Theorem 0.1
$S \subseteq V$ is affine, $\vec{0} \in S$, then S is affine if and only if S is a vector subspace of V.

Theorem 0.2 ($\widetilde{S} \coloneqq \{\vec{a} - \vec{b} \mid \vec{a}, \vec{b} \in S\}$)
For $\vec{x} \in S$, S is affine if and only if $S - \vec{x}$ is affine. If S is affine, $S - \vec{x} = \widetilde{S}$.

Theorem 0.3
Let $f : X \to Y$ be a function. f is sequentially continuous if and only if f is continuous.

Theorem 0.4 (Bolzano-Weierstrass Theorem for metric spaces)
A metric space (X, d) is compact if and only if (X, d) is sequentially compact.

Theorem 0.5 (Bolzano-Weierstrass Theorem for $\mathbb{R}^n$ space)
For $X \subseteq \mathbb{R}^n$, X is sequentially compact if and only if X is closed and bounded.

Theorem 0.6 (Heine-Borel Theorem)
For $X \subseteq \mathbb{R}^n$ with Euclidean metric, X is compact if and only if X is closed and bounded.

Theorem 0.7 (Chain Rule for Multivariate Differentiation)
Let $f : V \to W$ be a differentiable function, and let $g : im(f) \to Z$ be a differentiable function. For $\vec{a} \in V$, $D(g \circ f)(\vec{a}) = Dg(\vec{b}) \circ Df(\vec{a})$, where $f(\vec{a}) = \vec{b}$.

Theorem 0.8
Let A be an open subset of $\mathbb{R}^m$, and let $f : A \to \mathbb{R}^n \quad \vec{a} \mapsto (f_1(\vec{a}), f_2(\vec{a}), \cdots, f_n(\vec{a}))$ be a function. If $D_k f_j$ exists and is continuous, then $f \in C^1(A, \mathbb{R}^n)$.

Theorem 0.9
Given $f \in C^2(A, \mathbb{R})$, where A is an open subset of $\mathbb{R}^2$. Then we can write the following:

$$D_2 D_1 f(a, b) = \lim_{(h,k) \to (0,0)} \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk}$$

Theorem 0.10 (Inverse Function Theorem)
Let A be an open subset of $\mathbb{R}^n$, let $\vec{a} \in A$, let $f \in C^r(A, \mathbb{R}^n)$ with $r \geq 1$, and given $Df(\vec{a})$ is invertible. There exists an open neighborhood U of $\vec{a}$ such that $f|_U$ is a C^r-diffeomorphism, that is, f maps U bijectively to some open set in $\mathbb{R}^n$, and $f^{-1} : f(U) \to U$ is of C^r type.

Theorem 0.11
Given E as a open subset of $\mathbb{R}^n$, $f \in C^1(E, \mathbb{R}^n)$, and $\det(Df(\vec{x})) \neq 0$ for all $\vec{x} \in E$. Then $f(\vec{a}) \in Int(f(E))$ for all $\vec{a} \in E$, $f(E)$ is open in $\mathbb{R}^n$, and $f : E \to f(E)$ is an open map.

Theorem 0.12 (Implicit Function Theorem)
Let $\vec{G}$ in $\mathbb{R}^{k+n}$, $\vec{G} = (\vec{x}, \vec{y})$ with $\vec{x} \in \mathbb{R}^k$, $\vec{y} \in \mathbb{R}^n$. Fix $\vec{S}$ in $\mathbb{R}^{k+n}$, $\vec{S} = (\vec{a}, \vec{b})$ with $\vec{a} \in \mathbb{R}^k$, $\vec{b} \in \mathbb{R}^n$. For $f \in C^r(A, \mathbb{R}^n)$, where A is an open subset of $\mathbb{R}^{k+n}$. If we have $\vec{S} \in f^{-1}(\vec{0}) \coloneqq E$, and $\frac{\partial f}{\partial \vec{y}}\ \vec{S}$ is invertible. Then there exists a neighborhood U of $\vec{S}$ such that $E \cap U =$Graph(g) for a unique function $g \in C^r(B, \mathbb{R}^n)$, where $\vec{a} \in B$, and B is an open subset of $\mathbb{R}^k$. In other words, $\exists$ an open neighborhood B of $\vec{x}$ such that $\vec{y} = g(\vec{x})$ and $f(\vec{x}, g(\vec{x})) = \vec{0}$ for all $\vec{x} \in B$, with a unique function $g \in C^r(B, \mathbb{R}^n)$.

