Lemmas and Theorems

Math 295 - Honors Mathematics I
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Fall 2020

The Construction of the Unique Ordered Field

Axiom 1 (Peano Axioms)
There exists a triple $(\mathbb{N}', \sigma, 1_{\mathbb{N}})$ *such that the followings hold:*

1. $\mathbb{N}'$ *is a set, and the element* $1_{\mathbb{N}}$ *belongs to* $\mathbb{N}'$.
2. $\sigma : \mathbb{N}' \to \mathbb{N}'$ *is an injective function, and* $\forall n \in \mathbb{N}'$, $\sigma(n) \neq 1_{\mathbb{N}}$.
3. *For* $S \subset \mathbb{N}'$, *if* $1_{\mathbb{N}} \in S$ *and* $m \in S$, *then* $\sigma(m) \in S$, *and hence* $S = \mathbb{N}'$.

Theorem 1.1 (Principle of Recursive Definition)
Let X *be a set, let* $\varphi : X \to X$ *be a function, let* $a \in X$, *then there exists a unique function* $f : \mathbb{N}' \to X$ *such that* $f(1_{\mathbb{N}}) = a$, *and* $\forall n \in \mathbb{N}'$, *we have* $f(\sigma(n)) = \varphi(f(n))$.

Definition 1.1.0.0.1
Let $m \in \mathbb{N}'$, *let* $a = \sigma(m)$. *For the function* $\sigma : \mathbb{N}' \to \mathbb{N}'$ *given in the Peano Axiom, by using the Principle of Recursive Definition, we can define a unique function* $f_m : \mathbb{N}' \to \mathbb{N}'$ *such that* $f_m(1_{\mathbb{N}}) = \sigma(m) = a$, *and* $\forall n \in \mathbb{N}'$, *we have* $f_m(\sigma(n)) = \sigma(f_m(n))$.

Definition 1.1.0.0.2
Let $m, n \in \mathbb{N}'$, $m +_{\mathbb{N}} n \coloneqq f_m(n)$.

Definition 1.1.0.0.3
Let $m \in \mathbb{N}'$. *For the function* $f_m : \mathbb{N}' \to \mathbb{N}'$, *by the Principle of Recursive Definition, we can define a unique function* $\mu_m : \mathbb{N}' \to \mathbb{N}'$ *s.t.* $\mu_m(1_{\mathbb{N}}) = m$, *and* $\forall n \in \mathbb{N}'$, *we have* $\mu_m(\sigma(n)) = f_m(\mu_m(n))$.

Definition 1.1.0.0.4
Let $m, n \in \mathbb{N}'$, $m *_{\mathbb{N}} n \coloneqq \mu_m(n)$.

Lemma 1.1.1
For $m, n, q \in \mathbb{N}'$, *the followings hold:*

1. $(m = n) \iff (m +_{\mathbb{N}} q = n +_{\mathbb{N}} q)$
2. $f_1(n) = \sigma(n)$
3. $\mu_1(n) = n$

Lemma 1.1.2
For $m, n \in \mathbb{N}'$ *with* $m \neq n$, *exactly one of the followings holds:*

1. *If* $\exists!\ r \in \mathbb{N}'$ *s.t.* $m = n +_{\mathbb{N}} r$, *then we write* $m >_{\mathbb{N}} n$.
2. *If* $\exists!\ r \in \mathbb{N}'$ *s.t.* $n = m +_{\mathbb{N}} r$, *then we write* $n >_{\mathbb{N}} m$.

Corollary 1.1.2.1
For $m, n \in \mathbb{N}'$, *Trichotomy holds:*

1. m=n 2. $m >_{\mathbb{N}} n$ 3. $n >_{\mathbb{N}} m$

Definition 1.1.2.1.1
Let $m, n, l, q \in \mathbb{N}'$, *let* $\sim_{\mathbb{Z}}$ *be a relation on* $(\mathbb{N}' \times \mathbb{N}')$ *with* $(n, m) \sim_{\mathbb{Z}} (l, q) \iff n +_{\mathbb{N}} q = l +_{\mathbb{N}} m$.

Definition 1.1.2.1.2
$\mathbb{Z}' \coloneqq (\mathbb{N}' \times \mathbb{N}')/{\sim_{\mathbb{Z}}}$

Lemma 1.1.3
The relation $\sim_{\mathbb{Z}}$ *is an equivalence relation on the set* $\mathbb{N}' \times \mathbb{N}'$.

Definition 1.1.3.0.1
$0_{\mathbb{Z}} \coloneqq [(1_{\mathbb{N}}, 1_{\mathbb{N}})]$

Definition 1.1.3.0.2
$1_{\mathbb{Z}} \coloneqq [(1_{\mathbb{N}} +_{\mathbb{N}} 1_{\mathbb{N}}, 1_{\mathbb{N}})]$

Definition 1.1.3.0.3
Let $[(a, b)] \in \mathbb{Z}'$, $-[(a, b)] \coloneqq [(b, a)]$.

Definition 1.1.3.0.4
Let $[(n, m)], [(l, k)] \in \mathbb{Z}'$, $[(n, m)] +_{\mathbb{Z}} [(l, k)] \coloneqq [(n +_{\mathbb{N}} l, m +_{\mathbb{N}} k)]$.

Definition 1.1.3.0.5
Let $[(n, m)], [(l, k)] \in \mathbb{Z}'$, $[(n, m)] -_{\mathbb{Z}} [(l, k)] \coloneqq [(n, m)] +_{\mathbb{Z}} -[(l, k)]$.

Definition 1.1.3.0.6
Let $[(n, m)], [(l, k)] \in \mathbb{Z}'$, $[(n, m)] *_{\mathbb{Z}} [(l, k)] \coloneqq [(n *_{\mathbb{N}} l +_{\mathbb{N}} m *_{\mathbb{N}} k, m *_{\mathbb{N}} l +_{\mathbb{N}} n *_{\mathbb{N}} k)]$.

Lemma 1.1.4
For $[(m,n)],[(p,q)] \in \mathbb{Z}'$, exactly one of the followings hold:

1. $m +_{\mathbb{N}} q >_{\mathbb{N}} p +_{\mathbb{N}} n$, *then we write* $[(m,n)] >_{\mathbb{Z}} [(p,q)]$.
2. $m +_{\mathbb{N}} q <_{\mathbb{N}} p +_{\mathbb{N}} n$, *then we write* $[(m,n)] <_{\mathbb{Z}} [(p,q)]$.
3. $m +_{\mathbb{N}} q = p +_{\mathbb{N}} n$, *then we write* $[(m,n)] = [(p,q)]$.

Theorem 1.2
Let $i_{\mathbb{N}} : \mathbb{N}' \to \mathbb{Z}' \quad n \mapsto [(n+1_{\mathbb{N}}, 1_{\mathbb{N}})]$ *be a function, then* $i_{\mathbb{N}}(\mathbb{N}') \subseteq \mathbb{Z}'$*, with the followings hold:*

1. $i_{\mathbb{N}}$ *is an injection, and* $i_{\mathbb{N}}(1_{\mathbb{N}}) = 1_{\mathbb{Z}}$.
2. *For* $m,n \in \mathbb{N}'$, $i_{\mathbb{N}}(n +_{\mathbb{N}} m) = i_{\mathbb{N}}(n) +_{\mathbb{Z}} i_{\mathbb{N}}(m)$.
3. *For* $m,n \in \mathbb{N}'$, $i_{\mathbb{N}}(n *_{\mathbb{N}} m) = i_{\mathbb{N}}(n) *_{\mathbb{Z}} i_{\mathbb{N}}(m)$.
4. *For* $m,n \in \mathbb{N}'$, $(n >_{\mathbb{N}} m) \iff (i_{\mathbb{N}}(n) >_{\mathbb{Z}} i_{\mathbb{N}}(m))$.
5. *For* $m,n \in \mathbb{N}'$, $(n <_{\mathbb{N}} m) \iff (i_{\mathbb{N}}(n) <_{\mathbb{Z}} i_{\mathbb{N}}(m))$.

Definition 1.2.0.0.1
Let $a,b,c,d \in \mathbb{Z}'$, *let* $\sim_{\mathbb{Q}}$ *be a relation on* $(\mathbb{Z}' \times (\mathbb{Z}' \setminus \{0_{\mathbb{Z}}\}))$ *with* $(a,b) \sim_{\mathbb{Q}} (c,d) \iff a *_{\mathbb{Z}} d = b *_{\mathbb{Z}} c$.

Definition 1.2.0.0.2
$\mathbb{Q}' := (\mathbb{Z}' \times (\mathbb{Z}' \setminus \{0_{\mathbb{Z}}\}))/\sim_{\mathbb{Q}}$

Lemma 1.2.1
The relation $\sim_{\mathbb{Q}}$ *is an equivalence relation on the set* $(\mathbb{Z}' \times (\mathbb{Z}' \setminus \{0_{\mathbb{Z}}\}))$.

Definition 1.2.1.0.1
$0_{\mathbb{Q}} := [(0_{\mathbb{Z}}, 1_{\mathbb{Z}})]$

Definition 1.2.1.0.2
$1_{\mathbb{Q}} := [(1_{\mathbb{Z}}, 1_{\mathbb{Z}})]$

Definition 1.2.1.0.3
Let $[(a,b)] \in \mathbb{Q}'$, $-[(a,b)] := [(-a,b)]$.

Definition 1.2.1.0.4
Let $[(a,b)] \in \mathbb{Q}'$ *with* $a \neq 0_{\mathbb{Z}}$, $[(a,b)]^{-1} := [(b,a)]$.

Definition 1.2.1.0.5
Let $[(a,b)],[(c,d)] \in \mathbb{Q}'$, $[(a,b)] +_{\mathbb{Q}} [(c,d)] := [(a *_{\mathbb{Z}} d +_{\mathbb{Z}} b *_{\mathbb{Z}} c, b *_{\mathbb{Z}} d)]$.

Definition 1.2.1.0.6
Let $[(a,b)],[(c,d)] \in \mathbb{Q}'$, $[(a,b)] -_{\mathbb{Q}} [(c,d)] := [(a,b)] +_{\mathbb{Q}} -[(c,d)]$.

Definition 1.2.1.0.7
Let $[(a,b)],[(c,d)] \in \mathbb{Q}'$, $[(a,b)] *_{\mathbb{Q}} [(c,d)] := [(a *_{\mathbb{Z}} c, b *_{\mathbb{Z}} d)]$.

Definition 1.2.1.0.8
Let $[(a,b)],[(c,d)] \in \mathbb{Q}'$ *with* $c \neq 0_{\mathbb{Z}}$, $\frac{[(a,b)]}{[(c,d)]} := [(a,b)] *_{\mathbb{Q}} [(c,d)]^{-1}$.

Lemma 1.2.2
For $[(a,b)],[(c,d)] \in \mathbb{Q}'$*, exactly one of the followings hold:*

1. $a *_{\mathbb{Z}} d >_{\mathbb{Z}} b *_{\mathbb{Z}} c$, *and* $b *_{\mathbb{Z}} d >_{\mathbb{Z}} 0_{\mathbb{Z}}$, *then we write* $[(a,b)] >_{\mathbb{Q}} [(c,d)]$.
2. $a *_{\mathbb{Z}} d <_{\mathbb{Z}} b *_{\mathbb{Z}} c$, *and* $b *_{\mathbb{Z}} d <_{\mathbb{Z}} 0_{\mathbb{Z}}$, *then we write* $[(a,b)] >_{\mathbb{Q}} [(c,d)]$.
3. $a *_{\mathbb{Z}} d <_{\mathbb{Z}} b *_{\mathbb{Z}} c$, *and* $b *_{\mathbb{Z}} d >_{\mathbb{Z}} 0_{\mathbb{Z}}$, *then we write* $[(a,b)] <_{\mathbb{Q}} [(c,d)]$.
4. $a *_{\mathbb{Z}} d >_{\mathbb{Z}} b *_{\mathbb{Z}} c$, *and* $b *_{\mathbb{Z}} d <_{\mathbb{Z}} 0_{\mathbb{Z}}$, *then we write* $[(a,b)] <_{\mathbb{Q}} [(c,d)]$.
5. $a *_{\mathbb{Z}} d = b *_{\mathbb{Z}} c$, *then we write* $[(a,b)] = [(c,d)]$.

Definition 1.2.2.0.1

The function $|\ |_{\mathbb{Q}} : \mathbb{Q}' \to \mathbb{Q}' \quad x \mapsto \begin{cases} x & x >_{\mathbb{Q}} 0_{\mathbb{Q}} \\ -x & x <_{\mathbb{Q}} 0_{\mathbb{Q}} \\ 0_{\mathbb{Q}} & x = 0_{\mathbb{Q}} \end{cases}$ *is called the Absolute Value function on* $\mathbb{Q}'$.

