Note: Empty set does not have a least upper bound.
Note: If a least upper bound exists for $A \subseteq F$, then it is unique.

An ordered field (F, P) is said to have the **Least Upper Bound Property** provided that every nonempty bounded above subset A of F has a least upper bound.

An ordered field is said to be **Complete** provided that it has the least upper bound property.
Note: There exists a unique complete ordered field, called $\mathbb{R}$.

Let A be a subset of F, and let w be an element in F. w is called a **Lower Bound** for A provided that for all $a \in A$, we have $w \leq a$. If w is a lower bound for the set A, then we say the set A is **Bounded Below** by w, and A is bounded below in F.

Let A be a bounded below subset of F, and let β be an element in F. β is called a **Greatest Lower Bound** or **Infimum** for the set A provided that the followings hold:

1. β is a lower bound for A.
2. β is the greatest such, that is, if $w \in F$ is a lower bound for A, then $\beta \geq w$.

Note: Not every set has a greatest lower bound.
Note: Empty set does not have a greatest lower bound.
Note: If a greatest lower bound exists for $A \subseteq F$, then it is unique.

Let X be a subset of F, X is said to be **Inductive** provided that the followings hold:
• $1_F \in X$ • If $x \in X$, then $x + 1_F \in X$.

$\mathbb{N}_F := \{n \in F \mid n$ belongs to every inductive subset of $F\}$
$\mathbb{Z}_F := \{z \in F \mid |z| \in \mathbb{N}_F \cup \{0\}\}$
$\mathbb{Q}_F := \{q \in F \mid \exists\ z \in \mathbb{Z}_F$ and $n \in \mathbb{N}_F\ s.t.\ q = z * n^{-1}\}$
Note: When $F = \mathbb{R}$, then we have $\mathbb{N}_F = \mathbb{N},\ \mathbb{Z}_F = \mathbb{Z},\ \mathbb{Q}_F = \mathbb{Q}$.
Note: $\mathbb{N}$ is called the set of Natural Numbers, and $\mathbb{Z}$ is called the set of Integers.
Note: Rigorously, we define $\mathbb{Q} := (\mathbb{Z} \times \mathbb{N})/\!\sim_{\mathbb{Q}}$ with $(n_1, m_1) \sim_{\mathbb{Q}} (n_2, m_2) \iff (n_1 \cdot m_2 = n_2 \cdot m_1)$.
Note: $\mathbb{Q}$ is called the set of Rational Numbers.

Let B be a subset of F. B is said to be **Bounded** in F provided that the set B is both bounded above and bounded below in F.
Fact: Every nonempty bounded above subset of R has a Supremum in $\mathbb{R}$.
Fact: Every nonempty bounded below subset of R has a Infimum in $\mathbb{R}$.

Let A be a subset of F, let m be an element in F. The element m is called a **Maximal Element** for the set A provided that m belongs to A and m is an upper bound for A.

Let A be a subset of F, let u be an element in F. The element u is called a **Minimal Element** for the set A provided that u belongs to A and u is a lower bound for A.

Let A be a subset of F, let b be an element in F.
$-A := \{-x \mid x \in A\}$
$b + A := \{b + x \mid x \in A\}$

Let U be a subset of F. U is said to be **Well-ordered** provided that every nonempty subset of U has a minimal element.

Let S be a subset of $\mathbb{N}$. S is said to be **Weakly Inductive** provided that the followings hold:
• $1 \in S$ • If $k \in S$, then $(k+1) \in S$.

Let S be a subset of $\mathbb{N}$. S is said to be **Strongly Inductive** provided that the following holds:
For all $n \in \mathbb{N}$, if the set $\{k \in \mathbb{N} \mid k < n\}$ is a subset of S, then n is an element in S.

Let $d, b \in \mathbb{N}$. the element d is called a **Divisor** of b provided that there exists $m \in \mathbb{N}\ s.t.\ b = d * m$.
Notation: If $d \in \mathbb{N}$ is a divisor of $b \in \mathbb{N}$, then we write $d \mid b$.

Let $a, b \in \mathbb{N}$. The element $\gcd(a, b) \in \mathbb{N}$ is called the common divisor of a and b provided that $\gcd(a, b)$ is a divisor of both a and b, and $\gcd(a, b)$ is the greatest such, that is, $\forall d \in \mathbb{N}$, if d is a divisor of both a and b, then we have $\gcd(a, b) \geq d$.

The set $E := \{2 \cdot k \mid k \in \mathbb{Z}\}$ is called the set of **Even Numbers** in $\mathbb{R}$.
The set $O := \{2 \cdot k + 1 \mid k \in \mathbb{Z}\}$ is called the set of **Odd Numbers** in $\mathbb{R}$.
Each element in E is called an **Even Number**, and each element in O is called an **Odd Number**.

Let S be a set, S is said to be **Finite** provided that either S is an empty set, or there exists a bijection from the set $N_n = \{k \in \mathbb{N} \mid k < n\}$ to the set S for some $n \in \mathbb{N}$. If there exists a bijection from N_n to S for some $n \in \mathbb{N}$, then we say S has n element.
Note: Let S be a set, If there exists a bijection from N_n to S for some $n \in \mathbb{N}$, then n is unique.

Let S be a set. S is said to be **Infinite** provided that S is not finite.

Let S be a set. S is **Countably Infinite** provided that there exists a bijection from $\mathbb{N}$ to S.

Let S be a set. S is is said to be **Countable** provided that S is either finite or countably infinite.
Fact: The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ are all countable.
Fact: The sets $\mathbb{R}$ and $\mathbb{R} \setminus \mathbb{Q}$ are not countable.

$i^2 := -1$. The element i is called the **Imaginary Unit**.

Let $a, b \in \mathbb{R}$. $z := a + bi$ is called a **Complex Number**, where a is called the **Real Part** of z, b is called the **Imaginary Part** of z, and $a + bi$ is called the **Cartesian Form** of z.

$\mathbb{C} := \{a + bi \mid a, b \in \mathbb{R}\}$ is called the **Set of Complex Numbers**.
Note: The plane of complex numbers, called the **Complex Plane**, is a two dimensional plane.

Let $z = a + bi, w = c + di$ be complex numbers, and let $k \in \mathbb{R}$.
$z + w := (a + c) + (b + d)i$
$z \times w := (a \times c - b \times d) + (a \times d + b \times c)i$
$z \times k := (a \times k) + (b \times k)i$
$\bar{z} := a - bi$ is called the **Conjugate** of z.
$|z| := \sqrt{a^2 + b^2}$ is called the **Absolute Value** or the **Modulus** of z.
Fact: Every nonzero complex number z can be written as $z = |z|(cos(\theta) + i\sin(\theta))$ for some $\theta \in \mathbb{R}$.
Note: If z is a complex number, then $\bar{z} \in \mathbb{C}$, $|z| \in \mathbb{R}$, $|z|$ is unique, and the absolute value of $\frac{z}{|z|}$ is 1.

