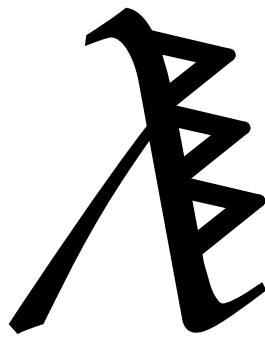


# Definitions and Notes

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# Sets

A **Set** is a collection of objects which are called **Elements**.

Note: Let  $S$  be a set, let  $x$  is an element, then we have either  $x \in S$  or  $x \notin S$ .

Note: Order does not matter in a set. Repeats are not detected in a set.

Let  $X, Y$  be sets.  $X$  is called a **Subset** of  $Y$ , denoted as  $X \subseteq Y$ , provided that every element of  $X$  is an element of  $Y$ , that is, if,  $X$  is a subset of  $Y$ , then for all  $x \in X$ , we have  $x \in Y$ . Moreover, if  $X$  is a subset of  $Y$ , and  $X \neq Y$ , then we say  $X$  is a **Proper Subset** of  $Y$ , denoted as  $X \subset Y$ .

A set with no element in it is called an **Empty Set**, denoted as  $\emptyset$ .

A set with exactly one element in it is called a **Singleton**.

Note: The empty set is a subset of every set.

Let  $a, b$  be elements.  $(a, b) := \{\{a\}, \{a, b\}\}$  is called an **Ordered Pair**.

Note: By definition, we have  $(a, b) = (c, d) \iff a = c \text{ and } b = d$ .

Let  $X$  be a set, let  $A$  and  $B$  be subsets of  $X$ :

$A \cup B := \{x \in X \mid x \in A \text{ or } x \in B\}$  is called the **Union** of  $A$  and  $B$ .

$A \cap B := \{x \in X \mid x \in A \text{ and } x \in B\}$  is called the **Intersection** of  $A$  and  $B$ .

$A - B = A \setminus B := \{x \in A \mid x \notin B\}$  is called the **Difference** between  $A$  and  $B$ .

$X \setminus A := \{x \in X \mid x \notin A\}$  is called the **Complement** of  $A$  in  $X$ .

$A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}$  is called the **Cartesian Product** of the sets  $A$  and  $B$ .

Let  $A, B$  be sets, let  $f$  be a subset of  $A \times B$ .  $f$  is called a **Function**, denoted as  $f : A \rightarrow B \quad x \mapsto f(x)$ , provided that for all  $a \in A$ , there exists a unique  $b \in B$  such that  $(a, b) \in f$ .

Note: The **Domain** of the function  $f : A \rightarrow B$  is  $A$ , and the **Codomain** of  $f$  is  $B$ .

Let  $X$  be a set.  $\mathcal{P}(X) := \{A \mid A \subseteq X\}$  is called the **Power Set** of  $X$ .

Note: The power set of a set  $X$  is the set of all subsets of  $X$ .

Let  $S$  be a set. A function  $\diamond : S \times S \rightarrow S$  is called a **Binary Operation** on  $S$ .

Notation: If  $\diamond$  is a binary operation on a set  $S$ , for  $s_1, s_2 \in S$ , we write  $s_1 \diamond s_2$  instead of  $\diamond(s_1, s_2)$ .

Note: If  $\diamond$  is a binary operation on a set  $S$ , then  $\diamond$  is a subset of  $(S \times S) \times S$ .

Note: Intersection and Union on a set  $X$  are binary operations from  $\mathcal{P}(X) \times \mathcal{P}(X)$  to  $\mathcal{P}(X)$ .

Let  $S$  be a set, let  $f : S \rightarrow S$  and  $g : S \rightarrow S$  be functions, let  $Fun(S) := \{h : S \rightarrow S\}$ .

$\circ : Fun(S) \times Fun(S) \rightarrow Fun(S) \quad (f, g) \mapsto f \circ g$  is a binary operation called **Function Composition**, and  $(f \circ g) : S \rightarrow S \quad (f \circ g)(x) \mapsto f(g(x))$  is a function from  $S$  to  $S$  that sends  $s \in S$  to  $f(g(s))$ .

A binary operation  $\diamond$  on a set  $S$  is called **Commutative** provided that  $\forall a, b \in S, a \diamond b = b \diamond a$ .

A binary operation  $\diamond$  on a set  $S$  is called **Associative** provided that  $\forall a, b, c \in S, (a \diamond b) \diamond c = a \diamond (b \diamond c)$ .

Let  $\diamond$  be a binary operation on  $S$ , the element  $e \in S$  is called an **Identity** for  $\diamond$  on  $S$ , denoted as  $\diamond$ -identity, provided that for all  $s \in S$  we have  $e \diamond s = s \diamond e = s$ .

Note: The empty set can be an identity element of a binary operation on a set.

Note: 0 is not the identity of subtraction on  $\mathbb{R}$ .

Let  $S$  be a set, let  $\diamond$  be a binary operation on  $S$ , let  $e$  be a  $\diamond$ -identity. Given  $s \in S$ , an element  $s' \in S$  is called an **Inverse** of  $s$  with respect to  $\diamond$  on  $S$ , denoted as  $\diamond$ -inverse, provided that  $s \diamond s' = s' \diamond s = e$ .

Let  $X$  be a set. The function  $Id_X : X \rightarrow X \quad x \mapsto x$  is called the **Identity Function** on  $X$ .

Let  $A$  and  $B$  be sets, let  $f : A \rightarrow B$  be a function.

$f$  is **Surjective** provided that  $\forall b \in B, \exists a \in A \text{ s.t. } f(a) = b$ .

$f$  is **Injective** provided that  $(f(a) = f(b)) \iff (a = b)$ .

$f$  is **Bijective** provided that  $f$  is both injective and surjective.

Note: Let  $f$  be a bijection from set  $A$  to set  $B$ , then  $\forall b \in B, \exists! a \in A, \text{ s.t. } f(a) = b$ .

Let  $A$  and  $B$  be sets. a function  $f : A \rightarrow B$  is said to be **Invertible** provided that  $f^{-1}$  is a function.

If  $f$  is invertible, the function  $f^{-1} : B \rightarrow A \quad f^{-1}(b) = a \iff f(a) = b$  is called the **Inverse** of  $f$ .

Note: Given sets  $A, B$  and  $f : A \rightarrow B, f^{-1} = \{(b, a) \in B \times A \mid (a, b) \in f\}$  is a subset of  $B \times A$ .

Let  $A$  and  $B$  be sets, and let  $f : A \rightarrow B$  be a function. The set  $f(A) := \{f(a) \in B \mid a \in A\}$  is called the **Image** of  $f$ .

Note: If  $f(A)$  is the image of the function  $f$  from set  $A$  to set  $B$ , then  $f(A)$  is a subset of  $B$ .

Let  $A, B$  be sets. The sets  $A$  and  $B$  are said to have the same **Cardinality** provided that exists a bijection from the set  $A$  to the set  $B$ .

Let  $S$  be a set. A **Relation** on  $S$  is a subset of  $S \times S$ .

Let  $R$  be a relation on the set  $S$ , and let  $x, y \in S$ . We write  $xRy$  whenever  $(x, y) \in R$ .

Let  $S$  be a set, let  $R$  be a relation on  $S$ .  $R$  is called an **Equivalence Relation** on  $S$  provided that the followings hold:

1.  $R$  is **Reflexive**, that is,  $\forall s \in S$ , we have  $sRs$ .
2.  $R$  is **Symmetric**, that is,  $\forall s, t \in S$ , if  $sRt$ , then  $tRs$ .
3.  $R$  is **Transitive**, that is,  $\forall s, t, u \in S$ , if  $sRt$  and  $tRu$ , then  $sRu$ .

If  $R$  is an equivalence relation on  $S$ , then for  $s, t \in S$ , we write  $s \sim t$  whenever  $(s, t) \in R$ .

