Proposition 0.0.1
Let A be an open subset of $\mathbb{R}^n$, and let ω be a 1-form defined on A.
For k-manifold $M \subseteq A$, the followings are equivalent:

1. $\mathcal{T}_{\vec{p}}(M) \subseteq \ker(\omega(\vec{p}))$ *for all* $\vec{p} \in M$
2. $\alpha^*\omega = 0$ *for all coordinate patches* α *for* M
3. $\int_C \omega = 0$ *for all* 1*-manifold* $C \subseteq M$.

Let ω be a 1-form defined on an open subset A of $\mathbb{R}^n$, for k-manifold $M \subseteq A$, M is called an integral manifold for ω provided that $\int_C \omega = 0$ for all 1-manifold $C \subseteq M$. Integral manifolds for ω are also integral manifold for $g\omega$ where g is a scalar function, because we have $\alpha^*(g\omega) = (\alpha^*g)(\alpha^*\omega)$.

Lemma 0.0.2
Let $f \in C^1(A, \mathbb{R})$ where A is an open subset of $\mathbb{R}^n$, with $df \neq 0$ on A.
Then, for $c \in \mathbb{R}$, the level set $f^{-1}(c)$ is an $(n-1)$-manifold without boundary.

Corollary 0.0.2.1
Let $f \in C^1(A, \mathbb{R})$ where A is an open subset of $\mathbb{R}^n$, with $df \neq 0$ on A.
Each level set of f is an integral manifold for df.

Let A be an open subset of $\mathbb{R}^n$, let $f : A \to \mathbb{C}$. $Y_\alpha \subseteq A$, we define $\int_{Y_\alpha} f\, dV = \int_{Y_\alpha} u\, dV + i \int_{Y_\alpha} v\, dV$.
Let $A \subseteq \mathbb{R}^n$ be open, let $\omega : A \to \mathbb{C}^n_{row}$, $\omega = \omega_1 + i\omega_2$, be a $\mathbb{C}$-valued 1-form. $\int_{Y_\alpha} \omega := (\int_{Y_\alpha} \omega_1) + (i \int_{Y_\alpha} \omega_2)$.
Let A be an open subset of $\mathbb{R}^n$, let $f : A \to \mathbb{C}$ with $f = u + iv$ for functions u and v. Define $D_j f := D_j u + i D_j v$.
If $f = u + iv$, then $f\, dz = (u + iv)(dx + idy) = (u + iv)dx + (iu - v)dy$. If the 1-form $f\, dz$ is closed, we have $D_1(if) = D_2(f)$, or the Cauchy-Riemann equation holds: $D_1 u = D_2 v$, $D_2 u = -D_1 v$. A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic provided that $f\, dz$ is closed, or the Cauchy-Riemann Equations hold for the function f.

Proposition 0.0.3
Let A be an open subset of $\mathbb{R}^n$, let $f : A \to \mathbb{C}$. we have $\left|\int_A f\right| \leq \int_A |f|$

Theorem 0.1
Let f be a holomorphic on open $A \subseteq \mathbb{C}$. If A is diffeomorphic to a convex set, then $f\, dz$ is exact.

Corollary 0.1.1 (Cauchy Integral Theorem)
Given a holomorphic function f defined on an open subset A of $\mathbb{C}$, where A is diffeomorphic to a convex set, and given $\alpha : [a, b] \to A$ being a piecewise C^1 function with $\alpha(a) = \alpha(b)$, we have $\int_{Y_\alpha} f\, dz = 0$.

Lemma 0.1.2
Let f and g be holomorphic functions. Then $f \cdot g$, $g \circ f$, $\frac{1}{g}$ and $\frac{f}{g}$ are holomorphic functions.
If f is holomorphic diffeomorphism, then f^{-1} is holomorphic.

Theorem 0.2 (Cauchy Integral Theorem)
Let C_1 and C_2 be disjoint circles in $\mathbb{C}$ with C_2 lying inside C_1, let A be an open set of points lying inside C_1 and outside of C_2, let U be an open subset of $\mathbb{C}$ containing $A \cup C_1 \cup C_2$, and let f be a holomorphic function on U, then we have $\int_{C_1} f\, dz = \int_{C_2} f\, dz$

Corollary 0.2.1
Let U be an open subset of $\mathbb{C}$ with some $z_0 \in U$, let f be a holomorphic function defined on $U \setminus \{z_0\}$,
then for $K = \{z \in \mathbb{C} \mid ||z - z_0|| = r\} \subseteq U$, we have $\frac{1}{2\pi i} \int_K f\, dz$ being independent of r.

Definition 0.2.1.0.1
Let U be an open subset of $\mathbb{C}$ with some $z_0 \in U$, let f be a holomorphic function defined on $U \setminus \{z_0\}$,
then for $K = \{z \in \mathbb{C} \mid ||z - z_0|| = r\} \subseteq U$. The residue of $f\, dz$ at z_0 is $Res(f\, dz, z_0) := \frac{1}{2\pi i} \int_K f\, dz$

Theorem 0.3
Let U be an open subset of $\mathbb{C}$, let D be a closed disc in U with $z_0 \in Int(D)$, and let f be a holomorphic function defined on $U \setminus \{z_0\}$. We have $\int_{Bd(D)} f\, dz = 2\pi i\, Res(f\, dz, z_0)$

Let g be a holomorphic function defined on an open subset U of $\mathbb{C}$ with $z_0 \in U$.
We have the following holds:

$$Res\left(\frac{g(z)}{z - z_0}\, dz, z_0\right) = \frac{1}{2\pi i} \int_{||z - z_0|| = r} \frac{g(z)}{z - z_0}\, dz = g(z_0)$$

Proposition 0.3.1 (ML-estimate in $\mathbb{R}^n$)
Let $Y_\alpha \subseteq \mathbb{R}^n$ be a parametrized 1-manifold parametrized by $\alpha : [a, b] \to Y_\alpha$. Let ω be a 1-form defined on an open subset of $\mathbb{R}^n$ containing Y_α. Then $\left|\left|\int_{Y_\alpha} \omega\right|\right| \leq \left(\sup_{\vec{v} \in Y_\alpha} ||\omega(\vec{v})||\right) \cdot length(Y_\alpha)$

Theorem 0.4 (ML-estimate in $\mathbb{C}$)
Let $Y_\alpha \subseteq \mathbb{C}$ be a parametrized 1-manifold parametrized by $\alpha : [a, b] \to Y_\alpha$. Let $f : A \to \mathbb{C}$ be a continuous function with $A \subseteq \mathbb{C}$ being an open and contains Y_α. Then $\left|\left|\int_{Y_\alpha} f\, dz\right|\right| \leq \left(\sup_{z \in Y_\alpha} |f(z)|\right) \cdot length(Y_\alpha)$

Definition 0.4.0.0.1
Let U be an open subset of $\mathbb{C}$, let $f : U \to \mathbb{C}$ be a function. f is said to be differentiable in real sense at $t \in U$ provided that $u : U \to \mathbb{R} \quad z \mapsto \Re(f(z))$ and $v : U \to \mathbb{R} \quad z \mapsto \Im(f(z))$ are both differentiable at t. Here we consider $\mathbb{C} \cong \mathbb{R}^2$ when evaluating the differentiability of u and v.