Theorem 0.13 (First Derivative Test for Higher Dimensions)
Let Ω be an open subset of $\mathbb{R}^n$, and let $h : \Omega \to \mathbb{R}$ be a function that achieves a local maximum, or minimum, at $\vec{p} \in \Omega$, then $Dh(\vec{p}) = 0$.

Theorem 0.14 (Lagrange Multiplier Theorem)
Let U be an open subset of $\mathbb{R}^{k+n}$, let the constraint function be $f \in C^1(U, \mathbb{R}^n)$ with $E = f^{-1}(\vec{0})$, let the objective function be $h \in C^1(U, \mathbb{R})$, with the property that $h|_E$ has a local maximum , or a local minimum, at $\vec{p} \in E$. Given rank$((Df(\vec{p})) = n$, we have $Dh(\vec{p})$ belongs to the row space of $Df(\vec{p})$, that is, we can write $Dh(\vec{p}) = \lambda_1 Df_1(\vec{p}) + \cdots \lambda_n Df_n(\vec{p})$ for $\lambda_j \in \mathbb{R}$.

Theorem 0.15 (Spectral Theorem)
Every symmetric square matrix admits an orthonormal basis of eigenvectors. The corresponding eigenvalues are all real.

Theorem 0.16 (Second Derivative Test for Higher Dimensions)
Let Ω be an open subset of $\mathbb{R}^n$, let $f \in C^2(\Omega, \mathbb{R})$, with $Hf(\vec{x}) \geq 0$ for all $\vec{x} \in \Omega$. If $Df(\vec{x}_0) = \vec{0}$ for some $\vec{x}_0 \in \Omega$, then $f(\vec{x}) \geq f(\vec{x}_0)$ for all $\vec{x} \in \Omega$.

Theorem 0.17 (Local Second Derivative Test)
Let A be an open subset of $\mathbb{R}^n$, let $f \in C^2(A, \mathbb{R})$, let $\vec{x}_0 \in A$ with $Df(\vec{x}_0) = \vec{0}$. Then we have the followings hold:

1. *If we have $Hf(\vec{x}_0) > 0$, then $Hf(\vec{x}) > 0$ for all $\vec{x} \in B_\delta(\vec{x}_0)$ with some $\delta > 0$, and the function f achieves a local minimum at $\vec{x}_0$*
2. *If we have $Hf(\vec{x}_0) \ngeq 0$, then the function f has a strict local maximum at the point $\vec{x}_0$ along some line in A, and hence f does not have a local minimum at the point $\vec{x}_0$.*
3. *If we have $Hf(\vec{x}_0) < 0$, then f has strict local maximum at $\vec{x}_0$*
4. *If we have $Hf(\vec{x}_0) \nleq 0$, then f does not have a local maximum at $\vec{x}_0$*
5. *If we have $Hf(\vec{x}_0)$ is not definite nor semi-definite, then the function f does not have a local max, nor local min, at the point $\vec{x}_0$.*

Proposition 0.18
Let $(X, \mathscr{T})$ be a topological space, the followings are equivalent:

1. *There exists $f : X \to \{0, 1\}$ that is a continuous surjective function.*
2. *There exists nonempty $A \subsetneq X$ that is open and closed in X*

Proposition 0.19
Let $(X, \mathscr{T})$ be a topological space. The function $f : X \to \mathbb{R}^n \quad x \mapsto (f_1(x), f_2(x), \cdots, f_n(x))$ is continuous if and only if each function $f_j : X \to \mathbb{R}$ is continuous.

Let f be a C^2 type function defined in a neighborhood of $\vec{x} \in \mathbb{R}^n$. The **Hessian** $Hf(\vec{x})$ of f at $\vec{x}$ is the $n \times n$ matrix whose entry at i-th row, j-th column is given by $D_k D_j f(\vec{x})$.

The **directional derivative of f at $\vec{a}$ in the direction of $\vec{u}$** is $f'(\vec{a}; \vec{u}) := \lim_{t \to 0} \frac{f(\vec{a}+t\vec{u}) - f(\vec{a})}{t}$.

Let V, W be normed vector spaces over the field $\mathbb{R}$ or the field $\mathbb{C}$, let $\vec{a} \in A$, where A is an open subset of V. For function $f : V \to W$, we write $Df(\vec{a}) = T$ provided that there exists $T \in B(V, W)$ that satisfies $\lim_{\vec{h} \to \vec{0}} \frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - T(\vec{h})}{||\vec{h}||} = \vec{0}$. If such T exists, then we write $Df(\vec{a})(\vec{h}) = T(\vec{h})$ and f is said to be **differentiable** at $\vec{a}$.