Let  $S$  be a set, let  $\sim$  be an equivalence relation on  $S$ , let  $x \in S$ .  $C(x) := \{y \in S \mid y \sim x\}$  is called the **Class** of  $x$  with respect to  $\sim$ , or the **Equivalence Class** of  $x$  with respect to  $\sim$ .

Notation: If  $C(x)$  is a class, then  $C(x)$  can also be denoted as  $[(x)]$ .

Let  $S$  be a set, and let  $\sim$  be an equivalence relation on  $S$ . The set  $S/\sim := \{C(x) \mid x \in X\}$  is called the **Quotient** of  $X$  with respect to  $\sim$ , or the **Factor Set** of  $X$  with respect to  $\sim$ .

# Fields

Let  $F$  be a set, let  $*$  and  $+$  be commutative and associative binary operations on  $F$ .  $(F, *, +)$  is called a **Field** provided that the followings hold:

1. There exists a  $+$ -identity in  $F$ , denoted as  $0_F$ .
2. There exists a  $*$ -identity in  $F$ , denoted as  $1_F$ .
3. For all  $f \in F$ , there exists a  $+$ -inverse of  $f$  in  $F$ , denoted as  $-f$ .
4. For all  $f \in (F \setminus \{0_F\})$ , there exists a  $*$ -inverse of  $f$  in  $F$ , denoted as  $f^{-1}$ .
5. For all  $a, b, c \in F$ , we have  $a * (b + c) = a * b + a * c$ .
6.  $0_F \neq 1_F$ .

Note: If  $(F, *, +)$  is a field, then  $F$  is not empty.

A field  $(F, +, *)$  is said to have an **Order Structure** provided that there exists a subset  $P$  of the set  $F$  such that the followings hold:

1.  $P$  is closed with respect to the binary operations  $+$  and  $*$ .
2. For  $a \in F$ , **Trichotomy** holds:  $\bullet a \in P \quad \bullet -a \in P \quad \bullet a = 0$

If  $P$  satisfies the requirements listed above, then we say the pair  $(F, P)$  has an ordered structure.

A field  $(F, +, *)$  is said to be **Ordered** provided that there exists a subset  $P$  of  $F$  such that  $(F, P)$  has an ordered structure, and we called the pair  $(F, P)$  an **Ordered Field**.

Note: Given  $(F, P)$  is an ordered field,  $P$  is always not empty.

Note: In general, a field can have more than one ordered structure.

Let  $(F, P)$  be an ordered field, let  $a, b \in F$ .

We say  $a$  is **Greater than**  $b$ , denoted as  $a > b$ , iff  $(a - b) \in P$ .

We say  $a$  is **Less than**  $b$ , denoted as  $a < b$ , iff  $-(a - b) \in P$ .

We say  $a$  is **Greater than or Equal to**  $b$ , denoted as  $a \geq b$ , iff  $(a - b) \in P \cup \{0_F\}$ .

We say  $a$  is **Less than or Equal to**  $b$ , denoted as  $a \leq b$ , iff  $-(a - b) \in P \cup \{0_F\}$ .

Let  $(F, P)$  be an ordered field.

The function  $| \cdot | : F \rightarrow F \quad x \mapsto \begin{cases} x & x \in P \\ -x & -x \in P \\ 0_F & x = 0_F \end{cases}$  is called the **Absolute Value Function** on  $F$

Let  $(F, P)$  be an ordered field, let  $a \in F$ ,  $|a|$  is called the **Absolute Value** of  $a$ .

Consider using the ordered field  $(F, P)$  from now on.

Let  $A$  be a subset of  $F$ , and let  $u$  be an element in  $F$ .  $u$  is called an **Upper Bound** for  $A$  provided that for all  $a \in A$ , we have  $u \geq a$ . If  $u$  is an upper bound for the set  $A$ , then we say the set  $A$  is **Bounded Above** by  $u$ , and  $A$  is bounded above in  $F$ .

Let  $A$  be a bounded above subset of  $F$ , and let  $\alpha$  be an element in  $F$ .  $\alpha$  is called a **Least Upper Bound** or **Supremum** for the set  $A$  provided that the followings hold:

1.  $\alpha$  is an upper bound for  $A$ .
2.  $\alpha$  is the least such, that is, if  $u \in F$  is an upper bound for  $A$ , then  $\alpha \leq u$ .

Note: Not every set has a least upper bound.

Note: Empty set does not have a least upper bound.

Note: If a least upper bound exists for  $A \subseteq F$ , then it is unique.

An ordered field  $(F, P)$  is said to have the **Least Upper Bound Property** provided that every nonempty bounded above subset  $A$  of  $F$  has a least upper bound.

An ordered field is said to be **Complete** provided that it has the least upper bound property.

Note: There exists a unique complete ordered field, called  $\mathbb{R}$ .

Let  $A$  be a subset of  $F$ , and let  $w$  be an element in  $F$ .  $w$  is called a **Lower Bound** for  $A$  provided that for all  $a \in A$ , we have  $w \leq a$ . If  $w$  is a lower bound for the set  $A$ , then we say the set  $A$  is **Bounded Below** by  $w$ , and  $A$  is bounded below in  $F$ .

Let  $A$  be a bounded below subset of  $F$ , and let  $\beta$  be an element in  $F$ .  $\beta$  is called a **Greatest Lower Bound** or **Infimum** for the set  $A$  provided that the followings hold:

1.  $\beta$  is a lower bound for  $A$ .
2.  $\beta$  is the greatest such, that is, if  $w \in F$  is a lower bound for  $A$ , then  $\beta \geq w$ .

Note: Not every set has a greatest lower bound.

Note: Empty set does not have a greatest lower bound.

Note: If a greatest lower bound exists for  $A \subseteq F$ , then it is unique.

Let  $X$  be a subset of  $F$ ,  $X$  is said to be **Inductive** provided that the followings hold:

- $1_F \in X$
- If  $x \in X$ , then  $x + 1_F \in X$ .

$\mathbb{N}_F := \{n \in F \mid n \text{ belongs to every inductive subset of } F\}$

$\mathbb{Z}_F := \{z \in F \mid |z| \in \mathbb{N}_F \cup \{0\}\}$

$\mathbb{Q}_F := \{q \in F \mid \exists z \in \mathbb{Z}_F \text{ and } n \in \mathbb{N}_F \text{ s.t. } q = z * n^{-1}\}$

Note: When  $F = \mathbb{R}$ , then we have  $\mathbb{N}_F = \mathbb{N}$ ,  $\mathbb{Z}_F = \mathbb{Z}$ ,  $\mathbb{Q}_F = \mathbb{Q}$ .

Note:  $\mathbb{N}$  is called the set of Natural Numbers, and  $\mathbb{Z}$  is called the set of Integers.

Note: Rigorously, we define  $\mathbb{Q} := (\mathbb{Z} \times \mathbb{N}) / \sim_{\mathbb{Q}}$  with  $(n_1, m_1) \sim_{\mathbb{Q}} (n_2, m_2) \iff (n_1 \cdot m_2 = n_2 \cdot m_1)$ .

Note:  $\mathbb{Q}$  is called the set of Rational Numbers.

Let  $B$  be a subset of  $F$ .  $B$  is said to be **Bounded** in  $F$  provided that the set  $B$  is both bounded above and bounded below in  $F$ .

Fact: Every nonempty bounded above subset of  $\mathbb{R}$  has a Supremum in  $\mathbb{R}$ .

Fact: Every nonempty bounded below subset of  $\mathbb{R}$  has a Infimum in  $\mathbb{R}$ .

Let  $A$  be a subset of  $F$ , let  $m$  be an element in  $F$ . The element  $m$  is called a **Maximal Element** for the set  $A$  provided that  $m$  belongs to  $A$  and  $m$  is an upper bound for  $A$ .

Let  $A$  be a subset of  $F$ , let  $u$  be an element in  $F$ . The element  $u$  is called a **Minimal Element** for the set  $A$  provided that  $u$  belongs to  $A$  and  $u$  is a lower bound for  $A$ .

Let  $A$  be a subset of  $F$ , let  $b$  be an element in  $F$ .