Let $z = a + bi = r(cos(\theta) + i\sin(\theta)) = re^{i\theta}$ be a complex number. θ is called the **Argument** of z. $r(cos(\theta) + i\sin(\theta))$ and $re^{i\theta}$ are called the **Polar Form** of z.
Note: If $z = re^{i\theta}$ is a complex number, then $r \in \mathbb{R}$ and $\theta \in \mathbb{R}$.
Note: If $z = re^{i\theta}$ is a complex number, then $r = |z|$ is unique, while θ is not unique.
Fact: Let $z \in \mathbb{C}$ and $z = re^{i\theta}$. If $\theta = \theta_0$ is one possibility, then the others are $\theta_0 + 2k\pi$ for $k \in \mathbb{Z}$.
Fact: If $z = re^{i\theta} = a + bi$ is a complex number, then $a = |z|\cos(\theta)$, $b = |z|\sin(\theta)$, $\theta = \tan^{-1}(\frac{b}{a})$.

Topology

Let A be a subset of $\mathbb{R}$. A is called an **Interval** provided that for all $x, y \in A$, if $z \in \mathbb{R}$ and $x < z < y$, then we have $z \in A$.

Let $a, b \in \mathbb{R}$ with $a \leq b$. The followings are intervals:
$(a, b) \coloneqq \{x \in \mathbb{R} \mid a < x < b\}$
$[a, b] \coloneqq \{x \in \mathbb{R} \mid a \leq x \leq b\}$
$(a, b] \coloneqq \{x \in \mathbb{R} \mid a < x \leq b\}$
$[a, b) \coloneqq \{x \in \mathbb{R} \mid a \leq x < b\}$
$\mathbb{R}_{>a} = (a, \infty) \coloneqq \{x \in \mathbb{R} \mid a < x\}$
$\mathbb{R}_{<b} = (-\infty, b) \coloneqq \{x \in \mathbb{R} \mid x < b\}$
$\mathbb{R}_{\geq a} = [a, \infty) \coloneqq \{x \in \mathbb{R} \mid a \leq x\}$
$\mathbb{R}_{\leq b} = (-\infty, b] \coloneqq \{x \in \mathbb{R} \mid x \leq b\}$
$(-\infty, \infty) \coloneqq \{x \in \mathbb{R}\}$

Let $a, r \in \mathbb{R}$ with $r > 0$. The set $B_r(a) \coloneqq \{x \in \mathbb{R} \mid (a - r) < x < (a + r)\}$ is called the **Ball** centered at a of radius r.

$\mathscr{T}_{EUC} \coloneqq \{A \subseteq \mathbb{R} \mid \forall a \in A,\ \exists\ r \in \mathbb{R}_{>0}\ s.t.\ B_r(a) \subseteq A\}$ is called the **Euclidean Topology** on $\mathbb{R}$.
Note: $\mathscr{T}_{EUC}$ is closed with respect to arbitrary unions and finite intersections.
Note: $\mathscr{T}_{EUC}$ is a set of sets, and $\mathscr{T}_{EUC}$ is a subset of $\mathscr{P}(\mathbb{R})$.
Note: The pair $(\mathbb{R}, \mathscr{T}_{EUC})$ is a topological space.
Note: We have $\mathbb{R} \in \mathscr{T}_{EUC}$ and $\emptyset \in \mathscr{T}_{EUC}$.

Let U be a subset of $\mathbb{R}$. The set U is said to be **Open** in the Euclidean topology on $\mathbb{R}$ provided that $\forall u \in U,\ \exists\ r \in \mathbb{R}_{>0}\ s.t.\ B_r(u) \subseteq U$. Let C be a subset of $\mathbb{R}$. The set C is said to be **Closed** in the Euclidean topology on $\mathbb{R}$ provided that $\mathbb{R} \setminus C$ is open in the Euclidean topology on $\mathbb{R}$.
Note: Arbitrary union of open sets and finite intersection of open sets in $\mathscr{T}_{EUC}$ are open.
Note: The collection of all open subsets of $\mathbb{R}$ in the Euclidean Topology forms $\mathscr{T}_{EUC}$.
Note: Each element in $\mathscr{T}_{EUC}$ is called an open subset of $\mathbb{R}$.
Note: The empty set and $\mathbb{R}$ are both open and closed in $\mathscr{T}_{EUC}$.
Note: All balls are openin the Euclidean topology on $\mathbb{R}$. In $(\mathbb{R}, \mathscr{T}_{EUC})$, balls are called **Open Balls**.

Consider using the topological space $(\mathbb{R}, \mathscr{T}_{EUC})$ from now on.

Let $p \in \mathbb{R}$. A **Neighborhood** of p in $\mathscr{T}_{EUC}$ is an open interval in $\mathscr{T}_{EUC}$ that contains p.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function, let $a \in \mathbb{R}$. We say f approaches l as x approaches a, denoted as $\lim_{x \to a} f = l$ for some $l \in \mathbb{R}$, provided that one of the followings holds:

1. For all $\epsilon > 0$, $\exists\ \delta > 0\ s.t.$ if $x \in B_\delta(a) \setminus \{a\}$, then $f(x) \in B_\epsilon(l)$.
2. For all $\epsilon > 0$, $\exists\ \delta > 0\ s.t.$ if $x \in \mathbb{R}$ satisfies $0 < |x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.
3. For $V \in \mathscr{T}_{EUC}$, if $l \in V$, then $\exists\ U \in \mathscr{T}_{EUC}\ s.t\ a \in U$ and $\forall x \in U \setminus \{a\}$, we have $f(x) \in V$.

Note: The three statements are equivalent.
Note: $\lim_{x \to a} f$ is called the **Limit** of the function $f : \mathbb{R} \to \mathbb{R}$ at $a \in \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function, let a be an element in $\mathbb{R}$. The function f is said to be **Continuous** at a provided that one of the followings holds:

1. $\lim_{x \to a} f = l$ for some $l \in \mathbb{R}$ and $f(a) = l$ for some $l \in \mathbb{R}$.
2. For all $\epsilon > 0$, $\exists\ \delta > 0\ s.t.$ if $x \in B_\delta(a)$, then $f(x) \in B_\epsilon(f(a))$.
3. For all $\epsilon > 0$, $\exists\ \delta > 0\ s.t.$ if $x \in \mathbb{R}$ satisfies $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.
4. For $V \in \mathscr{T}_{EUC}$, if $f(a) \in V$, then $\exists\ U \in \mathscr{T}_{EUC}\ s.t\ a \in U$ and $\forall x \in U$, we have $f(x) \in V$.

Note: The four statements are equivalent.

Let A, B be subsets of $\mathbb{R}$, let $f : A \to B$ be a function. f is **Continuous** on A provided that for all $a \in A$, the function f is continuous at a. If a function f is continuous on its domain, then we say the function f is **Continuous**.

Let A, B be subsets of $\mathbb{R}$, let S be a subset of A, let $f : A \to B$ be a function.
The **Restriction** of f to S is the function from S to B that sends $s \in S$ to $f(s)$, such function is denoted as $res_S f : S \to B$. Moreover, the image of $res_S f$ is denoted as $f([S])$ or $f(S)$ or $res_S f(S)$.

By Theorem, we can redefine the continuity of a function $f : \mathbb{R} \to \mathbb{R}$: The function $f : \mathbb{R} \to \mathbb{R}$ is said to be **Continuous** provided that for all $U \in \mathscr{T}_{EUC}$, we have $f^{-1}(U) \in \mathscr{T}_{EUC}$.

The assumption of using $(\mathbb{R}, \mathscr{T}_{EUC})$ is removed.

Let X be a set. A **Topology** $\mathscr{T}_X$ on the set X is a subset of $\mathscr{P}(X)$ that satisfies the followings:

1. The set X and the empty set are in $\mathscr{T}_X$.
2. $\mathscr{T}_X$ is closed with respect to arbitrary union.
3. $\mathscr{T}_X$ is closed with respect to finite intersection.

