The Construction of the Unique Ordered Field

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Axiom 1 (Peano Axioms)
There exists a triple $(\mathbb{N}', \sigma, 1_{\mathbb{N}})$ such that the followings hold:

1. *$\mathbb{N}'$ is a set, and the element $1_{\mathbb{N}}$ belongs to $\mathbb{N}'$*
2. *$\sigma : \mathbb{N}' \to \mathbb{N}'$ is an injective function, and $\forall n \in \mathbb{N}'$, $\sigma(n) \neq 1_{\mathbb{N}}$*
3. *For $S \subset \mathbb{N}'$, if $1_{\mathbb{N}} \in S$ and $m \in S$, then $\sigma(m) \in S$, and hence $S = \mathbb{N}'$*

Theorem 1.1 (Principle of Recursive Definition)
Let X be a set, let $\varphi : X \to X$ be a function, let $a \in X$, then there exists a unique function $f : \mathbb{N}' \to X$ such that $f(1_{\mathbb{N}}) = a$, and $\forall n \in \mathbb{N}'$, we have $f(\sigma(n)) = \varphi(f(n))$

Definition 1.1.0.0.1
Let $m \in \mathbb{N}'$, let $a = \sigma(m)$. By the Principle of Recursive Definition, for the function σ given in the Peano axiom, we can define a unique new function $f_m : \mathbb{N}' \to \mathbb{N}'$ such that $f_m(1_{\mathbb{N}}) = \sigma(m) = a$, and $\forall n \in \mathbb{N}'$, we have $f_m(\sigma(n)) = \sigma(f_m(n))$.

Definition 1.1.0.0.2
Let $m, n \in \mathbb{N}'$, $m +_{\mathbb{N}} n \coloneqq f_m(n)$

Definition 1.1.0.0.3
Let $m \in \mathbb{N}'$. By the Principle of Recursive Definition, for the function $f_m : \mathbb{N}' \to \mathbb{N}'$, we can define a unique new function $\mu_m : \mathbb{N}' \to \mathbb{N}'$ s.t. $\mu_m(1_{\mathbb{N}}) = m$, and $\forall n \in \mathbb{N}'$, we have $\mu_m(\sigma(n)) = f_m(\mu_m(n))$

Definition 1.1.0.0.4
Let $m, n \in \mathbb{N}'$, $m \cdot_{\mathbb{N}} n \coloneqq \mu_m(n)$. For notation, we write $m *_{\mathbb{N}} n = m \cdot_{\mathbb{N}} n$

Lemma 1.1.1
Let $m, n, q \in \mathbb{N}'$, we have $f_1(n) = \sigma(n)$, $\mu_1(n) = n$, and $(m = n) \iff (m +_{\mathbb{N}} q = n +_{\mathbb{N}} q)$

Lemma 1.1.2
For $m, n \in \mathbb{N}'$ with $m \neq n$, exactly one of the followings holds:

1. *If $\exists!\ r \in \mathbb{N}'$ s.t. $m = n +_{\mathbb{N}} r$, then we write $m >_{\mathbb{N}} n$*
2. *If $\exists!\ r \in \mathbb{N}'$ s.t. $n = m +_{\mathbb{N}} r$, then we write $n >_{\mathbb{N}} m$*

Corollary 1.1.2.1
For $m, n \in \mathbb{N}'$, we have Trichotomy holds:

1. *m=n* 2. *$m >_{\mathbb{N}} n$* 3. *$n >_{\mathbb{N}} m$*

Definition 1.1.2.1.1
Let S be a set, a Relation on S is a subset of $S \times S$. Let R be a relation on S, and let $x, y \in S$, we write xRy whenever $(x, y) \in \mathbb{R}$.

Definition 1.1.2.1.2
Let S be a set, let R be a relation on S, R is called an Equivalence Relation on S provided that the followings hold:

1. *R is reflexive, that is, $\forall s \in S$, we have sRs*
2. *R is symmetric, that is, $\forall s, t \in S$, if sRt, then tRs*
3. *R is transitive, that is, $\forall s, t, u \in S$, if sRt and tRu, then sRu*

If R is an equivalence relation on S, then for $s, t \in S$, we write $s \sim t$ whenever $(s, t) \in \mathbb{R}$

Definition 1.1.2.1.3
Let S be a set, let $\sim$ be an equivalence relation on S, let $x \in X$, $C(x) \coloneqq \{y \in S \mid y \sim x\}$ is called the Class of x, or the Equivalence Class of x. For notation, we write $C(x) = [(x)]$

Definition 1.1.2.1.4
Let S be a set, and let $\sim$ be an equivalence relation on S, $S/\!\sim \coloneqq \{C(x) \mid x \in X\}$ is called the Quotient of X by $\sim$, or the Factor Set of X by $\sim$.