$-A := \{-x \mid x \in A\}$

$b + A := \{b + x \mid x \in A\}$

Let  $U$  be a subset of  $F$ .  $U$  is said to be **Well-ordered** provided that every nonempty subset of  $U$  has a minimal element.

Let  $S$  be a subset of  $\mathbb{N}$ .  $S$  is said to be **Weakly Inductive** provided that the followings hold:

- $1 \in S$
- If  $k \in S$ , then  $(k + 1) \in S$ .

Let  $S$  be a subset of  $\mathbb{N}$ .  $S$  is said to be **Strongly Inductive** provided that the following holds:

For all  $n \in \mathbb{N}$ , if the set  $\{k \in \mathbb{N} \mid k < n\}$  is a subset of  $S$ , then  $n$  is an element in  $S$ .

Let  $d, b \in \mathbb{N}$ . the element  $d$  is called a **Divisor** of  $b$  provided that there exists  $m \in \mathbb{N}$  s.t.  $b = d * m$ .

Notation: If  $d \in \mathbb{N}$  is a divisor of  $b \in \mathbb{N}$ , then we write  $d \mid b$ .

Let  $a, b \in \mathbb{N}$ . The element  $\gcd(a, b) \in \mathbb{N}$  is called the common divisor of  $a$  and  $b$  provided that  $\gcd(a, b)$  is a divisor of both  $a$  and  $b$ , and  $\gcd(a, b)$  is the greatest such, that is,  $\forall d \in \mathbb{N}$ , if  $d$  is a divisor of both  $a$  and  $b$ , then we have  $\gcd(a, b) \geq d$ .

The set  $E := \{2 \cdot k \mid k \in \mathbb{Z}\}$  is called the set of **Even Numbers** in  $\mathbb{R}$ .

The set  $O := \{2 \cdot k + 1 \mid k \in \mathbb{Z}\}$  is called the set of **Odd Numbers** in  $\mathbb{R}$ .

Each element in  $E$  is called an **Even Number**, and each element in  $O$  is called an **Odd Number**.

Let  $S$  be a set,  $S$  is said to be **Finite** provided that either  $S$  is an empty set, or there exists a bijection from the set  $N_n = \{k \in \mathbb{N} \mid k < n\}$  to the set  $S$  for some  $n \in \mathbb{N}$ . If there exists a bijection from  $N_n$  to  $S$  for some  $n \in \mathbb{N}$ , then we say  $S$  has  $n$  element.

Note: Let  $S$  be a set, If there exists a bijection from  $N_n$  to  $S$  for some  $n \in \mathbb{N}$ , then  $n$  is unique.

Let  $S$  be a set.  $S$  is said to be **Infinite** provided that  $S$  is not finite.

Let  $S$  be a set.  $S$  is **Countably Infinite** provided that there exists a bijection from  $\mathbb{N}$  to  $S$ .

Let  $S$  be a set.  $S$  is said to be **Countable** provided that  $S$  is either finite or countably infinite.

Fact: The sets  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$  are all countable.

Fact: The sets  $\mathbb{R}$  and  $\mathbb{R} \setminus \mathbb{Q}$  are not countable.

$i^2 := -1$ . The element  $i$  is called the **Imaginary Unit**.

Let  $a, b \in \mathbb{R}$ .  $z := a + bi$  is called a **Complex Number**, where  $a$  is called the **Real Part** of  $z$ ,  $b$  is called the **Imaginary Part** of  $z$ , and  $a + bi$  is called the **Cartesian Form** of  $z$ .

$\mathbb{C} := \{a + bi \mid a, b \in \mathbb{R}\}$  is called the **Set of Complex Numbers**.

Note: The plane of complex numbers, called the **Complex Plane**, is a two dimensional plane.

Let  $z = a + bi, w = c + di$  be complex numbers, and let  $k \in \mathbb{R}$ .

$$z + w := (a + c) + (b + d)i$$

$$z \times w := (a \times c - b \times d) + (a \times d + b \times c)i$$

$$z \times k := (a \times k) + (b \times k)i$$

$\bar{z} := a - bi$  is called the **Conjugate** of  $z$ .

$|z| := \sqrt{a^2 + b^2}$  is called the **Absolute Value** or the **Modulus** of  $z$ .

Fact: Every nonzero complex number  $z$  can be written as  $z = |z|(\cos(\theta) + i \sin(\theta))$  for some  $\theta \in \mathbb{R}$ .

Note: If  $z$  is a complex number, then  $\bar{z} \in \mathbb{C}$ ,  $|z| \in \mathbb{R}$ ,  $|z|$  is unique, and the absolute value of  $\frac{z}{|z|}$  is 1.

Let  $z = a + bi = r(\cos(\theta) + i \sin(\theta)) = re^{i\theta}$  be a complex number.  $\theta$  is called the **Argument** of  $z$ .  $r(\cos(\theta) + i \sin(\theta))$  and  $re^{i\theta}$  are called the **Polar Form** of  $z$ .

Note: If  $z = re^{i\theta}$  is a complex number, then  $r \in \mathbb{R}$  and  $\theta \in \mathbb{R}$ .

Note: If  $z = re^{i\theta}$  is a complex number, then  $r = |z|$  is unique, while  $\theta$  is not unique.

Fact: Let  $z \in \mathbb{C}$  and  $z = re^{i\theta}$ . If  $\theta = \theta_0$  is one possibility, then the others are  $\theta_0 + 2k\pi$  for  $k \in \mathbb{Z}$ .

Fact: If  $z = re^{i\theta} = a + bi$  is a complex number, then  $a = |z| \cos(\theta)$ ,  $b = |z| \sin(\theta)$ ,  $\theta = \tan^{-1}(\frac{b}{a})$ .

# Topology

Let  $A$  be a subset of  $\mathbb{R}$ .  $A$  is called an **Interval** provided that for all  $x, y \in A$ , if  $z \in \mathbb{R}$  and  $x < z < y$ , then we have  $z \in A$ .

Let  $a, b \in \mathbb{R}$  with  $a \leq b$ . The followings are intervals:

$$\begin{aligned} (a, b) &:= \{x \in \mathbb{R} \mid a < x < b\} & [a, b] &:= \{x \in \mathbb{R} \mid a \leq x \leq b\} \\ (a, b] &:= \{x \in \mathbb{R} \mid a < x \leq b\} & [a, b) &:= \{x \in \mathbb{R} \mid a \leq x < b\} \\ \mathbb{R}_{>a} = (a, \infty) &:= \{x \in \mathbb{R} \mid a < x\} & \mathbb{R}_{<b} = (-\infty, b) &:= \{x \in \mathbb{R} \mid x < b\} \\ \mathbb{R}_{\geq a} = [a, \infty) &:= \{x \in \mathbb{R} \mid a \leq x\} & \mathbb{R}_{\leq b} = (-\infty, b] &:= \{x \in \mathbb{R} \mid x \leq b\} \\ (-\infty, \infty) &:= \{x \in \mathbb{R}\} \end{aligned}$$

Let  $a, r \in \mathbb{R}$  with  $r > 0$ . The set  $B_r(a) := \{x \in \mathbb{R} \mid (a - r) < x < (a + r)\}$  is called the **Ball** centered at  $a$  of radius  $r$ .

$\mathcal{T}_{EUC} := \{A \subseteq \mathbb{R} \mid \forall a \in A, \exists r \in \mathbb{R}_{>0} \text{ s.t. } B_r(a) \subseteq A\}$  is called the **Euclidean Topology** on  $\mathbb{R}$ .

Note:  $\mathcal{T}_{EUC}$  is closed with respect to arbitrary unions and finite intersections.

Note:  $\mathcal{T}_{EUC}$  is a set of sets, and  $\mathcal{T}_{EUC}$  is a subset of  $\mathcal{P}(\mathbb{R})$ .

Note: The pair  $(\mathbb{R}, \mathcal{T}_{EUC})$  is a topological space.

Note: We have  $\mathbb{R} \in \mathcal{T}_{EUC}$  and  $\emptyset \in \mathcal{T}_{EUC}$ .