Let (X, d) be a metric space, let $x_0 \in X$, and let $A \subseteq X$. x_0 is **interior** to A provided that $\exists\ \epsilon > 0$ such that $B_\epsilon(x_0) \subseteq A$. x_0 is **exterior** to A provided that $\exists\ \epsilon > 0$ such that $B_\epsilon(x_0) \cap A = \emptyset$. x_0 is a **boundary point** of A provided that $\forall \epsilon > 0$, we have $B_\epsilon(x_0) \cap A \neq \emptyset \neq B_\epsilon(x_0) \cap (X \setminus A)$.

$f : X \to Y$ is said to be **bi-Lipschitz** provided that $\exists\ C \in [0, +\infty), \widetilde{C} \in (0, \infty)$ such that $\widetilde{C} \cdot d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq C d_X(x_1, x_2)$ for all $x_1, x_2 \in X$.

In general, if $B \in Mat(n, n, \mathbb{R})$ is a symmetric matrix then we say that $B \geq 0$ if $\vec{a}^T B \vec{a} \geq 0$ for all $\vec{a} \in \mathbb{R}^n$. Here B admits a set of maximized real eigenvalues $\{\mu_1, \mu_2, \cdots, \mu_n\}$.

1. If $\mu_j \geq 0$ for all $1 \leq j \leq n$, then B is positive semi-definite, with $B > 0$
2. If $\mu_j > 0$ for all $1 \leq j \leq n$, then B is said to be positive definite, with $B \geq 0$.
3. If $\mu_j \leq 0$ for all $1 \leq j \leq n$, then B is said to be negative semi-definite, with $B \leq 0$.
4. If $\mu_j < 0$ for all $1 \leq j \leq n$, then B is said to be negative definite, with $B < 0$.

Note: For vector space V over $\mathbb{R}$. $S \subseteq V$ is convex if and only if S is connected.
Note: A function T is affine if and only if $T = \widetilde{T} + \vec{b}$ for some linear T and $\vec{b} = T(\vec{0})$.
Note: Let (X, d) be a metric space, let $A \subseteq X$, and let $x_0 \in X$. Int(A) is an open subset of A, and it contains all open subsets of A. Bd$(X \setminus A)$ =Bd(A). Bd(A) is closed.
Note: (X, d) is compact if and only if every sequence in X admits a convergent subsequence.
Note: Let (X, d) be a sequentially compact metric space, $\forall \epsilon > 0$, we can cover X by finitely many ϵ-ball, in other words, X is totally bounded.
Note: A path connected topological space is connected.
Note: Convex subset of $\mathbb{R}^n$ is connected and path-connected.
Note: Let V and W be normed vector spaces over field $\mathbb{R}$ or $\mathbb{C}$. For $T \in \hom(V, W)$, TFAE:
(1) T is Lipschitz, (2) T is continuous, (3) T is continuous at $\vec{0} \in V$,
(4) $\exists\ M \in [0, \infty)$ such that $||T\vec{v}|| \leq M||\vec{v}||\ \forall \vec{v} \in V$, in which we say T is bounded.
Note: Intersection of affine sets is affine, intersection of convex sets is convex.

Theorem 0.20
Let $f : V \to W$ be a linear map between normed vector space, TFAE:

1. $\exists\ M \in [0, \infty)$ *such that* $||T\vec{v}|| \leq M$ *with* $||\vec{v}|| \leq 1$.
2. $\exists\ M \in [0, \infty)$ *such that* $||T\vec{v}|| \leq M||\vec{v}||$ *for all* $\vec{v} \in V$.
3. T *satisfies* $d_W(T(\vec{v}_1), T(\vec{v}_2)) \leq M \cdot d_V(\vec{v}_1, \vec{v}_2)$ *for* $\vec{v}_1, \vec{v}_2 \in V$ *and some* $M \in [0, \infty)$.
4. T *is continuous on* V.
5. T *is continuous at* $\vec{0}$.

Theorem 0.21 (Contraction Mapping Theorem)
Let $f : X \to X$ be a contraction on a non-empty complete metric space X.
The equation $f(x) = x$ has exactly one solution $x \in X$.

Theorem 0.22
A linear map from a finite dimensional vector space to normed vector space is continuous.
If $T : \mathbb{R}^m \to W$ is a linear bijection, then T^{-1} is continuous.

Theorem 0.23
Let $|| \cdot ||$ and $| \cdot |$ be two norms on a finite dimensional vector space V over the field $\mathbb{R}$.
There exists $C_1, C_2 \in (0, \infty)$ such that $C_1^{-1}||\vec{v}|| \leq |\vec{v}| \leq C_2||\vec{v}||$ for all $\vec{v} \in V$.