Definition 1.1.2.1.5
Let $m, n, l, q \in \mathbb{N}'$, let $\sim_{\mathbb{Z}}$ be a relation on $(\mathbb{N}' \times \mathbb{N}')$ with $(n, m) \sim_{\mathbb{Z}} (l, q) \iff n +_{\mathbb{N}} q = l +_{\mathbb{N}} m$.
$\mathbb{Z}' \coloneqq (\mathbb{N}' \times \mathbb{N}')/\!\sim_{\mathbb{Z}}$

Lemma 1.1.3
The relation $\sim_{\mathbb{Z}}$ is an equivalence relation on the set $\mathbb{N}' \times \mathbb{N}'$

Definition 1.1.3.0.1
$0_{\mathbb{Z}} \coloneqq [(1_{\mathbb{N}}, 1_{\mathbb{N}})]$.

Definition 1.1.3.0.2
$1_{\mathbb{Z}} \coloneqq [(1_{\mathbb{N}} +_{\mathbb{N}} 1_{\mathbb{N}}, 1_{\mathbb{N}})]$.

Definition 1.1.3.0.3
Let $[(a,b)] \in \mathbb{Z}'$, $-[(a,b)] \coloneqq [(b,a)]$.

Definition 1.1.3.0.4
Let $[(n,m)], [(l,k)] \in \mathbb{Z}'$, $[(n,m)] +_{\mathbb{Z}} [(l,k)] \coloneqq [(n +_{\mathbb{N}} l, m +_{\mathbb{N}} k)]$.

Definition 1.1.3.0.5
Let $[(n,m)], [(l,k)] \in \mathbb{Z}'$, $[(n,m)] -_{\mathbb{Z}} [(l,k)] \coloneqq [(n,m)] +_{\mathbb{Z}} -[(l,k)]$.

Definition 1.1.3.0.6
Let $[(n,m)], [(l,k)] \in \mathbb{Z}'$, $[(n,m)] *_{\mathbb{Z}} [(l,k)] \coloneqq [(n *_{\mathbb{N}} l +_{\mathbb{N}} m *_{\mathbb{N}} k, m *_{\mathbb{N}} l +_{\mathbb{N}} n *_{\mathbb{N}} k)]$.

Lemma 1.1.4
Let $[(m,n)], [(p,q)] \in \mathbb{Z}'$, *exactly one of the followings hold:*

1. $m +_{\mathbb{N}} q >_{\mathbb{N}} p +_{\mathbb{N}} n$, *then we write* $[(m,n)] >_{\mathbb{Z}} [(p,q)]$
2. $m +_{\mathbb{N}} q <_{\mathbb{N}} p +_{\mathbb{N}} n$, *then we write* $[(m,n)] <_{\mathbb{Z}} [(p,q)]$
3. $[(m,n)] = [(p,q)]$

Theorem 1.2
Let $i_{\mathbb{N}} : \mathbb{N}' \to \mathbb{Z}' \quad n \mapsto [(n + 1_{\mathbb{N}}, 1_{\mathbb{N}})]$ *be a function, then* $i_{\mathbb{N}}(\mathbb{N}') \subseteq \mathbb{Z}'$, *with the followings hold:*

1. $i_{\mathbb{N}}$ *is an injection, and* $i_{\mathbb{N}}(1_{\mathbb{N}}) = 1_{\mathbb{Z}}$
2. *For* $m,n \in \mathbb{N}'$, $i_{\mathbb{N}}(n +_{\mathbb{N}} m) = i_{\mathbb{N}}(n) +_{\mathbb{Z}} i_{\mathbb{N}}(m)$
3. *For* $m,n \in \mathbb{N}'$, $i_{\mathbb{N}}(n *_{\mathbb{N}} m) = i_{\mathbb{N}}(n) *_{\mathbb{Z}} i_{\mathbb{N}}(m)$
4. *For* $m,n \in \mathbb{N}'$, $(n >_{\mathbb{N}} m) \iff (i_{\mathbb{N}}(n) >_{\mathbb{Z}} i_{\mathbb{N}}(m))$
5. *For* $m,n \in \mathbb{N}'$, $(n <_{\mathbb{N}} m) \iff (i_{\mathbb{N}}(n) <_{\mathbb{Z}} i_{\mathbb{N}}(m))$