Theorem 1.3
Let $i_{\mathbb{Z}} : \mathbb{Z}' \to \mathbb{Q}' \quad n \mapsto [(n, 1_{\mathbb{Z}})]$ *be a function, then we have* $i_{\mathbb{Z}}(\mathbb{Z}') \subseteq \mathbb{Q}'$*, with the followings hold:*

1. $i_{\mathbb{Z}}$ *is an injection, with* $i_{\mathbb{Z}}(1_{\mathbb{Z}}) = 1_{\mathbb{Q}}$ *and* $i_{\mathbb{Z}}(0_{\mathbb{Z}}) = 0_{\mathbb{Q}}$.
2. *For* $m,n \in \mathbb{Z}'$, $i_{\mathbb{Z}}(n +_{\mathbb{Z}} m) = i_{\mathbb{Z}}(n) +_{\mathbb{Q}} i_{\mathbb{Z}}(m)$.
3. *For* $m,n \in \mathbb{Z}'$, $i_{\mathbb{Z}}(n *_{\mathbb{Z}} m) = i_{\mathbb{Z}}(n) *_{\mathbb{Q}} i_{\mathbb{Z}}(m)$.
4. *For* $m,n \in \mathbb{Z}'$, $(n >_{\mathbb{Z}} m) \iff (i_{\mathbb{Z}}(n) >_{\mathbb{Q}} i_{\mathbb{Z}}(m))$.
5. *For* $m,n \in \mathbb{Z}'$, $(n <_{\mathbb{Z}} m) \iff (i_{\mathbb{Z}}(n) <_{\mathbb{Q}} i_{\mathbb{Z}}(m))$.

Definition 1.3.0.0.1
Any function of the form $seq : \mathbb{N}' \to \mathbb{Q}' \quad n \mapsto q_n$ *is called a Sequence in* $\mathbb{Q}'$*, the function seq is denoted as* (q_n) *or* $n \mapsto q_n$*.*

Definition 1.3.0.0.2
Let (q_n) *be a sequence in* $\mathbb{Q}'$*,* (q_n) *is said to be Cauchy provided that for all* $L \in \mathbb{N}'$*,* $\exists\ N \in \mathbb{N}'$ *s.t.* $\forall n, m \in \mathbb{N}'$ *with* $m >_{\mathbb{N}} N$ *and* $n >_{\mathbb{N}} N$*, we have* $|q_n -_{\mathbb{Q}} q_m|_{\mathbb{Q}} <_{\mathbb{Q}} \frac{1_{\mathbb{Q}}}{i_{\mathbb{Z}}(i_N(L))}$*.*

Definition 1.3.0.0.3
Let $(a_n), (b_n)$ *be Cauchy sequences in* $\mathbb{Q}'$*, we write* $\lim_{n\to\infty}(a_n -_{\mathbb{Q}} b_n) = 0_{\mathbb{Q}}$ *provided that for all* $L \in \mathbb{N}'$*,* $\exists\ N \in \mathbb{N}'$ *s.t.* $\forall n \in \mathbb{N}'$ *with* $n \geq N$*, we have* $|a_n -_{\mathbb{Q}} b_n|_{\mathbb{Q}} <_{\mathbb{Q}} \frac{1_{\mathbb{Q}}}{i_{\mathbb{Z}}(i_{\mathbb{N}}(L))}$*.*

Definition 1.3.0.0.4
$\mathscr{C}_{\mathbb{Q}} := \{(q_n) \mid (q_n)$ *is a Cauchy sequence in* $\mathbb{Q}'\ \}$

Definition 1.3.0.0.5
Let $\sim_{\mathbb{R}}$ *be a relation on* $\mathscr{C}_{\mathbb{Q}}$ *with* $(a_n) \sim_{\mathbb{R}} (b_n) \iff \lim_{n\to\infty}(a_n -_{\mathbb{Q}} b_n) = 0_{\mathbb{Q}}$*.*

Lemma 1.3.1
The relation $\sim_{\mathbb{R}}$ *is an equivalence relation on the set* $\mathscr{C}_{\mathbb{Q}}$*.*

Definition 1.3.1.0.1
$\mathbb{R} := \mathscr{C}_{\mathbb{Q}}/\sim_{\mathbb{R}}$ *is called the Set of Real Numbers, and each element in* $\mathbb{R}$ *is called a Real Number.*

Definition 1.3.1.0.2
Let $(\frac{1}{n})$ *denote the sequence* $seq : \mathbb{N}' \to \mathbb{Q}' \quad n \mapsto \frac{1_{\mathbb{Q}}}{i_{\mathbb{Z}}(i_{\mathbb{N}}(n))}$*,* $0_{\mathbb{R}} := [(\frac{1}{n})]$*.*

Definition 1.3.1.0.3
Let $(\frac{1+n}{n})$ *denote the sequence* $seq : \mathbb{N}' \to \mathbb{Q}' \quad n \mapsto \frac{1_{\mathbb{Q}} + i_{\mathbb{Z}}(i_{\mathbb{N}}(n))}{i_{\mathbb{Z}}(i_{\mathbb{N}}(n))}$*,* $1_{\mathbb{R}} := [(\frac{n+1}{n})]$*.*

Definition 1.3.1.0.4
Let $[(a_n)] \in \mathbb{R}$*,* $-[(a_n)] := [(-a_n)]$*.*

Definition 1.3.1.0.5
Let $[(a_n)], [(b_n)] \in \mathbb{R}$*,* $[(a_n)] +_{\mathbb{R}} [(b_n)] := [(a_n +_{\mathbb{Q}} b_n)]$*.*

Definition 1.3.1.0.6
Let $[(a_n)], [(b_n)] \in \mathbb{R}$*,* $[(a_n)] -_{\mathbb{R}} [(b_n)] := [(a_n)] +_{\mathbb{R}} -[(b_n)]$*.*

Definition 1.3.1.0.7
Let $[(a_n)], [(b_n)] \in \mathbb{R}$*,* $[(a_n)] *_{\mathbb{R}} [(b_n)] := [(a_n *_{\mathbb{Q}} b_n)]$*.*

Lemma 1.3.2
For $[(a_n)] \in \mathbb{R}$ *with* $[(a_n)] \neq 0_{\mathbb{R}}$*,* $\exists\ N \in \mathbb{N}'$ *s.t.* $\forall n \in \mathbb{N}'$ *with* $n >_{\mathbb{N}} N$ *we have* $a_n \neq 0_{\mathbb{Q}}$*. Let* (b_k) *be a sequence with* $b_k = 1_{\mathbb{Q}}$ *for* $1_{\mathbb{N}} \leq_{\mathbb{N}} k \leq_{\mathbb{N}} N$*, and* $b_k = \frac{1_{\mathbb{Q}}}{a_k}$ *for* $k >_{\mathbb{N}} N$*, then* (b_k) *belongs to* $\mathscr{C}_{\mathbb{Q}}$ *and we have* $[(a_n)] *_{\mathbb{R}} [(b_n)] = 1_{\mathbb{R}}$*, we write* $[(a_n)]^{-1} := [(b_n)]$*.*

Definition 1.3.2.0.1
Let $[(a_n)], [(b_n)] \in \mathbb{R}$ *with* $[(b_n)] \neq 0_{\mathbb{R}}$*,* $\frac{[(a_n)]}{[(b_n)]} := [(a_n)] *_{\mathbb{R}} [(b_n)]^{-1}$*.*

Lemma 1.3.3
For $[(a_n)], [(b_n)] \in \mathbb{R}$*, exactly one of the followings hold:*

1. $\exists\ N \in \mathbb{N}$ *s.t.* $\forall n \in \mathbb{N}$ *with* $n >_{\mathbb{N}} N$*, we have* $(a_n -_{\mathbb{Q}} b_n) >_{\mathbb{Q}} 0_{\mathbb{Q}}$*, then we write* $[(a_n)] >_{\mathbb{R}} [(b_n)]$*.*
2. $\exists\ N \in \mathbb{N}$ *s.t.* $\forall n \in \mathbb{N}$ *with* $n >_{\mathbb{N}} N$*, we have* $(b_n -_{\mathbb{Q}} a_n) >_{\mathbb{Q}} 0_{\mathbb{Q}}$*, then we write* $[(a_n)] <_{\mathbb{R}} [(b_n)]$*.*
3. $\lim_{n\to\infty}(a_n -_{\mathbb{Q}} b_n) = 0_{\mathbb{Q}}$*, then we write* $[(a_n)] = [(b_n)]$*.*

Definition 1.3.3.0.1
The function $|\ | : \mathbb{R} \to \mathbb{R} \quad x \mapsto \begin{cases} x & x >_{\mathbb{R}} 0_{\mathbb{R}} \\ -x & x <_{\mathbb{R}} 0_{\mathbb{R}} \\ 0_{\mathbb{R}} & x = 0_{\mathbb{R}} \end{cases}$ *is called the Absolute Value function on* $\mathbb{R}$*.*

Theorem 1.4
Let $i_{\mathbb{Q}} : \mathbb{Q}' \to \mathbb{R} \quad n \mapsto [(n, n, \cdots, n, \cdots)]$ *be a function, then* $i_{\mathbb{Q}}(\mathbb{Q}') \subseteq \mathbb{R}$*, with the followings hold:*

1. $i_{\mathbb{Q}}$ *is an injection, with* $i_{\mathbb{Q}.}(1_{\mathbb{Q}}) = 1_{\mathbb{R}}$ *and* $i_{\mathbb{Q}}(0_{\mathbb{Q}}) = 0_{\mathbb{R}}$*.*
2. *For* $m, n \in \mathbb{Q}'$*,* $i_{\mathbb{Q}}(n +_{\mathbb{Q}} m) = i_{\mathbb{Q}}(n) +_{\mathbb{R}} i_{\mathbb{Q}}(m)$*.*
3. *For* $m, n \in \mathbb{Q}'$*,* $i_{\mathbb{Q}}(n *_{\mathbb{Q}} m) = i_{\mathbb{Q}}(n) *_{\mathbb{R}} i_{\mathbb{Q}}(m)$*.*
4. *For* $m, n \in \mathbb{Q}'$*,* $(n >_{\mathbb{Q}} m) \iff (i_{\mathbb{Q}}(n) >_{\mathbb{R}} i_{\mathbb{Q}}(m))$*.*
5. *For* $m, n \in \mathbb{Q}'$*,* $(n <_{\mathbb{Q}} m) \iff (i_{\mathbb{Q}}(n) <_{\mathbb{R}} i_{\mathbb{Q}}(m))$*.*

Definition 1.4.0.0.1
The set $\mathbb{N} \coloneqq i_{\mathbb{Q}}(i_{\mathbb{Z}}(i_{\mathbb{N}}(N')))$ *is called the Set of Natural Numbers.*
The set $\mathbb{Z} \coloneqq i_{\mathbb{Q}}(i_{\mathbb{Z}}(Z'))$ *is called the Set of Integers.*
The set $\mathbb{Q} \coloneqq i_{\mathbb{Q}}(Q')$ *is called the Set of Rational Numbers.*

Definition 1.4.0.0.2
Each element in the set $\mathbb{N}$ *is called a Natural Number.*
Each element in the set $\mathbb{Z}$ *is called an Integer.*
Each element in the set $\mathbb{Q}$ *is called a Rational Number.*

Definition 1.4.0.0.3
$0 \coloneqq 0_{\mathbb{R}}$
$1 \coloneqq 1_{\mathbb{R}}$
$2 \coloneqq 1 + 1$

Definition 1.4.0.0.4
Let $m, n \in \mathbb{R}$*, we define the followings:*

1. $m + n \coloneqq m +_{\mathbb{R}} n$
2. $m * n \coloneqq m *_{\mathbb{R}} n$
3. $m - n \coloneqq m -_{\mathbb{R}} n$
4. $(m < n) \iff (m <_{\mathbb{R}} n)$
5. $(m > n) \iff (m >_{\mathbb{R}} n)$

Definition 1.4.0.0.5
Any function of the form $seq : \mathbb{N} \to \mathbb{R} \quad n \mapsto r_n$ *is called a Sequence in of Real Numbers, the function* seq *is denoted as* (r_n) *or* $n \mapsto r_n$*.*

Definition 1.4.0.0.6
Let (r_n) *be a sequence of real numbers,* (r_n) *is said to be Cauchy provided that for all* $\epsilon \in \mathbb{R}$ *with* $\epsilon > 0$*,* $\exists\ N \in \mathbb{N}$ *s.t.* $\forall n, m \in \mathbb{N}$ *with* $m > N$ *and* $n > N$*, we have* $|r_n - r_m| < \epsilon$*.*