Theorem 0.24
Let V be a vector space over a field F, a nonempty finite subset A of V is affinely independent if and only if for all $\vec{a} \in A$, $\vec{a}$ is not an affine combination of vectors in $A \setminus \{\vec{a}\}$.

Theorem 0.25
Let M be an $n \times n$ matrix and let $T : \mathbb{R}^n \to \mathbb{R}^n \quad \vec{x} \mapsto M\vec{x}$ be a function. T is isometry if and only if $< M\vec{x}, M\vec{y} >=< \vec{x}, \vec{y} >$ for $\vec{x}, \vec{y} \in \mathbb{R}^n$, if and only if $M^T M = I$.

Theorem 0.26
Any path connected set is connected.
Any connected open subset of $\mathbb{R}^n$ is path connected.
Any open subset of $\mathbb{R}^n$ is a countable disjoint union of connected open sets.

Theorem 0.27
Let Ω be an open subset of $\mathbb{R}^n$, let $\psi \in C^2(\Omega, \mathbb{R})$ with the property that $H\psi(\vec{x}) > 0\ \forall \vec{x} \in \Omega$.

$$\textit{epigraph}(\psi) = \bigcap_{\vec{x}_0 \in \Omega} \{(\vec{x}, y) \in \Omega \times \mathbb{R} \mid y \geq \psi(\vec{x}_0) + D\psi(\vec{x}_0)(\vec{x} - \vec{x}_0)\} \text{ is convex}$$

Theorem 0.28
Let Ω be a convex subset of $\mathbb{R}^n$. $f : \Omega \to \mathbb{R}$ is convex if and only if we have:
$f((1 - t)\vec{x}_0 + t\vec{x}_1) \leq (1 - t)f(\vec{x}_0) + tf(\vec{x}_1), \quad \forall \vec{x}_1, \vec{x}_0 \in \Omega,\ 0 \leq t \leq 1$

Theorem 0.29
Let A be an open subset of $\mathbb{R}^{k+n}$, and let $f : A \to \mathbb{R}^n$ be a differentiable function. Write f in the form $f(\vec{x}, \vec{y})$ for $\vec{x} \in \mathbb{R}^k$ and $\vec{y} \in \mathbb{R}^n$. If there exists a differentiable function $f : B \to \mathbb{R}^n$ defined on an open set B in $\mathbb{R}^k$ such that $f(\vec{x}, g(\vec{x})) = \vec{0}$ for all $\vec{x} \in B$, then for $\vec{x} \in B$, we have:

$$Dg(\vec{x}) = -\left[\frac{\partial f}{\partial \vec{y}}(\vec{x}, g(\vec{x}))\right]^{-1} \frac{\partial f}{\partial \vec{x}}(\vec{x}, g(\vec{x}))$$

Theorem 0.30
Let A be an open subset of $\mathbb{R}^n$, let $f : A \to \mathbb{R}^n$, and let $f(\vec{a}) = \vec{b}$. Let g be a function that maps an open neighborhood of $\vec{b}$ into $\mathbb{R}^n$ such that $g(\vec{b}) = \vec{a}$, and $g(f(\vec{x})) = \vec{x}$ for all $\vec{x}$ in a neighborhood of $\vec{a}$. If f is differentiable at $\vec{a}$ and if g is differentiable at $\vec{b}$, then we have $Dg(\vec{b}) = [Df(\vec{a})]^{-1}$.

Theorem 0.31
Let A be open in $\mathbb{R}^m$, let $f : A \to \mathbb{R}$ be differentiable on A. If A contains the line segment with end points $\vec{a}$ and $\vec{a} + \vec{h}$, then there is a point $\vec{c} = \vec{a} + t_0\vec{h}$ with $0 < t_0 < 1$ on such line segment such that $f(\vec{a} + \vec{h}) - f(\vec{a}) = Df(\vec{c}) \cdot \vec{h}$.

Theorem 0.32
Let A be a convex open subset of V and let $g : A \to W$ be a differentiable function satisfying $||Dg(\vec{a})|| \leq M$ for all $\vec{a} \in A$ and some $M \geq 0$. Then $||g(\vec{b}) - g(\vec{a})|| \leq M||\vec{b} - \vec{a}||$.