Definition 0.4.0.0.2
Let f be a function of C^1 type defined on U, where U is an open subset of $\mathbb{C}$. f is said to be complex differentiable, denoted as $\mathbb{C}$-differentiable, at $z_0 \in U$ provided that $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$. If f is $\mathbb{C}$-differentiable, $f'_{\mathbb{C}}(z_0) = \frac{\partial f}{\partial z}(z_0)$ is called the derivative of f at z_0.

Theorem 0.5
Let f be a function defined on U, where U is an open subset of $\mathbb{C}$. The followings are equivalent:

1. *f is holomorphic on U*
2. *f is of C^1 type on U, and $\frac{\partial f}{\partial \bar{z}} = 0$*
3. *f is $\mathbb{C}$-differentiable at each point in U, and $f'_{\mathbb{C}}$ is continuous.*

Corollary 0.5.1 (Differentiated Cauchy Integral Formula)
Let U be an open subset of $\mathbb{C}$, let $D \subseteq U$ be a closed disc with $z_0 \in Int(D)$, and let g be a holomorphic function defined on $U \setminus \{z_0\}$. Then we have:

$$g_{\mathbb{C}}^{(m)}(z_0) = \frac{m!}{2\pi i} \int_{z \in Bd(D)} \frac{g(z)\,dz}{(z - z_0)^{m+1}}$$

The function g is infinitely $\mathbb{C}$-differentiable. Here $0! = 1$.

Theorem 0.6 (Taylor's Theorem)
Let $z_0 \in \mathbb{C}$, let f be a holomorphic function defined on an open subset Ω of $\mathbb{C}$ that contains z_0.
For all $z \in \mathbb{C}$ that satisfies $|z - z_0| < \rho$ for some $d(z_0, Bd(\Omega)) > \rho > 0$, we have:

$$f(z) = \sum_{k=0}^{\infty} \frac{f_{\mathbb{C}}^{(k)}(z_0)}{k!}(z - z_0)^k$$

Here we denote $f^{(0)} := f$, $0! := 1$, and $(z - z_0)^0 := 1$ for $z = z_0$.

Theorem 0.7
Let f be a holomorphic function defined on a open subset Ω of $\mathbb{C}$. Denote $E := \bigcap_{k=0}^{\infty} (f_{\mathbb{C}}^{(k)})^{-1}(0)$. If we have Ω being connected, then we have either $E = \emptyset$ or $f(z) = 0$ for all $z \in \Omega$.

Corollary 0.7.1
Let Ω be a connected open subset of $\mathbb{C}$ that contains z_0, let f_1 and f_2 be holomorphic functions on Ω, with $f_1^{(k)}(z_0) = f_2^{(k)}(z_0)$ for all k. Then we have $f_1(z) = f_2(z)$ for all $z \in \Omega$.

Corollary 0.7.2
Let $Holo(A)$ denote the set of holomorphic functions defined on a set A. Let V be an open connected subset of $\mathbb{C}$, let U be a nonempty open proper subset of V. The restriction map from $Holo(V)$ to $Holo(U)$ is injective.

Theorem 0.8
For $z_0 \in \mathbb{C}$, consider the following power series $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$. If $\sum_{k=0}^{\infty} a_k(z - z_0)^k$ converges pointwise on $|z - z_0| < r$ for some $r > 0$. Then the function f is holomorphic on the set $\{z \in \mathbb{C} \mid |z - z_0| < r\}$.

Theorem 0.9
Let $z_0 \in \mathbb{C}$, let Ω be a connected open subset of $\mathbb{C}$ that contains z_0, let f be a holomorphic function defined on Ω and being not all zero on Ω. Then there exists $m \in \mathbb{N} \cup \{0\}$ such that, for $z \in \Omega$, $f(z) = (z - z_0)^m h(z)$ with some holomorphic function h defined on Ω and $h(z_0) \neq 0$.

In the settings of Theorem 0.9, m is called the order of f at z_0, denoted as $\mathrm{ord}_{z_0} f := m$.

Corollary 0.9.1
Let $z_0 \in \mathbb{C}$, let Ω be a connected open subset of $\mathbb{C}$ that contains z_0, let f be a holomorphic function defined on Ω with $f(z) \neq 0$ for some $z \in \Omega$. Then there exists $r > 0$ such that $f(z) \neq 0$ for all $z \in \Omega$ that satisfies $0 < |z - z_0| < r$.

Corollary 0.9.2
Let K be a compact set that is contained in some connected open subset of $\mathbb{C}$, let f be a holomorphic function defined on Ω with $f(z) \neq 0$ for some $z \in \Omega$. Then $\#(K \cap f^{-1}(0)) < \infty$.

Corollary 0.9.3
Let f_1 and f_2 be holomorphic functions on an open connected subset Ω of $\mathbb{C}$ with $f_1 = f_2$ on some infinite subset of a compact subset of Ω. Then $f_1 = f_2$ on Ω.

Corollary 0.9.3.1 (Persistence of Relations)
Let f_1 and f_2 be holomorphic functions defined on an open connected subset Ω of $\mathbb{C}$ that satisfies $\Omega \cap \mathbb{R} \neq \emptyset$. If with $f_1(z) = f_2(z)$ for all $z \in \Omega \cap \mathbb{R}$, then we have $f_1 = f_2$ on Ω.

Definition 0.9.3.1.1
A sequence on functions (f_j) defined on $\Omega \subseteq \mathbb{C}$ is said to converge almost uniformly to a function f defined on Ω provided that the sequence (f_j) converges uniformly to f on each compact subset K of Ω.

Theorem 0.10 (Weierstrass Convergence Theorem)
The limit of a almost uniformly convergent sequence of holomorphic functions is holomorphic.