Definition 1.2.0.0.1
Let $a,b,c,d \in \mathbb{Z}'$, *let* $\sim_{\mathbb{Q}}$ *be a relation on* $(\mathbb{Z}' \times (\mathbb{Z}' \setminus \{0_{\mathbb{Z}}\}))$ *with* $(a,b) \sim_{\mathbb{Q}} (c,d) \iff a *_{\mathbb{Z}} d = b *_{\mathbb{Z}} c$.
$\mathbb{Q}' \coloneqq (\mathbb{Z}' \times (\mathbb{Z}' \setminus \{0_{\mathbb{Z}}\}))/{\sim_{\mathbb{Q}}}$

Lemma 1.2.1
The relation $\sim_{\mathbb{Q}}$ *is an equivalence relation on the set* $\mathbb{Z}' \times (\mathbb{Z}' \setminus \{0_{\mathbb{Z}}\})$

Definition 1.2.1.0.1
$0_{\mathbb{Q}} \coloneqq [(0_{\mathbb{Z}}, 1_{\mathbb{Z}})]$.

Definition 1.2.1.0.2
$1_{\mathbb{Q}} \coloneqq [(1_{\mathbb{Z}}, 1_{\mathbb{Z}})]$.

Definition 1.2.1.0.3
Let $[(a,b)] \in \mathbb{Q}'$, $-[(a,b)] \coloneqq [(-a,b)]$.

Definition 1.2.1.0.4
Let $[(a,b)] \in \mathbb{Q}'$ *with* $a \neq 0_{\mathbb{Z}}$, $[(a,b)]^{-1} \coloneqq [(b,a)]$.

Definition 1.2.1.0.5
Let $[(a,b)], [(c,d)] \in \mathbb{Q}'$, $[(a,b)] +_{\mathbb{Q}} [(c,d)] \coloneqq [(a *_{\mathbb{Z}} d +_{\mathbb{Z}} b *_{\mathbb{Z}} c, b *_{\mathbb{Z}} d)]$.

Definition 1.2.1.0.6
Let $[(a,b)], [(c,d)] \in \mathbb{Q}'$, $[(a,b)] -_{\mathbb{Q}} [(c,d)] \coloneqq [(a,b)] +_{\mathbb{Q}} -[(c,d)]$.

Definition 1.2.1.0.7
Let $[(a,b)], [(c,d)] \in \mathbb{Q}'$, $[(a,b)] *_{\mathbb{Q}} [(c,d)] \coloneqq [(a *_{\mathbb{Z}} c, b *_{\mathbb{Z}} d)]$.

Definition 1.2.1.0.8
Let $[(a,b)], [(c,d)] \in \mathbb{Q}'$ *with* $c \neq 0_{\mathbb{Z}}$, $\frac{[(a,b)]}{[(c,d)]} \coloneqq [(a,b)] *_{\mathbb{Q}} [(c,d)]^{-1}$.

Lemma 1.2.2
Let $[(a,b)], [(c,d)] \in \mathbb{Q}'$, *exactly one of the followings hold:*

1. $a *_{\mathbb{Z}} d >_{\mathbb{Z}} b *_{\mathbb{Z}} c$, *and* $b *_{\mathbb{Z}} d >_{\mathbb{Z}} 0_{\mathbb{Z}}$, *then we write* $[(a,b)] >_{\mathbb{Q}} [(c,d)]$
2. $a *_{\mathbb{Z}} d <_{\mathbb{Z}} b *_{\mathbb{Z}} c$, *and* $b *_{\mathbb{Z}} d <_{\mathbb{Z}} 0_{\mathbb{Z}}$, *then we write* $[(a,b)] >_{\mathbb{Q}} [(c,d)]$
3. $a *_{\mathbb{Z}} d <_{\mathbb{Z}} b *_{\mathbb{Z}} c$, *and* $b *_{\mathbb{Z}} d >_{\mathbb{Z}} 0_{\mathbb{Z}}$, *then we write* $[(a,b)] <_{\mathbb{Q}} [(c,d)]$
4. $a *_{\mathbb{Z}} d >_{\mathbb{Z}} b *_{\mathbb{Z}} c$, *and* $b *_{\mathbb{Z}} d <_{\mathbb{Z}} 0_{\mathbb{Z}}$, *then we write* $[(a,b)] <_{\mathbb{Q}} [(c,d)]$
5. $[(a,b)] = [(c,d)]$

