

$$\vec{p} = m\vec{v} \quad \vec{v} = \vec{\omega} \times \vec{r} \quad \vec{l} = I\vec{\omega} = \vec{r} \times \vec{p} \quad \vec{N} = \frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} \quad U = \frac{1}{2}I\omega^2 \quad I = \int_V r^2 dm$$

Given  $m \frac{d^2x}{dx^2} = F(x)$ ,  $f(x_0) = 0$ , and  $F'(x_0) > 0$ . For small deviations from  $x_0$ :  $x(t) = x_0 + A \cos\left(\sqrt{\frac{F'(x_0)}{m}} t + \phi\right)$

In mathematical statement, Hamilton's Principle states that  $I = \int_{t_1}^{t_2} \mathcal{L} dt = \int_{t_1}^{t_2} (T - U) dt$  where we want to minimize  $I$ . We write  $\mathcal{L}(q_i, \dot{q}_i; t)$  and  $\mathcal{H}(q_i, p_i; t)$ . Denote the constraint in algebraic expression  $f(q_i; t) = 0$

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0 \quad \begin{cases} \frac{\partial \mathcal{L}}{\partial q_i} \text{ is called the generalized force component} \\ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \text{ is called the kinetic part} \end{cases} \quad \left( \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) + \sum_k \lambda_k \frac{\partial f_k}{\partial q_i} = 0$$

$$Q_i := \sum_{k=1}^m \lambda_k \frac{\partial f_k}{\partial q_i} \quad p_i := \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad \mathcal{H} := \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} \quad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} \quad -\dot{p}_i = \frac{\partial \mathcal{H}}{\partial q_i} \quad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$$

$Q_i$  is the generalized constrain force components.  $p_i$  is the generalized momentum associated with  $q_i$ . If  $p_i$  is invariant under time, the associated  $q_i$  is cyclic, and the generalized momentum  $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = p_i$  is the canonical momentum, or conjugate momentum. If the kinetic energy  $T$  is quadratic in  $\dot{q}_i$ , and  $U = U(q_i)$ , then  $\mathcal{H}$  gives the total energy of the system.

In the center of mass frame:

$$\mathcal{L} = \frac{\mu}{2} |\dot{\vec{r}}|^2 - U(r) = \frac{\mu}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r) \quad E = \frac{1}{2} \mu \dot{r}^2 + \frac{1}{2} \frac{l^2}{\mu r^2} + U(r) = \frac{1}{2} \mu \dot{r}^2 + U_{\text{eff}}(r)$$

$d_1, d_2$  are the distance of the two particles to the center of mass.  $l/(2\mu r^2)$  is the angular momentum barrier.

$$l = m_1 d_1^2 \dot{\theta} + m_2 d_2^2 \dot{\theta} = \mu r^2 \dot{\theta} = \text{constant} \quad \frac{d^2}{d\theta^2} \frac{1}{r} + \frac{1}{r} = -\frac{\mu r^2}{l^2} F(r) \quad \mu \ddot{r} + \frac{\partial}{\partial r} \left( U(r) + \frac{l^2}{2\mu r^2} \right) = 0$$

$$\int_0^t dt' = \int_{r(0)}^{r(t)} \left( \frac{2}{\mu} \left( E - U(r') - \frac{l^2}{2\mu(r')^2} \right) \right)^{1/2} dr' \quad \theta(t) - \theta(0) = \int_0^t \frac{1}{\mu(r(t'))^2} dt'$$

$$\theta(r) = \int_{r(0)}^{r(t)} \frac{d\theta}{dt} \frac{dt}{dr} dr' = \pm \int_{r(0)}^{r(t)} \frac{l/(\mu(r')^2) dr'}{\sqrt{\frac{2}{\mu} (E - U(r')) - \left( \frac{l^2}{\mu r'^2} \right)^2}}$$

**ASSUME**  $U(r) = -k/r$ .  $\epsilon$  is the eccentricity of the two-body system,  $u_0$  is constant,  $u = 1/r$ , Kepler's First Law states:

$$\alpha := \frac{l^2}{\mu k} \quad \epsilon = \frac{l^2 u_0}{\mu k} \quad \frac{\alpha}{r} = 1 + \epsilon \cos(\theta) \quad \epsilon = \sqrt{1 + \frac{2El^2}{\mu k^2}} \quad E = \frac{1}{2} \frac{l^2}{\mu} \frac{(1 + \epsilon)^2}{\alpha^2} - \frac{k(1 + \epsilon)}{\alpha}$$

$$u = \frac{\mu k}{l^2} \left( \frac{l^2 u_0}{\mu k} \cos(\theta) + 1 \right) \quad \alpha = a(1 + \epsilon)(1 - \epsilon) \quad a = \frac{\alpha}{1 - \epsilon^2} = \frac{k}{2|E|} \quad b = \sqrt{\alpha a} = \frac{\alpha}{\sqrt{1 - \epsilon^2}} = \frac{l}{\sqrt{2\mu|E|}}$$