Note: A set can have more than one topology.

Let X be a set, let $\mathscr{T}_X$ be a topology on X. The pair $(X, \mathscr{T}_X)$ is called a **Topological Space**.

Let $(X, \mathscr{T}_X)$ be a topological space. Each the element in $\mathscr{T}_X$ is called an **Open Subset** of X in $\mathscr{T}_X$, that is, let U be a subset of X, the set U is said to be **Open** if and only if U belongs to $\mathscr{T}_X$.

Let $(X, \mathscr{T}_X)$ be a topological space, let $p \in X$. A **Neighborhood** of p in $\mathscr{T}_X$ is an open set in $\mathscr{T}_X$ that contains p.

Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces. The function $f : X \to Y$ is said to be **Continuous** provided that for all $u \in \mathscr{T}_Y$, we have $f^{-1}(u) \in \mathscr{T}_X$.

Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces. The function $f : X \to Y$ is **Locally Constant** provided that $\forall x \in X$, $\exists$ a neighborhood U of x in $\mathscr{T}_X$ *s.t.* $\forall y \in U$, we have $f(y) = c$ for some $c \in Y$.

Let $(X, \mathscr{T})$ be a topological space, let A be a subset of X. The intersection of all closed subsets of X that contain A is called the **Topological Closure** of A in $\mathscr{T}$, denoted as $\bar{A}$.

Let $(X, \mathscr{T})$ be a topological space, let A be a subset of X. The union of all open sets that are contained in A is called the **Interior** of A in $\mathscr{T}$, denoted as $\texttt{int}(A)$.

Let X be a set. The **Indiscrete Topology** on X, denoted as $\mathscr{T}_{ind}$, contains only $\emptyset$ and X.
Let X be a set. The **Discrete Topology** on X, denoted as $\mathscr{T}_{dis}$, contains all subsets of X.
Note: In the discrete topology on the set X, every subset of X is both open and closed.
Note: Let X be a set. $\mathscr{T}_{ind} = \{\emptyset, X\}$ and $\mathscr{T}_{dis} = \mathscr{P}(X)$.

Let $(X, \mathscr{T}_X)$ be a topological space. If A is a subset of X, then A inherits the structure of the topological space from X, that is, $\mathscr{T}_A := \{U \cap A \mid U \in \mathscr{T}_X\}$ is the **Subspace Topology** on A, each element in $\mathscr{T}_A$ is called an **Open Subset** of A in the subspace topology of A.

In $(\mathbb{R}, \mathscr{T}_{EUC})$, let A be a subset of $\mathbb{R}$, let $\mathscr{T}_A$ denote the subspace topology on A, let $f : A \to \mathbb{R}$ be a function. let a be an element in A. We say f approaches l as x approaches a, denoted as $\lim_{x \to a} f = l$ for some $l \in \mathbb{R}$, provided that one of the followings holds:

1. For all $\epsilon > 0$, $\exists\, \delta > 0$ *s.t.* if $x \in (B_\delta(a) \setminus \{a\}) \cap A$, then $f(x) \in B_\epsilon(l)$.
2. For all $\epsilon > 0$, $\exists\, \delta > 0$, *s.t.* if $x \in A$ satisfies $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.
3. For $V \in \mathscr{T}_{EUC}$, if $l \in V$, then $\exists\, U \in \mathscr{T}_A$ *s.t.* $a \in U$ and $\forall x \in (U \setminus \{a\}) \cap A$, we have $f(x) \in V$.

Note: The three statements are equivalent.
Note: Let A be a subset of $\mathbb{R}$. $\lim_{x \to a} f$ is called the **Limit** of the function $f : A \to \mathbb{R}$ at $a \in A$.

In $(\mathbb{R}, \mathscr{T}_{EUC})$, let A be a subset of $\mathbb{R}$, let $\mathscr{T}_A$ denote the subspace topology on A, let $f : A \to \mathbb{R}$ be a function, let a be an element in A, f is **Continuous** at a provided that one of the followings holds:

1. $\lim_{x \to a} f = l$ for some $l \in \mathbb{R}$ and $f(a) = l$.
2. For all $\epsilon > 0$, $\exists\, \delta > 0$ *s.t.* if $x \in B_\delta(a) \cap A$, then $f(x) \in B_\epsilon(f(a))$.
3. For all $\epsilon > 0$, $\exists\, \delta > 0$ *s.t.* if $x \in A$ satisfies $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.
4. For $V \in \mathscr{T}_{EUC}$, if $f(a) \in V$, then $\exists\, U \in \mathscr{T}_A$ *s.t* $a \in U$ and $\forall x \in U \cap A$, we have $f(x) \in V$.

Note: The four statements are equivalent.
Note: By definition, if $f : A \to \mathbb{R}$ is continuous at $a \in A$, then $f : A \to f(A)$ is continuous at a.
Note: By definitions, if the function $f : A \to \mathbb{R}$ is continuous, then $f : A \to f(A)$ is continuous.

A topological space $(X, \mathscr{T}_X)$ is said to be **Disconnected** provided that there exist nonempty disjoint subsets $A, B \in \mathscr{T}_X$ such that $A \cup B = X$. A topological space $(Y, \mathscr{T}_Y)$ is said to be **Connected** provided that $(Y, \mathscr{T}_Y)$ is not disconnected.
Fact: Let X be a set. The topological space $(X, \mathscr{T}_{ind})$ is always connected.
Fact: In $(\mathbb{R}, \mathscr{T}_{EUC})$, let A be a subset of $\mathbb{R}$, let $\mathscr{T}_A$ denote the subspace topology on A. The topological space $(A, \mathscr{T}_A)$ is connected if and only if A is an interval.

Let $(X, \mathscr{T})$ be a topological space, let A be a subset of X. A collection of open sets $\mathscr{C} \subseteq \mathscr{T}$ is called an **Open Cover** of A in $\mathscr{T}$ provided that $A \subseteq \bigcup_{C \in \mathscr{C}} C$.

Let $(X, \mathscr{T})$ be a topological space, let A be a subset of X, and let $\mathscr{C}$ be an open cover of A in $\mathscr{T}$. A subcollection $\mathscr{C}'$ of $\mathscr{C}$ is called an **Open Subcover** of A in $\mathscr{T}$ provided that $A \subseteq \bigcup_{C' \in \mathscr{C}'} C'$.

Let $(X, \mathscr{T})$ be a topological space, let A be a subset of X, let $\mathscr{T}_A$ denote the subspace topology on A inherited from $\mathscr{T}$. The topological space $(A, \mathscr{T}_A)$ is said to be **Compact** provided that every open cover of A in $\mathscr{T}$ admits a finite open subcover in $\mathscr{T}$.
Fact: The topological space $(\mathbb{R}, \mathscr{T}_{EUC})$ is not compact.
Fact: If X is a finite set, then the topological space $(X, \mathscr{T})$ is always compact.
Fact: Let $a, b \in \mathbb{R}$ with $a < b$, let (a, b) be an interval, let $\mathscr{T}_{(a,b)}$ denote the subspace topology on (a, b). The topological space $((a, b), \mathscr{T}_{(a,b)})$ is not compact.