Definition 0.10.0.0.1
A k-tensor f defined on a vector space V is symmetric provided that $f(\vec{v}_1, \cdots, \vec{v}_{i+1}, \vec{v}_i, \cdots, \vec{v}_k) = f(\vec{v}_1, \cdots, \vec{v}_i, \vec{v}_{i+1}, \cdots, \vec{v}_k)$
A k-tensor f defined on a vector space V is alternating provided that $f(\vec{v}_1, \cdots, \vec{v}_{i+1}, \vec{v}_i, \cdots, \vec{v}_k) = -f(\vec{v}_1, \cdots, \vec{v}_i, \vec{v}_{i+1}, \cdots, \vec{v}_k)$

Theorem 0.11
Let V be an n-dimensional vector space with a basis $(\vec{a}_1, \vec{a}_2, \cdots .\vec{a}_n)$. Let $I = (i_1, i_2, \cdots, i_k)$ be a k-tuple of integers from the set $\{1, 2, \cdots, n\}$. There exists a unique k-tensor Φ_I on V such that for every k-tuple $M = (m_1, m_2, \cdots, m_k)$ of integers from the set $\{1, 2, \cdots, n\}$, we have $\Phi_I(\vec{a}_{m_1}, \vec{a}_{m_2}, \cdots, \vec{a}_{m_k},) = 1$ if and only if $I = M$, and $\Phi_I(\vec{a}_{m_1}, \vec{a}_{m_2}, \cdots, \vec{a}_{m_k},) = 0$ otherwise.
For $f \in \mathscr{L}^k(V)$, we have $f = \sum_I f(\vec{a}_I)\Phi_I$, where we write $\vec{a}_I := (\vec{a}_{m_1}, \vec{a}_{m_2}, \cdots, \vec{a}_{m_k},)$.

For $f \in \mathscr{L}^k(V)$ and $g \in \mathscr{L}^l(V)$, $f \otimes g : V^{k+l} \to \mathbb{R} \quad (\vec{v}_1, \vec{v}_2, \cdots \vec{v}_{k+l}) \mapsto f(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k) \cdot g(\vec{v}_{k+1}, \vec{v}_{k+2}, \cdots, \vec{v}_{k+l})$
For $f \in \mathscr{L}^k(V)$, $h \in \mathscr{L}^m(V)$, and $g \in \mathscr{L}^l(V)$, and $c \in \mathbb{R}$, we have $(f \otimes g) \otimes h = f \otimes (g \otimes h)$,
$(c \cdot f) \otimes g = c \cdot (f \otimes g) = f \otimes (c \cdot g)$, $(f + g) \otimes h = f \otimes h + g \otimes h$, $f \otimes (g + h) = f \otimes g + f \otimes h$.

Let V and W be vector spaces, let $T : V \to W$ be a linear transformation. For $f \in \mathscr{L}^k(W)$, and $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k \in V$, we define $T^* f : V^k \to \mathbb{R} \quad (\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k) \mapsto f(T(\vec{v}_1), T(\vec{v}_2), \cdots, T(\vec{v}_k))$.
$T^*(f \otimes g) = T^* f \otimes T^* g$ for all $f, g \in \mathscr{L}^k(W)$. $(S \circ T)^* f = T^*(S^* f)$ for all $f \in \mathscr{L}^k(W)$

Theorem 0.12
Let V be a vector space, there exists a function $W : \mathscr{A}^k(V) \times \mathscr{A}^l(V) \to \mathscr{A}^{k+l}(V) \quad (f, g) \mapsto f \wedge g$ such that $f \wedge g \in \mathscr{A}^{k+l}(V)$ for $f \in \mathscr{A}^k(V)$, $g \in \mathscr{A}^l(V)$, and satisfies all of the followings:

1. *For $f \in \mathscr{A}^k(V)$, $g \in \mathscr{A}^l(V)$, and $h \in \mathscr{A}^m(V)$, we have $f \wedge (g \wedge h) = (f \wedge g) \wedge h$*
2. *For $f \in \mathscr{A}^k(V)$, $g \in \mathscr{A}^l(V)$, and scalar c, we have $(c \cdot f) \wedge g = c \cdot (f \wedge g) = f \wedge (c \cdot g)$*
3. *For $f, g \in \mathscr{A}^k(V)$ and $h \in \mathscr{A}^l(V)$, we have $h \wedge (f + g) = h \wedge f + h \wedge g$, and $(f + g) \wedge h = f \wedge h + g \wedge h$*
4. *For $f \in \mathscr{A}^k(V)$ and $g \in \mathscr{A}^l(V)$, we have $g \wedge f = (-1)^{kl} \cdot f \wedge g$*
5. *Given a finite basis of V, let $(\Phi_i \mid 1 \le i \le n)$ be the corresponding dual basis for V^*, and let $(\Psi_I \mid I$ is an ascending k-tuple of integers in $\{1, 2, \cdots, n\})$ be the corresponding family of elementary alternating tensors. For ascending k-tuple $I = (i_1, i_2, \cdots, i_k)$ of integers in $\{1, 2, \cdots, n\}$, we have $\Psi_I = \Phi_{i_1} \wedge \Phi_{i_2} \wedge \cdots \wedge \Phi_{i_k}$.*
6. *Let $T : V \to W$ be a linear transformation with W being a vector space, let f and g be alternating tensors on W, then we have $T^*(f \wedge g) = T^*f \wedge T^*g$.*

Let $[I]$ denote the set of ascending k-tuples of integers from $\{1, 2, \cdots, n\}$. A k-form defined on an open subset U of $\mathbb{R}^n$ is a continuous function $\omega : U \to \mathscr{A}^k(\mathbb{R}^n) \quad \vec{x} \mapsto \sum_{I \in [I]} b_I(\vec{x})\Psi_I$ where b_I are continuous functions from U to $\mathbb{R}$. The degree of a k-form is k, denoted as $\deg(\omega)$.

Let U be a subset of $\mathbb{R}^n$ and let V be a subset of $\mathbb{R}^l$, let $\Phi : U \to V$ be a C^1 function, let ω be a k-form defined on V, then $\Phi^*\omega$ is a k-form defined on U given by $\Phi^*\omega : U \to \mathscr{A}^k(U) \quad \vec{x} \mapsto (D\Phi(\vec{x}))^*\omega(\Phi(\vec{x}))$ where we have $(D\Phi(\vec{x}))^*\omega(\Phi(\vec{x})) : U^k \to \mathbb{R} \quad (\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_k) \mapsto \omega(\Phi(\vec{x}))(D\Phi(\vec{x})(\vec{u}_1), D\Phi(\vec{x})(\vec{u}_2), \cdots, D\Phi(\vec{x})(\vec{u}_k))$.

$$d\,(\alpha\, dx_1 + \beta\, dx_2) = \left(\frac{\partial \beta}{\partial x_1} - \frac{\partial \alpha}{\partial x_2}\right) dx_1 \wedge dx_2 \qquad d\left(\sum_j b_j(\vec{x}) dx_j\right) = \sum_{j<k} \left(\frac{\partial b_j}{\partial x_j} - \frac{\partial b_j}{\partial x_k}\right) dx_j \wedge dx_k$$

A k-form ω is sad to be closed provided that we have $d\omega = 0$.