Definition 1.4.0.0.7
Let (r_n) *be a sequence of real numbers,* (r_n) *converges to some* $l \in \mathbb{R}$ *provided that for all* $\epsilon \in \mathbb{R}$ *with* $\epsilon > 0$*,* $\exists\ N \in \mathbb{N}$ *s.t.* $\forall n \in \mathbb{N}$ *with* $n > N$*, we have* $|r_n - l| < \epsilon$*. If* (r_n) *converges, then we say* (r_n) *is a Convergent Sequence of real numbers in the Euclidean topology. If* (r_n) *converges to some* $l \in \mathbb{R}$*, then* $\lim_{n\to\infty} r_n \coloneqq l$ *is called the limit of* (r_n)*.*

Theorem 1.5
Let $r \in \mathbb{R}$*, for all* $w \in \mathbb{Q}$ *with* $w > 0$*,* $\exists\ q \in \mathbb{Q}$ *s.t.* $|r - q| < w$*.*

Lemma 1.5.1
Q *is Archimedean, that is, the followings hold:*

1. *For* $q \in \mathbb{Q}$*,* $\exists\ N \in \mathbb{N}$ *s.t.* $N > q$*.*
2. *For* $q \in \mathbb{Q}$ *with* $q > 0$*,* $\exists\ N \in \mathbb{N}$ *s.t.* $\frac{1}{N} < q$*.*

Lemma 1.5.2
Let (x_n) *be a Cauchy sequence of real numbers, then* (x_n) *converges to some* $l \in \mathbb{R}$*.*

Definition 1.5.2.0.1
Let (t_m) *be a sequence of real numbers,* (t_m) *is Monotonic Increasing provided that for all* $a, b \in \mathbb{N}$ *with* $a > b$*, we have* $t_a \geq t_b$*.*

Definition 1.5.2.0.2
Let (t_m) *be a sequence of real numbers,* (t_m) *is Monotonic Decreasing provided that for all* $a, b \in \mathbb{N}$ *with* $a > b$*, we have* $t_a \leq t_b$*.*

Definition 1.5.2.0.3
Let (t_m) *be a sequence of real numbers,* (t_m) *is Monotonic provided that* (t_m) *is either monotonic increasing or monotonic decreasing.*

Definition 1.5.2.0.4
Let (t_m) *be a sequence of real numbers,* (t_m) *is Bounded provided that for all* $s \in \{t_n \mid m \in \mathbb{N}\}$*, there exist* $M, N \in \mathbb{R}$ *such that* $N < s < M$*.*

Lemma 1.5.3
Let (x_n) *be a sequence of real numbers. If* (x_n) *is monotonic and bounded, then* (x_n) *is Cauchy.*

Corollary 1.5.3.1
All bounded monotonic sequences of real numbers converge.

Theorem 1.6
The ordered field $(\mathbb{R}, +, *, 1, 0, <)$ *has the least upper bound property.*

Definition 1.6.0.0.1
Let $(F_1, +_1, *_1, 0_1, 1_1)$ *and* $(F_2, +_2, *_2, 0_2, 1_2)$ *be fields, the field* $(F_1, +_1, *_1, 0_1, 1_1)$ *is isomorphic to the field* $(F_2, +_2, *_2, 0_2, 1_2)$ *provided that there exists a bijection* $\varphi : F_1 \to F_2$ *such that* $\forall x, y \in F_1$, *we have the followings hold:*

1. $\varphi(x +_1 y) = \varphi(x) +_2 \varphi(y)$
2. $\varphi(x *_1 y) = \varphi(x) *_2 \varphi(y)$
3. $\varphi(1_1) = 1_2$

Definition 1.6.0.0.2
Let $(F_1, +_1, *_1, 0_1, 1_1, <_1)$ *and* $(F_2, +_2, *_2, 0_2, 1_2, <_2)$ *be ordered fields, we say* $(F_1, +_1, *_1, 0_1, 1_1, <_1)$ *is isomorphic to* $(F_2, +_2, *_2, 0_2, 1_2, <_2)$ *provided that there exists a bijection* $\varphi : F_1 \to F_2$ *such that* $\forall x, y \in F_1$, *we have the followings hold:*

1. *If* $x <_1 y$, *then* $\varphi(x) <_2 \varphi(y)$.
2. *If* $x >_1 y$, *then* $\varphi(x) >_2 \varphi(y)$.
3. $\varphi(x +_1 y) = \varphi(x) +_2 \varphi(y)$
4. $\varphi(x *_1 y) = \varphi(x) *_2 \varphi(y)$
5. $\varphi(1_1) = 1_2$

Theorem 1.7
Let $(F_1, +_F, *_F, 0_F, 1_F, <_F)$ *be an ordered field. If* $(F_1, +_F, *_F, 0_F, 1_F, <_F)$ *is a complete ordered field, then* $(F_1, +_F, *_F, 0_F, 1_F, <_F)$ *is isomorphic to* $(\mathbb{R}, +, *, 1, 0, <)$.

Definition 1.7.0.0.1
The ordered field $(\mathbb{R}, +, *, 1, 0, <)$ *is called the Unique Complete Ordered field.*

Fields

Theorem 2.1 (De Morgan's Law)
Let S be a set, let A, B be subsets of S, then the followings hold:

1. $(A \cap B)^C = A^C \cup B^C$
2. $S(A \cup B)^C = A^C \cap B^C$

Lemma 2.1.1
Let S be a set, let $\diamond$ be a binary operation on S. If $\diamond$-identity exists, then the $\diamond$-identity is unique.

Lemma 2.1.2
Let S be a set, let $\diamond$ be a binary operation on S.
If $\diamond$ is associative and $s' \in S$ is the $\diamond$-inverse of $s \in S$, then s' is the unique $\diamond$-inverse of s.

Lemma 2.1.3
Let F be a field, for $a, b, c, d \in F$, the followings hold:

1. $-(-a) = a$
2. $-(a + b) = (-a) + (-b)$
3. $(a + b = c + b) \iff (a = c)$
4. $a * 0 = 0$
5. $(a * b = 0) \iff (a = 0 \text{ or } b = 0)$
6. $(-a) * b = -(a * b)$
7. $(-a) * (-b) = a * b$
8. If $a \neq 0_F$, then $(a^{-1})^{-1} = a$.
9. If $b \neq 0_F$ and $c \neq 0_F$, then $a * b^{-1} = (a * c) * (b * c)^{-1}$.
10. If $b \neq 0_F$ and $d \neq 0_F$, then $a * d^{-1} + c * d^{-1} = (a * d + b * c) * (b * d)^{-1}$.
11. If $a \neq 0_F$ and $b \neq 0_F$, then $(a * b)^{-1} = a^{-1} * b^{-1}$.
12. If $b \neq 0_F$ and $d \neq 0_F$, then $(a * d = c * b) \iff (\frac{a}{b} = \frac{c}{d})$.

Theorem 2.2
Let F be a field, the followings hold:

1. For $a \in F$, $(a * a = 0) \iff (a = 0)$.
2. For $a \in F$, if $\exists\, b \in F$ s.t. $b^2 = a$, then $(-b)^2 = a$.

Consider using the ordered field (F, P) from now on.

Lemma 2.2.1
For $a, b, c, d \in F$, the followings hold:

1. If $a < b$, then $(-b) < (-a)$.
2. If $a > b$ and $b > c$, then $a > c$.
3. If $a < b$ and $c < d$, then $(a + c) < (b + d)$.
4. If $a < b$ and $c > d$, then $(a - c) < (b - d)$.
5. If $a < b$ and $c > 0_F$, then $(a * c) < (b * d)$.
6. If $a < b$ and $c < 0_F$, then $(a * c) > (b * d)$.
7. If $a > 1_F$, then $a^2 > a$.
8. If $0_F \leq a < b$, then $a^2 < b^2$.
9. If $0_F < a < 1_F$, then $a^2 < a$.
10. If $0_F \leq a < b$ and $0_F \leq c < d$, then $a * c < b * d$.
11. If $a \geq 0_F$, $b \geq 0_F$, and $a^2 < b^2$, then $a < b$.

Lemma 2.2.2
For $a, b, c \in F$ with $a, b \in (P \cup \{0_F\})$, the followings hold:

1. $(c \in P) \iff (c^{-1} \in P)$
2. $(a > 0_F) \iff (a^2 > 0_F)$
3. $(a = b) \iff (a^2 = b^2)$
4. $(a > b) \iff (a^2 > b^2)$

Corollary 2.2.2.1
For $a \in F \setminus \{0_F\}$, we have $a^2 \in P$.

Corollary 2.2.2.2
Given the ordered field (F, P), we have $1_F > 0_F$.

Corollary 2.2.2.3
Given the ordered field (F, P), $2_F \coloneqq 1_F + 1_F$ belongs to P.

Lemma 2.2.3
For $a, b \in F$, we have $(a \leq b) \iff (x < y + \epsilon \; \forall \epsilon \in P)$.

Lemma 2.2.4
For $x \in F$, we have $|x|^2 = x^2$.

Lemma 2.2.5
For $x \in F$, we have $x \leq |x|$.

Lemma 2.2.6
For $a, b \in F$, we have $(-b \leq a \leq b) \iff (|a| < b)$.

Theorem 2.3
For $a, b \in F$, the followings hold:

1. $|a * b| = |a| * |b|$
2. $|a - b| \leq (|a| + |b|)$
3. $(|a| - |b|) \leq |a - b|$
4. $|(|x| - |y|)| \leq |x - y|$
5. *If $b \neq 0_F$, then $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.*

Theorem 2.4 (Triangle Inequality)
For $a, b \in F$, we have $|a + b| \leq |a| + |b|$.

Theorem 2.5 (Schwarz Inequality)
For $a, b, c, d \in F$, we have $a \cdot b + c \cdot d \leq \sqrt{a^2 + c^2} \cdot \sqrt{b^2 + d^2}$.

Lemma 2.5.1
Let A be a subset of F. If A has a supremum, then the supremum is unique.

Corollary 2.5.1.1
Let $a \in F$, and let $A = \{x \in F \mid x < a\}$ be a subset of F, then a is the least upper bound for A.

Lemma 2.5.2
Let A be a subset of F. If A has an infimum, then the infimum is unique.

Corollary 2.5.2.1
Let $a \in F$, and let $A = \{x \in F \mid x > a\}$ be a subset of F, then a is the greatest lower bound for A.

Theorem 2.6
(The ordered field F has the lest upper bound property) iff (F has the greatest upper bound property)

Lemma 2.6.1
$A \subseteq F$ has a maximal element iff the followings hold:

1. *A has a supremum.*
2. *The supremum of A belongs to A.*

Lemma 2.6.2
$B \subseteq F$ has a minimal element iff the followings hold:

1. *B has an infimum.*
2. *The infimum of B belongs to B.*

Theorem 2.7
Every nonempty finite subset of $\mathbb{R}$ contains a maximal element and a minimal element.

Lemma 2.7.1
For $a, b \in \mathbb{N}_F$, we have $a + b \in \mathbb{N}_F$, and $a \cdot b \in \mathbb{N}_F$,
where $\mathbb{N}_F := \{x \in F \mid x$ belongs to every inductive subset of $F\}$.

Lemma 2.7.2
There exists a unique inductive set $\mathbb{N}_F \subseteq F$ that contains all inductive sets in F.

Lemma 2.7.3
For $n \in N_F$, we have $n \geq 1$.

Lemma 2.7.4
For $m, n \in \mathbb{N}_F$, we have $(n < m) \iff m = n + r$ for some $r \in \mathbb{N}_F$.

Lemma 2.7.5
For $a, b \in \mathbb{N}_F$. If $a > b$, then $a - b \in \mathbb{N}_F$.

Theorem 2.8
The only strongly inductive subset of $\mathbb{N}_F$ is $\mathbb{N}_F$.

Theorem 2.9
Every nonempty subset of $\mathbb{N}_F$ has a minimal element.

The assumption of using the ordered field (F, P) is now removed.

From now on, we use $(\mathbb{R}, \mathbb{R}_{>0})$ to denote the unique complete ordered field.

Lemma 2.9.1
In the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, let A, B be nonempty subsets of $\mathbb{R}$. If $\forall a \in A,\ \forall b \in B$, we have $a < b$, then the followings hold:

1. $\sup(A) \leq \inf(B)$
2. $(\sup(A) = \inf(B)) \iff (\forall n \in \mathbb{N},\ \exists\, b \in B$ *and* $a \in A$ *s.t.* $b - a < \frac{1}{n})$

Lemma 2.9.2
In the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, the subset $\mathbb{N} \subseteq \mathbb{R}$ is not bounded above.

Lemma 2.9.3
Let (F, P) be an ordered field, for $a, b \in P$, we have $(a < b) \iff (a^{-1} > b^{-1})$.