Let $(X, \mathscr{T})$ be a topological space, let $x, y \in X$. A **Path** connecting x and y is a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Let $(X, \mathscr{T})$ be a topological space. The topological space $(X, \mathscr{T})$ is said to be **Path Connected** provided that for all $x, y \in X$, there exists a path connecting x and y.
Note: If the topological space $(X, \mathscr{T})$ is path connected, then $(X, \mathscr{T})$ is connected.
Note: The connectedness of a topological space does not imply its path connectedness.

$\mathbb{R}^2 := \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$
$\mathbb{R}^3 := (\mathbb{R} \times \mathbb{R}) \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

Let X be a set. $d : X \times X \to \mathbb{R}_{\geq 0}$ is called a **Metric** on X provided that the followings hold:

1. For all $x, y \in X$, we have $d(x, y) = d(y, x)$.
2. For $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.
3. For all $x, y, z \in X$, we have $d(x, y) \leq (d(x, z) + d(z, y))$.

The function $d_3 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ $((x_1, y_1, z_1), (x_2, y_2, z_2)) \mapsto \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$ is called the Euclidean metric, or Euclidean distance, on $\mathbb{R}^3$.
The function $d_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ $((x_1, y_1), (x_2, y_2)) \mapsto \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is called the Euclidean metric, or Euclidean distance, on $\mathbb{R}^2$.
The function $d_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ $x \mapsto \mid x \mid$ is called the Euclidean metric, or Euclidean distance, on $\mathbb{R}$.

Let X be a set, let d_X be a metric on X. The pair (X, d_X) is called a **Metric Space**.

Let (X, d) be a metric space, let $a \in X$, and let $r > 0$. The set $B_r := \{x \in X \mid d(x, a) < r\}$ is called a **Ball** centered at a of radius r.

Let (X, d) be a metric space. $\mathscr{T}_d := \{A \subseteq X \mid \forall a \in A,\ \exists\ r > 0\ s.t.\ B_r(a) \subseteq A\}$ is the **Topology on X associated to d**. In the topological space $(X, \mathscr{T}_d)$, let A be a subset of X. The set A is said to be **Open** in $\mathscr{T}_d$ provided that for all $a \in A$, there exists $r > 0$ such that $B_r(a)$ is a subset of A.

A topological space $(X, \mathscr{T})$ is said to be **Metrizable** provided that $\exists$ a metric d on X $s.t$ $\mathscr{T}_d = \mathscr{T}$.
Note: Let (X, d) be a metric space, and let $a \in X$, then $B_r(a) \in \mathscr{T}_d$.
Note: Indiscrete topology is not metrizable.
Note: Discrete topology is metrizable.

Let d_1, d_2, d_3 denote the Euclidean metric on $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$, respectively, and let $r \in \mathbb{R}$.
$S^0 := \{x \in \mathbb{R} \mid d_1(x, 0) = r\}$ is a zero-dimensional sphere of radius r in $\mathbb{R}$.
$S^1 := \{(x, y) \in \mathbb{R}^2 \mid d_2((x, y), (0, 0)) = r\}$ is an one-dimensional sphere of radius r in $\mathbb{R}^2$.
$S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid d_2((x, y, z), (0, 0, 0)) = r\}$ is a two-dimensional sphere of radius r in $\mathbb{R}^3$.
Note: A zero-dimensional sphere is often called an open ball in $\mathbb{R}$.
Note: A one-dimensional sphere is often called a circle in $\mathbb{R}^2$.
Note: A two-dimensional sphere is often called a sphere in $\mathbb{R}^3$.

Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces, let $f : X \to Y$ be a function. f is called an **Open Map** from $(X, \mathscr{T}_X)$ to $(Y, \mathscr{T}_Y)$ provided that for all $V \in \mathscr{T}_X$, we have $f(V) \in \mathscr{T}_Y$.

Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces, let $f : X \to Y$ be a function. The function f is called a **Homeomorphism** from $(X, \mathscr{T}_X)$ to (Y, T_Y) provided that the followings hold:

1. $f : X \to Y$ is bijective.
2. $f : X \to Y$ is continuous.
3. $f^{-1} : Y\ toX$ is continuous.

Fact: Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces. If $f : X \to Y$ is a homeomorphism from $(X, \mathscr{T}_X)$ to $(Y, \mathscr{T}_Y)$, then the inverse of f, $f^{-1} : Y \to X$ is a homeomorphism from $(Y, \mathscr{T}_Y)$ to $(X, \mathscr{T}_X)$.
Fact: Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be homeomorphic topological spaces, let $h : X \to Y$ be a homeomorphism, let A be a subset of X, let $\mathscr{T}_A$ and $\mathscr{T}_{h(A)}$ be subspace topology, the restriction of h on A, $res_A h : A \to h(A)$, is a homeomorphism from $(A, \mathscr{T}_A)$ to $(h(A), \mathscr{T}_{h(A)})$.

Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces. The topological space $(X, \mathscr{T}_X)$ is said to be **Homeomorphic** to $(Y, \mathscr{T}_Y)$ provided that there exists a homeomorphism from $(X, \mathscr{T}_X)$ to $(Y, \mathscr{T}_Y)$.
Note: Being homeomorphic defines an equivalence relation on topological spaces.
Fact: Let $\mathscr{T}_{EUC_2}$ denote the topology on $\mathbb{R}^2$ associated to the Euclidean metric on $\mathbb{R}^2$. The topological space $(\mathbb{R}^2, \mathscr{T}_{EUC_2})$ is not homeomorphic to the topological space $(\mathbb{R}, \mathscr{T}_{EUC})$.

Let $(X, \mathscr{T})$ be a connected topological space, let $x \in X$, let $\mathscr{T}_x$ denote the subspace topology on the set $X \setminus \{x\}$ inherited from $\mathscr{T}$. x is called a **Cut Point** of X provided that $(X \setminus \{x\}, \mathscr{T}_x)$ is disconnected, and $cut(X) \coloneqq \{x \in X \mid x \text{ is a cut point of } X\}$ denotes the set of cut point of X.
Fact: Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be connected homeomorphic topological spaces, any homeomorphism between $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ sends cut point to cut point, that is, if $h : X \to Y$ is a homeomorphism from $(X, \mathscr{T}_X)$ to $(Y, \mathscr{T}_Y)$, then the restriction of h on $cut(X)$, $res_{cut(X)} h : cut(X) \to cut(Y)$, is a bijection from $cut(X)$ to $cut(Y)$.

Let $(X, \mathscr{T}_X)$ be a topological space. $(X, \mathscr{T}_X)$ is said to be **Totally Disconnected** provided that the only connected subsets of X are the singletons and the empty set.
Note: $(\mathbb{R}, \mathscr{T}_{dis})$, $(\mathbb{Z}, \mathscr{T}_{EUC})$, $(\mathbb{N}, \mathscr{T}_{EUC})$, $(\mathbb{Q}, \mathscr{T}_{EUC})$, $(\mathbb{R} \setminus \mathbb{Q}, \mathscr{T}_{EUC})$ are totally disconnected.

