For complex number z_1, z_2, $\overline{z_1/z_2} = \bar{z}_1/\bar{z}_2$. $|z_1| = \sqrt{z_1\bar{z}_1}$. $1/z = \bar{z}/|z|^2$. $e^{i\theta} = \sin(\theta) + i\cos(\theta)$.
$\cos(iz) = \cosh(z) = (e^z + e^{-z})/2$. $\sin(iz) = i\sin(z) = i(e^x - e^{-x})/2$.
The equation $z^n = 1$ has n solutions $e^{i2\pi k/n}$ with $0 \le k \le n-1$.

Series sum: $\sum_{n=0}^{N} a^n = (1-a^N)/(1-a)$. $\sum_{n=0}^{N} a + bn = (N+1)(a+Nb/2)$.
Taylor series expanding around x_0: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{n!}(x-x_0)^k$.
Binomial Expansion: $(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2 \pm \frac{n(n-1)(n-2)}{3!}x^3 \cdots$.

Given $m\frac{d^2x}{dx^2} = f(x)$ with $f(x_0) = 0$ and $f'(x_0) > 0$, for small deviations from x_0:

$$x(t) = x_0 + A\cos(\sqrt{\frac{f'(x_0)}{m}}\ t + \phi)$$

with A and ϕ depending on initial conditions.

$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$. $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$. $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$. $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$.
$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$. $\frac{d\hat{\theta}}{dt} = -\frac{d\theta}{dt}\hat{\rho}$. $\frac{d\hat{\rho}}{dt} = \frac{d\theta}{dt}\hat{\theta}$. $\nabla f = \frac{\partial f}{\partial x}\hat{\imath}+\frac{\partial f}{\partial y}\hat{\jmath}+\frac{\partial f}{\partial z}\hat{k}$. $\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho}+\frac{1}{\rho}\frac{\partial f}{\partial \theta}\hat{\theta}+\frac{\partial f}{\partial z}\hat{k}$.

For maximizing $\Phi(x_1,x_2,x_3)$ with constraints $f(x_1,x_2,x_3) = 0$ and $g(x_1,x_2,x_3) = 0$.
We need Lagrange multipliers λ and μ. $\frac{\partial\Phi}{\partial x_i} = \lambda\frac{\partial f}{\partial x_i} + \mu\frac{\partial g}{\partial x_i}$, $f(x_1,x_2,x_3) = 0$, and $g(x_1,x_2,x_3) = 0$.

For vector field $\vec{F}(\vec{r}) = F_x(\vec{r})\hat{\imath} + F_y(\vec{r})\hat{\jmath} + F_z(\vec{r})\hat{k}$, and scalar function f:

$$Divergence = \nabla\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}$$

$$Curl = \nabla\times\vec{F} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (F_x, F_y, F_z) = \frac{1}{\rho}(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial z}) \times (F_\rho, \rho F_\theta, F_z)$$

$$Laplacian = \nabla^2 f = \nabla\cdot\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

For matrices A, B, $(AB)_{ij} = \sum_k A_{il}B_{kj}$. $(AB)^T = B^T A^T$. The Hermitian conjugate of A is $A^\dagger$.
$(A^\dagger)_{ij} = \bar{A_{ji}}$. A matrix A is hermitian provided that $A_{ij} = \bar{A_{ji}}$.

For matrix M, $f(M) = \sum_{n=1}^{\infty}\frac{f(0)}{n!}M^n \qquad \Rightarrow \qquad e^M = I + M + \frac{1}{2}M^2 + \cdots = \sum_{k=0}^{\infty}\frac{1}{k!}M^k$.

For $n\times n$ matrices A, B and scalar λ, $\det(\lambda A) = \lambda^n\det(A)$, $\det(A^T) = \det(A)$, $\det(AB) = \det(A)\det(B)$. Swap two rows or two columns of A to get A', $\det(A') = -\det(A)$. Subtracting one row, or column, of A to another row, or columns, of A to get A'', $\det(A'') = \det(A)$.

Given $u(t)$ as a function, $\frac{du}{dt} = Mu$, where M is a constant matrix. $u(t) = e^{Mt}u(0)$.