Let  $U$  be a subset of  $\mathbb{R}$ . The set  $U$  is said to be **Open** in the Euclidean topology on  $\mathbb{R}$  provided that  $\forall u \in U, \exists r \in \mathbb{R}_{>0} \text{ s.t. } B_r(u) \subseteq U$ . Let  $C$  be a subset of  $\mathbb{R}$ . The set  $C$  is said to be **Closed** in the Euclidean topology on  $\mathbb{R}$  provided that  $\mathbb{R} \setminus C$  is open in the Euclidean topology on  $\mathbb{R}$ .

Note: Arbitrary union of open sets and finite intersection of open sets in  $\mathcal{T}_{EUC}$  are open.

Note: The collection of all open subsets of  $\mathbb{R}$  in the Euclidean Topology forms  $\mathcal{T}_{EUC}$ .

Note: Each element in  $\mathcal{T}_{EUC}$  is called an open subset of  $\mathbb{R}$ .

Note: The empty set and  $\mathbb{R}$  are both open and closed in  $\mathcal{T}_{EUC}$ .

Note: All balls are open in the Euclidean topology on  $\mathbb{R}$ . In  $(\mathbb{R}, \mathcal{T}_{EUC})$ , balls are called **Open Balls**.

Consider using the topological space  $(\mathbb{R}, \mathcal{T}_{EUC})$  from now on.

Let  $p \in \mathbb{R}$ . A **Neighborhood** of  $p$  in  $\mathcal{T}_{EUC}$  is an open interval in  $\mathcal{T}_{EUC}$  that contains  $p$ .

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function, let  $a \in \mathbb{R}$ . We say  $f$  approaches  $l$  as  $x$  approaches  $a$ , denoted as  $\lim_{x \rightarrow a} f = l$  for some  $l \in \mathbb{R}$ , provided that one of the followings holds:

1. For all  $\epsilon > 0, \exists \delta > 0$  s.t. if  $x \in B_\delta(a) \setminus \{a\}$ , then  $f(x) \in B_\epsilon(l)$ .
2. For all  $\epsilon > 0, \exists \delta > 0$  s.t. if  $x \in \mathbb{R}$  satisfies  $0 < |x - a| < \delta$ , then  $|f(x) - f(a)| < \epsilon$ .
3. For  $V \in \mathcal{T}_{EUC}$ , if  $l \in V$ , then  $\exists U \in \mathcal{T}_{EUC}$  s.t.  $a \in U$  and  $\forall x \in U \setminus \{a\}$ , we have  $f(x) \in V$ .

Note: The three statements are equivalent.

Note:  $\lim_{x \rightarrow a} f$  is called the **Limit** of the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  at  $a \in \mathbb{R}$ .

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function, let  $a$  be an element in  $\mathbb{R}$ . The function  $f$  is said to be **Continuous** at  $a$  provided that one of the followings holds:

1.  $\lim_{x \rightarrow a} f = l$  and  $f(a) = l$  for some  $l \in \mathbb{R}$ .
2. For all  $\epsilon > 0, \exists \delta > 0$  s.t. if  $x \in B_\delta(a)$ , then  $f(x) \in B_\epsilon(f(a))$ .
3. For all  $\epsilon > 0, \exists \delta > 0$  s.t. if  $x \in \mathbb{R}$  satisfies  $|x - a| < \delta$ , then  $|f(x) - f(a)| < \epsilon$ .
4. For  $V \in \mathcal{T}_{EUC}$ , if  $f(a) \in V$ , then  $\exists U \in \mathcal{T}_{EUC}$  s.t.  $a \in U$  and  $\forall x \in U$ , we have  $f(x) \in V$ .

Note: The four statements are equivalent.

Let  $A, B$  be subsets of  $\mathbb{R}$ , let  $f : A \rightarrow B$  be a function.  $f$  is **Continuous** on  $A$  provided that for all  $a \in A$ , the function  $f$  is continuous at  $a$ . If a function  $f$  is continuous on its domain, then we say the function  $f$  is **Continuous**.

Let  $A, B$  be subsets of  $\mathbb{R}$ , let  $S$  be a subset of  $A$ , let  $f : A \rightarrow B$  be a function.

The **Restriction** of  $f$  to  $S$  is the function from  $S$  to  $B$  that sends  $s \in S$  to  $f(s)$ , such function is denoted as  $res_S f : S \rightarrow B$ . Moreover, the image of  $res_S f$  is denoted as  $f([S])$  or  $f(S)$  or  $res_S f(S)$ .

By Theorem, we can redefine the continuity of a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ : The function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is said to be **Continuous** provided that for all  $U \in \mathcal{T}_{EUC}$ , we have  $f^{-1}(U) \in \mathcal{T}_{EUC}$ .

The assumption of using  $(\mathbb{R}, \mathcal{T}_{EUC})$  is removed.

Let  $X$  be a set. A **Topology**  $\mathcal{T}_X$  on the set  $X$  is a subset of  $\mathcal{P}(X)$  that satisfies the followings:

1. The set  $X$  and the empty set are in  $\mathcal{T}_X$ .
2.  $\mathcal{T}_X$  is closed with respect to arbitrary union.
3.  $\mathcal{T}_X$  is closed with respect to finite intersection.

Note: A set can have more than one topology.

Let  $X$  be a set, let  $\mathcal{T}_X$  be a topology on  $X$ . The pair  $(X, \mathcal{T}_X)$  is called a **Topological Space**.

Let  $(X, \mathcal{T}_X)$  be a topological space. Each the element in  $\mathcal{T}_X$  is called an **Open Subset** of  $X$  in  $\mathcal{T}_X$ , that is, let  $U$  be a subset of  $X$ , the set  $U$  is said to be **Open** if and only if  $U$  belongs to  $\mathcal{T}_X$ .

Let  $(X, \mathcal{T}_X)$  be a topological space, let  $p \in X$ . A **Neighborhood** of  $p$  in  $\mathcal{T}_X$  is an open set in  $\mathcal{T}_X$  that contains  $p$ .

Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. The function  $f : X \rightarrow Y$  is said to be **Continuous** provided that for all  $u \in \mathcal{T}_Y$ , we have  $f^{-1}(u) \in \mathcal{T}_X$ .

Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. The function  $f : X \rightarrow Y$  is **Locally Constant** provided that  $\forall x \in X, \exists$  a neighborhood  $U$  of  $x$  in  $\mathcal{T}_X$  s.t.  $\forall y \in U$ , we have  $f(y) = c$  for some  $c \in Y$ .

Let  $(X, \mathcal{T})$  be a topological space, let  $A$  be a subset of  $X$ . The intersection of all closed subsets of  $X$  that contain  $A$  is called the **Topological Closure** of  $A$  in  $\mathcal{T}$ , denoted as  $\bar{A}$ .

Let  $(X, \mathcal{T})$  be a topological space, let  $A$  be a subset of  $X$ . The union of all open sets that are contained in  $A$  is called the **Interior** of  $A$  in  $\mathcal{T}$ , denoted as  $\text{int}(A)$ .

Let  $X$  be a set. The **Indiscrete Topology** on  $X$ , denoted as  $\mathcal{T}_{ind}$ , contains only  $\emptyset$  and  $X$ .

Let  $X$  be a set. The **Discrete Topology** on  $X$ , denoted as  $\mathcal{T}_{dis}$ , contains all subsets of  $X$ .

Note: In the discrete topology on the set  $X$ , every subset of  $X$  is both open and closed.

Note: Let  $X$  be a set.  $\mathcal{T}_{ind} = \{\emptyset, X\}$  and  $\mathcal{T}_{dis} = \mathcal{P}(X)$ .

Let  $(X, \mathcal{T}_X)$  be a topological space. If  $A$  is a subset of  $X$ , then  $A$  inherits the structure of the topological space from  $X$ , that is,  $\mathcal{T}_A := \{U \cap A \mid U \in \mathcal{T}_X\}$  is the **Subspace Topology** on  $A$ , each element in  $\mathcal{T}_A$  is called an **Open Subset** of  $A$  in the subspace topology of  $A$ .