Let U be a subset of $\mathbb{R}^k$ that is open in either $\mathbb{R}^k$ or $\mathbb{H}^k$, and let ω be a k-form defined on an open subset U of $\mathbb{R}^k$ given by $\omega : U \to \mathscr{A}^k(\mathbb{R}^k) \quad \vec{x} \mapsto f(\vec{x})\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$, with f being continuous function on U. Then $\int_U \omega := \int_U f$ whenever $\int_U f$ exists. Let Y be a parametrized k-manifold in $\mathbb{R}^n$ parametrized by $\alpha : U \to Y$, let ω be a k-form defined on open subset of $\mathbb{R}^n$ containing Y, we define $\int_{Y_\alpha} \omega := \int_U \alpha^*\omega$.

Lemma 0.12.1
Let U be a subset of $\mathbb{R}^l$ and let V be a subset of $\mathbb{R}^n$, let $\Phi : U \to V$ be a C^1 function, let ω be a k-form defined on V given by equation (W), we have $d(\Phi^*\omega) = \Phi^* d\omega$.

Let M be a k-manifold in $\mathbb{R}^n$. Given coordinate path $\alpha_i : U_i \to V_i$ on M for $i = 0, 1$, we say α_1, α_0 overlap if $V_0 \cap V_1 \ne \emptyset$. We say α_1, α_0 overlap positively provided that the transition function $\alpha_1^{-1} \circ \alpha_0$ is orientation preserving. Let M be a k-manifold in $\mathbb{R}^n$. M is said to be orientable provided that M can be covered by a collection of coordinate patches such that each pair of coordinate patches overlap positively, if they overlap at all. M is said to be non-orientable if it cannot be covered by such collection of coordinate patches. Given a collection of coordinate patches covering M that overlap positively, adjoin to this collection all other coordinate patches on M that overlap these patches positively, denote such collection as O. O is called an orientation on M. A coordinate patch α on M is said to be orientation preserving provided that α overlaps any one of the coordinate patches in O positively. Otherwise α is said to be orientation reversing.

Let M be an oriented 1-manifold in $\mathbb{R}^n$. Choose a coordinate patch $\alpha_{\vec{p}} : U \to V$ on M about $\vec{p}$ belonging to the given orientation of M, $\vec{T} : M \to \mathbb{R}^n \times \mathbb{R}^n \quad \vec{p} \mapsto (\vec{p}\,;\, \frac{D\alpha_{\vec{p}}(t_0)}{||D\alpha_{\vec{p}}(t_0)||})$, where $\alpha_{\vec{p}}(t_0) = \vec{p}$. $\vec{T}$ is called the unit tangent field corresponding to the orientation of M.

Let M be an oriented $(n-1)$-manifold in $\mathbb{R}^n$, let $\vec{p} \in M$, let $\alpha : U \to V$ be a coordinate patch on M about $\vec{p}$ belonging to the given orientation of M, denote $\alpha(\vec{x}) = \vec{p}$. Let $(\vec{p}; \vec{n}(\vec{p}))$ be a unit vector in the n-dimensional vector space $\mathscr{T}_{\vec{p}}(\mathbb{R}^n)$ that is orthogonal to the $(n-1)$-dimensional linear subspace $\mathscr{T}_{\vec{p}}(M)$ such that the matrix $\begin{bmatrix} \vec{n}(\vec{p}) & D\alpha(\vec{x}) \end{bmatrix}$ has positive determinant. $\vec{N} : M \mapsto \mathbb{R}^n \times \mathbb{R}^n \quad \vec{p} \mapsto (\vec{p}; \vec{n}(\vec{p}))$ is called the unit normal field to M corresponding to the orientation of M.

Let M be an n-manifold in $\mathbb{R}^n$. The natural orientation of M consists of all coordinate patches α on M for which $\det(D\alpha(\vec{x})) > 0$ for all $\vec{x}$ in the definition of domain of α.

Let M be an orientable k-manifold with nonempty manifold boundary ∂M. If k is even, the corresponding induced orientation of ∂M is the orientation obtained by restricting coordinate patches belonging to O. If k is odd, the corresponding induced orientation of ∂M is the opposite of the orientation of ∂M obtained by restricting coordinate patches belonging to O.

Let M be an oriented k-manifold in $\mathbb{R}^n$, let $\alpha : U \to V$ be a coordinate patch on M belonging to the given orientation, with $\alpha(\vec{q}) = \vec{p} \in M$, let ω be a k-form defined on an open subset of $\mathbb{R}^n$ containing M. We can write $\alpha^*\omega = f(\vec{x})\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k$ for some 0-form f defined on the definition of domain of ω. ω is said to be positive for M at $\vec{p}$ provided that $f(\vec{p}) > 0$, ω is said to be negative for M at $\vec{p}$ provided that $f(\vec{p}) < 0$, and ω is said to be integral for M at $\vec{p}$ provided that $f(\vec{p}) = 0$. M is integral manifold for ω provided that ω is integral for M at $\vec{p}$ for all $\vec{p} \in M$.

Theorem 0.13 (Theorem 36.2 on Munkres)
Let M be a compact oriented k-manifold in $\mathbb{R}^n$, let ω be a k-form defined in a open subset of $\mathbb{R}^n$ containing M, and let λ be the scalar function on M defined by $\lambda : M \to \mathbb{R} \quad \vec{p} \mapsto \omega(\vec{p})((\vec{p}; \vec{a}_1), (\vec{p}; \vec{a}_2), \cdots, (\vec{p}; \vec{a}_k))$ where, for $\vec{p} \in M$, the family $((\vec{p}; \vec{a}_1), (\vec{p}; \vec{a}_2), \cdots, (\vec{p}; \vec{a}_k))$ forms an orthonormal frame in the linear space $\mathscr{T}_{\vec{p}}(M)$ belonging to the given orientation of M. Then λ is continuous, and we have $\int_M \omega = \int_M \lambda\, dV$.

Lemma 0.13.1 (Lemma 25.2 on Munkres)
Let M be a compact k-manifold in $\mathbb{R}^n$ of class C^r. Given a covering $\mathscr{C}$ of M by coordinate patches, there exists a finite collection of C^∞ functions from $\mathbb{R}^n$ to $\mathbb{R}$, denoted as $P = \{\phi_1, \phi_2, \cdots, \phi_l\}$, such for each $1 \le i \le l$, ϕ_i has compact support and there exists a coordinate patch $\alpha_i : U_i \to V_i$ in the collection $\mathscr{C}$ such that we have $supp(\phi_i) \cap M \subseteq V_i$, $\phi_i(\vec{x}) \ge 0$ for all $\vec{x} \in \mathbb{R}^n$, and $\sum_{i=1}^l \phi_i(\vec{x}) = 1$ for all $\vec{x} \in M$.