Corollary 2.9.3.1 (Archimedean Property for $\mathbb{R}$)
In the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, $\forall \epsilon \in \mathbb{R}$ with $\epsilon > 0,\ \exists\, n \in \mathbb{N}$ s.t. $n^{-1} < \epsilon$.

Corollary 2.9.3.2 (The Density of $\mathbb{Q}$ in $\mathbb{R}$)
In the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, for $a, b \in \mathbb{R}$ with $a < b$, $\exists\, q \in \mathbb{Q}$ s.t. $a < q < b$.

Theorem 2.10 (Characterization of Supremum)
In the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, let A be a nonempty bounded above subset of $\mathbb{R}$, let $\alpha \in \mathbb{R}$ be an upper bound for A, then we have $(\alpha = \sup(A)) \iff (\forall \epsilon > 0,\ \exists\, a \in A$ s.t. $\alpha - \epsilon < a \leq \alpha)$.

Theorem 2.11 (Characterization of Infimum)
In the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, let B be a nonempty bounded below subset of $\mathbb{R}$, let $\beta \in \mathbb{R}$ be a lower bound for B, then we have $(\beta = \inf(B)) \iff (\forall \epsilon > 0,\ \exists\, b \in B$ s.t. $\beta \leq b < \beta + \epsilon)$.

Lemma 2.11.1
In the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, let A be a nonempty subset of $\mathbb{R}$.
If A is bounded above, then $\forall z \in \mathbb{R}$, $(z \geq \sup(A)) \iff (z \geq a$ for all $a \in A)$.
If A is bounded below, then $\forall s \in \mathbb{R}$, $(s \leq \inf(A)) \iff (s \leq a$ for all $a \in A)$.

Theorem 2.12
In the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, let A be a subset of $\mathbb{R}$.
If A is bounded above in $(\mathbb{R}, \mathbb{R}_{>0})$, then A has an supremum.
If A is bounded below in $(\mathbb{R}, \mathbb{R}_{>0})$, then A has an infimum.

Lemma 2.12.1
In the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, let A be a subset of $\mathbb{R}$, let $b \in \mathbb{R}$,

1. *(A is bounded above) $\iff$ ($-A$ is bounded below)*
2. *(A is bounded above) $\iff$ ($b + A$ is bounded above)*
3. *(A is bounded below) $\iff$ ($b + A$ is bounded below)*
4. *(A is not empty and bounded above) $\iff$ $(-\sup(A) = \inf(-A))$*
5. *If A is nonempty and bounded above, then $\sup(b + A) = b + \sup(A)$.*
6. *If A is nonempty and bounded below, then $\inf(b + A) = b + \inf(A)$.*

Lemma 2.12.2
In the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, let A, B be subsets of $\mathbb{R}$. If both $\sup(A)$ and $\sup(B)$ exist, then we have $\sup(A) + \sup(B) = \sup(A + B)$, that is, $\sup(A) + \sup(B)$ is the supremum of the set $A + B$.

Theorem 2.13
Let (F, P) be an ordered field, let $a \in P$, let $S_a = \{x \in F \mid x^2 \leq a\}$, then S_a is not empty and bounded above, and ($x \in F$ is the supremum of S_a) $\iff$ ($x^2 = a$ has a solution in F).

Corollary 2.13.1
In the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, $\forall a \in P,\ \exists\, x \in \mathbb{R}$ s.t. $x^2 = a$, that is, all positive real numbers in $\mathbb{R}$ have at least one square root.

Corollary 2.13.2 (The Density of Irrational Numbers in $\mathbb{R}$)
In the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, for $a, b \in \mathbb{R}$ with $a < b$, $\exists\, r \in (\mathbb{R} \setminus \mathbb{Q})$, s.t. $a < r < b$.

Lemma 2.13.3
In the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, for $a, b \in \mathbb{R}$ with $a > 0$ and $b > 0$, the followings hold:

1. *If $a < b$, then for all $n \in \mathbb{N}$, we have $a^n \leq b^n$.*
2. *If $a^n \leq b^n$ for some $n \in \mathbb{N}$, then $a < b$.*

Consider using the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$ from now on.

Theorem 2.14
Let S be a subset of $\mathbb{N}$. If S is weakly inductive, then $S = \mathbb{N}$.

Lemma 2.14.1
$1 \in \mathbb{R}$ *is the minimal element of* $\mathbb{N}$.

Lemma 2.14.2
For $n, m \in \mathbb{N}$, we have $(m > n) \iff (m \geq n + 1)$.

Corollary 2.14.2.1
For $n, m \in \mathbb{N}$, we have $(n < m \leq n + 1) \iff (m = n + 1)$.

Theorem 2.15
$\mathbb{N}$ is well ordered, that is, if $S \subset \mathbb{N}$ is nonempty, then $\exists\, l \in S$ such that $t \leq s$ for all $s \in S$.

Lemma 2.15.1
Every nonempty bounded above subset of $\mathbb{N}$ has a maximal element, that is, if $S \subset \mathbb{N}$ is nonempty and bounded above, then $\exists\, l \in S$ such that $t \geq s$ for all $s \in S$.

Theorem 2.16
Let S be a subset of $\mathbb{N}$. If S is strongly inductive, then $S = \mathbb{N}$.

Lemma 2.16.1
For all $n, m \in \mathbb{N}$, we have $2 \cdot n \neq 2 \cdot m - 1$.

Lemma 2.16.2
For $n \in \mathbb{N}$, $\exists!\ m \in \mathbb{N}$ s.t. either one of the followings hold:

1. $n = 2 \cdot m$
2. $n = 2 \cdot m - 1$

Corollary 2.16.2.1
A natural number is either an odd number or an even number.

Lemma 2.16.3
The sum of a rational number and an irrational number is an irrational number.
The product of a non-zero rational number and an irrational number is an irrational number.

Lemma 2.16.4
Let A be a infinite set, then there exist a proper subset B of A and a function $g : A \to B$ such that g is a bijection.

Lemma 2.16.5
Given $a \in \mathbb{N}$, we denote $N_a \coloneqq \{n \in \mathbb{N} \mid n \leq a\}$.
Let $m, l \in \mathbb{N}$. then (there exists a bijection from $\mathbb{N}_l$ to $\mathbb{N}_m$) $\iff$ ($l = m$).

Lemma 2.16.6
Given $a \in \mathbb{N}$, we denote $N_a \coloneqq \{n \in \mathbb{N} \mid n \leq a\}$. Let $m \in \mathbb{N}$.
If S is a nonempty subset of $\mathbb{N}_m$, then $\exists$ a bijection from S to N_l for some $l \in \mathbb{N}$ with $l \leq n$.

Corollary 2.16.6.1
Given $a \in \mathbb{N}$, we denote $N_a \coloneqq \{n \in \mathbb{N} \mid n \leq a\}$.
Let A be a set. If A is finite, then $\exists!\ n \in \mathbb{N}$ such that there exists a bijection from $\mathbb{N}_n$ to A.

Corollary 2.16.6.2
Every subset of $\mathbb{N}$ is countable.

Lemma 2.16.7
Let A be a nonempty set, then (A is countable) $\iff$ ($\exists$ an injection from A to $\mathbb{N}$).

Lemma 2.16.8
Let A be a nonempty set, then (A is countable) $\iff$ ($\exists$ a surjection from $\mathbb{N}$ to A).

Theorem 2.17
Let A be a nonempty set, the followings are equivalent:

1. *The set A is countable.*
2. *There exists a surjection from $\mathbb{N}$ to A.*
3. *There exists an injection from A to $\mathbb{N}$.*

Corollary 2.17.1
A subset of a countable set is countable.

Corollary 2.17.2
A union of finitely many countable sets is countable.

Corollary 2.17.3
The set $\mathbb{N} \times \mathbb{N}$ *is countable.*

Lemma 2.17.4
The set $\mathbb{Z}$ *is countably infinite.*

Corollary 2.17.4.1
The set $\mathbb{Z}$ *is countable.*

Proposition 2.17.5
The set $\mathbb{Q}$ *is countably infinite.*

Corollary 2.17.5.1
The set $\mathbb{Q}$ *is countable.*

Proposition 2.17.6
Let $X = \{x_1, x_2\}$ *be a set with* $x_1 \neq x_2$*,*
then the set $S \coloneqq \{f \mid f \text{ is a function from } \mathbb{N} \text{ to } X\}$ *is not countable.*

Lemma 2.17.7
Every nonempty bounded below subset of $\mathbb{Z}$ *has a minimal element, that is, if* $S \subset \mathbb{Z}$ *is nonempty and bounded below, then there exists* $l \in S$ *such that* $t \leq s$ *for all* $s \in S$*.*

Lemma 2.17.8
Every nonempty bounded above subset of $\mathbb{Z}$ *has a maximal element, that is, if* $S \subset \mathbb{Z}$ *is nonempty and bounded above, then there exists* $l \in S$ *such that* $t \geq s$ *for all* $s \in S$*.*

Theorem 2.18
For $a, b \in \mathbb{R}$ *with* $b - a > 1$*, there exists* $m \in \mathbb{Z}$ *such that* $a < m < b$*.*

Corollary 2.18.1
For $a, b \in \mathbb{R}$ *with* $b - a \geq 1$*, there exists* $m \in \mathbb{Z}$ *such that* $a < m < b$*.*

Corollary 2.18.2
For $b \in \mathbb{R}$*, there exists a unique* $m \in \mathbb{Z}$ *such that* $b - 1 < m < b$*.*

Lemma 2.18.3
Let X *be a set, let* $\sim$ *be an equivalence relation on* X*, then the followings hold:*

1. *For all* $y \in X$*, we have* $y \in C(y)$*.*
2. *For all* $x, y \in X$*, if* $x \sim y$*, then* $C(x) = C(y)$*.*
3. *For all* $x, y \in X$*, we have* $(x \sim y) \iff (C(x) \cap C(y) \neq \emptyset)$*.*
4. *We have* $X = \bigcup_{x \in X} C(x)$ *and* $X = \bigsqcup_{C \in X/\sim} C$*.*

Theorem 2.19 (Division Algorithm for Natural Numbers)
For $D, d \in \mathbb{N},\ \exists!\ q, r \in \mathbb{N} \cup \{0\}$ *with* $r < d$ *s.t.* $D = q \cdot d + r$*.*

Theorem 2.20 (Division Algorithm)
For $D, d \in \mathbb{Z}$ *with* $d \neq 0$*,* $\exists!\ q, r \in \mathbb{Z}$ *with* $0 \leq r < |b|$ *such that* $D = q \cdot d + r$*.*

Lemma 2.20.1 (Bezout's Theorem)
For $a, b \in \mathbb{N}$*,* $\exists\ m, n \in \mathbb{Z}$ *s.t.* $a \cdot m + b \cdot n = \gcd(a, b)$*.*

Topology

From now on, consider using the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$.

Lemma 3.0.1
Let $a, b \in \mathbb{R}$ with $a \leq b$, the followings are intervals:
$(a, b) \coloneqq \{x \in \mathbb{R} \mid a < x < b\}$ $\quad [a, b] \coloneqq \{x \in \mathbb{R} \mid a \leq x \leq b\}$ $\quad (-\infty, \infty) \coloneqq \{x \in \mathbb{R}\}$
$(a, b] \coloneqq \{x \in \mathbb{R} \mid a < x \leq b\}$ $\quad [a, b) \coloneqq \{x \in \mathbb{R} \mid a \leq x < b\}$ $\quad (a, \infty) \coloneqq \{x \in \mathbb{R} \mid a < x\}$
$(-\infty, b) \coloneqq \{x \in \mathbb{R} \mid x < b\}$ $\quad [a, \infty) \coloneqq \{x \in \mathbb{R} \mid a \leq x\}$ $\quad (-\infty, b] \coloneqq \{x \in \mathbb{R} \mid x \leq b\}$

Consider using the topological space $(\mathbb{R}, \mathcal{T}_{EUC})$ from now on.

Lemma 3.0.2
Let $a, b \in \mathbb{R}$ with $a \leq b$, the following intervals are open in the Euclidean topology:
(a, b) $\quad (a, \infty)$ $\quad (-\infty, b)$ $\quad (-\infty, \infty)$

Corollary 3.0.2.1
Let $a, r \in \mathbb{R}$ with $r > 0$, then $B_r(a) = (a - r, a + r)$ is an open interval in the Euclidean topology.

Corollary 3.0.2.2
Let $a, b \in \mathbb{R}$ with $a \leq b$, the following intervals are closed in the Euclidean topology:
$[a, b]$ $\quad [a, \infty)$ $\quad (-\infty, b]$ $\quad (-\infty, \infty)$

Lemma 3.0.3
Let A be a subset of $\mathbb{R}$, let $f : A \to \mathbb{R}$ be a function. If the limit of f at $a \in A$ exists, then the limit of f at a is unique, that is, if $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} f(x) = m$, then $l = m$.