In  $(\mathbb{R}, \mathcal{T}_{EUC})$ , let  $A$  be a subset of  $\mathbb{R}$ , let  $\mathcal{T}_A$  denote the subspace topology on  $A$ , let  $f : A \rightarrow \mathbb{R}$  be a function. let  $a$  be an element in  $A$ . We say  $f$  approaches  $l$  as  $x$  approaches  $a$ , denoted as  $\lim_{x \rightarrow a} f = l$  for some  $l \in \mathbb{R}$ , provided that one of the followings holds:

1. For all  $\epsilon > 0, \exists \delta > 0$  s.t. if  $x \in (B_\delta(a) \setminus \{a\}) \cap A$ , then  $f(x) \in B_\epsilon(l)$ .
2. For all  $\epsilon > 0, \exists \delta > 0$ , s.t. if  $x \in A$  satisfies  $|x - a| < \delta$ , then  $|f(x) - f(a)| < \epsilon$ .
3. For  $V \in \mathcal{T}_{EUC}$ , if  $l \in V$ , then  $\exists U \in \mathcal{T}_A$  s.t.  $a \in U$  and  $\forall x \in (U \setminus \{a\}) \cap A$ , we have  $f(x) \in V$ .

Note: The three statements are equivalent.

Note: Let  $A$  be a subset of  $\mathbb{R}$ .  $\lim_{x \rightarrow a} f$  is called the **Limit** of the function  $f : A \rightarrow \mathbb{R}$  at  $a \in A$ .

In  $(\mathbb{R}, \mathcal{T}_{EUC})$ , let  $A$  be a subset of  $\mathbb{R}$ , let  $\mathcal{T}_A$  denote the subspace topology on  $A$ , let  $f : A \rightarrow \mathbb{R}$  be a function, let  $a$  be an element in  $A$ ,  $f$  is **Continuous** at  $a$  provided that one of the followings holds:

1.  $\lim_{x \rightarrow a} f = l$  and  $f(a) = l$ .
2. For all  $\epsilon > 0, \exists \delta > 0$  s.t. if  $x \in B_\delta(a) \cap A$ , then  $f(x) \in B_\epsilon(f(a))$ .
3. For all  $\epsilon > 0, \exists \delta > 0$  s.t. if  $x \in A$  satisfies  $|x - a| < \delta$ , then  $|f(x) - f(a)| < \epsilon$ .
4. For  $V \in \mathcal{T}_{EUC}$ , if  $l \in V$ , then  $\exists U \in \mathcal{T}_A$  s.t.  $a \in U$  and  $\forall x \in U \cap A$ , we have  $f(x) \in V$ .

Note: The four statements are equivalent.

Note: By definition, if  $f : A \rightarrow \mathbb{R}$  is continuous at  $a \in A$ , then  $f : A \rightarrow f(A)$  is continuous at  $a$ .

Note: By definitions, if the function  $f : A \rightarrow \mathbb{R}$  is continuous, then  $f : A \rightarrow f(A)$  is continuous.

A topological space  $(X, \mathcal{T}_X)$  is said to be **Disconnected** provided that there exist nonempty disjoint subsets  $A, B \in \mathcal{T}_X$  such that  $A \cup B = X$ . A topological space  $(Y, \mathcal{T}_Y)$  is said to be **Connected** provided that  $(Y, \mathcal{T}_Y)$  is not disconnected.

Fact: Let  $X$  be a set. The topological space  $(X, \mathcal{T}_{ind})$  is always connected.

Fact: In  $(\mathbb{R}, \mathcal{T}_{EUC})$ , let  $A$  be a subset of  $\mathbb{R}$ , let  $\mathcal{T}_A$  denote the subspace topology on  $A$ . The topological space  $(A, \mathcal{T}_A)$  is connected if and only if  $A$  is an interval.



Let  $(X, \mathcal{T})$  be a topological space, let  $A$  be a subset of  $X$ . A collection of open sets  $\mathcal{C} \subseteq \mathcal{T}$  is called an **Open Cover** of  $A$  in  $\mathcal{T}$  provided that  $A \subseteq \bigcup_{C \in \mathcal{C}} C$ .

Let  $(X, \mathcal{T})$  be a topological space, let  $A$  be a subset of  $X$ , and let  $\mathcal{C}$  be an open cover of  $A$  in  $\mathcal{T}$ . A subcollection  $\mathcal{C}'$  of  $\mathcal{C}$  is called an **Open Subcover** of  $A$  in  $\mathcal{T}$  provided that  $A \subseteq \bigcup_{C' \in \mathcal{C}'} C'$ .

Let  $(X, \mathcal{T})$  be a topological space, let  $A$  be a subset of  $X$ , let  $\mathcal{T}_A$  denote the subspace topology on  $A$  inherited from  $\mathcal{T}$ . The topological space  $(A, \mathcal{T}_A)$  is said to be **Compact** provided that every open cover of  $A$  in  $\mathcal{T}$  admits a finite open subcover in  $\mathcal{T}$ .

Fact: The topological space  $(\mathbb{R}, \mathcal{T}_{EUC})$  is not compact.

Fact: If  $X$  is a finite set, then the topological space  $(X, \mathcal{T})$  is always compact.

Fact: Let  $a, b \in \mathbb{R}$  with  $a < b$ , let  $(a, b)$  be an interval, let  $\mathcal{T}_{(a,b)}$  denote the subspace topology on  $(a, b)$ . The topological space  $((a, b), \mathcal{T}_{(a,b)})$  is not compact.

Let  $(X, \mathcal{T})$  be a topological space, let  $x, y \in X$ . A **Path** connecting  $x$  and  $y$  is a continuous function  $f : [0, 1] \rightarrow X$  such that  $f(0) = x$  and  $f(1) = y$ .

Let  $(X, \mathcal{T})$  be a topological space. The topological space  $(X, \mathcal{T})$  is said to be **Path Connected** provided that for all  $x, y \in X$ , there exists a path connecting  $x$  and  $y$ .

Note: If the topological space  $(X, \mathcal{T})$  is path connected, then  $(X, \mathcal{T})$  is connected.

Note: The connectedness of a topological space does not imply its path connectedness.

$$\mathbb{R}^2 := \mathbb{R} \times \mathbb{R} = \{(x, y) \mid x, y \in \mathbb{R}\}$$

$$\mathbb{R}^3 := (\mathbb{R} \times \mathbb{R}) \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

Let  $X$  be a set.  $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$  is called a **Metric** on  $X$  provided that the followings hold:

1. For all  $x, y \in X$ , we have  $d(x, y) = d(y, x)$ .
2. For  $x, y \in X$ ,  $d(x, y) = 0$  if and only if  $x = y$ .
3. For all  $x, y, z \in X$ , we have  $d(x, y) \leq (d(x, z) + d(z, y))$ .

The function  $d_3 : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}_{\geq 0}$   $((x_1, y_1, z_1), (x_2, y_2, z_2)) \mapsto \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$  is called the Euclidean metric, or Euclidean distance, on  $\mathbb{R}^3$ .

The function  $d_2 : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}_{\geq 0}$   $((x_1, y_1), (x_2, y_2)) \mapsto \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$  is called the Euclidean metric, or Euclidean distance, on  $\mathbb{R}^2$ .

The function  $d_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$   $x \mapsto |x|$  is called the Euclidean metric, or Euclidean distance, on  $\mathbb{R}$ .

Let  $X$  be a set, let  $d_X$  be a metric on  $X$ . The pair  $(X, d_X)$  is called a **Metric Space**.

Let  $(X, d)$  be a metric space, let  $a \in X$ , and let  $r > 0$ . The set  $B_r := \{x \in X \mid d(x, a) < r\}$  is called a **Ball** centered at  $a$  of radius  $r$ .