Definition 1.2.2.0.1

The function $|\ |_{\mathbb{Q}} : \mathbb{Q}' \to \mathbb{Q}' \quad x \mapsto \begin{cases} x & x >_{\mathbb{Q}} 0_{\mathbb{Q}} \\ -x & x <_{\mathbb{Q}} 0_{\mathbb{Q}} \\ 0_{\mathbb{Q}} & x = 0_{\mathbb{Q}} \end{cases}$ *is called the Absolute Value function on* $\mathbb{Q}'$

Theorem 1.3

Let $i_{\mathbb{Z}} : \mathbb{Z}' \to \mathbb{Q}' \quad n \mapsto [(n, 1_{\mathbb{Z}})]$ *be a function, then we have* $i_{\mathbb{Z}}(\mathbb{Z}') \subseteq \mathbb{Q}'$*, with the followings hold:*

1. $i_{\mathbb{Z}}$ *is an injection, with* $i_{\mathbb{Z}}(1_{\mathbb{Z}}) = 1_{\mathbb{Q}}$ *and* $i_{\mathbb{Z}}(0_{\mathbb{Z}}) = 0_{\mathbb{Q}}$
2. *For* $m, n \in \mathbb{Z}'$, $i_{\mathbb{Z}}(n +_{\mathbb{Z}} m) = i_{\mathbb{Z}}(n) +_{\mathbb{Q}} i_{\mathbb{Z}}(m)$
3. *For* $m, n \in \mathbb{Z}'$, $i_{\mathbb{Z}}(n *_{\mathbb{Z}} m) = i_{\mathbb{Z}}(n) *_{\mathbb{Q}} i_{\mathbb{Z}}(m)$
4. *For* $m, n \in \mathbb{Z}'$, $(n >_{\mathbb{Z}} m) \iff (i_{\mathbb{Z}}(n) >_{\mathbb{Q}} i_{\mathbb{Z}}(m))$
5. *For* $m, n \in \mathbb{Z}'$, $(n <_{\mathbb{Z}} m) \iff (i_{\mathbb{Z}}(n) <_{\mathbb{Q}} i_{\mathbb{Z}}(m))$

Definition 1.3.0.0.1

Any function of the form $seq : \mathbb{N}' \to \mathbb{Q}' \quad n \mapsto q_n$ *is called a Sequence in* $\mathbb{Q}'$*, the function seq is denoted as* (q_n) *or* $n \mapsto q_n$

Definition 1.3.0.0.2

Let (q_n) *be a sequence in* $\mathbb{Q}'$*,* (q_n) *is said to be Cauchy provided that for all* $L \in \mathbb{N}'$*,* $\exists\ N \in \mathbb{N}'$ *s.t.* $\forall n, m \in \mathbb{N}'$ *with* $m >_{\mathbb{N}} N$ *and* $n >_{\mathbb{N}} N$*, we have* $|q_n -_{\mathbb{Q}} q_m|_{\mathbb{Q}} <_{\mathbb{Q}} \frac{1_{\mathbb{Q}}}{i_{\mathbb{Z}}(i_N(L))}$

Definition 1.3.0.0.3

Let $(a_n), (b_n)$ *be Cauchy sequences in* $\mathbb{Q}'$*, we write* $\lim_{n\to\infty}(a_n -_{\mathbb{Q}} b_n) = 0_{\mathbb{Q}}$ *provided that for all* $L \in \mathbb{N}'$*,* $\exists\ N \in \mathbb{N}'$ *s.t.* $\forall n \in \mathbb{N}'$ *with* $n \geq N$*, we have* $|a_n -_{\mathbb{Q}} b_n|_{\mathbb{Q}} <_{\mathbb{Q}} \frac{1_{\mathbb{Q}}}{i_{\mathbb{Z}}(i_{\mathbb{N}}(L))}$