Theorem 0.33
Let $f : Q \to \mathbb{R}$ be a bounded function with Q being a box of $\mathbb{R}^n$.
Given $\epsilon > 0$ and some $k \in \mathbb{N}$, the followings are equivalent:

1. *The function f is Riemann integrable on Q*
2. *There exists a partition P such that $U(f, P) < L(f, P) + \epsilon$*
3. *$\exists$ some subboxes $R_1, R_2, \cdots, R_j$ of Q s.t. $\mathscr{D}_k(f) \subseteq R_1 \cup R_2 \cup \cdots \cup R_j$ and $\sum_{i=1}^{j} V(R_i) < \epsilon$*
4. *There exists some subboxes R_p of Q such that $\mathscr{D}(f) \subseteq \bigcup_{p=1}^{\infty} R_p$, with $\sum_{p=1}^{\infty} V(R_p) < \epsilon$*
5. *There exists some subboxes R_p of Q such that $\mathscr{D}(f) \subseteq \bigcup_{p=1}^{\infty} rInt(R_p)$ with $\sum_{p=1}^{\infty} V(R_p) < \epsilon$*

In the context of this theorem, a box R is called a subbox of Q provided that R is a box in $\mathbb{R}^n$ contained in Q, and here R is allowed to have zero volume, or in other words, measure zero.

Theorem 0.34 (Fubini's Theorem)
Let A be a box in $\mathbb{R}^k$, B be a box in $\mathbb{R}^n$, let $Q = A \times B$, and let $f : Q \to \mathbb{R}$ be a bounded function. Then we can write:

$$\underline{\int_Q} f \le \underline{\int_{\vec{x}\in A}} \underline{\int_{\vec{y}\in B}} f(\vec{x},\vec{y}) \le \left\{ \begin{array}{l} \underline{\int_{\vec{x}\in A}} \overline{\int_{\vec{y}B}} f(\vec{x},\vec{y}) \\ \overline{\int_{\vec{x}\in A}} \underline{\int_{\vec{y}\in B}} f(\vec{x},\vec{y}) \end{array} \right\} \le \overline{\int_{\vec{x}\in A}} \overline{\int_{\vec{y}\in B}} f(\vec{x},\vec{y}) \le \overline{\int_Q} f$$

Corollary 0.34.1
Let A be a box in $\mathbb{R}^k$, B be a box in $\mathbb{R}^n$, let $Q = A \times B$, and let $f : Q \to \mathbb{R}$ be a bounded function. If f is integrable on Q, then we have:

$$\underline{\int_Q} f = \underline{\int_{\vec{x}\in A}} \underline{\int_{\vec{y}\in B}} f(\vec{x},\vec{y}) = \left\{ \begin{array}{l} \underline{\int_{\vec{x}\in A}} \overline{\int_{\vec{y}B}} f(\vec{x},\vec{y}) \\ \overline{\int_{\vec{x}\in A}} \underline{\int_{\vec{y}\in B}} f(\vec{x},\vec{y}) \end{array} \right\} = \overline{\int_{\vec{x}\in A}} \overline{\int_{\vec{y}\in B}} f(\vec{x},\vec{y}) = \overline{\int_Q} f$$

and we can write:

$$\int_Q f = \int_{\vec{x}\in A} \overline{\int_{\vec{y}\in B}} f(\vec{x},\vec{y}) = \int_{\vec{x}\in A} \underline{\int_{\vec{y}\in B}} f(\vec{x},\vec{y})$$

Theorem 0.35
Let S be a bounded subset of $\mathbb{R}^n$, let $f : S \to \mathbb{R}$ be a bounded continuous function, let $E = \{\vec{x}_0 \in Bd(S) \mid \lim_{\vec{x}\in S,\ \vec{x}\to\vec{x}_0} f(\vec{x}) \ne 0\}$. If we have $m^*(E) = 0$, then f is Riemann integrable on S.

Theorem 0.36 (Theorem 15.2 on Munkres)
Let A be an open subset of $\mathbb{R}^n$, let $E_1 \subseteq E_2 \subseteq \cdots \subseteq A$ be compact rectifiable sets with $\bigcup_j Int(E_j) = A$, then we have ext $\int_A f = \lim_{j\to\infty} \int_{E_j} f$ when ext $\int_A f$ exists for continuous function $f : A \to \mathbb{R}$.

Theorem 0.37 (Theorem 15.6 on Munkres)
Let A be an open subset of $\mathbb{R}^n$, let $U_1 \subseteq U_2 \subseteq \cdots \subseteq A$ be open subsets of A with $\bigcup_j U_j = A$, then we have ext $\int_A f = \lim_{j\to\infty}$ ext $\int_{U_j} f$ whenever ext $\int_A f$ exists for the continuous function $f : A \to \mathbb{R}$.