Let  $(X, d)$  be a metric space.  $\mathcal{T}_d := \{A \subseteq X \mid \forall a \in A, \exists r > 0 \text{ s.t. } B_r(a) \subseteq A\}$  is the **Topology on  $X$  associated to  $d$** . In the topological space  $(X, \mathcal{T}_d)$ , let  $A$  be a subset of  $X$ . The set  $A$  is said to be **Open** in  $\mathcal{T}_d$  provided that for all  $a \in A$ , there exists  $r > 0$  such that  $B_r(a)$  is a subset of  $A$ .

A topological space  $(X, \mathcal{T})$  is said to be **Metrizable** provided that  $\exists$  a metric  $d$  on  $X$  s.t.  $\mathcal{T}_d = \mathcal{T}$ .

Note: Let  $(X, d)$  be a metric space, and let  $a \in X$ , then  $B_r(a) \in \mathcal{T}_d$ .

Note: Indiscrete topology is not metrizable.

Note: Discrete topology is metrizable.

Let  $d_1, d_2, d_3$  denote the Euclidean metric on  $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$ , respectively, and let  $r \in \mathbb{R}$ .

$S^0 := \{x \in \mathbb{R} \mid d_1(x, 0) = r\}$  is a zero-dimensional sphere of radius  $r$  in  $\mathbb{R}$ .

$S^1 := \{(x, y) \in \mathbb{R}^2 \mid d_2((x, y), (0, 0)) = r\}$  is an one-dimensional sphere of radius  $r$  in  $\mathbb{R}^2$ .

$S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid d_3((x, y, z), (0, 0, 0)) = r\}$  is a two-dimensional sphere of radius  $r$  in  $\mathbb{R}^3$ .

Note: A zero-dimensional sphere is often called an open ball in  $\mathbb{R}$ .

Note: A one-dimensional sphere is often called a circle in  $\mathbb{R}^2$ .

Note: A two-dimensional sphere is often called a sphere in  $\mathbb{R}^3$ .

Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces, let  $f : X \rightarrow Y$  be a function.  $f$  is called an **Open Map** from  $(X, \mathcal{T}_X)$  to  $(Y, \mathcal{T}_Y)$  provided that for all  $V \in \mathcal{T}_X$ , we have  $f(V) \in \mathcal{T}_Y$ .

Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces, let  $f : X \rightarrow Y$  be a function. The function  $f$  is called a **Homeomorphism** from  $(X, \mathcal{T}_X)$  to  $(Y, \mathcal{T}_Y)$  provided that the followings hold:

1.  $f : X \rightarrow Y$  is bijective.
2.  $f : X \rightarrow Y$  is continuous.
3.  $f^{-1} : Y \rightarrow X$  is continuous.

Fact: Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. If  $f : X \rightarrow Y$  is a homeomorphism from  $(X, \mathcal{T}_X)$  to  $(Y, \mathcal{T}_Y)$ , then the inverse of  $f$ ,  $f^{-1} : Y \rightarrow X$  is a homeomorphism from  $(Y, \mathcal{T}_Y)$  to  $(X, \mathcal{T}_X)$ .

Fact: Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be homeomorphic topological spaces, let  $h : X \rightarrow Y$  be a homeomorphism, let  $A$  be a subset of  $X$ , let  $\mathcal{T}_A$  and  $\mathcal{T}_{h(A)}$  be subspace topology, the restriction of  $h$  on  $A$ ,  $res_A h : A \rightarrow h(A)$ , is a homeomorphism from  $(A, \mathcal{T}_A)$  to  $(h(A), \mathcal{T}_{h(A)})$ .

Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be topological spaces. The topological space  $(X, \mathcal{T}_X)$  is said to be **Homeomorphic** to  $(Y, \mathcal{T}_Y)$  provided that there exists a homeomorphism from  $(X, \mathcal{T}_X)$  to  $(Y, \mathcal{T}_Y)$ .

Note: Being homeomorphic defines an equivalence relation on topological spaces.

Fact: Let  $\mathcal{T}_{EUC_2}$  denote the topology on  $\mathbb{R}^2$  associated to the Euclidean metric on  $\mathbb{R}^2$ . The topological space  $(\mathbb{R}^2, \mathcal{T}_{EUC_2})$  is not homeomorphic to the topological space  $(\mathbb{R}, \mathcal{T}_{EUC})$ .

Let  $(X, \mathcal{T})$  be a connected topological space, let  $x \in X$ , let  $\mathcal{T}_x$  denote the subspace topology on the set  $X \setminus \{x\}$  inherited from  $\mathcal{T}$ .  $x$  is called a **Cut Point** of  $X$  provided that  $(X \setminus \{x\}, \mathcal{T}_x)$  is disconnected, and  $cut(X) := \{x \in X \mid x \text{ is a cut point of } X\}$  denotes the set of cut point of  $X$ .

Fact: Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  be connected homeomorphic topological spaces, any homeomorphism between  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  sends cut point to cut point, that is, if  $h : X \rightarrow Y$  is a homeomorphism from  $(X, \mathcal{T}_X)$  to  $(Y, \mathcal{T}_Y)$ , then the restriction of  $h$  on  $cut(X)$ ,  $res_{cut(X)} h : cut(X) \rightarrow cut(Y)$ , is a bijection from  $cut(X)$  to  $cut(Y)$ .

Let  $(X, \mathcal{T}_X)$  be a topological space.  $(X, \mathcal{T}_X)$  is said to be **Totally Disconnected** provided that the only connected subsets of  $X$  are the singletons and the empty set.

Note:  $(\mathbb{R}, \mathcal{T}_{dis})$ ,  $(\mathbb{Z}, \mathcal{T}_{EUC})$ ,  $(\mathbb{N}, \mathcal{T}_{EUC})$ ,  $(\mathbb{Q}, \mathcal{T}_{EUC})$ ,  $(\mathbb{R} \setminus \mathbb{Q}, \mathcal{T}_{EUC})$  are totally disconnected.

Let  $(X, \mathcal{T}_X)$  be a topological space, let  $A$  be a subset of  $X$ . The element  $a \in A$  is called an **Isolated Point** of  $A$  provided that  $\exists U \in \mathcal{T}_X$  s.t.  $a \in U$  and  $U \cap (A \setminus \{a\}) = \emptyset$ , that is, there exists an open subset  $U$  of  $X$  such that  $U$  contains  $a$  and  $U$  contains no other elements of  $A$ .

Note: In  $(\mathbb{R}, \mathcal{T}_{EUC})$ , all elements in  $\mathbb{N}$  are isolated points of  $\mathbb{N}$ .

Note: In  $(\mathbb{R}, \mathcal{T}_{EUC})$ , all elements in  $\mathbb{Z}$  are isolated points of  $\mathbb{Z}$ .

Note: In  $(\mathbb{R}, \mathcal{T}_{EUC})$ ,  $\mathbb{Q}$  has no isolated point.

Fact: Let  $(X, \mathcal{T}_X)$  be a topological space, let  $A$  be a subset of  $X$ . If  $a \in A$  is an isolated point of  $A$ , then  $\{a\}$  is open in the subspace topology on the set  $A$  inherited from  $\mathcal{T}_X$ .

Let  $(X, \mathcal{T}_X)$  be a topological space, let  $A$  be a subset of  $X$ . The set  $A$  is said to be **Perfect** provided that  $A$  is closed in  $\mathcal{T}_X$  and  $A$  has no isolated point.

Let  $(X, \mathcal{T}_X)$  be a topological space, let  $A$  be a subset of  $X$ . The element  $a \in X$  is called a **Limit Point** of  $A$ , or an **Accumulation Point** of  $A$ , provided that  $\forall U \in \mathcal{T}_X$ , if  $a \in U$ , then  $\exists a' \in A \setminus \{a\}$  s.t.  $a' \in U$ , that is, for all open subsets  $U$  of  $X$  that contain  $a$ , we have  $U \cap (A \setminus \{a\}) \neq \emptyset$ .

Note: In  $(\mathbb{R}, \mathcal{T}_{EIC})$  any singleton subset of  $\mathbb{R}$  has no accumulation point.

