$$\vec{p} = m\vec{v} \qquad \vec{v} = \vec{\omega} \times \vec{r} \qquad \vec{l} = I\vec{\omega} = \vec{r} \times \vec{p} \qquad \vec{N} = \frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} \qquad U = \frac{1}{2}I\omega^2 \qquad I = \int_V r^2\, dm$$

Given $m\frac{d^2x}{dx^2} = F(x)$, $f(x_0) = 0$, and $F'(x_0) > 0$. For small deviations from x_0: $x(t) = x_0 + A\cos\left(\sqrt{\frac{F'(x_0)}{m}}\; t + \phi\right)$

In mathematical statement, Hamilton's Principle states that $I = \int_{t_1}^{t_2} \mathscr{L}\, dt = \int_{t_1}^{t_2} (T - U)\, dt$ where we want to minimize I. We write $\mathscr{L}(q_i, \dot{q}_i; t)$ and $\mathscr{H}(q_1, p_i; t)$. Denote the constraint in algebraic expression $f(q_i; t) = 0$

$$\frac{\partial \mathscr{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_i} = 0 \qquad \begin{cases} \frac{\partial \mathscr{L}}{\partial q_i} \text{ is called the generalized force component} \\ \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_i} \text{ is called the kinetic part} \end{cases} \qquad \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right) + \sum_k \lambda_k \frac{\partial f_k}{\partial q_i} = 0$$

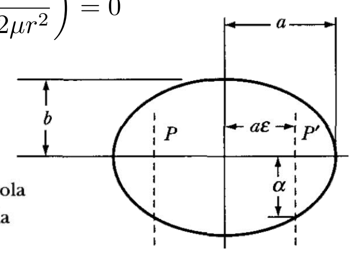
$$Q_i := \sum_{k=1}^{m} \lambda_k \frac{\partial f_k}{\partial q_i} \qquad p_i := \frac{\partial \mathscr{L}}{\partial \dot{q}_i} \qquad \mathscr{H} := \sum_i \dot{q}\frac{\partial \mathscr{L}}{\partial \dot{q}} - \mathscr{L} \qquad \dot{q}_i = \frac{\partial \mathscr{H}}{\partial p_i} \qquad -\dot{p}_i = \frac{\partial \mathscr{H}}{\partial q_i} \qquad \frac{\partial \mathscr{H}}{\partial t} = -\frac{\partial \mathscr{L}}{\partial t}$$

Q_i is the generalized constrain force components. p_i is the generalized momentum associated with q_i. If p_i is invariant under time, the associated q_i is cyclic, and the generalized momentum $\frac{\partial \mathscr{L}}{\partial \dot{q}_i} = p_i$ is the canonical momentum, or conjugate momentum. If the kinetic energy T is quadratic in $\dot{q}_i$, and $U = U(q_i)$, then $\mathscr{H}$ gives the total energy of the system.
In the center of mass frame:

$$\mathscr{L} = \frac{\mu}{2}|\dot{\vec{r}}|^2 - U(r) = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - U(r) \qquad E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r) = \frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r)$$

d_1, d_2 are the distance of the two particles to the center of mass. $l/(2\mu r^2)$ is the angular momentum barrier.

$$l = m_1 d_1^2\dot{\theta} + m_2 d_2^2\dot{\theta} = \mu r^2\dot{\theta} = \text{constant} \qquad \frac{d^2}{d\theta^2}\frac{1}{r} + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r) \qquad \mu\ddot{r} + \frac{\partial}{\partial r}\left(U(r) + \frac{l^2}{2\mu r^2}\right) = 0$$



$$\int_0^t dt' = \int_{r(0)}^{r(t)} \left(\frac{2}{\mu}\left(E - U(r') - \frac{l^2}{2\mu(r')^2}\right)\right)^{1/2} dr' \qquad \theta(t) - \theta(0) = \int_0^t \frac{1}{\mu(r(t'))^2}\, dt'$$

$$\theta(r) = \int_{r(0)}^{r(t)} \frac{d\theta}{dt}\frac{dt}{dr}\, dr' = \pm\int_{r(0)}^{r(t)} \frac{l/(\mu(r')^2)\, dr'}{\sqrt{\frac{2}{\mu}(E - U(r')) - \left(\frac{l^2}{\mu r'}\right)^2}}$$

$\varepsilon > 1$,	$E > 0$	Hyperbola
$\varepsilon = 1$,	$E = 0$	Parabola
$0 < \varepsilon < 1$,	$V_{\min} < E < 0$	Ellipse
$\varepsilon = 0$,	$E = V_{\min}$	Circle