Theorem 0.38 (Fubini's Theorem for Simple Regions)
Let $S := \{(x,t) \mid x \in C,\ \phi(x) \le t \le \psi(x)\}$ be a simple region in $\mathbb{R}^n$, where C is a compact rectifiable set in $\mathbb{R}^{n-1}$ for $n \ge 2$, $\phi : C \to \mathbb{R}$ and $\psi : C \to \mathbb{R}$ are continuous functions with the property $\phi(x) \le \psi(x)$ for all $x \in C$, let $f : S \to \mathbb{R}$ be a continuous function. Then f is integrable over S and we have the following holds:

$$\int_S f = \int_{x\in C} \int_{t=\phi(x)}^{t=\psi(x)} f(x,t)\,dt\,dx$$

Theorem 0.39 (Change of Variable Theorem)
Let A be an open subset of $\mathbb{R}^n$, let B be an open subset of $\mathbb{R}^n$, and let g be a diffeomorphism from A to B. For continuous function $f : B \to \mathbb{R}$, f is integrable over B if and only if the function $(f \circ g) \cdot |\det Dg|$ is integrable over A. Moreover, if f is integrable over B, we have

$$ext \int_B f = ext \int_A (f \circ g) \cdot |\det Dg|$$

That is, for continuous function $f : B \to \mathbb{R}$, we have either ext $\int_B f =$ ext $\int_A (f \circ g) \cdot |\det Dg|$, or neither ext $\int_B f$ nor ext $\int_A (f \circ g) \cdot |\det Dg|$ exists.

Theorem 0.40 (Partition of Unity Theorem)
Let Ω be an open subset of $\mathbb{R}^n$. If $\Omega = \bigcup_{\alpha\in A} U_\alpha$ for some open subsets U_α of $\mathbb{R}^n$, then there exist some functions $\phi_1, \phi_2, \cdots \in C^\infty(\Omega, [0,\infty))$ such that the followings hold:

1. *Each $supp(\phi_j)$ is compact*
2. *Each $supp(\phi_j)$ is contained in some U_α*

3. Each $\vec{x} \in \Omega$ has an open neighborhood that intersects only finitely many supp(ϕ_j)
4. $\sum_{j=1}^{\infty} \phi_j(\vec{x}) = 1$ for all $\vec{x} \in \Omega$, such sum is called the locally finite sum.

Theorem 0.41 (Theorem 21.2 from Munkres)
For $k \leq n$, let $T : \mathbb{R}^k \to \mathbb{R}^n \quad \vec{x} \mapsto A\vec{x} + \vec{b}$ be an affine injection for some matrix A and $\vec{b} \in \mathbb{R}^n$. One can pick an orthogonal $n \times n$ matrix B such that we have:

$$B \cdot A = \begin{bmatrix} M \\ Z \end{bmatrix}$$

for some $k \times k$ matrix M and zero matrix Z.

Theorem 0.42
For $M \subseteq \mathbb{R}^n$, the followings are equivalent:
1. *For all $\vec{p} \in M$, there exist a set $U \subseteq \mathbb{R}^k$ open in $\mathbb{R}^k$, a set $V \subseteq M$ open in M that contains $\vec{p}$, and a homeomorphism $\alpha \in C^r(U, V)$ with rank $D\alpha(\vec{x}) = k$ for all $\vec{x} \in U$.*
2. *For all $\vec{p} \in M$, there exist a set $A \subseteq \mathbb{R}^k$ open in $\mathbb{R}^k$, a set $V \subseteq M$ open in M that contains $\vec{p}$, a function $g \in C^r(A, \mathbb{R}^{n-k})$, and a coordinate permutation $\rho : \mathbb{R}^n \to \mathbb{R}^n$, such that we have $\rho(V) = Graph(g)$.*

Theorem 0.43
Let U be an open subset of $\mathbb{R}^n$, let $F \in C^r(U, \mathbb{R}^{n-k})$, let $M = F^{-1}(\vec{0})$. If $rank(DF(\vec{x})) = n - k$ for all $\vec{x} \in M$. Then M is a k-manifolds without boundary of class C^r.

Theorem 0.44 (Theorem 24.4 on Munkres)
Let M be a k-manifold, ∂M is a C^r $k-1$ manifold without boundary.

Theorem 0.45
Every k-manifold M can be decomposed uniquely as a disjoint union of open connected k-manifolds, which are called the components of M.

Theorem 0.46
Every connected 1-manifold of class C^r is C^r-diffeomorphic to an interval in $\mathbb{R}$ or to the circle S^1.