Lemma 3.0.4
Let A be a subset of $\mathbb{R}$, let $h : A \to \mathbb{R}$ be a function, let $a \in A$. If $\lim_{x \to a} h(x) = l$ for some $l \in \mathbb{R} \setminus \{0\}$, then $\exists\, \delta > 0$ s.t. if $x \in \mathbb{R}$ satisfies $0 < |x - a| < \delta$, we have $|h(x)| > \frac{|l|}{2}$.

Lemma 3.0.5
Let A be a subset of $\mathbb{R}$, let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions, let $a \in A$. If $\lim_{x \to a} f(x) = m$ for some $m \in \mathbb{R}$, and $\lim_{x \to a} g(x) = n$ for some $n \in \mathbb{R}$, then the followings hold:

1. $\lim_{x \to a}(c \cdot f)(x) = \lim_{x \to a}[c \cdot f(x)] = c \cdot m$
2. $\lim_{x \to a}(f + g)(x) = \lim_{x \to a}[f(x) + g(x)] = m + n$
3. $\lim_{x \to a}(f \cdot g)(x) = \lim_{x \to a}[f(x) \cdot g(x)] = m \cdot n$
4. *If $n \neq 0$, then* $\lim_{x \to a}(\frac{f}{g})(x) = \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{m}{n}$

Lemma 3.0.6
Let A, B be subsets of $\mathbb{R}$, let $f : A \to B$ and $g : B \to \mathbb{R}$ be functions, let $a \in A$ and $l \in B$, let $\circ$ denote function composition. If $\lim_{x \to a} f(x) = l$, $\lim_{y \to l} g(y) = m$ for some $m \in \mathbb{R}$, and $m = g(l)$, then we have $\lim_{x \to a}(g \circ f)(x) = m$.

Lemma 3.0.7
Let A, B be sets, let $S_1 \subseteq S_2 \subseteq A$, $T_1 \subseteq T_2 \subseteq B$, let $g : A \to B$ be a function, the followings hold:

1. $g(S_1 \cap S_2) \subseteq g(S_1) \cap g(S_2)$
2. $g(S_1) \subseteq g(S_2)$
3. $g(S_1 \cup S_2) = g(S_1) \cup g(S_2)$
4. $g^{-1}(T_1) \subseteq g^{-1}(T_2)$
5. $g^{-1}(T_1 \cap T_2) = g^{-1}(T_1) \cap g^{-1}(T_2)$
6. $S_1 \subseteq g^{-1}(g(S_1))$
7. $g(g^{-1}(T_1)) \subseteq T_1$

Theorem 3.1
Let $f : \mathbb{R} \to \mathbb{R}$ be a function. (f is continuous) $\iff$ (for all $U \in \mathcal{T}_{EUC}$, we have $f^{-1}(U) \in \mathcal{T}_{EUC}$).

Lemma 3.1.1
The topological space $(\mathbb{R}, \mathcal{T}_{EUC})$ is connected.

Corollary 3.1.1.1
Let $A \subseteq \mathbb{R}$ be an interval, let $\mathcal{T}_A$ denote the subspace topology on A inherited from $\mathcal{T}_{EUC}$, then the topological space $(A, \mathcal{T}_A)$ is connected.

Theorem 3.2
*Let A be a subset of $\mathbb{R}$, let $\mathcal{T}_A$ denote the subspace topology on A inherited from $\mathcal{T}_{EUC}$,
then (A is an interval) $\iff$ (the topological space $(A, \mathcal{T}_A)$ is connected).*

The assumption of using $(\mathbb{R}, \mathcal{T}_{EUC})$ is now removed.

Lemma 3.2.1
Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y), (Z, \mathcal{T}_Z)$ be topological spaces, let $f : X \to Y$ and $g : Y \to Z$ be continuous functions, let $\circ$ denote function composition, then the function $(g \circ f) : X \to Z$ is also continuous.

Lemma 3.2.2
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, let A be a subset of X, let $\mathcal{T}_A$ denote the subspace topology on A inherited from $\mathcal{T}_X$, let $f : X \to Y$ be a continuous function, then the restriction of f on A, $res_A f : A \to Y$ is a continuous function from $(A, \mathcal{T}_A)$ to $(Y, \mathcal{T}_Y)$.

Lemma 3.2.3
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, let $f : X \to Y$ be a continuous function, let $\mathcal{T}_{f(X)}$ denote the subspace topology on $f(X)$ inherited from $\mathcal{T}_Y$, then the function $f : X \to f(X)$ is a continuous function from $(X, \mathcal{T}_X)$ to $(f(A), \mathcal{T}_{f(A)})$.

Theorem 3.3
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, let $f : X \to Y$ be a continuous function, let $\mathcal{T}_{f(X)}$ denote the subspace topology on $f(X)$ inherited from $\mathcal{T}_Y$. If $(X, \mathcal{T}_X)$ is connected, then the topological space $(f(X), \mathcal{T}_{f(X)})$ is connected.

Corollary 3.3.1 (Intermediate Value Theorem)
In $(\mathbb{R}, \mathcal{T}_{EUC})$, let $a, b \in \mathbb{R}$ with $a < b$, let $f : [\,a, b\,] \to \mathbb{R}$ be a function, let y be an element between $f(a)$ and $f(b)$, then there exists an element $c \in [\,a, b\,]$ such that $y = f(c)$.

Lemma 3.3.2
Let $(X, \mathcal{T}_X)$ be a topological space. If $(X, \mathcal{T}_X)$ is path connected, then $(X, \mathcal{T}_X)$ is connected.

Lemma 3.3.3
In $(\mathbb{R}, \mathcal{T}_{EUC})$, let A be a subset of $\mathbb{R}$, let $\mathcal{T}_A$ denote the subspace topology on A inherited from $\mathcal{T}_{EUC}$. (The topological space $(A, \mathcal{T}_A)$ is connected) $\iff$ (the topological space $(A, \mathcal{T}_A)$ is path connected)

Theorem 3.4
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, let $f : X \to Y$ be a continuous function, let $\mathcal{T}_{f(X)}$ denote the subspace topology on $f(X)$ inherited from $\mathcal{T}_Y$. If $(X, \mathcal{T}_X)$ is path connected, then the topological space $(f(X), \mathcal{T}_{f(X)})$ is path connected.

Theorem 3.5 (Heine–Borel Theorem)
In $(\mathbb{R}, \mathcal{T}_{EUC})$, let A be a subset of $\mathbb{R}$, let $\mathcal{T}_A$ denote the subspace topology on A inherited from $\mathcal{T}_{EUC}$. (A is closed and bounded) $\iff$ (the topological space $(A, \mathcal{T}_A)$ is compact).

Corollary 3.5.1
In $(\mathbb{R}, \mathcal{T}_{EUC})$, let $a, b \in \mathbb{R}$ with $a \leq b$, let $[\,a, b\,]$ be an interval, let $\mathcal{T}_{ab}$ denote the subspace topology on $[\,a, b\,]$ inherited from $\mathcal{T}_{EUC}$, then the topological space $([\,a, b\,], \mathcal{T}_{ab})$ is compact.

Lemma 3.5.2
Let $(X, \mathcal{T})$ be a topological space, let subspace topology be given, a union of finitely many compact subsets of X is compact.

Theorem 3.6
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, let $f : X \to Y$ be a continuous function, let $\mathcal{T}_{f(X)}$ denote the subspace topology on $f(X)$ inherited from $\mathcal{T}_Y$. If $(X, \mathcal{T}_X)$ is compact, then the topological space $(f(X), \mathcal{T}_{f(X)})$ is compact.

Lemma 3.6.1
In $(\mathbb{R}, \mathcal{T}_{EUC})$, let A be a nonempty bounded subset of $\mathbb{R}$. If A is closed, then A contains a maximal element and a minimal element.

Theorem 3.7 (Extreme Value Theorem in $(\mathbb{R}, \mathcal{T}_{EUC})$)
In $(\mathbb{R}, \mathcal{T}_{EUC})$, let $a, b \in \mathbb{R}$ with $a \leq b$, let $f : [\,a, b\,] \to \mathbb{R}$ be a continuous function, then $\exists\ x_m \in [\,a, b\,]$ s.t. $\forall x \in [\,a, b\,]$, we have $f(x) \geq f(x_m)$, and $\exists\ x_M \in [\,a, b\,]$, s.t. $\forall x \in [\,a, b\,]$, we have $f(x) \leq f(x_M)$.

Theorem 3.8 (Extreme Value Theorem)
Let $(X, \mathcal{T})$ be a nonempty compact topological space, let $f : X \to \mathbb{R}$ be a continuous function, then the set $f(X) \subseteq \mathbb{R}$ contains a maximal element and a minimal element.

Lemma 3.8.1
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, let A be a subset of X, let $f : X \to Y$ be a homeomorphism let $\mathcal{T}_A$ denote the subspace topology on A inherited from $\mathcal{T}_X$, let $\mathcal{T}_{f(A)}$ denote the subspace topology on $f(A)$ inherited from $\mathcal{T}_Y$, then the restriction of f on A, $res_A f : A \to f(A)$ is a homeomorphism from $(A, \mathcal{T}_A)$ to $(f(A), \mathcal{T}_{f(A)})$.

Lemma 3.8.2
In $(\mathbb{R}, \mathcal{T}_{EUC})$*, let* I *be a subset of* $\mathbb{R}$*, let* $\mathcal{T}_I$ *denote the subspace topology on* I *inherited from* $\mathcal{T}_{EUC}$*, let* $f : I \to \mathbb{R}$ *be a function. If the topological space* $(I, \mathcal{T}_I)$ *is connected and* f *is locally constant and continuous on* I*, then* f *is a constant function.*

Lemma 3.8.3
In $(\mathbb{R}, \mathcal{T}_{EUC})$*, let* I *be a subset of* $\mathbb{R}$*, let* $\mathcal{T}_I$ *denote the subspace topology on* I *inherited from* $\mathcal{T}_{EUC}$*. If every continuous, locally constant function* $f : I \to \mathbb{R}$ *is a constant function, then the topological space* $(I, \mathcal{T}_I)$ *is connected.*

Lemma 3.8.4
Let $(X, \mathcal{T})$ *be a topological space, let* A, B *be subsets of* X*, let* $\bar{A} \subset X$ *denote the topological closure of* A*, let* $\bar{B} \subset X$ *denote the topological closure of* B*, the followings hold:*

1. $\bar{A}$ *is the smallest closed subset of* X *that contains* A*.*
2. $(x \in \bar{A}) \iff (\forall C \in \mathcal{T},$ *if* $x \in C$*, then* $C \cap A \neq \emptyset)$*.*
3. *If* A *is a subset of* B*, then* $\bar{A}$ *is a subset of* $\bar{B}$*.*

Lemma 3.8.5
Let $(X, \mathcal{T})$ *be a topological space, let* A *be a subset of* X*, then the interior of* A *is the largest open subset of* X *that is contained in* A*.*

Lemma 3.8.6
Let (X, d) *be a metric space,* $\mathcal{T}_d := \{A \subseteq X \mid \forall a \in A, \ \exists \ r > 0 \ s.t. \ B_r(a) \subseteq A\}$ *is a topology on* X*.*

Lemma 3.8.7
Let (X, d) *be a metric space, let* $\mathcal{T}_d$ *denote the topology on* X *associated to* d*, then the topological space* $(X, \mathcal{T}_d)$ *is Hausdorff, that is, every metric space is Hausdorff.*

Corollary 3.8.7.1
Every metrizable topological space is Hausdorff.

Lemma 3.8.8
In $(\mathbb{R}, \mathcal{T}_{EUC})$*, let* S *be a subset of* $\mathbb{R}$*.*
$(S$ *is closed*$) \iff ($*the set of accumulation points of* S *is a subset of* $S)$

Theorem 3.9
In $(\mathbb{R}, \mathcal{T}_{EUC})$*, let* S *be a subset of* $\mathbb{R}$*, let* $\mathcal{T}_S$ *denote the subspace topology on* S *inherited from* $\mathcal{T}_{EUC}$*, then the following statements are equivalent:*

1. *The topological space* $(S, \mathcal{T}_S)$ *is compact.*
2. S *is closed in* $\mathcal{T}_{EUC}$ *and bounded in* $\mathbb{R}$*.*
3. *Every sequence in* $(S, \mathcal{T}_S)$ *has a convergent subsequence whose limit belongs to* S*.*
4. S *is bounded in* $\mathbb{R}$ *and* $\nexists$ *sequence in* $(S, \mathcal{T}_S)$ *whose limit belongs to* $\mathbb{R} \setminus S$*.*

Functions

Lemma 4.0.1
Let A, B be sets, let $f : A \to B$ be a function, then (f is invertible) $\iff$ (f is bijective).