Let $(X, \mathscr{T}_X)$ be a topological space, let A be a subset of X. The element $a \in A$ is called an **Isolated Point** of A provided that $\exists\ U \in \mathscr{T}_X\ s.t.\ a \in U$ and $U \cap (A \setminus \{a\}) = \emptyset$, that is, there exists an open subset U of X such that U contains a and U contains no other elements of A.
Note: In $(\mathbb{R}, \mathscr{T}_{EUC})$, all elements in $\mathbb{N}$ are isolated points of $\mathbb{N}$.
Note: In $(\mathbb{R}, \mathscr{T}_{EUC})$, all elements in $\mathbb{Z}$ are isolated points of $\mathbb{Z}$.
Note: In $(\mathbb{R}, \mathscr{T}_{EUC})$, $\mathbb{Q}$ has no isolated point.
Fact: Let $(X, \mathscr{T}_X)$ be a topological space, let A be a subset of X. If $a \in A$ is an isolated point of A, then $\{a\}$ is open in the subspace topology on the set A inherited from $\mathscr{T}_X$.

Let $(X, \mathscr{T}_X)$ be a topological space, let A be a subset of X. The set A is said to be **Perfect** provided that A is closed in $\mathscr{T}_X$ and A has no isolated point.

Let $(X, \mathscr{T}_X)$ be a topological space, let A be a subset of X. The element $a \in X$ is called a **Limit Point** of A, or an **Accumulation Point** of A, provided that $\forall\ U \in \mathscr{T}_X$, if $a \in U$, then $\exists\ a' \in A \setminus \{a\}\ s.t.\ a' \in U$, that is, for all open subsets U of X that contain a, we have $U \cap (A \setminus \{a\}) \neq \emptyset$.
Note: In $(\mathbb{R}, \mathscr{T}_{EIC})$ any singleton subset of $\mathbb{R}$ has no accumulation point.

Let $(X, \mathscr{T}_X)$ be a topological space. The topological space $(X, \mathscr{T}_X)$ is said to be **Hausdorff** provided that for all distinct $x, y \in X$, there exist disjoint $U, V \in \mathscr{T}_X\ s.t.\ x \in U$ and $y \in V$.
Note: In a Hausdorff space, we can separate distinct points with open sets.

Let (X, d_X) and (Y, d_Y) be metric spaces, let $f : X \to Y$ be a function. f is said to be **Lipschitzian** provided that there exists $c \in \mathbb{R}$ such that $\forall x, y \in X$, we have $d_Y(f(x), f(y)) \leq c \cdot d_X(x, y)$.

Topological Invariants are the properties of topological spaces that are preserved by homeomorphisms between topological spaces.

Groups

A set G with a binary operation $\diamond$ on G is called a **Group** provided that the followings hold:

1. The binary operation $\diamond$ is associative.
2. There exists $\diamond$-identity $e \in G$.
3. For all $g \in G$, there exists $h \in G$ such that $g \diamond h = h \diamond g = e$.

Note: If $(G, \diamond)$ is a group, then the $\diamond$-inverse of each $g \in G$ is unique.
Note: The followings are groups: $(\mathbb{R}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{R} \setminus \{0\}, *)$, $((0, \infty), *)$
Note: The followings are not group: $(\mathbb{R}, *)$, $(\mathbb{N}, +)$, $([0, \infty), *)$
Note: Let $(F, +, *)$ be a filed, $(F, +)$ and $(F \setminus \{0_F\}, *)$ are groups.

Let $(G, \diamond_G), (H, \diamond_H)$ be groups, let $\varphi : G \to H$ be a function, φ is called a **Group Homomorphism** from $(G, \diamond_G)$ to $(H, \diamond_H)$ provided that for $a, b \in G$, we have $\varphi(a \diamond_G b) = \varphi(a) \diamond_H \varphi(b)$.
Fact: Let $(G, \diamond_G), (H, \diamond_H)$ be groups. The function $\varphi : G \to H$ that sends all elements in G to the $\diamond_H$-identity in H is a group homomorphism from $(G, \diamond_G)$ to $(H, \diamond_H)$.

Let $(G, \diamond_G), (H, \diamond_H)$ be groups, let $\varphi : G \to H$ be a group homomorphism from $(G, \diamond_G)$ to $(H, \diamond_H)$. φ is called a **Group Isomorphism** from $(G, \diamond_G)$ to $(H, \diamond_H)$ provided that φ is a bijection from G to H. If φ is a group isomorphism from $(G, \diamond_G)$ to $(H, \diamond_H)$, then $(G, \diamond_G)$ is Isomorphic to $(H, \diamond_H)$.
Fact: Let $(G, \diamond_G), (H, \diamond_H)$ be groups. If $\varphi : G \to H$ is a group isomorphism from $(G, \diamond_G)$ to $(H, \diamond_H)$, then the inverse of φ, $\varphi^{-1} : H \to G$, is a group isomorphism from $(H, \diamond_H)$ to $(G, \diamond_G)$.
Fact: The group $((0, \infty), *)$ is isomorphic to the group $(\mathbb{R}, +)$.

Let $(G, \diamond_G), (H, \diamond_H)$ be groups, let $e_H \in H$ be the $\diamond_H$-identity, let $\varphi : G \to H$ be a group homomorphism from $(G, \diamond_G)$ to $(H, \diamond_H)$. $im(\varphi) \coloneqq \{\varphi(g) \mid g \in G\}$ is called the **Image** of φ. $ker(\varphi) \coloneqq \{g \in G \mid \varphi(g) = e_H\}$ is called the **Kernel** of φ.

A group $(G, \diamond)$ is said to be **Abelian** provided that $\diamond$ is commutative.

Let $n \in \mathbb{N}$, let $X_n = \{x_1, x_2, x_3, \cdots, x_n\}$ be a finite set. A bijection $f : X_n \to X_n$ is called a **Permutation** of the set X_n, and the set $s_n \coloneqq \{f : X_n \to X_n \mid f \text{ is bijective}\}$ is called the **Collection of the Permutations** of the finite set X_n.
Note: Let $n \in \mathbb{N}$, let $\circ$ denote function composition. $S_n \coloneqq (s_n, \circ)$ is a group.
Fact: Let $n \in \mathbb{N}$, the set s_n has $n!$ element.

Let $n \in \mathbb{N}$, $S_n \coloneqq (s_n, \circ)$ is called the **Symmetric Group on n Letters**.
Fact: For $n \in \mathbb{N}$ with $n \geq 3$, the group S_n is not abelian.

Let $n \in \mathbb{N}$ with $n \geq 3$, given a regular polygon with n sides. a **Dihedral Group**, denoted as D_n is the group of symmetries of the n-sided regular polygon, which includes rotations and reflections of the n-sided regular polygon.
Fact: Let $n \in \mathbb{N}$ with $n \geq 3$, the group D_n has $2n$ elements.
Fact: Let $n \in \mathbb{N}$ with $n \geq 3$, the group D_n is not abelian.
Fact: Let $n \in \mathbb{N}$ with $n > 3$, D_n is not isomorphic to S_n.

Calculus

Consider using the topological space $(\mathbb{R}, \mathscr{T}_{EUC})$ from now on.
Fix $a, b \in \mathbb{R}$ with $a < b$, $n \in \mathbb{N}$, consider using the interval $[a, b]$ from now on.

