Example: Consider the equality $y^5 + xy + z = 0$, call it $f(x, z, y) = 0$.
Here we order the variables as (x, z, y). Then we can write the following:

$$Df = \begin{bmatrix} y & 1 & 5y^4 + x \end{bmatrix}$$

We can use Implicit Function Theorem when we have $5y^4 + x \neq 0$. We will identity the "folding curve" for the $f(x, z, y) = 0$. Here we need $g(x, z, y) = 0$ for $g(x, z, y) = \begin{bmatrix} y^5 + xy + z \\ 5y^4 + x \end{bmatrix}$.

$$Dg(x, z, y) = \begin{bmatrix} y & 1 & 5y^4 + x \\ 1 & 0 & 20y^3 \end{bmatrix}$$

Note here:

$$\begin{bmatrix} y & 1 \\ 1 & 0 \end{bmatrix} \text{ is always invertible}$$

We can choose (x, z) being dependent variables and y being independent variable.
Then $x = -5y^4,\ y = 4z^5$.
From Wednesday handout:

$$f : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto y^5 + xy + 1$$

$$Df = \begin{bmatrix} y & 5y^4 + x \end{bmatrix}$$

$$E := f^{-1}(0) \text{ is locally } y = g(x) \text{except possibily when } 5y^4 + x = 0$$

Assuming y is differentiable with respect to x, then we have

$$5y^4 \frac{dy}{dx} + y + x\frac{dy}{dx} = 0 \qquad \Rightarrow \qquad \frac{dy}{dx} = -\frac{y}{5y^4 + x}$$

$$g'(x) = -\frac{g(x)}{5(g(x))^4 + x}$$

$$\text{let } \widetilde{g}(x) = \begin{bmatrix} x \\ g(x) \end{bmatrix}$$

$$f \circ \widetilde{g} = 0 \text{ because we assume that the graph of g lies in E}$$

$$Df(\widetilde{g}(x))(\widetilde{g}'(x)) = 0$$

$$\begin{bmatrix} \frac{\partial f}{\partial x}\begin{bmatrix} x \\ g(x) \end{bmatrix} & \frac{\partial f}{\partial y}\begin{bmatrix} x \\ g(x) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ g'(x) \end{bmatrix} \end{bmatrix} = \frac{\partial f}{\partial x}\begin{bmatrix} x \\ g(x) \end{bmatrix} + \frac{\partial f}{\partial y}\begin{bmatrix} x \\ g(x) \end{bmatrix} \cdot g'(x) = 0$$

$$g'(x) = \frac{-\frac{\partial f}{\partial x}\begin{bmatrix} x \\ g(x) \end{bmatrix}}{\frac{\partial f}{\partial y}\begin{bmatrix} x \\ g(x) \end{bmatrix}}$$

In higher dimension, let $\widetilde{g}(\vec{x}) = \begin{bmatrix} \vec{x} \\ g(\vec{x}) \end{bmatrix}$, $f \circ \widetilde{g} = 0$. $(Df \circ \widetilde{g})D\widetilde{g} = 0$

$$\Rightarrow \begin{bmatrix} \frac{\partial f}{\partial \vec{x}} \circ \widetilde{g} & \frac{\partial f}{\partial \vec{y}}\widetilde{g} \end{bmatrix} \begin{bmatrix} I \\ Dg \end{bmatrix} = \frac{\partial f}{\partial \vec{x}} \circ \widetilde{g} + (\frac{\partial f}{\partial \vec{y}} \circ \widetilde{g})Dg = 0$$

$$Dg(\vec{x}) = -\left(\left(\frac{\partial f}{\partial \vec{y}}\right)^{-1} \frac{\partial f}{\partial \vec{x}}\right)(\widetilde{g}(\vec{x}))$$