Lemma 4.0.2
Let A be a set, let $f : A \to A$ be an involution, then f is bijective.

Lemma 4.0.3
Let A be a finite set, let $f : A \to A$ be an involution.
If A is finite and has odd number of element, then $\exists\ a \in A$ s.t. $f(a) = a$.

Lemma 4.0.4
Let A, B, C be sets, let $f : A \to B$ and $g : B \to C$ be functions, then the function $(g \circ f) : A \to C$ is injective provided that f is injective and $res_{f(A)} g : f(A) \to C$ is injective.

Lemma 4.0.5
Let A, B, C be sets, let $f : A \to B$ and $g : B \to C$ be functions, then the function $(g \circ f) : A \to C$ is surjective provided that $res_{f(A)} g : f(A) \to C$ is surjective.

Theorem 4.1
Let A, B, C be sets, let $f : A \to B$ and $g : B \to C$ be functions, then the function $(g \circ f) : A \to C$ is bijective provided that f is injective and $res_{f(A)} g : f(A) \to C$ is bijective.

From now on, consider using the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, and the topological space $(\mathbb{R}, \mathscr{T}_{EUC})$.
The term IWIMP refers to Interval With Infinitely Many Points.

Lemma 4.1.1
Let A, B be subsets of $\mathbb{R}$, let $f : A \to B$ be a function. If f is strictly increasing on A, then the inverse of f, $f^{-1} : f(A) \to A$, is strictly increasing on $f(A)$.

Lemma 4.1.2
Let A, B be subsets of $\mathbb{R}$, let $f : A \to B$ be a function. If f is strictly decreasing on A, then the inverse of f, $f^{-1} : f(A) \to A$, is strictly decreasing on $f(A)$.

Lemma 4.1.3
Let A be subsets of $\mathbb{R}$, let $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$ be functions that are strictly increasing on A, then the function $(f + g) : A \to \mathbb{R} \quad x \mapsto f(x) + g(x)$ is strictly increasing on A.

Lemma 4.1.4
Let A, B be subsets of $\mathbb{R}$, let $f : A \to B$, $g : B \to \mathbb{R}$ be functions that are strictly increasing on their domains, then the function $(g \circ f) : A \to \mathbb{R} \quad x \mapsto g(f(x))$ is strictly increasing on A.

Lemma 4.1.5
Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is bounded on $\mathbb{R}$, then f is locally bounded.

Lemma 4.1.6
Let A be a subset of $\mathbb{R}$, let $f : A \to \mathbb{R}$ be a function that is continuous at $a \in A$. If $f(a) \neq 0$, then there exists an interval $I \in \mathscr{T}_{EUC}$ such that $f(x) \neq 0$ for $x \in I$.

Lemma 4.1.7
Let A be a subset of $\mathbb{R}$, let $a \in A$, $c \in \mathbb{R}$, let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions that are continuous at a, then the followings hold:

1. *The function $(f + g) : A \to \mathbb{R} \quad x \mapsto f(x) + g(x)$ is continuous at a.*
2. *The function $(f \cdot g) : A \to \mathbb{R} \quad x \mapsto f(x) \cdot g(x)$ is continuous at a.*
3. *The function $(c \cdot f) : A \to \mathbb{R} \quad x \mapsto c \cdot f(x)$ is continuous at a.*
4. *The function $|f| : A \to \mathbb{R} \quad x \mapsto |f(x)|$ is continuous at a.*
5. *If $g(a) \neq 0$, then $\frac{f}{g} : A \to \mathbb{R} \quad x \mapsto \frac{f(x)}{g(x)}$ is continuous at a.*

Lemma 4.1.8
Let $a, b \in \mathbb{R}$, let $[a, b]$ be an interval, let $f : [a, b] \to \mathbb{R}$ be a function that is continuous on $[a, b]$, then there exists a function $g : \mathbb{R} \to \mathbb{R}$ that is continuous on $\mathbb{R}$ with $g(x) = f(x)$ for all $x \in [a, b]$.

Lemma 4.1.9
Given $n \in \mathbb{N}$, the function $f : \mathbb{R} \to \mathbb{R} \quad x \mapsto x^n$ is continuous.

Proposition 4.1.10
Every polynomial function is continuous.

Lemma 4.1.11
Let $I \subseteq \mathbb{R}$ be an IWIMP, let $f : I \to \mathbb{R}$ be a function. If f is injective and continuous on I, then f is either strictly increasing or strictly decreasing on I.

Lemma 4.1.12 (a version of the Inverse Function Theorem)
Let $I \subseteq \mathbb{R}$ be an IWIMP, let $f : I \to \mathbb{R}$ be a function. If f is injective and continuous on I, then the inverse of f, $f^{-1} : f(I) \to I$, is continuous on $f(I)$.

Lemma 4.1.13 (Principle of Uniformly Continuity)
Let $a, b \in \mathbb{R}$ with $a < b$, let $[a, b]$ be an interval, and let $g : [a, b] \to \mathbb{R}$ be a continuous function, then g is uniformly continuous on $[a, b]$.

Lemma 4.1.14
Let A be a subset of $\mathbb{R}$, let $f : A \to \mathbb{R}$ be a function. If f is uniformly continuous on A, then f is continuous on A

Lemma 4.1.15
Let A be a subset of $\mathbb{R}$, let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions that are uniformly continuous on A, then the function $(f + g) : A \to \mathbb{R} \quad x \mapsto f(x) + g(x)$ is uniformly continuous on A.

Lemma 4.1.16
Let A be a subset of $\mathbb{R}$, let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions that are uniformly continuous on A. If f and g are bounded on A, then the function $(f \cdot g) : A \to \mathbb{R} \quad x \mapsto f(x) \cdot g(x)$ is uniformly continuous on A.

Lemma 4.1.17
Let A, B be a subset of $\mathbb{R}$, let $f : A \to B$ and $g : B \to \mathbb{R}$ be functions, let $\circ$ denote function composition. If f is uniformly continuous on A and g is uniformly continuous on B, then the function $(g \circ f) : A \to \mathbb{R}$ is uniformly continuous on A.

Proposition 4.1.18
For some $\alpha \in (0, \infty)$, let $f : [0, \infty) \to \mathbb{R} \quad x \mapsto x^\alpha$ be a function, the followings hold:

1. *If $0 < \alpha \leq 1$, then f is uniformly continuous on $[0, \infty)$.*
2. *If $\alpha > 1$, then f is not uniformly continuous on $[0, \infty)$.*

Polynomials

From now on, consider using the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, and the topological space $(\mathbb{R}, \mathscr{T}_{EUC})$
The term IWIMP refers to Interval With Infinitely Many Points.

Theorem 5.1
Let $I \subseteq \mathbb{R}$ be an IWIMP, let $a \in I$, $n \in \mathbb{N}$, and let $f : I \to \mathbb{R}$ be a function, then we have:

1. *(Lagrange Remainder) If f is $(n+1)$-times differentiable on I,*
   then $\forall x \in I$, $\exists\, t \in I$ between a and x s.t. $R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x - a)^{n+1}$.
2. *(Cauchy Remainder) If f is $(n+1)$-times differentiable on I,*
   then $\forall x \in I$, $\exists\, t \in I$ between a and x s.t. $R_{n,a} = \frac{f^{(n+1)}(t)}{n!}(x - t)^n (x - a)$.
3. *If f is $(n+1)$-times differentiable at a, and f is continuous on I,*
   then we have $R_{n,a} = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x - t)^n \, dt$.

Corollary 5.1.1
Given $n \in \mathbb{N}$, the best degree n polynomial approximation of a degree n polynomial function is the polynomial function itself. That is, let $f : \mathbb{R} \to \mathbb{R} \quad x \mapsto \sum_{j=0}^{n} b_j x^j$ be a polynomial function of degree n, let $a \in \mathbb{R}$, let $P_{n,a}$ denote the Taylor polynomial of degree n centered at a associated to f, then for all $x \in \mathbb{R}$, we have $P_{n,a}(x) = f(x)$.

Groups

From now on, consider using the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$

Lemma 6.0.1
Let $(G, \diamond_G)$ and $(H, \diamond_H)$ be groups, let $\varphi : G \to H$ be a group isomorphism from $(G, \diamond_G)$ to $(H, \diamond_H)$, then $\varphi^{-1} : H \to G$ is a group isomorphism from $(H, \diamond_H)$ to $(G, \diamond_G)$.

Proposition 6.0.2
Let $(G, \diamond_G), (H, \diamond_H)$ be groups, let $\varphi : G \to H$ be a group homomorphism from $(G, \diamond_G)$ to $(H, \diamond_H)$, then $(im(\varphi), \diamond_H)$ and $(ker(\varphi), \diamond_G)$ are groups.

Lemma 6.0.3
The set s_n has $n!$ elements.

Lemma 6.0.4
Given $n \in \mathbb{N}$, S_n is a group.

Lemma 6.0.5
For $n \in \{1, 2\}$, the group S_n is abelian, while for $n \in \mathbb{N}$ with $n \geq 3$, the group S_n is not abelian.

Proposition 6.0.6
The group D_3 is isomorphic to the group S_3, while $\forall n \in \mathbb{N}$ with $n > 3$, D_n is not isomorphic to S_n.

Lemma 6.0.7
The function $\ln : (0, \infty) \to \mathbb{R}$ is a group homomorphism from $((0, \infty), *)$ to $(\mathbb{R}, +)$.

Corollary 6.0.7.1
The group $((0, \infty), *)$ is isomorphic to the group $(\mathbb{R}, +)$.

Lemma 6.0.8
Let $(G, \diamond)$ be a group, let $e \in G$ be the $\diamond$-identity, let $x \in G$, then $(x \diamond x = x) \iff (x = e)$.

Lemma 6.0.9
Let $(G, \diamond_G)$ and $(H, \diamond_H)$ be groups, let $\varphi : G \to H$ be a group homomorphism from $(G, \diamond_G)$ to $(H, \diamond_H)$, let $e_G \in G$ be the $\diamond_G$-identity, and let $e_H \in H$ be the $\diamond_H$-identity, then the followings hold:

1. $\varphi(e_G) = e_H$
2. $\forall x \in G$, *we have* $\varphi(x^{-1}) = (\varphi(x))^{-1}$.
3. $\forall x \in G$, *and* $\forall n \in \mathbb{N}$, *we have* $\varphi(x^n) = (\varphi(x))^n$.

Calculus

From now on, consider using the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$, and the topological space $(\mathbb{R}, \mathcal{T}_{EUC})$. IWIMP refers to Interval With Infinitely Many Points. Fix $a, b \in \mathbb{R}$, let $[a, b]$ be an interval.

Lemma 7.0.1
Let P, Q be partitions of $[a, b]$ with $P \leq Q$, let $f : [a, b] \to \mathbb{R}$ be a bounded function, then we have $U(f, P) \geq U(f, Q)$.

Lemma 7.0.2
Let P, Q be partitions of $[a, b]$, let $f : [a, b] \to \mathbb{R}$ be a bounded function, then $L(f, P) \leq U(f, Q)$.

Lemma 7.0.3
Any constant function of the form $f : [a, b] \to \mathbb{R}$ is Darboux integrable on $[a, b]$.

Lemma 7.0.4 (Darboux Integrability Criterion)
Let $f : [a, b] \to \mathbb{R}$ be a bounded function.
(f is Darboux integrable on $[a, b]$) iff ($\forall \epsilon > 0$, $\exists$ partitions P, Q of $[a, b]$ s.t. $U(f, P) - L(f, Q) < \epsilon$).

Lemma 7.0.5
Let $f : [a, b] \to \mathbb{R}$ be a function. If f is Riemann integrable on $[a, b]$, then f is bounded on $[a, b]$.

Lemma 7.0.6
Let $f : [a, b] \to \mathbb{R}$ be a bounded function, let P denote partition of $[a, b]$, and let P_n denote regular partition of $[a, b]$. If $\lim_{||P|| \to 0} U(f, P) = l$ for some $l \in \mathbb{R}$, then $\lim_{n \to \infty} U(f, P_n) = l$. Similarly, if $\lim_{||P|| \to 0} L(f, P) = m$ for some $m \in \mathbb{R}$, then $\lim_{n \to \infty} L(f, P_n) = m$.

Lemma 7.0.7
Let $f : [a, b] \to \mathbb{R}$ be a bounded function, let P, Q be partitions of $[a, b]$,
then we have $\lim_{||P|| \to 0} U(f, P) = U(f)$, and $\lim_{||Q|| \to 0} L(f, P) = L(f)$.

Corollary 7.0.7.1
Let $f : [a, b] \to \mathbb{R}$ be a bounded function,
then (f is Darboux integrable on $[a, b]$) iff (f is Darboux Integrable on $[a, b]$).
Moreover, if f is Darboux integrable on $[a, b]$ or S-integrable on $[a, b]$, then we have $I_S = I_D$.