Definition 0.13.1.0.1
Let M be a compact oriented k-manifold in $\mathbb{R}^n$, along with orientation O on M. Take $\mathscr{C}$ to be a finite collection of coordinate patches in O that cover M, denoted as $C = \{\alpha_1, \alpha_2, \cdots, \alpha_N\}$. One can use partition of unity to write $\omega = \sum_{j=1}^{N} \omega_j$ such that the support of each ω_j is a subset of V_j, where V_j is the codomain of a coordinate patch $\alpha_j : U_j \to V_j$ in C. Here we define $\int_M \omega = \sum_{j=1}^{N} (\int_{(V_j)_{\alpha_j}} \omega_j)$

Theorem 0.14 (The Generalized Stokes' Theorem)
Let $k > 1$, let M be a compact oriented k-manifold in $\mathbb{R}^n$, with ∂M having the induced orientation if ∂M is not empty, let ω be a $(k-1)$-form defined in an open set of $\mathbb{R}^n$ containing M, then we have $\int_M d\omega = \int_{\partial M} \omega$ if ∂M is not empty, and we have $\int_{\partial M} \omega = 0$ if ∂M is empty.

Exterior Calculus	Vector Calculus
Exterior derivative operator d	Del operator $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$
0-form k define on $\mathbb{R}^2$	Scalar field k of C^1 type defined on $\mathbb{R}^2$
1-forms $\omega = \alpha\, dx + \beta\, dy$	Vector field $\vec{F}$
2-forms $f\, dx \wedge dy$ defined on $\mathbb{R}^2$	Scalar field f
1-form ω wedged with 1-form η	Scalar field $\det\left(\left[\vec{F}_1 \vec{F}_2\right]\right)$ with $\vec{F}_1, \vec{F}_2$ being vector fields
$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$	Gradient of f, $\text{grad}(f) \coloneqq \nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$
$d(\alpha\, dx + \beta\, dy) = \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\right) dx \wedge dy$	Curl of $\vec{F}$, $\text{curl}(\vec{F}) \coloneqq \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\right)$
$\int_{M_1} \omega$	$\int_{M_1} \left\langle \vec{F}, d\vec{l} \right\rangle = \int_{M_1} \left\langle \vec{F}, \vec{T} \right\rangle dV$ [1]
$\int_{M_1} df = \Delta_{M_1} f$	$\int_{M_1} \left\langle \nabla f, \vec{T} \right\rangle = \Delta_{M_1} f$
$\int_{M_2} f\, dx \wedge dy = \int_{M_2} f$	$\int_{M_2} f$
$\int_{\partial M_2} \omega = \int_{M_2} d\omega$	Circulation of $\vec{F}$ along ∂M_2, $\int_{M_2} \text{curl}(\vec{F})$

[1] Here we define: $d\vec{l} \coloneqq (dx, dy)$. Since we have $\vec{F}(\vec{x}) = (\alpha(\vec{x}), \beta(\vec{x}))$, so we have $d\vec{l} = \vec{T}\, dV$.

Lemma 0.14.1 (Lemma 38.1 on Munkres)
Let M be a compact oriented 1-manifold in $\mathbb{R}^n$, and let $\vec{T}$ be the unit tangent vector to M corresponding to the given orientation of M. Let $\vec{F}$ be a vector field defined in $\mathbb{R}^n$ and let ω be the 1-form corresponds to $\vec{F}$. Then $\int_M \omega = \int_M \left\langle \vec{F}, \vec{T} \right\rangle dV$.
Let M be a compact oriented $(n-1)$-manifold in $\mathbb{R}^n$, and let $\vec{N}$ be the corresponding unit normal vector field, let $\vec{F}$ be a vector field defined on open $A \subseteq \mathbb{R}^n$ that contains M, and let ω be the $(n-1)$-form corresponds to $\vec{F}$, then $\int_M \omega = \int_M \left\langle \vec{F}, \vec{N} \right\rangle dV$.
Let M be a compact n-manifold in $\mathbb{R}^n$, oriented naturally, and let $\omega = h\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ be an n-form defined on an open set of $\mathbb{R}^n$ containing M, with h being the scalar field corresponds to ω, then $\int_M \omega = \int_M h\, dV$.

Theorem 0.15 (The Divergence Theorem)
Let M be a compact n-manifold in $\mathbb{R}^n$, let $\vec{N}$ e the unit normal vector field to ∂M that points outwards from M, and let $\vec{F}$ be a vector field defined on an open subset of $\mathbb{R}^n$ containing M, then we have $\int_M div(\vec{F})\, dV = \int_{\partial M} \left\langle \vec{F}, \vec{N} \right\rangle dV$

Theorem 0.16 (Stokes' Theorem for 2-manifold in $\mathbb{R}^3$)
Let M be a compact oriented 2-manifold in $\mathbb{R}^3$, let $\vec{N}$ be a unit normal field to M corresponding to the orientation of M, and let $\vec{F}$ be a vector field of C^∞ type defined on an open subset of $\mathbb{R}^3$ containing M. If ∂M is empty, then $\int_M \left\langle curl(\vec{F}), \vec{N} \right\rangle dV = 0$. If ∂M is nonempty, let $\vec{T}$ be the unit tangent vector field to ∂M chosen such that $\vec{N}(\vec{p}) \times \vec{T}(\vec{p})$ points into M from $\vec{p} \in \partial M$, then $\int_M \left\langle curl(\vec{F}), \vec{N} \right\rangle dV = \int_{\partial M} \left\langle \vec{F}, \vec{N} \right\rangle dV$

Proposition 0.16.1
Let ω be an alternating k-tensor with k being an odd number. For any alternating m-tensor $\hat{\omega}$, we have $\omega \wedge \hat{\omega} \wedge \omega = 0$.

Theorem 0.17 (Cauchy's Estimate)
Let $z_0 \in \mathbb{C}$ be given, let $r > 0$ be given, let f be a holomorphic function defined on $D = \{z \in \mathbb{C} \mid |z - z_0| < r\}$ with $|f(z)| < M$ for all $z \in D$. Then we have $|f'_{\mathbb{C}}(z_0)| \leq \frac{M}{r}$.

Theorem 0.18 (Liouville's Theorem)
Let f be a holomorphic function on $\mathbb{C}$. If $f(\mathbb{C})$ is a bounded set, then f is a constant function.