Vector space V over F has properties:
$\vec{u}+\vec{v}\in V$, $\vec{u}+\vec{v} = \vec{v}+\vec{u}$, $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$,
$\lambda\vec{u}\in V$, $\lambda(\vec{u}+\vec{v}) = \lambda\vec{v}+\lambda\vec{u}$, $(\lambda\vec{u})+(\mu\vec{u}) = (\lambda+\mu)\vec{u}$, where $\vec{u},\vec{v},\vec{w}\in V$, and $\lambda,\mu\in F$,
zero vector $\vec{0}\in V$ with $\vec{0}+\vec{u} = \vec{u}$, $1\in F$ with $1\vec{u} = \vec{u}$. Each $\vec{u}$ has additive inverse $\vec{w}$, $\vec{u}+\vec{w} = \vec{0}$.

L is a linear transformation on vector space V over F provided that for $\vec{v},\vec{w}\in V$ and $a,b\in F$, we have $L(a\vec{v}+b\vec{w}) = aL(\vec{v})+bL(\vec{w})$.

When two distinct eigenvectors have the same eigenvalues they are said to be degenerate. When the number of eigenvectors is less than n, the $n\times n$ matrix is said to be defective.

For Hermitian matrice:
Eigenvectors with different eigenvalues are orthogonal and have real-valued eigenvalues. Eigenvectors with the same eigenvalues can be made orthogonal by Gram-Schmidt orthonormalization.

For basis $\{\vec{e}_k\}$ and $\{\vec{e}'_k\}$, let the coordinate of $\vec{u}$ be $\vec{v}$ in basis $\{\vec{e}_k\}$ and v' in $\{\vec{e}'_k\}$. One can find P such that $v' = Pv$ and $v = P^{-1}v'$, with P_{ij} satisfies $\vec{e}'_j = \sum_i^n P_{ij}\vec{e}_i$. Let the representation of a linear operator be the matrix M in basis $\{\vec{e}_k\}$, and M' in basis $\{\vec{e}'_k\}$, we have $M' = PMP^{-1}$.

Let $(\vec{v}_i)$ be the eigenvectors of M corresponds to eigenvalues (λ_i). Let $S = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{bmatrix}$.
Then $D = S^{-1}MS$ is a diagonal matrix with (λ_i) on the diagonal. Here we have $M^k = SD^kS$.
For Hermitian matrix, $S^{-1} = S^\dagger$.

Separable first order ODE: $\frac{dy}{dx} = f(x)g(y) \quad \Rightarrow \quad \int \frac{1}{g(y)}dy = \int f(x)dx$

Exact differentials: $f(x,y)dx + g(x,y)dy = 0 \quad \Rightarrow \quad d\Phi = \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy$
Guess $f(x,y) = \frac{\partial \Phi}{\partial x}$ and $g(x,y) = \frac{\partial \Phi}{\partial y}$. Check $\frac{pdf}{\partial y} = \frac{\partial x}{\partial g}$. Solve Φ. If not exact:
Try $\mu(x,y) = \mu(x)$, if we have $\frac{1}{\mu}\frac{d\mu}{dx} = \frac{1}{g}(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x})$ depends only on x, proceed with μ.
Try $\mu(x,y) = \mu(y)$, if we have $\frac{1}{\mu}\frac{d\mu}{dy} = \frac{1}{f}(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y})$ depends only on y, proceed with μ.
Write $\mu(x,y)f(x,y)dx + \mu(x,y)g(x,y)dy = 0$, and solve with exact differentials.

System of first-order equations:
λ_1, λ_2 are eigenvalues of M, $\vec{v}_1, \vec{v}_2$ are eigenvectors correspond to λ_1, λ_2 respectively.

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = ae^{\lambda_1 t}\vec{v}_1 + be^{\lambda_2 t}\vec{v}_2$$

Constant coefficient: $\alpha y'' + \beta y' + \gamma y = 0$, guess $y = e^{\lambda t}$, with $\lambda = -\frac{\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$.

Reduction of Order and Variation of Parameter:
For $d(x)y'' + b(x)y + cy = h(x)$, general solution is $y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$, with $c_1, c_2 \in \mathbb{R}$.

$$y_2(x) = y_1(x)\int^x \frac{e^{-\int^s \frac{b(t)}{d(t)}dt}}{(\,y_1(s)\,)^2}\,ds \qquad y_p(x) = \int^x h(s)\frac{y_1(s)y_2(x) - y_1(x)y_2(s)}{W(s)}\,ds$$
$$Wronskian = W(s) = \det\left(\begin{bmatrix} y_1(s) & y_2(s) \\ y_1'(s) & y_2'(s) \end{bmatrix}\right)$$

Normal Modes: Taylor expand energy E with displacement ϵ: $E \approx \dot{\epsilon}^T M\dot{\epsilon} + \epsilon^T K\epsilon$ with some matrix M and K. With $\frac{dE}{dt} = 0$, we get $M\ddot{\epsilon} + k\epsilon = 0$. Assume that $\epsilon = e^{i\omega t}\eta$ with some constant vector η, we get $(-\omega^2 M + K)\eta = 0$. Solve for ω_i by $\det(-{\omega_i}^2 M + K) = 0$. For each ω_i, find η_i such that $(-\omega_i^2 M + K)\eta_i = 0$. Each η_i is a normal mode with a frequency of oscillation ω_i.
Now write: $\epsilon = \sum_{k=1}^N c_k(\cos(\omega_k t + \theta_i))\eta_k$. Find c_k and θ_i with initial conditions.