Lemma 7.0.8
Let $f : [a, b] \to \mathbb{R}$ be a bounded function. If f is S-integrable on $[a, b]$, then f is Riemann Integrable on $[a, b]$. Moreover, if f is S-integrable on $[a, b]$, then we have $I_R = I_S$.

Lemma 7.0.9
Let $f : [a, b] \to \mathbb{R}$ be a function. If f is Riemann integrable on $[a, b]$, then f is S-integrable $[a, b]$. Moreover, if f is Riemann integrable $[a, b]$, then we have $I_S = I_R$.

Theorem 7.1
Let $f : [a, b] \to \mathbb{R}$ be a function, the followings are equivalent:

1. *f is bounded and Darboux integrable on $[a, b]$.*
2. *f is bounded and S-integrable on $[a, b]$.*
3. *f is Riemann integrable on $[a, b]$.*

Moreover, if any one of the three holds, then we have $I_S = I_D = I_R$.

Lemma 7.1.1
Let the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, let $c \in \mathbb{R}$, then we have:

1. *The function $(f + g) : [a, b] \to \mathbb{R} \quad x \mapsto f(x) + g(x)$ is integrable on $[a, b]$.*
2. *The function $(c \cdot f) : [a, b] \to \mathbb{R} \quad x \mapsto c \cdot f(x)$ is integrable on $[a, b]$.*
3. *The function $f^2 : [a, b] \to \mathbb{R} \quad x \mapsto (f(x))^2$ is integrable on $[a, b]$.*
4. *The function $(f \cdot g) : [a, b] \to \mathbb{R} \quad x \mapsto f(x) \cdot g(x)$ is integrable on $[a, b]$.*

Moreover, we have $\int_a^b (f + g) = (\int_a^b f) + (\int_a^b g)$ and $\int_a^b (c \cdot f) = c \cdot \int_a^b f$.

Theorem 7.2
Let $f : [a, b] \to \mathbb{R}$ be a function. If f is continuous, then f is integrable on $[a, b]$.

Lemma 7.2.1
Let $c \in (a, b)$, let $[a, c]$, $[c, b]$ be intervals, let $f : [a, b] \to \mathbb{R}$ be a function.
(f is integrable on $[a, b]$) iff ($res_{[a,c]} f$ is integrable on $[a, c]$ and $res_{[c,b]} f$ is integrable on $[c, b]$)

Corollary 7.2.1.1
Let $l, m, n \in [a, b]$, let $f : [a, b] \to \mathbb{R}$ be a function.
If f is integrable on $[a, b]$, then we have $\int_l^m f + \int_m^n f = \int_l^n f$.

Lemma 7.2.2
The function $\ln : (0,\infty) \to \mathbb{R} \quad x \mapsto \ln(x)$ *is strictly increasing.*

Lemma 7.2.3
Let $f : [a,b] \to \mathbb{R}$ *be a function that is bounded by* $M \in \mathbb{R}$*, that is,* $\forall x \in [a,b]$*, we have* $|f(x)| \leq M$*.*
If f *is integrable on* $[a,b]$*, then* $-M \cdot (b-a) \leq \int_a^b f \leq M \cdot (b-a)$*.*

Lemma 7.2.4
Let $f : [a,b] \to \mathbb{R}$ *be a function. If* f *is integrable on* $[a,b]$*, then* $|f| : [a,b] \to \mathbb{R} \quad x \mapsto |f(x)|$ *is integrable on* $[a,b]$*. Moreover, if* f *is integrable on* $[a,b]$*, then we have* $|\int_a^b f| \leq \int_a^b |f|$*.*

Lemma 7.2.5
Let $f : \mathbb{R} \to \mathbb{R}$ *be a function, let* $m \in [0,\infty)$*.*
If f *is odd, and* f *is integrable on* $[-m,m]$*, and we have* $\int_{-m}^m f = 0$*.*

Lemma 7.2.6
Let $f : [a,b] \to \mathbb{R}$ *be a function that is integrable on* $[a,b]$*. If* $f(x) \geq 0 \ \forall x \in [a,b]$*, then* $\int_a^b f \geq 0$*.*

Lemma 7.2.7
Let $f : [a,b] \to \mathbb{R}$ *and* $g : [a,b] \to \mathbb{R}$ *be functions that are integrable on* $[a,b]$*.*
If $f(x) \geq g(x) \ \forall x \in [a,b]$*, then* $\int_a^b f \geq \int_a^b g$*.*

Lemma 7.2.8
Let $f : [a,b] \to \mathbb{R}$ *be a function that is integrable on* $[a,b]$*, let* $c \in \mathbb{R}$*,*
then $g : [a+c, b+c] \to \mathbb{R} \quad x \mapsto f(x-c)$ *is integrable on* $[a+c, b+c]$*, and we have* $\int_a^b f = \int_{a+c}^{b+c} g$*.*

Lemma 7.2.9
Let $c \in \mathbb{R}_{>0}$*, let* $f : [c \cdot a, c \cdot b] \to \mathbb{R}$ *be a function that is integrable on* $[c \cdot a, c \cdot b]$*,*
then $g : [a,b] \to \mathbb{R} \quad x \mapsto f(c \cdot x)$ *is integrable on the interval* $[a,b]$*, and we have* $\int_{c \cdot a}^{c \cdot b} f = c \cdot \int_a^b g$*.*

Lemma 7.2.10
Let $f : [a,b] \to \mathbb{R}$ *be a function that is integrable on* $[a,b]$*, let* $F : [a,b] \to \mathbb{R} \quad x \mapsto \int_a^x f$ *be a function, then* F *is continuous on* $[a,b]$*.*

Corollary 7.2.10.1
The function $\ln : (0,\infty) \to \mathbb{R} \quad x \mapsto \ln(x)$ *is continuous.*

Theorem 7.3
Let $\mathscr{T}_0$ *denote the subspace topology on the interval* $(,\infty)$ *inherited from* $\mathscr{T}_{EUC}$*.*
The function $\ln : (0,\infty) \to \mathbb{R} \quad x \mapsto \ln(x)$ *is a homeomorphism from* $((0,\infty), \mathscr{T}_0)$ *to* $(\mathbb{R}, \mathscr{T}_{EUC})$*.*

The assumptions, $a, b \in \mathbb{R}$, $[a,b]$ is an interval, are now removed.
We continue using the ordered field $(\mathbb{R}, \mathbb{R}_{>0})$ and the topological space $(\mathbb{R}, \mathscr{T}_{EUC})$.

Lemma 7.3.1
Let A *be a subset of* $\mathbb{R}$*, let* $f : A \to \mathbb{R}$ *be a function, let* $a \in A$*,*
then ($\lim_{x \to a} f(x) = l$ *for some* $l \in \mathbb{R}$*)* $\iff$ *(*$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$*).*

Lemma 7.3.2
Let A *be a subset of* $\mathbb{R}$*, let* $f : A \to \mathbb{R}$ *be a function, let* $a \in A$*,*
then ($\lim_{x \to a} f(x) = f(a)$*)* $\iff$ *(*$\lim_{h \to 0}[f(a+h) - f(a)] = 0$*).*

Lemma 7.3.3
Let $I \subseteq \mathbb{R}$ *be an IWIMP, let* $f : I \to \mathbb{R}$ *and* $g : I \to \mathbb{R}$ *be functions that are differentiable on* I*, let* $c \in \mathbb{R}$*, then the followings hold:*

1. *The function* $(f+g) : I \to \mathbb{R} \quad x \mapsto f(x) + g(x)$ *is differentiable on* I*.*
2. *The function* $(c \cdot f) : I \to \mathbb{R} \quad x \mapsto c \cdot f(x)$ *is differentiable on* I*.*

Moreover, we have $(f+g)' = f' + g'$ *and* $(c \cdot f)' = c \cdot (f')$*.*

Lemma 7.3.4
Given $n \in \mathbb{N}$*, let* $g : \mathbb{R} \to \mathbb{R} \quad x \mapsto x^n$ *be a function, let* $a \in \mathbb{R}$*, then* g *is differentiable at* a*.*

Corollary 7.3.4.1
All polynomial functions are differentiable on $\mathbb{R}$*.*

Lemma 7.3.5
Let $I \subseteq \mathbb{R}$ *be an IWIMP, let* $f : I \to \mathbb{R}$ *be a function that is differentiable at* $c \in I$*. If* f *attains a local minimal or a local maximum at* c*, then we have* $f'(c) = 0$*.*

Lemma 7.3.6
Let $I \subseteq \mathbb{R}$ be an IWIMP, let $f : I \to \mathbb{R}$ be a function, let $c \in I$. If f is differentiable at c, then f is continuous at c.

Lemma 7.3.7 (Chain Rule)
Let $I, W \subseteq \mathbb{R}$ be IWIMP, let $g : I \to W$ be a function that is differentiable at $a \in I$, let $f : W \to \mathbb{R}$ be a function that is differentibale at $g(a)$, let $\circ$ denote function composition, then the function $f \circ g : I \to \mathbb{R}$ is differentiable at a, and we have $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.

Theorem 7.4 (Rolle's Theorem)
Let $a, b \in \mathbb{R}$ with $a < b$, let $[a, b]$ be an interval, let $f : [a, b] \to \mathbb{R}$ be a function with $f(a) = f(b)$. If f is differentiable on (a, b) and continuous on $[a, b]$, then $\exists\ c \in (a, b)$ s.t. $f'(c) = 0$.

Theorem 7.5 (Cauchy's Mean Value Theorem)
Let $a, b \in \mathbb{R}$ with $a < b$, let $[a, b]$ be an interval, let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be functions. If both f and g are differentiable on (a, b), and both f and g are continuous on $[a, b]$, then $\exists\ c \in (a, b)$ s.t. $f'(c) \cdot [g(b) - g(a)] = g'(c) \cdot [f(b) - f(a)]$.

Corollary 7.5.1 (Mean Value Theorem)
Let $a, b \in \mathbb{R}$ with $a < b$, let $[a, b]$ be an interval, let $f : [a, b] \to \mathbb{R}$ be a function. If f is continuous on $[a, b]$ and differentiable on (a, b), then $\exists\ c \in (a, b)$ s.t. $f'(c) \cdot (b - a) = f(b) - f(a)$.

Corollary 7.5.2
Let $I \in \mathscr{T}_{EUC}$ be an IWIMP, let $f : I \to \mathbb{R}$ be a function that is differentiable on I. If $f' \equiv 0$, that is, if $f' : I \to \mathbb{R} \quad x \mapsto 0$, then f is a constant function.

Corollary 7.5.3
Let $a, b \in \mathbb{R}$ with $a < b$, let $[a, b]$ be an interval, let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be functions. If f and g are both continuous on $[a, b]$ and differentiable on (a, b), and $f' = g'$ on $[a, b]$ then $\exists\ c \in \mathbb{R}$ s.t. $\forall x \in [a, b]$, we have $f(x) = g(x) + c$.

Lemma 7.5.4
Let A be a subset of $\mathbb{R}$, let $f : A \to \mathbb{R}$ be a function, let $a \in A$, then $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h)$.