Theorem 0.47
Let A be an open connected subset of $\mathbb{R}^n$, let $f \in C^1(A, \mathbb{R})$. $df(x) = 0$ for $x \in A$ if and only if f is a constant function.

Theorem 0.48 (Fundamental Theorem of Calculus I(a) for 1-forms)
Let ω be a 1-form on A, where A is a connected open subset of $\mathbb{R}^m$. The followings are equivalent:
1. *$\omega = df$ for some $f \in C^1(A, \mathbb{R})$, in which case ω is said to be exact on A.*
2. *For $\alpha \in C^1_{pw}([a, b], A)$ with $\alpha(a) = \alpha(b)$, we have $\int_{Y_\alpha} \omega = 0$.*
3. *For $\alpha_j \in C^1_{pw}([a_j, b_j], A)$ with $\alpha_1(a_1) = \alpha_2(a_2)$ and $\alpha_1(b_1) = \alpha_2(b_2)$, we have $\int_{Y_{\alpha_1}} \omega = \int_{Y_{\alpha_2}} \omega$, in which case ω is said to be path independent.*

Theorem 0.49 (Fundamental Theorem of Calculus I(b) for 1-forms)
Let ω be a closed 1-form on $A \subseteq \mathbb{R}^m$. If A is a convex open subset of $\mathbb{R}^m$, then ω is exact on A.

Theorem 0.50 (Fundamental Theorem of Calculus II for one-forms)
For C^1 type function $\alpha : [a, b] = I \to A$ where A is an open subset of $\mathbb{R}^n$, with C^1 function $f : A \to \mathbb{R}$, we can write the following:

$$\int_{Y_\alpha} df = \int_I \alpha^* df = \int_I d(f \circ \alpha) = \int_I (f \circ \alpha)' = (f \circ \alpha)(b) - (f \circ \alpha)(a) = f(\alpha(b)) - f(\alpha(a)) := \Delta_{Y_\alpha} f$$

Lemma 0.50.1
Let ω be a closed 1-form of C^1 type, and let α be a C^2 type function. Then $\alpha^*\omega$ is closed.

Lemma 0.50.2 (Green's Theorem for Two-dimensional Boxes)
Let ω be a 1-form defined on an open set $A \subseteq \mathbb{R}^2$ which contains a box R of $\mathbb{R}^2$, then we have:

$$\int_{\substack{Bd(R) \\ \text{counter-clockwise orientation}}} \omega = \int_R (D_1\omega_2 - D_2\omega_1)$$

where ω_1, ω_2 are component functions of ω. The counter-clockwise orientation of $Bd(R)$ refers to a path which maps an interval in $\mathbb{R}$ to $Bd(R)$ that goes in counter-clockwise direction on $Bd(R)$.

Note: Consider a bounded function defined on a box Q.

1. If $f^{-1}(0)$ is dense in Q, then all $L(f,P) \le 0$, all $U(f,P) \ge 0$, which implies $\underline{\int}_Q f \le 0 \le \overline{\int}_Q f$.
2. If $f^{-1}(0)$ is dense in Q and f is integrable, then $\int_Q f = 0$.
3. If $f \ge 0$, $f(\vec{a}) > 0$, f is continuous at $\vec{a}$, then $\underline{\int}_Q f > 0$.
4. If $f \ge 0$, $f(\vec{a}) > 0$, and $\underline{\int}_Q f = 0$, then f is discontinuous.
5. If $f \ge 0$ and f is integrable on Q, with $\int_Q f = 0$, then $Q \setminus f^{-1}(0)$ has measure zero.

A bounded set $S \subseteq \mathbb{R}^n$ is said to be rectifiable provided that any one of the following holds:

1. The function $\mathbb{I} : S \to \mathbb{R} \quad \vec{x} \mapsto 1$ is Riemann integrable on S
2. The indicator function $\mathbb{I}_S$ is integrable on some box $Q \subseteq \mathbb{R}^n$ that contains S.
3. $m^*(Bd(S)) = 0$
4. $m^{*,J}(Bd(S)) = 0$

Lebesgue outer measure: $m^*(E) := \inf\left\{\sum_{j=1}^{\infty} V(Q_j) \mid E \subseteq \bigcup_{j=1}^{\infty} Q_j, \text{ where } Q_j \text{ are boxes in } \mathbb{R}^n\right\}$

Jordan outer measure: $m^{*,J}(E) := \inf\left\{\sum_{j=1}^{k} V(Q_j) \mid E \subseteq \bigcup_{j=1}^{k} Q_j, \text{ where } Q_j \text{ are boxes in } \mathbb{R}^n\right\}$

The Q_j can be replaced by $Int(Q_j)$. The two measures are equal when E is compact.