Orthogonal Integrals: $\int_{-\pi}^{\pi} e^{imx} \cdot \overline{e^{inx}} = \int_{-\pi}^{\pi} e^{i(m-n)x}dx = 2\pi\delta_{mn}$, $\delta_{mn} = 1$ if $m = n$ else $\delta_{mn} = 0$.
When $n \neq m \neq 0$: $\int_{-\pi}^{\pi} \cos(mx)\cos(nx) = \int_{-\pi}^{\pi} \sin(mx)\cos(nx) = \int_{-\pi}^{\pi} \cos(mx)\sin(nx) = 0$,
$\int_{-\pi}^{\pi} \cos(mx) = \int_{-\pi}^{\pi} \sin(mx) = 0 \qquad \int_{-\pi}^{\pi} \cos^2(mx) = \int_{-\pi}^{\pi} \sin^2(mx) = \pi$

Fourier Series: For f defined on $[a,b]$, $f(x) = \sum_{m=-\infty}^{\infty} c_m e^{i2\pi mx/L}$ with $L = b - a$.
If f is real-valued defined on $[a,b]$, $f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos(2\pi nx/L) + b_m \sin(2\pi nx/L)$.

$$c_n = \frac{1}{b-a}\int_a^b f(x)e^{-i2\pi nx/L}\,dx \qquad a_0 = \frac{1}{b-a}\int_a^b f(x)dx$$
$$a_m = \frac{2}{b-a}\int_a^b f(x)\cos(2\pi nx/L)\,dx \qquad b_m = \frac{2}{b-a}\int_a^b f(x)\sin(2\pi nx/L)\,dx$$

Fourier series of **even functions** on $[-L/2, L/2]$ has the form $f(x) = a_0 + \sum_{m=1}^{\infty} a_m \cos(2\pi nx/L)$, with $a_0 = \frac{2}{L}\int_0^{L/2} f(x)\,dx$, $a_m = \frac{4}{L}\int_0^{L/2} f(x)\cos(2\pi nx/L)\,dx$. Fourier series of **odd functions** on $[-L/2, L/2]$ has the form $f(x) = \sum_{m=0}^{\infty} b_m \sin(2\pi nx/L)$ with $b_m = \frac{4}{L}\int_0^{L/2} f(x)\sin(2\pi nx/L)\,dx$.

Separable PDE: Solve $\frac{\partial^2}{\partial x^2}T(x,y) + \frac{\partial^2}{\partial y^2}T(x,y) = 0$.
Assume that $T(x,y) = f(x)g(y)$. Write $g(y)f''(x) + f(x)g''(y) = \frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} = 0$.
Then $\frac{f''(x)}{f(x)} = \alpha = -\frac{g''(y)}{g(y)} \Rightarrow f''(x) - \alpha f(x) = 0$ and $g''(y) + \alpha g(y) = 0$. Use boundary conditions or initial conditions to solve for α, then g and f, and assemble. Note here if we have $e^{\lambda L} - e^{-\lambda L} = 0$ for some non-zero L and ω, we can write $e^{2\lambda L} = 1 \Rightarrow 2\lambda L = 2mi\pi$ for $m \in \mathbb{Z}$.

$$e^{\lambda i} + e^{-\lambda i} = \cos(\lambda) + i\sin(\lambda) + \cos(-\lambda) + i\sin(-\lambda) = 2\cos(\lambda)$$
$$e^{\lambda i} - e^{-\lambda i} = \cos(\lambda) + i\sin(\lambda) - \cos(-\lambda) - i\sin(-\lambda) = 2\sin(\lambda)$$

Rotation matrix: Rotate vectors counterclockwise through an angle θ.

$$\text{2-D}: \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \qquad \text{3-D}: \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Integration by parts: } \int u\,v\,dx = u\left(\int v\,dx\right) - \int u'\left(\int v\,dx\right)dx$$