Let M, B be a subset of $\mathbb{R}$, let A be a subset of M, let $f : M \to B$ be a function.
f is **Strictly Increasing** on A provided that if $x, y \in A$ with $x < y$, then $f(x) < f(y)$.
f is **Strictly Decreasing** on A provided that if $x, y \in A$ with $x < y$, then $f(x) > f(y)$.
f is **Non-Decreasing** on A provided that if $x, y \in A$ with $x < y$, then $f(x) \le f(y)$.
f is **Non-Increasing** on A provided that if $x, y \in A$ with $x < y$, then $f(x) \ge f(y)$.
f is called a **Constant Function** on A provided that $\exists\, k \in \mathbb{R}$ $s.t.$ $\forall c \in f(A),\ c = k$.
f is **Bounded** on A provided that the set $f(A)$ is bounded.
f is **Non-Negative** on A provided that $\forall c \in f(A),\ c \ge 0$.
f is **Non-Positive** on A provided that $\forall c \in f(A),\ c \le 0$.
f is **Negative** on A provided that $\forall c \in f(A),\ c < 0$.
f is **Positive** on A provided that $\forall c \in f(A),\ c > 0$.
Fact: Let A, B be subsets of $\mathbb{R}$, let $f : A \to B$ be a function. If f is bounded on A, then there exists $m \in \mathbb{R}$ such that for all $x \in A$, we have $|f(x)| \le m$.

Let A, B be subsets of $\mathbb{R}$, let $f : A \to B$ be a function. f is said to be **Locally Bounded** on A provided that for all $x \in A$, there exists a neighborhood U of x such that f is bounded on U.

Let A, B be subsets of $\mathbb{R}$, let $f : A \to B$ be a function. f is said to be **Locally Constant** on A provided that for all $x \in A$, $\exists\ \epsilon > 0$ such that for all $y \in B_\epsilon(x)$, we have $f(y) = d$ for some $d \in \mathbb{R}$.

Let A, B be subsets of $\mathbb{R}$, let $f : A \to B$ be a function. f is said to be **Uniformly Continuous** on A provided that $\forall \epsilon > 0$, $\exists\ \delta > 0$ $s.t.$ if $x, y \in A$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
Fact: Let $A \subseteq \mathbb{R}$. If $f : A \to \mathbb{R}$ is uniformly continuous on A, then f is continuous on A.

Let $a, b \in \mathbb{R}$, let $[a, b]$ be an interval, $l \coloneqq b - a$ is called the **Length** of $[a, b]$.

An ordered set $P \coloneqq \{t_0, t_1, t_2, \cdots, t_n\}$ of points in $\mathbb{R}$, where $a = t_0 < t_1 < t_2 < \cdots < t_n = b$, is called a **Partition** of $[a, b]$. For $i \in \mathbb{N},\ 1 \le i \le n$, the interval $[t_{i-1}, t_i]$ is called a **Subinterval** of P.

Let $P = \{t_0, t_1, t_2, \cdots, t_n\}$ be a partition of $[a, b]$. If the length of each subinterval of P is $\frac{b-a}{n}$, that is, for $i \in \mathbb{N},\ 0 \le i \le n,\ t_i = a + \frac{b-a}{n} i$, then P is called a **Regular Partition** on $[a, b]$.

Let P, Q be partitions of $[a, b]$, Q is called a **Refinement** of P provided that P as a set is a subset of the set Q, and we write $P \le Q$ if Q is a refinement of P.

Let $P = \{t_0, t_1, t_2, \cdots, t_n\}$ be a partition of $[a, b]$.
$||P|| \coloneqq max\{t_i - t_{i-1} \mid i \in \mathbb{N},\ 1 \le i \le n\}$ is called the **Norm** of P
Note: The norm of a partition P is the maximum length of the subintervals of P

Let $P = \{t_0, t_1, t_2, \cdots, t_n\}$ be a partition of $[a, b]$, let $f : [a, b] \to \mathbb{R}$ be a bounded function.
For $i \in \mathbb{N},\ 1 \le i \le n$, $\mathscr{M}_i \coloneqq \sup\{f(x) \mid x \in [t_{i-1}, t_i]\}$. $U(f, P) \coloneqq \sum_{i=1}^{n} [(t_i - t_{i-1}) \cdot \mathscr{M}_i]$ is called the **Upper Darboux Sum** of the function f on the interval $[a, b]$ with respect to the partition P.
Note: By definition, $U(f, P)$ is the sum of the area of rectangles of height $\mathscr{M}_i$ and base $(t_i - t_{i-1})$.

Let $P = \{t_0, t_1, t_2, \cdots, t_n\}$ be a partition of $[a, b]$, let $f : [a, b] \to \mathbb{R}$ be a bounded function.
For $i \in \mathbb{N},\ 1 \le i \le n$, $\mu_i \coloneqq \inf\{f(x) \mid x \in [t_{i-1}, t_i]\}$. $L(f, P) \coloneqq \sum_{i=1}^{n} [(t_i - t_{i-1}) \cdot \mu_i]$ is called the **Lower Darboux Sum** of the function f on the interval $[a, b]$ with respect to the partition P.
Note: By definition, $U(f, P)$ is the sum of the area of rectangles of height μ_i and base $(t_i - t_{i-1})$.
Fact: Let $f : [a, b] \to \mathbb{R}$ be a bounded function, $\forall$ partitions P of $[a, b]$, we have $L(f, P) \le U(f, P)$.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function.
$U(f) \coloneqq inf\{U(f, P) \mid P$ is a partition of $[a, b]\}$
$L(f) \coloneqq sup\{L(f, P) \mid P$ is a partition of $[a, b]\}$
Note: By definitions, $\{U(f, P) \mid P$ is a partition of $[a, b]\}$ is bounded below by $L(f, \{a, b\})$.
Note: By definitions, $\{L(f, P) \mid P$ is a partition of $[a, b]\}$ is bounded above by $U(f, \{a, b\})$.
Note: By definitions, we have $L(f) \le U(f)$.

Let $f : [\,a,b\,] \to \mathbb{R}$ be a bounded function. f is said to be **Darboux Integrable** on the interval $[\,a,b\,]$ provided that $U(f) = L(f)$. If f is Darboux integrable on $[\,a,b\,]$, $I_D \coloneqq U(f)$ is called the **Darboux Integral** of f on $[\,a,b\,]$. Furthermore, if f is Darboux integrable and non-negative on $[\,a,b\,]$, I_D is called the Area Bounded by the graph of f on $[\,a,b\,]$.
Note: Not all functions are Darboux integrable.
Fact: If $f : [\,a,b\,] \to \mathbb{R}$ is a constant function, then f is Darboux integrable on $[\,a,b\,]$.

Let P be partitions of $[\,a,b\,]$, let $f : [\,a,b\,] \to \mathbb{R}$ be a bounded function.
We can write $\lim_{||P|| \to 0} U(f,P) = l$ for some $l \in \mathbb{R}$ provided that for all $\epsilon > 0$, there exists $\delta > 0$ such that for all partitions P of $[\,a,b\,]$ with $||P|| < \delta$, we have $|U(f,P) - l| < \epsilon$.

Let Q be partitions of $[\,a,b\,]$, let $f : [\,a,b\,] \to \mathbb{R}$ be a bounded function.
We can write $\lim_{||Q|| \to 0} L(f,Q) = l$ for some $l \in \mathbb{R}$ provided that for all $\epsilon > 0$, there exists $\delta > 0$ such that for all partitions Q of $[\,a,b\,]$ with $||Q|| < \delta$, we have $|L(f,Q) - l| < \epsilon$.