1-form ω	Integrating factor $B(x, y)$
$-\alpha(x)\beta(y)\, dx + dy$	$1/\beta(y)$
$-(\beta(x)y + \gamma(x))\, dx + dy$	$\exp(-\int \beta(x)\, dx)$
$-\beta(y/x)\, dx + dy$	$1/(y - x\beta(y/x))$

Lemma 0.18.1
Let M be a nonempty compact orientable k-manifold in $\mathbb{R}^n$ with nonempty manifold boundary ∂M. There exists an $(k-1)$-form of C^∞ type defined on $\mathbb{R}^n$ such that $\int_{\partial M} \omega \neq 0$.

Lemma 0.18.2
Let M be a k-manifold in $\mathbb{R}^n$ with nonempty manifold boundary ∂M, and let ω be an $(k-1)$-form defined on an open subset of $\mathbb{R}^n$ containing M. If $R : M \to \partial M$ is a C^1 retraction, then ∂M is integral for the $(k-1)$-form $\omega - R^*\omega$.

Lemma 0.18.3
Let M be a k-manifold in $\mathbb{R}^n$ with nonempty manifold boundary ∂M. Let $R : M \to \partial M$ be a function of C^1 type and let η be a k-form defined on an open subset of $\mathbb{R}^n$ containing ∂M, then M is an integral for $R^*\eta$.

Theorem 0.19 (Non-retraction Theorem)
Let M be a nonempty compact orientable k-manifold in $\mathbb{R}^n$. There is no retraction of C^1 type from M to ∂M.

Theorem 0.20 (Brouwer Fixed Point Theorem)
Let $B^n := \{\vec{x} \in \mathbb{R}^n \mid ||\vec{x}|| \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$. If $f : B^n \to B^n$ is a function of C^1 type, then there exists $\vec{x} \in B^n$ such that $f(\vec{x}) = \vec{x}$, and such $\vec{x}$ is called a fixed point of f.

Theorem 0.21
Let B^n denote the closed unit ball in $\mathbb{R}^n$. If $\vec{v}$ points inwards at all points $\vec{p}$ on the boundary ∂B^n, then there is an equilibrium point in B^n.

Theorem 0.22 (Rectification Theorem)
Let $\vec{x}_0 \in A$ where A is an open subset of $\mathbb{R}^n$, let v be a vector field of C^∞ type defined on A, with $v(\vec{x}_0) \neq \vec{0}$. Then there exists an open neighborhood U of $\vec{x}_0$ contained in A, and a C^∞-diffeomorphism $\alpha : U \to V$ such that $D\alpha(\vec{x})(v(\vec{x})) = \vec{e}_1$, where $\vec{e}_1 = (1, 0, 0, \cdots, 0)$ is the first element in the standard basis of $\mathbb{R}^n$.

$F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ defined for $\vec{z} \in \mathbb{R}^n$, $t \in \mathbb{R}$, such that $D_{n+1}F(\vec{z}, t) = v(F(\vec{z}, t))$ and $F(\vec{z}, 0) = \vec{z}$. Denote $F^s(\vec{z}, t) := F(\vec{z}, t+s)$, then $D_{n+1}F^s(\vec{z}, t) = v(F^s(\vec{z}, t))$, and $F^s(\vec{z}, 0) = F(\vec{z}, s)$, so we get $F(\vec{z}, t+s) = F^s(\vec{z}, t) = F(F(\vec{z}, s), t)$. Then $g^t : \mathbb{R}^n \to \mathbb{R}^n \quad \vec{z} \mapsto F(\vec{z}, t)$, has the property $g^{t+s}(\vec{z}) = g^t(g^s(\vec{z}))$ for $\vec{z} \in \mathbb{R}^n$, $t, s \in \mathbb{R}$, and g^0 being the identity transformation.

Theorem 0.23
Let M be a closed k-manifold without boundary in $\mathbb{R}^n$ of C^∞ class, let v be a vector field that is tangent to M at all $\vec{p} \in M$, that is, we have $v(\vec{p}) \in \mathcal{T}_{\vec{p}}(M)$ for all $\vec{p} \in M$. Then each $g^t|_M$ induced by v belongs to Diffeo(M).

Corollary 0.23.1
Let M be a compact k-manifold without boundary in $\mathbb{R}^n$ of C^∞ class, and let U be an open subset of $\mathbb{R}^n$ containing M. For vector field $v \in C^2(U, \mathbb{R}^n)$ such that v is tangent to M, each $g^t|_M$ induced by v belongs to Diffeo(M), and there exists Lipchitz $\hat{v} \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ such that $\hat{v} = v$ on some neighborhood of M, here the flow for $\hat{v}$ along M is also a flow for v along M.

Theorem 0.24 (Hairy Billiard Ball Theorem)
Any tangential vector field of C^1 type defined on an even-dimensional sphere S^{2n} vanishes at some point $\vec{p}$ in S^{2n}. There is no $v : S^{2n} \to \mathbb{R}^{2n+1} \setminus \{0\}$ with $\langle v(\vec{p}), \vec{p} \rangle = 0$ for all $\vec{p} \in S^{2n}$

Theorem 0.25
Let v be a C^1 inward-pointing vector field on B^n, then v must vanish some point on B^n.

Theorem 0.26 (Cauchy Integral Theorem)
Let M be a naturally oriented compact 2-manifold in $\mathbb{C}$, let $f \in C^1(M, \mathbb{C})$ be holomorphic on $M \setminus \partial M$, and let ∂M obtain the induced orientation from M. then we have $\int_{\partial M} f\, dz = 0$

Theorem 0.27 (Residue Theorem)
Let M be a compact 2-manifold in $\mathbb{C}$, let $E = \{z_1, z_2, \cdots, z_k\} \subseteq M \setminus \partial M$, let $f \in C^1(M \setminus E, \mathbb{C})$ be holomorphic on $M \setminus (\partial M \cup E)$, then we have the following holds:

$$\int_{\partial M} f\, dz = 2\pi i \sum_{j=1}^{k} Res(f\, dz, z_j)$$

Theorem 0.28 (Rouche's Theorem)
Let M be a compact 2-manifold in Ω, where Ω is an open subset of $\mathbb{C}$, let f, h be holomorphic functions on Ω, with $|h(x)| < |f(x)|$ for $x \in \partial M$, then the number of zeros of $f + h$ in M is equal to the number of zeros of f in M.