Theorem 7.6 (L'Hopital's Rule)
Let $I \in \mathscr{T}_{EUC}$ be an IWIMP, let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at $a \in I$, then the followings hold:

1. *If $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$ and $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = l$ where $l \in \mathbb{R}$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = l$.*
2. *If $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} g(x) = 0$ and $\lim_{x \to a^-} \frac{f'(x)}{g'(x)} = l$ where $l \in \mathbb{R}$, then $\lim_{x \to a^-} \frac{f(x)}{g(x)} = l$.*
3. *If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ and $\lim_{x \to a} \frac{f'(x)}{g'(x)} = l$ for some $l \in \mathbb{R}$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = l$.*
4. *If $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$ and $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = -\infty$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = -\infty$.*
5. *If $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} g(x) = 0$ and $\lim_{x \to a^-} \frac{f'(x)}{g'(x)} = -\infty$, then $\lim_{x \to a^-} \frac{f(x)}{g(x)} = -\infty$.*
6. *If $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$ and $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \infty$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \infty$.*
7. *If $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} g(x) = 0$ and $\lim_{x \to a^-} \frac{f'(x)}{g'(x)} = \infty$, then $\lim_{x \to a^-} \frac{f(x)}{g(x)} = \infty$.*
8. *If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ and $\lim_{x \to a} \frac{f'(x)}{g'(x)} = -\infty$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = -\infty$.*
9. *If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ and $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \infty$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \infty$.*

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions that are differentiable on $\mathbb{R}$, then the followings hold:

1. *If $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} g(x) = 0$ and $\lim_{x \to -\infty} \frac{f'(x)}{g'(x)} = l$ where $l \in \mathbb{R}$, then $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = l$.*
2. *If $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0$ and $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = l$ where $l \in \mathbb{R}$, then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = l$.*
3. *If $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} g(x) = 0$ and $\lim_{x \to -\infty} \frac{f'(x)}{g'(x)} = -\infty$, then $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = -\infty$.*
4. *If $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} g(x) = 0$ and $\lim_{x \to -\infty} \frac{f'(x)}{g'(x)} = \infty$, then $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = \infty$.*
5. *If $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0$ and $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = -\infty$, then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = -\infty$.*
6. *If $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0$ and $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \infty$, then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty$.*

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions that are differentiable on $\mathbb{R}$, then the followings hold:

1. *If $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} g(x) = \infty$ and $\lim_{x \to -\infty} \frac{f'(x)}{g'(x)} = l$ where $l \in \mathbb{R}$, then $\lim_{x \to -\infty} \frac{f(x)}{g(x)} = l$.*
2. *If $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty$ and $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = l$ where $l \in \mathbb{R}$, then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = l$.*

** There are still many other cases for L'Hopital's Rule, while those cases that have been proven in 2020 Fall Semester Math 295 either in class or in homework 10 are listed above.*

Proposition 7.6.1
Let $I \in \mathscr{T}_{EUC}$ be an IWIMP, let $f : I \to \mathbb{R}$ be a function that is twice differentiable on I. If there exist three distinct elements $a, b, c \in I$ s.t. $f(a) = f(b) = f(c) = 0$, then $\exists\ k \in I$ s.t. $f''(k) = 0$.

Theorem 7.7 (Fundamental Theorem of Calculus I)
Let $a, b \in \mathbb{R}$ with $a < b$, let $[a, b]$ be an interval, let $f : [a, b] \to \mathbb{R}$ be a continuous function, let $F : [a, b] \to \mathbb{R} \quad x \mapsto \int_a^x f$ be a function, then F is differentiable on $[a, b]$ and we have $F'(x) = f(x)$.

Proposition 7.7.1
Let $\ln : (0, \infty) \to \mathbb{R} \quad x \mapsto \ln(x)$ be a function, then $\lim_{x\to\infty} \ln(x) = \infty$ and $\lim_{x\to 0^+} \ln(x) = -\infty$.

Theorem 7.8 (Inverse Function Theorem)
Let $I \subseteq \mathbb{R}$ be an IWIMP, let $f : I \to \mathbb{R}$ be a continuous injective function, let $f^{-1} : f(I) \to I$ denote the inverse of f, and let $b \in f(I)$. If f is differentiable at $f^{-1}(b)$ with $f'(f^{-1}(b)) \neq 0$, then f^{-1} is differentiable at b, and we have $(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$.

Lemma 7.8.1 (Product Rule)
Let $A \subseteq \mathbb{R}$ be an IWIMP, let $a \in A$, let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions that are differentiable at a, then the function $(f \cdot g) : A \to \mathbb{R} \quad x \mapsto f(x) \cdot g(x)$ is differentiable at a, and we have $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.

Lemma 7.8.2
Let $a \in \mathbb{R} \setminus \{0\}$, let $h : \mathbb{R} \setminus \{0\} \to \mathbb{R} \quad x \mapsto \frac{1}{x}$ be a function, then h is differentiable at a, and we have $h'(a) = -\frac{1}{a^2}$.

Corollary 7.8.2.1 (Quotient Rule)
Let $A \subseteq \mathbb{R}$ be an IWIMP, let $a \in A$, let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions that are differentiable at a with $g(a) \neq 0$, then the function $(\frac{f}{g}) : A \to \mathbb{R} \quad x \mapsto \frac{f(x)}{g(x)}$ is differentiable at a and we have $(\frac{f}{g})'(a) = \frac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{(g(a))^2}$.

Lemma 7.8.3
Let $A \subseteq \mathbb{R}$ be an IWIMP, let $f : A \to \mathbb{R}$ be a function that is differentiable on A. If $f' : A \to \mathbb{R}$ is positive on A, then f is strictly increasing on A. Similarly, if $f' : A \to \mathbb{R}$ is negative on A, then f is strictly decreasing on A.

Theorem 7.9 (Second Derivative Test)
Let $A \subseteq \mathbb{R}$ be an IWIMP, let $a \in A$, let $f : A \to \mathbb{R}$ be a function that is twice differentiable on A. If $f'(a) = 0$ and $f''(a) > 0$, then f attains a local minimum at a. Similarly, if $f'(a) = 0$ and $f''(a) < 0$, then f attains a local maximum at a.

Theorem 7.10 (Fundamental Theorem of Calculus II)
Let $a, b \in \mathbb{R}$ with $a < b$, let $[a, b]$ be an interval, let f be a function that is integrable on $[a, b]$. If $f = g'$ for some function $g : [a, b] \to \mathbb{R}$, then we have $\int_a^b f = g(b) - g(a)$.

Theorem 7.11 (Squeeze Theorem)
Let A be a subset of $\mathbb{R}$, let $f : A \to \mathbb{R}$, $g : A \to \mathbb{R}$, $h : A \to \mathbb{R}$ be functions with $f(x) \leq g(x) \leq h(x)$ $\forall x \in A$, let $a \in A$. If $\lim_{x\to a} f(x) = \lim_{t\to a} h(t) = l$ for some $l \in \mathbb{R}$, then $\lim_{s\to a} g(s) = l$.

Theorem 7.12 (The Cauchy-Schwarz Inequality)
Let $a, b \in \mathbb{R}$ with $a < b$, let $[a, b]$ be an interval, let f and g be functions that are integrable on $[a, b]$, then the inequality holds: $(\int_a^b f \cdot g)^2 \leq (\int_a^b f^2) \cdot (\int_a^b g^2)$

Sequences

Lemma 8.0.1
In $(\mathbb{R}, \mathscr{T}_{EUC})$, let $c \in \mathbb{R}$, any sequence of the form $n \mapsto \frac{c}{n}$ converges to 0.

Lemma 8.0.2
In $(\mathbb{R}, \mathscr{T}_{EUC})$, let $n \mapsto a_n$ and $n \mapsto b_n$ be sequences of real numbers with $a_n < b_n \ \forall n \in \mathbb{N}$. If both (a_n) and (b_n) converge, then we have $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

Lemma 8.0.3
Let $(X, \mathscr{T})$ be a Hausdorff topological space, let (x_n) be a sequence in $(X, \mathscr{T})$. If (x_n) converges, then the limit $\lim_{n\to\infty} x_n$ is unique, that is, limits of sequences in Hausdorff spaces are unique.

Lemma 8.0.4
Let $(X, \mathscr{T})$ be a topological space, let (x_n) be a sequence in $(X, \mathscr{T})$, let (x_{j_k}) be a subsequence of (x_n). If (x_n) converges in $(X, \mathscr{T})$, then (x_{j_k}) converges in $(X, \mathscr{T})$. Moreover, if (x_n) converges to $l \in X$, then (x_{j_k}) converges to l.

Corollary 8.0.4.1
Let $(X, \mathscr{T})$ be a topological space, let (x_n) be a sequence on $(X, \mathscr{T})$.
(The sequence (x_n) converges to $l \in X$) $\iff$ (every subsequence of (x_n) converges to l)

Theorem 8.1
Every bounded monotonic sequence of real numbers in the Euclidean topology converges.

Theorem 8.2 (Bolzano-Weierstrass Theorem)
Every bounded sequence of real numbers in the Euclidean topology has a convergent subsequence.

Lemma 8.2.1 (Squeeze Theorem)
Let $(s_n), (t_n), (u_n)$ be sequences of real numbers in the Euclidean topology, let $l \in \mathbb{R}$. If we have $\lim_{n\to\infty} s_n = l = \lim_{n\to\infty} t_n$, and $\exists\ M \in \mathbb{R}$ s.t. $\forall m \in \mathbb{R}$ with $m > M$, we have $s_m \le u_m \le t_m$, then we have $\lim_{n\to\infty} u_m = l$.

Lemma 8.2.2
Let (X, d) be a metric space, let $\mathscr{T}_d$ denote the topology on X associated to d, then every convergent sequence in $(X, \mathscr{T}_d)$ is a Cauchy sequence.

Lemma 8.2.3
Let $a, b \in \mathbb{R}$ with $a \ge b$, let $[\,a, b\,]$ be an interval, let $\mathscr{T}_{[\,a,b\,]}$ denote the subspace topology on $[\,a, b\,]$ inherited from $\mathscr{T}_{EUC}$, then the topological space $([\,a, b\,], \mathscr{T}_{[\,a,b\,]})$ is Cauchy complete.

Lemma 8.2.4
Let (X, d) be a metric space, let $\mathscr{T}_d$ denote the topology on X associated to d, let (x_n) be a Cauchy sequence in $(X, \mathscr{T}_d)$, then the set $S = \{x_n \mid x \in \mathbb{N}\}$ is bounded in X.

Corollary 8.2.4.1
Let (X, d) be a metric space, let $\mathscr{T}_d$ denote the topology on X associated to d, let (x_n) be a convergent sequence in $(X, \mathscr{T}_d)$, then the set $S = \{x_n \mid x \in \mathbb{N}\}$ is bounded in X.

Theorem 8.3
Every Cauchy sequence of real numbers in the Euclidean topology converges.

Lemma 8.3.1
Let $(a_n), (b_n)$ be Cauchy sequences of real numbers in the Euclidean topology, let $c \in \mathbb{R}$, then the followings hold:

1. *The sequence $n \mapsto (c \cdot a_n)$ is a Cauchy sequence, and $\lim_{n\to\infty}(c \cdot a_n) = c \cdot \lim_{n\to\infty} a_n$.*
2. *$n \mapsto (a_n + b_n)$ is a Cauchy sequence, and $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.*
3. *$n \mapsto (a_n \cdot b_n)$ is a Cauchy sequence, and $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$.*
4. *If $\lim_{n\to\infty} b_n \ne 0$ and $\forall n \in \mathbb{N},\ b_n \ne 0$, then $n \mapsto \frac{a_n}{b_n}$ is a Cauchy, and $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.*

Theorem 8.4
Let $(X, \mathscr{T}_X), (Y, \mathscr{T}_Y)$ be topological spaces, let $f : X \to Y$ be a continuous function, let $n \mapsto x_n$ be a sequence in $(X, \mathscr{T}_X)$. If $n \mapsto x_n$ converges in $(X, \mathscr{T}_X)$, then the sequence $n \mapsto f(x_n)$ converges in $(Y, \mathscr{T}_Y)$. Moreover, if $n \mapsto x_n$ converges to some $l \in X$, then $n \mapsto f(x_n)$ converges to $f(l) \in Y$.

Proposition 8.4.1
Let $(X, d_X), (Y, d_Y)$ be metric spaces, let $\mathscr{T}_{dX}$ denote the topology on X associated to d_X, let $\mathscr{T}_{dY}$ denote the topology on Y associated to d_Y, let $f : X \to Y$ be a function. [f is continuous at $x \in X$] iff [for each sequence $n \mapsto x_n$ in $(X, \mathscr{T}_{dX})$ that converges to x, the sequence $n \mapsto f(x_n)$ in $(Y, \mathscr{T}_{dY})$ converges to $f(x) \in Y$]

Lemma 8.4.2

In $(\mathbb{R}, \mathscr{T}_{EUC})$*, let* $S \subseteq \mathbb{R}$*,* $x \in \mathbb{R}$*, let* $\mathscr{T}_{S_x}$ *denote the subspace topology on* $S \setminus \{x\}$ *inherited from* $\mathscr{T}_{EUC}$*.*
(x *is an accumulation poing of* S*) iff (* $\exists$ *a sequence* $n \mapsto s_n$ *in* $(S \setminus \{x\}, \mathscr{T}_{S_x})$ *s.t.* $\lim_{n\to\infty} s_n = x$*)*

Theorem 8.5

In $(\mathbb{R}, \mathscr{T}_{EUC})$*, let* S *be a subset of* $\mathbb{R}$*, let* $\mathscr{T}_S$ *denote the subspace topology on* S *inherited from* $\mathscr{T}_{EUC}$*, then the following statements are equivalent:*

1. *The topological space* $(S, \mathscr{T}_S)$ *is compact.*
2. S *is closed in* $\mathscr{T}_{EUC}$ *and bounded in* $\mathbb{R}$*.*
3. *Every sequence in* $(S, \mathscr{T}_S)$ *has a convergent subsequence whose limit belongs to* S*.*
4. S *is bounded in* $\mathbb{R}$ *and* $\nexists$ *sequence in* $(S, \mathscr{T}_S)$ *whose limit belongs to* $\mathbb{R} \setminus S$*.*