Let  $(X, \mathcal{T}_X)$  be a topological space. The topological space  $(X, \mathcal{T}_X)$  is said to be **Hausdorff** provided that for all distinct  $x, y \in X$ , there exist disjoint  $U, V \in \mathcal{T}_X$  s.t.  $x \in U$  and  $y \in V$ .

Note: In a Hausdorff space, we can separate distinct points with open sets.

Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces, let  $f : X \rightarrow Y$  be a function.  $f$  is said to be **Lipschitzian** provided that there exists  $c \in \mathbb{R}$  such that  $\forall x, y \in X$ , we have  $d_Y(f(x), f(y)) \leq c \cdot d_X(x, y)$ .

**Topological Invariants** are the properties of topological spaces that are preserved by homeomorphisms between topological spaces.

# Groups

A set  $G$  with a binary operation  $\diamond$  on  $G$  is called a **Group** provided that the followings hold:

1. The binary operation  $\diamond$  is associative.
2. There exists  $\diamond$ -identity  $e \in G$ .
3. For all  $g \in G$ , there exists  $h \in G$  such that  $g \diamond h = h \diamond g = e$ .

Note: If  $(G, \diamond)$  is a group, then the  $\diamond$ -inverse of each  $g \in G$  is unique.

Note: The followings are groups:  $(\mathbb{R}, +)$ ,  $(\mathbb{Z}, +)$ ,  $(\mathbb{R} \setminus \{0\}, *)$ ,  $((0, \infty), *)$

Note: The followings are not group:  $(\mathbb{R}, *)$ ,  $(\mathbb{N}, +)$ ,  $([0, \infty), *)$

Note: Let  $(F, +, *)$  be a field,  $(F, +)$  and  $(F \setminus \{0_F\}, *)$  are groups.

Let  $(G, \diamond_G), (H, \diamond_H)$  be groups, let  $\varphi : G \rightarrow H$  be a function,  $\varphi$  is called a **Group Homomorphism** from  $(G, \diamond_G)$  to  $(H, \diamond_H)$  provided that for  $a, b \in G$ , we have  $\varphi(a \diamond_G b) = \varphi(a) \diamond_H \varphi(b)$ .

Fact: Let  $(G, \diamond_G), (H, \diamond_H)$  be groups. The function  $\varphi : G \rightarrow H$  that sends all elements in  $G$  to the  $\diamond_H$ -identity in  $H$  is a group homomorphism from  $(G, \diamond_G)$  to  $(H, \diamond_H)$ .

Let  $(G, \diamond_G), (H, \diamond_H)$  be groups, let  $\varphi : G \rightarrow H$  be a group homomorphism from  $(G, \diamond_G)$  to  $(H, \diamond_H)$ .  $\varphi$  is called a **Group Isomorphism** from  $(G, \diamond_G)$  to  $(H, \diamond_H)$  provided that  $\varphi$  is a bijection from  $G$  to  $H$ . If  $\varphi$  is a group isomorphism from  $(G, \diamond_G)$  to  $(H, \diamond_H)$ , then  $(G, \diamond_G)$  is Isomorphic to  $(H, \diamond_H)$ .

Fact: Let  $(G, \diamond_G), (H, \diamond_H)$  be groups. If  $\varphi : G \rightarrow H$  is a group isomorphism from  $(G, \diamond_G)$  to  $(H, \diamond_H)$ , then the inverse of  $\varphi$ ,  $\varphi^{-1} : H \rightarrow G$ , is a group isomorphism from  $(H, \diamond_H)$  to  $(G, \diamond_G)$ .

Fact: The group  $((0, \infty), *)$  is isomorphic to the group  $(\mathbb{R}, +)$ .

Let  $(G, \diamond_G), (H, \diamond_H)$  be groups, let  $e_H \in H$  be the  $\diamond_H$ -identity, let  $\varphi : G \rightarrow H$  be a group homomorphism from  $(G, \diamond_G)$  to  $(H, \diamond_H)$ .  $im(\varphi) := \{\varphi(g) \mid g \in G\}$  is called the **Image** of  $\varphi$ .  $ker(\varphi) := \{g \in G \mid \varphi(g) = e_H\}$  is called the **Kernel** of  $\varphi$ .

A group  $(G, \diamond)$  is said to be **Abelian** provided that  $\diamond$  is commutative.

Let  $n \in \mathbb{N}$ , let  $X_n = \{x_1, x_2, x_3, \dots, x_n\}$  be a finite set. A bijection  $f : X_n \rightarrow X_n$  is called a **Permutation** of the set  $X_n$ , and the set  $s_n := \{f : X_n \rightarrow X_n \mid f \text{ is bijective}\}$  is called the **Collection of the Permutations** of the finite set  $X_n$ .

Note: Let  $n \in \mathbb{N}$ , let  $\circ$  denote function composition.  $S_n := (s_n, \circ)$  is a group.

Fact: Let  $n \in \mathbb{N}$ , the set  $s_n$  has  $n!$  element.

Let  $n \in \mathbb{N}$ ,  $S_n := (s_n, \circ)$  is called the **Symmetric Group on  $n$  Letters**.

Fact: For  $n \in \mathbb{N}$  with  $n \geq 3$ , the group  $S_n$  is not abelian.

Let  $n \in \mathbb{N}$  with  $n \geq 3$ , given a regular polygon with  $n$  sides. a **Dihedral Group**, denoted as  $D_n$  is the group of symmetries of the  $n$ -sided regular polygon, which includes rotations and reflections of the  $n$ -sided regular polygon.

Fact: Let  $n \in \mathbb{N}$  with  $n \geq 3$ , the group  $D_n$  has  $2n$  elements.

Fact: Let  $n \in \mathbb{N}$  with  $n \geq 3$ , the group  $D_n$  is not abelian.

Fact: Let  $n \in \mathbb{N}$  with  $n > 3$ ,  $D_n$  is not isomorphic to  $S_n$ .

# Calculus

Consider using the topological space  $(\mathbb{R}, \mathcal{T}_{EUC})$  from now on.

Fix  $a, b \in \mathbb{R}$  with  $a < b$ ,  $n \in \mathbb{N}$ , consider using the interval  $[a, b]$  from now on.

Let  $M, B$  be a subset of  $\mathbb{R}$ , let  $A$  be a subset of  $M$ , let  $f : M \rightarrow B$  be a function.

$f$  is **Strictly Increasing** on  $A$  provided that if  $x, y \in A$  with  $x < y$ , then  $f(x) < f(y)$ .

$f$  is **Strictly Decreasing** on  $A$  provided that if  $x, y \in A$  with  $x < y$ , then  $f(x) > f(y)$ .

$f$  is **Non-Decreasing** on  $A$  provided that if  $x, y \in A$  with  $x < y$ , then  $f(x) \leq f(y)$ .

$f$  is **Non-Increasing** on  $A$  provided that if  $x, y \in A$  with  $x < y$ , then  $f(x) \geq f(y)$ .

$f$  is called a **Constant Function** on  $A$  provided that  $\exists k \in \mathbb{R}$  s.t.  $\forall c \in f(A)$ ,  $c = k$ .

$f$  is **Bounded** on  $A$  provided that the set  $f(A)$  is bounded.

$f$  is **Non-Negative** on  $A$  provided that  $\forall c \in f(A)$ ,  $c \geq 0$ .

$f$  is **Non-Positive** on  $A$  provided that  $\forall c \in f(A)$ ,  $c \leq 0$ .

$f$  is **Negative** on  $A$  provided that  $\forall c \in f(A)$ ,  $c < 0$ .

$f$  is **Positive** on  $A$  provided that  $\forall c \in f(A)$ ,  $c > 0$ .

Fact: Let  $A, B$  be subsets of  $\mathbb{R}$ , let  $f : A \rightarrow B$  be a function. If  $f$  is bounded on  $A$ , then there exists  $m \in \mathbb{R}$  such that for all  $x \in A$ , we have  $|f(x)| \leq m$ .