**ASSUME**  $U(r) = -k/r$ . Kepler's Second and Third Law:

$$\frac{dA}{dt} = \frac{r^2}{2} \frac{d\theta}{dt} = \frac{r^2 \dot{\theta}}{2} = \frac{l}{2\mu} = \text{constant} \quad \tau = \frac{2\mu}{l} A = \frac{2\mu}{l} (\pi ab) = \pi k \sqrt{\frac{\mu}{2}} |E|^{-3/2} \quad \tau^2 = \frac{4\pi^2}{k} \mu a^3$$

For the transfer from the ellipse to the circular orbit of radius  $r_2$ ,  $\Delta v_2 = v_2 - v_{t2}$ .

$$v_{t2} = \sqrt{\frac{2k}{mr_2} \left( \frac{r_1}{r_1 + r_2} \right)} \quad T_t = \frac{\tau_t}{2} = \pi \sqrt{\frac{m}{k}} a_t^{3/2}$$

**ASSUME ELASTIC COLLISION** of  $m_1$  and  $m_2$ . Center of mass parameters:  $\vec{R}, \vec{V}, M$ . Initial KE  $T_0$  in LAB.

Moving particle: initial speed  $u_1$  in LAB, initial speed  $u'_1$  in CM, final speed  $v_1$  in LAB, final speed  $v'_1$  in CM.

Particle at rest in LAB: initial speed  $u_2$  in LAB, initial speed  $u'_2$  in CM, final speed  $v_2$  in LAB, final speed  $v'_2$  in CM.

$$\vec{R} = \frac{\int_M \vec{r} dm}{\int_M dm} \quad \vec{F}_{ext} = M \frac{d\vec{V}_{CM}}{dt} = \frac{d\vec{P}_{CM}}{dt} \quad \vec{V} = \frac{m_1 \vec{u}_1}{m_1 + m_2} = -\vec{u}'_2 \quad \frac{V}{v'_1} = \frac{m_1 u_1 / (m_1 + m_2)}{m_2 u_1 / (m_1 + m_2)} = \frac{m_1}{m_2}$$

$$\tan(\psi) = \frac{\sin(\theta)}{\cos(\theta) + (V/v'_1)} = \frac{\sin(\theta)}{\cos(\theta) + (m_1/m_2)} \quad \tan(\zeta) = \frac{\sin(\theta)}{1 - \cos(\theta)} = \cot\left(\frac{\theta}{2}\right) \quad 2\zeta = \pi - \theta$$

$$\frac{T_1}{T_0} := \frac{m_1 v_1^2}{m_1 u_1^2} = 1 - \frac{2m_1 m_2 (1 - \cos(\theta))}{(m_1 + m_2)^2} = \frac{m_1^2}{(m_1 + m_2)^2} \left( \cos(\psi) \pm \sqrt{\left( \frac{m_2}{m_1} \right)^2 - \sin^2(\psi)} \right)^2 \quad (*)$$

In (\*), take plus sign for the radical unless  $m_1 > m_2$ . If  $m_1 > m_2$  evaluated using  $\psi$ , the result is double-valued.

$$\frac{T_2}{T_0} := \frac{m_2 v_2^2}{m_1 u_1^2} = 1 - \frac{T_1}{T_0} = \frac{4m_1 m_2 \cos^2(\zeta)}{(m_1 + m_2)^2} \quad \text{if } \zeta \leq \frac{\pi}{2}$$

$$\sin(\zeta) = \sqrt{\frac{m_1 T_1}{m_2 T_2}} \sin(\psi) \quad \tan(\psi) = \frac{\sin(2\zeta)}{(m_1/m_2) - \cos(2\zeta)} \quad \tan(\psi) = \frac{\sin(\phi)}{(m_1/m_2) - \cos(\phi)}$$

If  $m_1 > m_2$ , the following specifies the maximum value allowed for  $\psi$ :

$$\psi_{max} = \sin^{-1} \left( \frac{v'_1}{V} \right) = \sin^{-1} \left( \frac{m_2}{m_1} \right)$$

If we have  $m_1 = m_2$ , we have  $\theta = 2\psi$ , and we get the followings:

$$\frac{T_1}{T_0} = \cos^2(\psi) \quad \frac{T_2}{T_0} = \sin^2(\psi) \quad \zeta + \psi = \frac{\pi}{2} \quad \text{if } m_1 = m_2$$