Let P, Q be partitions of $[\,a,b\,]$, let $f : [\,a,b\,] \to \mathbb{R}$ be a bounded function. The function f is said to be **S-Integrable** on the interval $[\,a,b\,]$ provided that $\lim_{||P|| \to 0} U(f,P) = \lim_{||Q|| \to 0} L(f,Q)$. If f is S-integrable on $[\,a,b\,]$, $I_S \coloneqq \lim_{||Q|| \to 0} L(f,Q)$ is called the **S-Integral** of f on $[\,a,b\,]$.

Let X^* be a subset of $\mathbb{R}$. Given $n \in \mathbb{N}$, X^* is called an ***n*-Tuple** provided that X^* has n element.

Let $P = \{t_0, t_1, t_2, \cdots, t_n\}$ be a partition of $[\,a,b\,]$, let n-tuple $X^* = \{x_1^*, x_2^*, \cdots, x_n^*\}$ be a subset of $\mathbb{R}$. X^* is **Compatible** with the partition P provided that for $i \in \mathbb{N}$, $1 \le i \le n$, we have $x_i^* \in [\,t_{i-1}, t_i\,]$.

Let $P = \{t_0, t_1, t_2, \cdots, t_n\}$ be a partition of $[\,a,b\,]$, let $f : [\,a,b\,] \to \mathbb{R}$ be a function, let n-tuple $X^* = \{x_1^*, x_2^*, \cdots, x_n^*\}$ be a subset of $\mathbb{R}$ compatible with P. $R(f,P,X^*) \coloneqq \sum_{i=1}^{n} [f(x_i^*)(t_i - t_{i-1})]$ is called the **Riemann Sum** of the function f on the interval $[\,a,b\,]$ with respect to P and X^*.

Let $P = \{t_0, t_1, t_2, \cdots, t_n\}$ be a partition of $[\,a,b\,]$, let $f : [\,a,b\,] \to \mathbb{R}$ be a function, let n-tuple $X^* = \{x_1^*, x_2^*, \cdots, x_n^*\}$ be a subset of $\mathbb{R}$ compatible with P. We can write $\lim_{||P|| \to 0} R(f,P,X^*) = l$ for some $l \in \mathbb{R}$ provided that for all $\epsilon > 0$, there exists $\delta > 0$ such that for all partitions P of $[\,a,b\,]$ with $||P|| < \delta$, we have $|R(f,P,X^*) - l| < \epsilon$.

Let $P = \{t_0, t_1, t_2, \cdots, t_n\}$ be a partition of $[\,a,b\,]$, let $f : [\,a,b\,] \to \mathbb{R}$ be a function, let n-tuple $X^* = \{x_1^*, x_2^*, \cdots, x_n^*\}$ be a subset of $\mathbb{R}$ compatible with P. f is said to be **Riemann Integrable** on the interval $[\,a,b\,]$ provided that $\lim_{||P|| \to 0} R(f,P,X^*) = l$ for some $l \in \mathbb{R}$. If f is Riemann Integrable on $[\,a,b\,]$, $I_R \coloneqq \lim_{||P|| \to 0} R(f,P,X^*)$ is called the **Riemann Integral** of f on $[\,a,b\,]$.
Fact: If a function $f : [\,a,b\,] \to \mathbb{R}$ is Riemann Integrable on $[\,a,b\,]$, then f is bounded on $[\,a,b\,]$.

Let $f : [\,a,b\,] \to \mathbb{R}$ be a function.
By Theorem, f is said to be **Integrable** on $[\,a,b\,]$ provided that any one of the followings holds:

1. f is bounded on $[\,a,b\,]$ and f is Darboux integrable on $[\,a,b\,]$.
2. f is bounded on $[\,a,b\,]$ and f is S-integrable on $[\,a,b\,]$.
3. f is Riemann integrable on $[\,a,b\,]$.

If f is integrable on $[\,a,b\,]$, $\int_a^b f$ is called the **Integral** of f on $[\,a,b\,]$.

Let $f : [\,a,b\,] \to \mathbb{R}$ be a function that is integrable on $[\,a,b\,]$. Given $a < b$, $\int_b^a f \coloneqq -\int_a^b f$.

The function $\ln : (0,\infty) \to \mathbb{R} \quad x \mapsto \int_1^x \frac{1}{t}$ is called the **Natural Logarithm Function**.
Notation: For $x \in \mathbb{R}$, $0 < x < 1$ we write $\ln(x) = \int_1^x \frac{1}{t} = -\int_x^1 \frac{1}{t}$.
Fact: The function $\gamma : (0,\infty) \to \mathbb{R} \quad t \mapsto \frac{1}{t}$ is continuous on $(0,\infty)$, hence integrable on $(0,\infty)$.
Fact: The Natural Logarithm function is strictly increasing and continuous on $(0,\infty)$.
Fact: The Natural Logarithm function is a bijection from $(0,\infty)$ to $\mathbb{R}$.
Fact: Given $\ln : (0,\infty) \to \mathbb{R} \quad x \mapsto \int_1^x \frac{1}{t}$, we have $\ln(1) = 0$, and $\ln((0,\infty)) = \mathbb{R}$.

The inverse function of the Natural Logarithm function, $\exp : \mathbb{R} \to (0,\infty) \quad \exp(t) = s \iff t = \ln(s)$, is called the **Natural Exponential Function**.
Note: The Natural Logarithm function is a bijection from $(0,\infty)$ to $\mathbb{R}$, hence invertible.

The assumptions, $a, b \in \mathbb{R}$ with $a < b$, $n \in \mathbb{N}$, $[\,a,b\,]$ is an interval, are now removed.

We continue using the topological space $(\mathbb{R}, \mathcal{T}_{EUC})$.
The term IWIMP refers to Interval With Infinitely Many Points.

Let A, B be subsets of $\mathbb{R}$, let $f : A \to B$ be a function, let $a \in A$. We say that the limit of f at a when f approaches a from the right is l, denoted as $\lim_{x \to a^+} f(x) = l$ for some $l \in \mathbb{R}$, provided that $\forall \epsilon > 0,\ \exists\, \delta > 0\ s.t.$ if $x \in \mathbb{R}$ satisfies $0 < x - a < \delta$, then $|f(x) - l| < \epsilon$.

Let A, B be subsets of $\mathbb{R}$, let $f : A \to B$ be a function, let $a \in A$. We say that the limit of f at a when f approaches a from the left is m, denoted as $\lim_{x \to a^-} f(x) = m$ for some $m \in \mathbb{R}$, provided that $\forall \epsilon > 0,\ \exists\, \delta > 0\ s.t.$ if $x \in \mathbb{R}$ satisfies $0 < a - x < \delta$, then $|f(x) - m| < \epsilon$.

Let $A \subseteq \mathbb{R}$ be an interval. If A has a minimal element a, then a is called a **End Point** of A. If A has a maximal element b, then b is called a **End Point** of A.

Let $A \subseteq \mathbb{R}$ be an IWIMP, let B be a subset of $\mathbb{R}$, let $a \in A$, let $f : A \to B$ be a function.
If a is not an end point of A, then f is **Differentiable** at a provided that there exists $l \in \mathbb{R}$ such that $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = l$. If a is the minimal element of A, then f is **Differentiable** at a provided that there exists $l \in \mathbb{R}$ such that $\lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h} = l$. If a is the maximal element of A, then f is **Differentiable** at a provided that there exists $l \in \mathbb{R}$ such that $\lim_{h \to 0^-} \frac{f(a+h)-f(a)}{h} = l$.
If f is **Differentiable** at a, then $f'(a) \coloneqq l$ is called the **Derivative** of f at a.
Note: Continuity of a function does not imply differentiability of the function.