Definition 0.28.0.0.1
Let $f : \mathbb{R} \to \mathbb{C}$ be a bounded function such that the set $\{x \in \mathbb{R} \mid f \text{ is discontinuous at } x\}$ has measure zero, and ext $\int_{\mathbb{R}} |f| < \infty$. The Fourier Transform of f, denoted as $\hat{f}$, is the function defined by $\hat{f} : \mathbb{R} \to \mathbb{R} \quad t \mapsto$ ext $\int_{\mathbb{R}} f(x)e^{-ixt}\, dx$

Definition 0.28.0.0.2
Let M be compact oriented 1-manifold without boundary in $\mathbb{R}^2 \simeq \mathbb{C}$. We define $\mathbb{W}_M : \mathbb{C} \setminus M \to \mathbb{C} \quad z \mapsto \frac{1}{2\pi i} \int_{\zeta \in M} \frac{d\zeta}{\zeta - z}$

For compact 1-manifold $M \subseteq \mathbb{R}$, there exists $U \subseteq \mathbb{C}$ containing M such that $U \setminus M$ has two components L, R. Denote the winding number of $w \in L$ as $\mathbb{W}(L)$ and denote the winding number of $z \in R$ as $\mathbb{W}(R)$, we have: $\mathbb{W}(L) = \mathbb{W}(R) + 1$

Theorem 0.29 (Residue theorem for Winding Numbers)
Let M be a compact 1-manifold without boundary in $\mathbb{C}$, let K denote the union of the bounded components on $\mathbb{C} \setminus M$, let U be an open subset of $\mathbb{C}$ containing $M \cup K$, let $\{z_1, z_2, \cdots, z_m\} \in U \setminus M$, and let f be a function being holomorphic on $U \setminus \{z_1, z_2, \cdots, z_m\}$, then we have:

$$\int_M f\, dz = 2\pi i \sum_{j=1}^{m} \mathbb{W}_M(z_j) \cdot Res(f\, dz, z_j)$$

Let U be a subset of $\mathbb{R}^k$, let $f : U \to \mathbb{R}^n$ be a function. f is called an immersion provided that $Df(\vec{x})$ is injective for all $\vec{x} \in U$. f is called a submersion provided that $Df(\vec{x})$ is surjective for all $\vec{x} \in U$.
For immersion $f : \mathbb{R} \to \mathbb{R}^2$. $\frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}} = \kappa_f =$ curvature of f

Theorem 0.30
Let Ω be a compact 2-manifold in $\mathbb{R}^2$, we have area$(\Omega) \leq \frac{1}{4\pi} (\text{length}(\partial\Omega))^2$

Let A be an open subset of $\mathbb{R}^k$. A Riemannian metric G on A is a smooth function defined by: $G : A \to \text{Pos}(k)$. Given $\psi \in C^2([a, b], A)$, we define the G-length of Y_ψ, or the length of Y_ψ in G metric, as $l_G(Y_\psi) := \int_a^b \sqrt{\psi'(t)^T \cdot G(\psi(t)) \cdot \psi'(t)}\, dt$

Theorem 0.31
Vertical segment and Arc in $\mathbb{H}^2_+$ of circle centered on x-axis minimize Poincare length among curves with the same end points.

Let A be an open subset of $\mathbb{R}^n$, $\Omega^k(A) := \{\omega \mid \omega \text{ is a } k\text{-form of } C^\infty \text{ type defined on } A\}$,
$Cl^k(A) := \{\omega \in \Omega^k(A) \mid d\omega = 0\}$, $E^k(A) := \{d\eta \mid \eta \in \Omega^{k-1}(A)\}$, $H^k_{dR}(A) := Cl^k(A)/E^k(A)$

Theorem 0.32
Let E be a nonempty affine subset of $\mathbb{R}^n$, we have: $\dim(H^k_{dR}(\mathbb{R} \setminus E)) = \begin{cases} 1 & k = n - \dim(E) \text{ or } k = 0 \\ 0 & \text{otherwise} \end{cases}$
with an exception where $\dim(E) = n - 1$ and $k = 0$, in which case $\dim(H^0_{dR}(\mathbb{R}^n \setminus E)) = 2$.

Corollary 0.32.1
Let E_1 and E_2 be nonempty affine subsets of $\mathbb{R}^n$, and if $R^n \setminus E_1$ is diffeomorphic to $\mathbb{R}^n \setminus E_2$, then we have $\dim(E_1) = \dim(E_2)$.

Definition 0.32.1.0.1
Let M be a manifold in $\mathbb{R}^n$, and let U be an open subset of $\mathbb{R}^n$ containing M such that M is closed in U. $\Omega^k(M) := \Omega^k(U)/\{\omega \in \Omega^k(U) \mid M \text{ is integral for } \omega\}$. Equivalently, $\Omega^k(M)$ is a set consisting of smooth ω defined on U that maps $\vec{p} \in M$ to $\omega(\vec{p}) \in \mathscr{A}^k(\mathscr{T}_{\vec{p}}(M))$.

Definition 0.32.1.0.2
For s-manifold M in $\mathbb{R}^n$, $Cl^k(M) := \ker(d_k)$, $E^k(M) := Im(d_{k-1}) \subseteq Cl^k(M)$, $H^k_{dR}(M) := Cl^k(M)/E^k(M)$

Theorem 0.33
Let M be a compact oriented s-manifold without boundary, $\dim(H^s_{dR}(M)) = \#\{\text{connected componenents of } M\} = \dim(H^0_{dR}(M))$

Theorem 0.34
Let M be a compact connected non-orientable s-manifold, we have $H^s_{dR}(M) = 0$.
Let M be a non-compact connected s-manifold, then $H^s_{dR}(M) = 0$.
Let M be a compact connected s-manifold with $\partial M \neq 0$, then $H^s_{dR}(M) = 0$.
Let M be a compact oriented s-manifold without boundary, then $H^s_{dR}(M) \neq 0$.

Definition 0.34.0.0.1
Consider an open subset A of $\mathbb{R}^n$, for ascending k-tuple I of integers in $\{1, 2, \cdots, n\}$, let I' be the $(n-k)$-tuple complementary to I, we define the Hodge star operator $*$ as the following:

$$* : \Omega^k(A) \to \Omega^{n-k}(A) \qquad \left(\sum_{[I]} b_I(\vec{x}) dx_I \right) \mapsto \sum_{[I]} sgn(I, I')\, b_I(\vec{x}) dx_{I'}$$

Here the notation $sgn(I, I')$ denotes $sgn(\sigma_{II'})$, where $\sigma_{II'}$ is a permutation that sorts the concatenated n-tuple (I, I').

Consider 1-form $\alpha\, dx + \beta\, dy + \gamma\, dz$, 2-form $\alpha\, dy \wedge dz + \beta\, dz \wedge dx + \gamma\, dx \wedge dy$.