Corollary 0.50.3

Let S be a bounded subset of $\mathbb{R}^n$, let $f : S \to \mathbb{R}$ be a bounded continuous function.
If $m^*(Bd(S)) = 0$, then f is Riemann integrable on S.

Definition 0.50.3.0.1

For $f \in C(A, \mathbb{R})$ where A is an open subset of $\mathbb{R}^n$,
$ext \int_A f$ exists provided that at least one of $ext \int_A f_+$ and $ext \int_A f_-$ is finite.
Take supremum of integrating f on compact rectifiable set when f is non-negative.

$avg_A f := \frac{\int_A f}{V(A)}$ and for a special case where $f : \mathbb{R}^n \to \mathbb{R}^n \quad \vec{x} \mapsto \vec{x}$, $avg_A f$ is the centroid of A.

Definition 0.50.3.0.2

Let Q be a box in $\mathbb{R}^k$, and let T be an affine injection from $\mathbb{R}^k$ to $\mathbb{R}^n$ of the form $\vec{x} \mapsto A\vec{x} + \vec{b}$ for some matrix A and vector $\vec{b} \in \mathbb{R}^n$. We define $V_k(T(Q)) = \sqrt{\det(A^T A)} \cdot V(Q)$.

Definition 0.50.3.0.3

Let $k, n, r \in \mathbb{N}$ with $k \le n$, let $\alpha \in C^r(U, \mathbb{R}^n)$, where U is an open subset of $\mathbb{R}^k$. The set $Y := \alpha(U)$, equipped with the map α, constitute a parametrized k-manifold of class C^r, denoted as Y_α.

$$V_k(Y_\alpha) := ext \int_U \mathcal{V}(D\alpha) \qquad \int_{Y_\alpha} f\, dV := ext \int_U (f \circ \alpha)\mathcal{V}(D\alpha)$$

Definition 0.50.3.0.4

Given $r, k, n \in \mathbb{N}$, a set $M \subseteq \mathbb{R}^n$ is called a k-manifold without boundary of class C^r provided that for all $\vec{p} \in M$, there exist a set $V \subseteq M$ that contains $\vec{p}$, a set $U \subseteq \mathbb{R}^k$, with V being open in M and U being open in $\mathbb{R}^k$, and a homeomorphism $\alpha \in C^r(U, V)$, with $rank(D\alpha(\vec{x})) = k$ for all $\vec{x} \in U$. The map α is called a coordinate patch on M about $\vec{p}$.

Definition 0.50.3.0.5

Given $r, k, n \in \mathbb{N}$, a set $M \subseteq \mathbb{R}^n$ is called a k-manifold of class C^r provided that, for all $\vec{p} \in M$, there exist a subset U of $\mathbb{R}^k$ open in either $\mathbb{R}^k$ or $\mathbb{H}^k$, a subset V of M open in M, and a homeomorphism $\alpha \in C^r(U, V)$ with $rank(D\alpha(\vec{x})) = k$ for all $\vec{x} \in U$. If such α exists for $\vec{p} \in M$, then α is called the coordinate patch on M about $\vec{p}$, and M is also called a C^r manifold which might have boundary.

Definition 0.50.3.0.6

Let M be a k-manifold. For $\vec{p} \in M$, $\vec{p}$ is called a boundary point of M provided that there exists a coordinate patch $\alpha : U \to V$ on M about $\vec{p}$ such that U is open in $\mathbb{H}^k$, V is open in M, and $\vec{p} = \alpha((x_1, x_2, \cdots, x_{k-1}, 0))$. The set of boundary points of M is called the manifold boundary of M. denoted as ∂M. For $\vec{q} \in M \setminus \partial M$, $\vec{q}$ is called an interior point of M.

Definition 0.50.3.0.7

Let A be an open subset of $\mathbb{R}^n$, B be an open subset of $\mathbb{R}^m$, let $\alpha \in C^1(B, A)$, and let ω be an 1-form defined on A. $\alpha^*\omega := (\omega \circ \alpha) \cdot D\alpha$ is called the pullback of ω by α.

$$\int_{Y_\alpha} df = \int_I \alpha^* df = \int_I d(f \circ \alpha) = \int_I (f \circ \alpha)' = (f \circ \alpha)(b) - (f \circ \alpha)(a) = f(\alpha(b)) - f(\alpha(a)) := \Delta_{Y_\alpha} f$$

If we have $\omega \in C^1$ and is exact, then $D_k\omega_j = D_j\omega_k$, in which case ω is said to be closed on A.