Definition 1.3.0.0.4

$\mathscr{C}_{\mathbb{Q}} \coloneqq \{(q_n) \mid (q_n)$ *is a Cauchy sequence in* $\mathbb{Q}'\ \}$

Definition 1.3.0.0.5

Let $\sim_{\mathbb{R}}$ *be a relation on* $\mathscr{C}_{\mathbb{Q}}$ *with* $(a_n) \sim_{\mathbb{R}} (b_n) \iff \lim_{n\to\infty}(a_n -_{\mathbb{Q}} b_n) = 0_{\mathbb{Q}}$

Lemma 1.3.1

The relation $\sim_{\mathbb{R}}$ *is an equivalence relation on the set* $\mathscr{C}_{\mathbb{Q}}$

Definition 1.3.1.0.1

$\mathbb{R} \coloneqq \mathscr{C}_{\mathbb{Q}}/\sim_{\mathbb{R}}$ *is called the set of real numbers*

Definition 1.3.1.0.2

Let $(\frac{1}{n})$ *denote the sequence* $seq : \mathbb{N}' \to \mathbb{Q}' \quad n \mapsto \frac{1_{\mathbb{Q}}}{i_{\mathbb{Z}}(i_{\mathbb{N}}(n))}$, $0_{\mathbb{R}} \coloneqq [(\frac{1}{n})]$

Definition 1.3.1.0.3

Let $(\frac{1+n}{n})$ *denote the sequence* $seq : \mathbb{N}' \to \mathbb{Q}' \quad n \mapsto \frac{1_{\mathbb{Q}} + i_{\mathbb{Z}}(i_{\mathbb{N}}(n))}{i_{\mathbb{Z}}(i_{\mathbb{N}}(n))}$, $1_{\mathbb{R}} \coloneqq [(\frac{n+1}{n})]$

Definition 1.3.1.0.4

Let $[(a_n)] \in \mathbb{R}$, $-[(a_n)] \coloneqq [(-a_n)]$

Definition 1.3.1.0.5

Let $[(a_n)], [(b_n)] \in \mathbb{R}$, $[(a_n)] +_{\mathbb{R}} [(b_n)] \coloneqq [(a_n +_{\mathbb{Q}} b_n)]$

Definition 1.3.1.0.6

Let $[(a_n)], [(b_n)] \in \mathbb{R}$, $[(a_n)] -_{\mathbb{R}} [(b_n)] \coloneqq [(a_n)] +_{\mathbb{R}} -[(b_n)]$

Definition 1.3.1.0.7

Let $[(a_n)], [(b_n)] \in \mathbb{R}$, $[(a_n)] *_{\mathbb{R}} [(b_n)] \coloneqq [(a_n *_{\mathbb{Q}} b_n)]$

Lemma 1.3.2

Let $[(a_n)] \in \mathbb{R}$ *with* $[(a_n)] \neq 0_{\mathbb{R}}$*, then* $\exists\ N \in \mathbb{N}$ *s.t.* $\forall n \in \mathbb{N}$*, we have* $a_n \neq 0_{\mathbb{Q}}$*. Let* (b_k) *be a subsequence of* (a_n) *with* $b_k = 1_{\mathbb{Q}}$ *for* $1 \leq k \leq N$*, and* $b_k = \frac{1_{\mathbb{Q}}}{a_k}$ *for* $k > N$*, then* (b_k) *belongs to* $\mathscr{C}_{\mathbb{Q}}$ *and we have* $[(a_n)] *_{\mathbb{R}} [(b_n)] = 1_{\mathbb{R}}$*, we write* $[(a_n)]^{-1} \coloneqq [(b_n)]$.