1. $*(\alpha\, dx + \beta\, dy + \gamma\, dz) = \alpha dy \wedge dz + \beta dz \wedge dx + \gamma dx \wedge dy$
2. $*(\alpha\, dy \wedge dz + \beta\, dz \wedge dx + \gamma\, dx \wedge dy) = \alpha\, dx + \beta\, dy + \gamma\, dz$

Lemma 0.34.1
For 0-form f and k-forms ω, l-form $\widetilde{\omega}$ defined on an open subset of $\mathbb{R}^n$, denote $\omega = \sum_{[I]} b_I(\vec{x}) dx_I$, $\widetilde{\omega} = \sum_{[J]} \widetilde{b}_J(\vec{x}) dx_J$.
$*(f\omega) = f * \omega \qquad *(*(\omega)) = (-1)^{k(n-k)}\omega \qquad *(\omega_1 + \omega_2) = *(\omega_1) + *(\omega_2)$
$\omega \wedge *(\omega) = \sum_{[I]} b_I^2 dx_1 \wedge \cdots \wedge dx_n$ *If $\deg(\omega) = \deg(\widetilde{\omega})$, then $\omega \wedge *(\widetilde{\omega}) = \widetilde{\omega} \wedge *(\omega) = \sum_{[I]} b_I \widetilde{b}_I\, dx_1 \wedge \cdots \wedge dx_n$*

Theorem 0.35
Let A, B be open subsets of $\mathbb{R}^n$, if $\Phi : A \to B$ defines an orientation preserving isometry, then we have $\Phi^*(*(\omega)) = *(\Phi^*(\omega))$ holds for all k-forms ω defined on an open subset of $\mathbb{R}^n$ containing B, with $k \leq n$.

Definition 0.35.0.0.1
Let A be an open subset of $\mathbb{R}^n$, $\Delta : \Omega^k(A) \to \Omega^k(A) \qquad \omega \mapsto (-1)^{kn} * d * d\omega + (-1)^n d * d * \omega$

For k-form $\omega = \sum_I b_I dx_I$, we have $\Delta\left(\sum_I b_I dx_I\right) = \sum_I (\Delta b_I)\, dx_I$. For 0-form f, $\Delta f = 0 \iff d{*}df = 0 \iff \sum_j D_j D_j = 0$.

Lemma 0.35.1 (Green's First Identities)
Let M be a compact n-manifold in $\mathbb{R}^n$, let $f, g \in C^2(M, \mathbb{R})$, then we have:

$$\int_M f\, \Delta g = \int_M f * \Delta g = \int_M f d * dg = \int_M d(f \wedge *dg) - df \wedge *dg = \int_{\partial M} f \wedge *dg - \int_M df \wedge *dg == \int_{\partial M} f \wedge *dg - \int_M \langle df,\, dg \rangle$$

Corollary 0.35.1.1 (Green's Second Identity)
Let M be a compact n-manifold in $\mathbb{R}^n$, let $f, g \in C^2(M, \mathbb{R})$, then we have: $\int_M (f\Delta g - g\Delta f) = \int_{\partial M} (f * dg - g * df)$

For all $f \in C^2(\bar{B}_1(\vec{0}), \mathbb{R})$, with $n > 2$. We have $\text{avg}_{||\vec{x}||=1} f = f(\vec{0}) + \frac{\int_{\bar{B}_1(\vec{0}) \setminus \{0\}} \left((||\vec{x}||^{2-n} - 1)\Delta f \right)}{(n-2) V_{n-1}(S^{n-1})}$, $\text{avg}_A f = \frac{\int_A f}{V(A)}$

Theorem 0.36 (Gauss' Mean Value Theorem)
Let $f \in C^2(A, \mathbb{R})$ where A is an open subset of $\mathbb{R}^n$, f is harmonic on A if and only if the mean value property holds for all closed balls in A: $avg_{||\vec{x}-\vec{x}_0||=r} f = f(\vec{x}_0)$. Here $\{\vec{x} \mid ||\vec{x} - \vec{x}_0|| \leq r\} \subseteq A$ is a closed ball of radius r centered at $\vec{x}_0 \in A$.

Corollary 0.36.1
Let $\vec{a} \in \mathbb{R}^n$, let $f \in C^2(\bar{B}_r(\vec{a}), \mathbb{R})$, then we get the followings:

1. *If $\Delta f(\vec{x}) \geq 0$ for $\vec{x} \in \bar{B}_r(\vec{a})$, and $\exists\, \vec{p} \in \bar{B}_r(\vec{a})$ such that $\Delta f(\vec{p}) > 0$, then $avg_{\partial \bar{B}_r(\vec{a})} f > f(\vec{a})$*
2. *If $\Delta f(\vec{x}) \leq 0$ for $\vec{x} \in \bar{B}_r(\vec{a})$, and $\exists\, \vec{p} \in \bar{B}_r(\vec{a})$ such that $\Delta f(\vec{p}) < 0$, then $avg_{\partial \bar{B}_r(\vec{a})} f < f(\vec{a})$*
3. *If $\Delta f(\vec{x}) = 0$ for $\vec{x} \in \bar{B}_r(\vec{a})$, then $avg_{\partial \bar{B}_r(\vec{a})} f = f(\vec{a})$*

Lemma 0.36.2
For k-forms μ, ζ defined on $\mathbb{R}^4$, and 0-form f defined on $\mathbb{R}^4$, we get $\circledast(f\mu) = f \circledast \mu$, $\circledast(\mu + \zeta) = \circledast\mu + \circledast\zeta$, $\circledast\circledast\mu = (-1)^{1+\deg(\mu)}\mu$

Definition 0.36.2.0.1
For k-form μ defined on $\mathbb{R}^4$, we defined $\Box\mu$ as $\Box\mu := -\circledast d \circledast d\mu - d \circledast d \circledast \mu$

Theorem 0.37
Let $M \subseteq \mathbb{R}^n$ be an oriented k-manifold. There exists a k-form $\omega_{\vec{p}}$ defined on some open neighborhood of M with the property that ω is positive at every point of M.

Riemannian metric G_f induced by f. Let $\alpha = 1 + (D_1 f)^2$, $\beta = D_1 f \cdot D_2 f$, $\gamma = 1 + (D_2 f)^2$. $\text{length}(Y_\eta) = \int_a^b \sqrt{\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}}$

Theorem 0.38 (Borsuk-Ulam Theorem for Low-dimensional Smooth Case)
For all $f \in C^1(S^2, \mathbb{R}^2)$, there exists some $\vec{x} \in S^2$ such that $f(-\vec{x}) = f(\vec{x})$.

$$\int_0^{2\pi} \sin^2(\theta) = \int_0^{2\pi} \cos^2(\theta) = \pi \qquad (-1)^{k+1} \int_M d\omega \wedge \eta = \int_M \omega \wedge d\eta \qquad \int u\, dv = uv - \int v\, du$$