ASSUME $U(r) = -k/r$. ϵ is the eccentricity of the two-body system, u_0 is constant, $u = 1/r$, Kepler's First Law states:

$$\alpha := \frac{l^2}{\mu k} \qquad \epsilon = \frac{l^2 u_0}{\mu k} \qquad \frac{\alpha}{r} = 1 + \epsilon\cos(\theta) \qquad \epsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}} \qquad E = \frac{1}{2}\frac{l^2}{\mu}\frac{(1+\epsilon)^2}{\alpha^2} - \frac{k(1+\epsilon)}{\alpha}$$

$$u = \frac{\mu k}{l^2}\left(\frac{l^2 u_0}{\mu k}\cos(\theta) + 1\right) \qquad \alpha = a(1+\epsilon)(1-\epsilon) \qquad a = \frac{\alpha}{1-\epsilon^2} = \frac{k}{2|E|} \qquad b = \sqrt{\alpha a} = \frac{\alpha}{\sqrt{1-\epsilon^2}} = \frac{l}{\sqrt{2\mu|E|}}$$

ASSUME $U(r) = -k/r$. Kepler's Second and Third Law:

$$\frac{dA}{dt} = \frac{r^2}{2}\frac{d\theta}{dt} = \frac{r^2\dot{\theta}}{2} = \frac{l}{2\mu} = \text{constant} \qquad \tau = \frac{2\mu}{l}A = \frac{2\mu}{l}(\pi ab) = \pi k\sqrt{\frac{\mu}{2}}|E|^{-3/2} \qquad \tau^2 = \frac{4\pi^2}{k}\mu a^3$$

For the transfer from the ellipse to the circular orbit of radius r_2, $\Delta v_2 = v_2 - v_{t2}$.

$$v_{t2} = \sqrt{\frac{2k}{mr_2}\left(\frac{r_1}{r_1 + r_2}\right)} \qquad T_t = \frac{\tau_t}{2} = \pi\sqrt{\frac{m}{k}}a_t^{3/2}$$

ASSUME INELASTIC COLLISION of m_1 and m_2. Center of mass parameters: $\vec{R}, \vec{V}, M$. Initial KE T_0 in LAB.
Moving particle: initial speed u_1 in LAB, initial speed u_1' in CM, final speed v_1 in LAB, final speed v_1' in CM.
Particle at rest in LAB: initial speed u_2 in LAB, initial speed u_2' in CM, final speed v_2 in LAB, final speed v_2' in CM.

$$\vec{R} = \frac{\int_M \vec{r}\, dm}{\int_M dm} \qquad \vec{F}_{ext} = M\frac{d\vec{V}_{CM}}{dt} = \frac{d\vec{P}_{CM}}{dt} \qquad \vec{V} = \frac{m_1\vec{u}_1}{m_1 + m_2} = -\vec{u}_2' \qquad \frac{V}{v_1'} = \frac{m_1 u_1/(m_1 + m_2)}{m_2 u_1/(m_1 + m_2)} = \frac{m_1}{m_2}$$

$$\tan(\psi) = \frac{\sin(\theta)}{\cos(\theta) + (V/v_1')} = \frac{\sin(\theta)}{\cos(\theta) + (m_1/m_2)} \qquad \tan(\zeta) = \frac{\sin(\theta)}{1 - \cos(\theta)} = \cot\left(\frac{\theta}{2}\right) \qquad 2\zeta = \pi - \theta$$

$$\frac{T_1}{T_0} := \frac{m_1 v_1^2}{m_1 u_1^2} = 1 - \frac{2m_1 m_2(1 - \cos(\theta))}{(m_1 + m_2)^2} = \frac{m_1^2}{(m_1 + m_2)^2}\left(\cos(\psi) \pm \sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2(\psi)}\right)^2 \qquad (*)$$

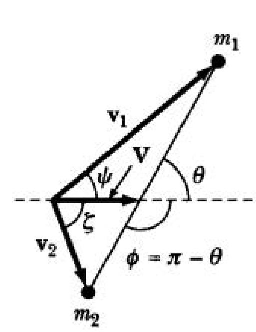
In (*), take plus sign for the radical unless $m_1 > m_2$. If $m_1 > m_2$ evaluated using ψ, the result is double-valued.

$$\frac{T_2}{T_0} := \frac{m_2 v_2^2}{m_1 u_1^2} = 1 - \frac{T_1}{T_0} = \frac{4m_1 m_2\cos^2(\zeta)}{(m_1 + m_2)^2} \qquad \text{if } \zeta \le \frac{\pi}{2}$$

$$\sin(\zeta) = \sqrt{\frac{m_1 T_1}{m_2 T_2}}\sin(\psi) \qquad \tan(\psi) = \frac{\sin(2\psi)}{(m_1/m_2) - \cos(2\psi)} \qquad \tan(\psi) = \frac{\sin(\phi)}{(m_1/m_2) - \cos(\phi)}$$

If $m_1 > m_2$, the following specifies the maximum value allowed for ψ:

$$\psi_{max} = \sin^{-1}\left(\frac{v_1'}{V}\right) = \sin^{-1}\left(\frac{m_2}{m_1}\right)$$



If we have $m_2 = m_2$, we have $\theta = 2\psi$, and we get the followings:

$$\frac{T_1}{T_0} = \cos^2(\psi) \qquad \frac{T_2}{T_0} = \sin^2(\psi) \qquad \zeta + \quad = \frac{\pi}{2} \qquad \text{if } m_1 = m_2$$

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