Let $A \subseteq \mathbb{R}$ be an IWIMP, let B be a subset of $\mathbb{R}$, let $a \in A$, let $f : A \to B$ be a function. f is **Differentiable** on A provided that $\forall a \in A$, f is differentiable at a. If f is differentiable on A, then the function $f' : A \to \mathbb{R} \quad a \mapsto f'(a)$ is called the **Derivative** of of f. If the function f is differentiable on its domain, then the function is said to be **Differentiable**.

Let $A \subseteq \mathbb{R}$ be an IWIMP, let B be a subset of $\mathbb{R}$, let $f : A \to B$ be a function, let $a, b \in A$.
a is called a **Local Maximum** of f provided that $\exists\, \delta > 0\ s.t.\ \forall x \in B_\delta(a) \cap A$, we have $f(a) \geq f(x)$.
b is called a **Local Minimum** of f provided that $\exists\, \lambda > 0\ s.t.\ \forall x \in B_\lambda(b) \cap A$, we have $f(b) \leq f(x)$.

Let $A \subset \mathbb{R}$ be an IWIMP, let B be a subset of $\mathbb{R}$, let $f : A \to B$ be a function, let $a \in A$. a is called a **Critical Point** of f on A provided that either one of the followings holds:

1. f is not differentiable at a.
2. f is differentiable at a and $f'(a) = 0$.

If a is a critical point of f on A, then $f(a)$ is called a **Critical Value** of f on A.
Fact: Let $A \subseteq \mathbb{R}$ be an IWIMP, let B be a subset of $\mathbb{R}$, let $f : A \to B$ be a function. If f has a local maximum or a local minimum at $a \in A$, and if f is differentiable at a, then $f'(a) = 0$.

Let $r \in \mathbb{R}$, let $x \in \mathbb{R}_{>0}$, $x^r \coloneqq \exp(r \cdot \ln(x))$ is called x **raised to the power of** n, in this case, x is called the **Base**, n is called the **Exponent** or the **Power**.

Sequences

Let $(X, \mathscr{T})$ be a topological space, a function $seq : \mathbb{N} \to X \quad n \mapsto x_n$ is called a **Sequence** in $(X, \mathscr{T})$.
Notation: A sequence $seq : \mathbb{N} \to X \quad n \mapsto x_n$ in $(X, \mathscr{T})$ can be denoted as $n \mapsto x_n$ or (x_n).

Let $(X, \mathscr{T})$ be a topological space, let (x_n) be a sequence in $(X, \mathscr{T})$. We say (x_n) **Converges** to $l \in X$ provided that $\forall\ U \in \mathscr{T}$, if $l \in U$, then $\exists\ N \in \mathbb{N}\ s.t.\ \forall n \in \mathbb{N}$ with $n \geq N$, we have $x_n \in U$. If (x_n) converges to $l \in X$, (x_n) is called a **Convergent Sequence**, and we write $\lim_{n\to\infty} x_n = l$.
Note: In $(\mathbb{R}, \mathscr{T}_{ind})$, all sequences converge and they converge to every real number.
Note: In $(\mathbb{R}, \mathscr{T}_{dis})$, only those sequences that are eventually constant converge.
Fact: If a sequence in a Hausdorff topological space converges, then the limit of the sequence is unique, that is, the sequence converges to a unique element in that Hausdorff topological space.

A sequence (x_n) in $(\mathbb{R}, \mathscr{T}_{EUC})$ is called a **Sequence of Real Numbers in the Euclidean Topology**. We say the sequence (x_n) **Converges** to some $l \in \mathbb{R}$ provided that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have $|x_n - l| < \epsilon$.
Note: Not all sequence of real numbers in the Euclidean topology converges.

Let $(X, \mathscr{T})$ be a topological space, let $\mathscr{A} : \mathbb{N} \to X \quad n \mapsto a_n$ be a sequence in $(X, \mathscr{T})$, let $\mathscr{J} : \mathbb{N} \to \mathbb{N} \quad k \mapsto j_k$ be a strictly increasing function, let $\circ$ denote function composition. The function $\mathscr{A} \circ \mathscr{J} : \mathbb{N} \to X \quad n \mapsto a_{j_n}$ is called a **Subsequence** of the sequence $\mathscr{A}$.

Let (t_m) be a sequence of real numbers in the Euclidean topology.
(t_m) is said to be M**onotonic Increasing** provided that $\forall a, b \in \mathbb{N}$ with $a > b$, we have $t_a \geq t_b$.
(t_m) is said to be **Monotonic Decreasing** provided that $\forall a, b \in \mathbb{N}$ with $a > b$, we have $t_a \leq t_b$.

Let (t_m) be a sequence of real numbers in the Euclidean topology. (t_m) is said to be **Monotonic** provided that (t_m) is either monotonic increasing or monotonic decreasing.

Let (X, d) be a metric space, let (x_n) be a sequence in $(X, \mathscr{T}_d)$. (x_n) is called a **Cauchy Sequence** in X provided that $\forall \epsilon > 0,\ \exists\ N \in \mathbb{N}\ s.t.\ \forall n, m \in \mathbb{N}$ with $n > \mathbb{N}$ and $m > \mathbb{N}$, we have $d(x_n, x_m) < \epsilon$.

Let (X, d) be a metric space. The metric space (X, d) is said to be **Cauchy Complete** provided that every Cauchy sequence in $(X, \mathscr{T}_d)$ converges to some $l \in X$.
Note: Cauchy Complete is not a topological invariant.
Fact: $(\mathbb{R}, d_{EUC})$ is Cauchy complete.
Fact: $([0, 1], d_{EUC})$ is Cauchy complete.

Polynomials

Consider using the topological space $(\mathbb{R}, \mathscr{T}_{EUC})$ from now on.
The term IWIMP refers to Interval With Infinitely Many Points.

A **Polynomial Function of degree** n, denoted as $P : \mathbb{R} \to \mathbb{R}$, is a function from $\mathbb{R}$ to $\mathbb{R}$ of the form $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, with $a_n, a_{n-1}, \cdots, a_1, a_0 \in \mathbb{R}$, $n \in \mathbb{N}$, and $a_n \neq 0$.

Let $A \subseteq \mathbb{R}$ be an IWIMP, let B be a subset of $\mathbb{R}$, let $a \in A$, let $n \in \mathbb{N}$, let $f : A \to B$ be a function that is n-times differentiable at a. The function $P_{n,a} : \mathbb{R} \to \mathbb{R} \quad x \mapsto \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is called the **Taylor Polynomial** of degree n, centered at a, associated to f.
Note: By definition, given $n \in \mathbb{N}$, $a \in A$, $P_{n,a}$ is unique and n-times differnetiable.

Let $A \subseteq \mathbb{R}$ be an IWIMP, let B be a subset of $\mathbb{R}$, let $a \in A$, let $n \in \mathbb{N}$, let $f : A \to B$ be a function that is n-times differentiable at a, let $P_{n,a} : \mathbb{R} \to \mathbb{R}$ be the Taylor polynomial of degree n, centered at a, associated to f. $R_{n,a} : \mathbb{R} \to \mathbb{R} \quad x \mapsto f(x) - P_{n,a}(x)$ is called the **Remainder Term** of $P_{n,a}$.