Definition 1.3.2.0.1

Let $[(a_n)], [(b_n)] \in \mathbb{R}$ *with* $[(b_n)] \neq 0_{\mathbb{R}}$, $\frac{[(a_n)]}{[(b_n)]} \coloneqq [(a_n)] *_{\mathbb{R}} [(b_n)]^{-1}$

Lemma 1.3.3

Let $[(a_n)], [(b_n)] \in \mathbb{R}$*, exactly one of the followings hold:*

1. *If* $\exists\ N \in \mathbb{N}$ *s.t.* $\forall n \in \mathbb{N}$ *with* $n >_{\mathbb{N}} N$*, we have* $(a_n -_{\mathbb{Q}} b_n) >_{\mathbb{Q}} 0_{\mathbb{Q}}$*, then we write* $[(a_n)] >_{\mathbb{R}} [(b_n)]$
2. *If* $\exists\ N \in \mathbb{N}$ *s.t.* $\forall n \in \mathbb{N}$ *with* $n >_{\mathbb{N}} N$*, we have* $(b_n -_{\mathbb{Q}} a_n) >_{\mathbb{Q}} 0_{\mathbb{Q}}$*, then we write* $[(a_n)] <_{\mathbb{R}} [(b_n)]$
3. $[(a_n)] = [(b_n)]$

Definition 1.3.3.0.1

The function $| \ | : \mathbb{R} \to \mathbb{R} \quad x \mapsto \begin{cases} x & x >_{\mathbb{R}} 0_{\mathbb{R}} \\ -x & x <_{\mathbb{R}} 0_{\mathbb{R}} \\ 0_{\mathbb{R}} & x = 0_{\mathbb{R}} \end{cases}$ *is called the Absolute Value function on* $\mathbb{R}$

Theorem 1.4

Let $i_{\mathbb{Q}} : \mathbb{Q}' \to \mathbb{R} \quad n \mapsto [(n, n, \cdots, n, \cdots)]$ *be a function, then* $i_{\mathbb{Q}}(\mathbb{Q}') \subseteq \mathbb{R}$*, with the followings hold:*

1. $i_{\mathbb{Q}}$ *is an injection, with* $i_{\mathbb{Q}}(1_{\mathbb{Q}}) = 1_{\mathbb{R}}$ *and* $i_{\mathbb{Q}}(0_{\mathbb{Q}}) = 0_{\mathbb{R}}$
2. *For* $m, n \in \mathbb{Q}'$, $i_{\mathbb{Q}}(n +_{\mathbb{Q}} m) = i_{\mathbb{Q}}(n) +_{\mathbb{R}} i_{\mathbb{Q}}(m)$
3. *For* $m, n \in \mathbb{Q}'$, $i_{\mathbb{Q}}(n *_{\mathbb{Q}} m) = i_{\mathbb{Q}}(n) *_{\mathbb{R}} i_{\mathbb{Q}}(m)$
4. *For* $m, n \in \mathbb{Q}'$, $(n >_{\mathbb{Q}} m) \iff (i_{\mathbb{Q}}(n) >_{\mathbb{R}} i_{\mathbb{Q}}(m))$
5. *For* $m, n \in \mathbb{Q}'$, $(n <_{\mathbb{Q}} m) \iff (i_{\mathbb{Q}}(n) <_{\mathbb{R}} i_{\mathbb{Q}}(m))$

Definition 1.4.0.0.1

The set $\mathbb{N} := i_{\mathbb{Q}}(i_{\mathbb{Z}}(i_{\mathbb{N}}(N')))$ *is called the set of Natural Numbers*
The set $\mathbb{Z} := i_{\mathbb{Q}}(i_{\mathbb{Z}}(Z'))$ *is called the set of Integers*
The set $\mathbb{Q} := i_{\mathbb{Q}}(Q')$ *is called the set of Rational Numbers*

Definition 1.4.0.0.2

$0 := 0_{\mathbb{R}}$
$1 := 1_{\mathbb{R}}$
$2 := 1 + 1$

Definition 1.4.0.0.3

Let $m, n \in \mathbb{R}$
$m + n := m +_{\mathbb{R}} n$
$m * n := m *_{\mathbb{R}} n$
$m - n := m -_{\mathbb{R}} n$
$(m < n) \iff (m <_{\mathbb{R}} n)$
$(m > n) \iff (m >_{\mathbb{R}} n)$

Definition 1.4.0.0.4

Any function of the form $seq : \mathbb{N} \to \mathbb{R} \quad n \mapsto r_n$ *is called a Sequence in of Real Numbers, the function* seq *is denoted as* r_n *or* $n \mapsto r_n$

