Q4.

$$\int_0^L f_n^*(x) F(x)\,dx = \int_0^L f_n^*(x) \underbrace{\sum_{m=-\infty}^{\infty} a_m f_m(x)}_{\text{using Fourier Series}} dx$$

$$= \int_0^L \sum_{m=-\infty}^{\infty} f_n^*(x)\, a_m f_m(x)\,dx$$

$$= \underbrace{\sum_{m=-\infty}^{\infty} a_m \int_0^L}_{\text{by the linearity of integral.}} f_n^*(x) f_m(x)\,dx$$

$$\int_0^L f_n^*(x) f_m(x)\,dx = \begin{cases} 0 & \text{when } m \neq 0 \\ 1 & \text{when } m = n \end{cases}$$

$$= \sum_{m=-\infty}^{\infty} a_m \int_0^L f_n^*(x) f_m(x)\,dx$$

$$= a_n + \sum_{m \neq n} a_m \int_0^L f_n^*(x) f_m(x)\,dx$$

$$= a_n + 0$$

$$\int_0^L f_n^*(x) F(x)\,dx = a_n$$

Q5

Note: $F_1(x) = \frac{1}{6} - \frac{1}{\pi^2} \cos(2\pi m x)$

$f_m(x) = \frac{1}{\sqrt{L}} e^{i\left(\frac{2\pi m x}{L}\right)}$ with $L = 1$

$$\hookrightarrow f_m(x) = e^{i2\pi m x}$$

$$a_n = \int_0^1 f_n^*(x) F(x)\, dx$$

$$f_0(x) = e^{i2\pi \cdot 0x} = e^0 = 1$$

$$a_0 = \int_0^1 f_0(x)\, x(1-x)\, dx = \int_0^1 x - x^2\, dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$f_1(x) = e^{i2\pi x} \quad , \quad f_{-1}(x) = e^{-i2\pi x}$$

$$a_1 = \int_0^1 f_1(x) F(x)\, dx = \int_0^1 e^{i2\pi x}\, x(1-x)\, dx = -\frac{1}{2\pi^2}$$

$$a_{-1} = \int_0^1 f_{-1}(x) F(x)\, dx = \int_0^1 e^{-i2\pi x}\, x(1-x)\, dx = -\frac{1}{2\pi^2}$$

Concluding the above. we get :

$$F_1(x) = a_{-1} f_{-1}(x) + a_0 f_0(x) + a_1 f_1(x)$$

$$= -\frac{1}{2\pi^2} e^{-i(2\pi x)} + \frac{1}{6} e^{i(2\pi\, 0x)} - \frac{1}{2\pi^2} e^{i(2\pi x)}$$

$$= \frac{1}{6} - \left(\frac{1}{2\pi^2}\right)\left(e^{-i2\pi x} + e^{i2\pi x}\right)$$

$$= \frac{1}{6} - \frac{1}{\pi^2} \cos(2\pi x)$$

Q8

$$f_0(x) = e^{i2\pi 0x} = e^0 = 1$$

$f_0(x) = e \qquad = e \quad = 1$

$$a_0 = \int_0^1 f_0(x)\, F(x)\, dx = \int_0^{1/2} T_0\, dx = \frac{1}{2} T_0$$

$f_1(x) = e^{i2\pi x}$, $f_{-1}(x) = e^{-i2\pi x}$

$$a_{-1} = \int_0^1 f_{-1}^*(x) F(x)\, dx = \int_0^{1/2} e^{i2\pi x}\, T_0\, dx = \frac{iT_0}{\pi}$$

$$a_1 = \int_0^1 f_1^*(x) F(x)\, dx = \int_0^{1/2} e^{-i2\pi x}\, T_0\, dx = -\frac{iT_0}{\pi}$$

$$F_1(x) = \frac{iT_0}{\pi} e^{-i2\pi x} + \frac{1}{2}T_0 - \frac{iT_0}{\pi} e^{i2\pi x}$$

$$= \frac{iT_0}{\pi}\left(e^{-i2\pi x} - e^{i2\pi x}\right) + \frac{1}{2}T_0$$

$$= \frac{iT_0}{\pi}\Big(\cos(-2\pi x) + i\sin(-2\pi x) - \cos(2\pi x) - i\sin(2\pi x)\Big) + \frac{1}{2}T_0$$

$$= \frac{iT_0}{\pi}\Big(-i\sin(2\pi x) - i\sin(2\pi x)\Big) + \frac{1}{2}T_0$$

$$= \frac{2T_0}{\pi}\sin(2\pi x) + \frac{1}{2}T_0$$

Q2 $\quad F\left(\frac{1}{4}\right) = \frac{1}{4}\left(1 - \frac{1}{4}\right) = \frac{1}{4}\,\frac{3}{4} = \frac{3}{16}$

$$\frac{3}{16} = \frac{1}{6} - \frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(2\pi m \cdot \frac{1}{4}\right)$$

$$\frac{3}{16} = \frac{1}{6} - \frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(m\frac{\pi}{2}\right)$$

$m = 2n$

$2 \quad 1 \quad 1 \quad \sum^{\infty} \quad (-1)^n$

$$16 \quad 6 \quad \pi^2 \quad \sum_{m=1}$$

$m = 2n$

$$\frac{3}{16} = \frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)^2}$$

$$\frac{3}{16} = \frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{4}\frac{(-1)^n}{n^2}$$

$$\frac{3}{16} = \frac{1}{6} - \frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\left(\frac{3}{16} - \frac{1}{6}\right)\cdot 4\pi^2 = -\left(\frac{3}{4} - \frac{2}{3}\right)\pi^2 = -\frac{\pi^2}{12}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{\pi^2}{12}$$