Definition 1.4.0.0.5

Let (r_n) *be a sequence of real numbers,* (r_n) *is said to be Cauchy provided that for all* $\epsilon \in \mathbb{R}$ *with* $\epsilon > 0$, $\exists\ N \in \mathbb{N}$ *s.t.* $\forall n, m \in \mathbb{N}$ *with* $m > N$ *and* $n > N$*, we have* $|r_n - r_m| < \epsilon$

Definition 1.4.0.0.6

Let (r_n) *be a sequence of real numbers,* (r_n) *converges to some* $l \in \mathbb{R}$ *provided that for all* $\epsilon \in \mathbb{R}$ *with* $\epsilon > 0$, $\exists\ N \in \mathbb{N}$ *s.t.* $\forall n \in \mathbb{N}$ *with* $n > N$*, we have* $|r_n - l| < \epsilon$*. If* (r_n) *converges, then we say* (r_n) *is a Convergent Sequence of real numbers in the Euclidean topology. If* (r_n) *converges to some* $l \in \mathbb{R}$*, then* $\lim_{n\to\infty} r_n := l$ *is called the limit of* (r_n)

Theorem 1.5

Let $r \in \mathbb{R}$*, for* $w \in \mathbb{Q}$ *with* $w > 0$, $\exists\ q \in \mathbb{Q}$ *s.t.* $|r - q| < w$

Lemma 1.5.1

Q *is Archimedean, that is, the followings hold:*

1. *For* $q \in \mathbb{Q}$, $\exists\ N \in \mathbb{N}$ *s.t.* $N > q$
2. *For* $q \in \mathbb{Q}$ *with* $q > 0$, $\exists\ N \in \mathbb{N}$ *s.t.* $\frac{1}{N} < q$

Lemma 1.5.2

Let (x_n) *be a Cauchy sequence of real numbers, then* (x_n) *converges to some* $l \in \mathbb{R}$

Lemma 1.5.3

Let (x_n) *be a sequence of real numbers. If* (x_n) *is monotonic and bounded, then* (x_n) *is Cauchy.*

Corollary 1.5.3.1

All bounded monotonic sequences of real numbers converge

Theorem 1.6

The ordered field $(\mathbb{R}, +, *, 1, 0, <)$ *has the least upper bound property*

Definition 1.6.0.0.1

Let $(F_1, +_1, *_1, 0_1, 1_1)$ and $(F_2, +_2, *_2, 0_2, 1_2)$ be fields, the field $(F_1, +_1, *_1, 0_1, 1_1)$ is isomorphic to the field $(F_2, +_2, *_2, 0_2, 1_2)$ provided that there exists a bijection $\varphi : F_1 \to F_2$ such that $\forall x, y \in F_1$, we have the followings hold:

1. $\varphi(x +_1 y) = \varphi(x) +_2 \varphi(y)$
2. $\varphi(x *_1 y) = \varphi(x) *_2 \varphi(y)$
3. $\varphi(1_1) = 1_2$

Definition 1.6.0.0.2

Let $(F_1, +_1, *_1, 0_1, 1_1, <_1)$ and $(F_2, +_2, *_2, 0_2, 1_2, <_2)$ be ordered fields, we say $(F_1, +_1, *_1, 0_1, 1_1, <_1)$ is isomorphic to $(F_2, +_2, *_2, 0_2, 1_2, <_2)$ provided that there exists a bijection $\varphi : F_1 \to F_2$ such that $\forall x, y \in F_1$, we have the followings hold:

1. $\varphi(x +_1 y) = \varphi(x) +_2 \varphi(y)$
2. $\varphi(x *_1 y) = \varphi(x) *_2 \varphi(y)$
3. $\varphi(1_1) = 1_2$
4. *If $x <_1 y$, then $\varphi(x) <_2 \varphi(y)$*

Theorem 1.7

Let $(F_1, +_F, *_F, 0_F, 1_F, <_F)$ be an ordered field. If $(F_1, +_F, *_F, 0_F, 1_F, <_F)$ is a complete ordered field, then $(F_1, +_F, *_F, 0_F, 1_F, <_F)$ is isomorphic to $(\mathbb{R}, +, *, 1, 0, <)$

Definition 1.7.0.0.1

The ordered field $(\mathbb{R}, +, *, 1, 0, <)$ is called the Unique Complete Ordered field