Let  $A, B$  be subsets of  $\mathbb{R}$ , let  $f : A \rightarrow B$  be a function.  $f$  is said to be **Locally Bounded** on  $A$  provided that for all  $x \in A$ , there exists a neighborhood  $U$  of  $x$  such that  $f$  is bounded on  $U$ .

Let  $A, B$  be subsets of  $\mathbb{R}$ , let  $f : A \rightarrow B$  be a function.  $f$  is said to be **Locally Constant** on  $A$  provided that for all  $x \in A$ ,  $\exists \epsilon > 0$  such that for all  $y \in B_\epsilon(x)$ , we have  $f(y) = d$  for some  $d \in \mathbb{R}$ .

Let  $A, B$  be subsets of  $\mathbb{R}$ , let  $f : A \rightarrow B$  be a function.  $f$  is said to be **Uniformly Continuous** on  $A$  provided that  $\forall \epsilon > 0$ ,  $\exists \delta > 0$  s.t. if  $x, y \in A$  with  $|x - y| < \delta$ , then  $|f(x) - f(y)| < \epsilon$ .

Fact: Let  $A \subseteq \mathbb{R}$ . If  $f : A \rightarrow \mathbb{R}$  is uniformly continuous on  $A$ , then  $f$  is continuous on  $A$ .

Let  $a, b \in \mathbb{R}$ , let  $[a, b]$  be an interval,  $l := b - a$  is called the **Length** of  $[a, b]$ .

An ordered set  $P := \{t_0, t_1, t_2, \dots, t_n\}$  of points in  $\mathbb{R}$ , where  $a = t_0 < t_1 < t_2 < \dots < t_n = b$ , is called a **Partition** of  $[a, b]$ . For  $i \in \mathbb{N}$ ,  $1 \leq i \leq n$ , the interval  $[t_{i-1}, t_i]$  is called a **Subinterval** of  $P$ .

Let  $P = \{t_0, t_1, t_2, \dots, t_n\}$  be a partition of  $[a, b]$ . If the length of each subinterval of  $P$  is  $\frac{b-a}{n}$ , that is, for  $i \in \mathbb{N}$ ,  $0 \leq i \leq n$ ,  $t_i = a + \frac{b-a}{n}i$ , then  $P$  is called a **Regular Partition** on  $[a, b]$ .

Let  $P, Q$  be partitions of  $[a, b]$ ,  $Q$  is called a **Refinement** of  $P$  provided that  $P$  as a set is a subset of the set  $Q$ , and we write  $P \leq Q$  if  $Q$  is a refinement of  $P$ .

Let  $P = \{t_0, t_1, t_2, \dots, t_n\}$  be a partition of  $[a, b]$ .

$\|P\| := \max\{t_i - t_{i-1} \mid i \in \mathbb{N}, 1 \leq i \leq n\}$  is called the **Norm** of  $P$

Note: The norm of a partition  $P$  is the maximum length of the subintervals of  $P$

Let  $P = \{t_0, t_1, t_2, \dots, t_n\}$  be a partition of  $[a, b]$ , let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function.

For  $i \in \mathbb{N}$ ,  $1 \leq i \leq n$ ,  $\mathcal{M}_i := \sup\{f(x) \mid x \in [t_{i-1}, t_i]\}$ .  $U(f, P) := \sum_{i=1}^n [(t_i - t_{i-1}) \cdot \mathcal{M}_i]$  is called the **Upper Darboux Sum** of the function  $f$  on the interval  $[a, b]$  with respect to the partition  $P$ .

Note: By definition,  $U(f, P)$  is the sum of the area of rectangles of height  $\mathcal{M}_i$  and base  $(t_i - t_{i-1})$ .

Let  $P = \{t_0, t_1, t_2, \dots, t_n\}$  be a partition of  $[a, b]$ , let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function.

For  $i \in \mathbb{N}$ ,  $1 \leq i \leq n$ ,  $\mu_i := \inf\{f(x) \mid x \in [t_{i-1}, t_i]\}$ .  $L(f, P) := \sum_{i=1}^n [(t_i - t_{i-1}) \cdot \mu_i]$  is called the **Lower Darboux Sum** of the function  $f$  on the interval  $[a, b]$  with respect to the partition  $P$ .

Note: By definition,  $U(f, P)$  is the sum of the area of rectangles of height  $\mu_i$  and base  $(t_i - t_{i-1})$ .

Fact: Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function,  $\forall$  partitions  $P$  of  $[a, b]$ , we have  $L(f, P) \leq U(f, P)$ .

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function.

$U(f) := \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$

$L(f) := \sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$

Note: By definitions,  $\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$  is bounded below by  $L(f, \{a, b\})$ .

Note: By definitions,  $\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$  is bounded above by  $U(f, \{a, b\})$ .

Note: By definitions, we have  $L(f) \leq U(f)$ .

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function.  $f$  is said to be **Darboux Integrable** on the interval  $[a, b]$  provided that  $U(f) = L(f)$ . If  $f$  is Darboux integrable on  $[a, b]$ ,  $I_D := U(f)$  is called the **Darboux Integral** of  $f$  on  $[a, b]$ . Furthermore, if  $f$  is Darboux integrable and non-negative on  $[a, b]$ ,  $I_D$  is called the Area Bounded by the graph of  $f$  on  $[a, b]$ .

Note: Not all functions are Darboux integrable.

Fact: If  $f : [a, b] \rightarrow \mathbb{R}$  is a constant function, then  $f$  is Darboux integrable on  $[a, b]$ .

Let  $P$  be partitions of  $[a, b]$ , let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function.

We can write  $\lim_{||P|| \rightarrow 0} U(f, P) = l$  for some  $l \in \mathbb{R}$  provided that for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all partitions  $P$  of  $[a, b]$  with  $||P|| < \delta$ , we have  $|U(f, P) - l| < \epsilon$ .

Let  $Q$  be partitions of  $[a, b]$ , let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function.

We can write  $\lim_{||Q|| \rightarrow 0} L(f, Q) = l$  for some  $l \in \mathbb{R}$  provided that for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all partitions  $Q$  of  $[a, b]$  with  $||Q|| < \delta$ , we have  $|L(f, Q) - l| < \epsilon$ .

Let  $P, Q$  be partitions of  $[a, b]$ , let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded function. The function  $f$  is said to be **S-Integrable** on the interval  $[a, b]$  provided that  $\lim_{||P|| \rightarrow 0} U(f, P) = \lim_{||Q|| \rightarrow 0} L(f, Q)$ . If  $f$  is S-integrable on  $[a, b]$ ,  $I_S := \lim_{||Q|| \rightarrow 0} L(f, Q)$  is called the **S-Integral** of  $f$  on  $[a, b]$ .

Let  $X^*$  be a subset of  $\mathbb{R}$ . Given  $n \in \mathbb{N}$ ,  $X^*$  is called an **n-Tuple** provided that  $X^*$  has  $n$  element.

Let  $P = \{t_0, t_1, t_2, \dots, t_n\}$  be a partition of  $[a, b]$ , let n-tuple  $X^* = \{x_1^*, x_2^*, \dots, x_n^*\}$  be a subset of  $\mathbb{R}$ .  $X^*$  is **Compatible** with the partition  $P$  provided that for  $i \in \mathbb{N}$ ,  $1 \leq i \leq n$ , we have  $x_i^* \in [t_{i-1}, t_i]$ .

Let  $P = \{t_0, t_1, t_2, \dots, t_n\}$  be a partition of  $[a, b]$ , let  $f : [a, b] \rightarrow \mathbb{R}$  be a function, let n-tuple  $X^* = \{x_1^*, x_2^*, \dots, x_n^*\}$  be a subset of  $\mathbb{R}$  compatible with  $P$ .  $R(f, P, X^*) := \sum_{i=1}^n [f(x_i^*)(t_i - t_{i-1})]$  is called the **Riemann Sum** of the function  $f$  on the interval  $[a, b]$  with respect to  $P$  and  $X^*$ .

Let  $P = \{t_0, t_1, t_2, \dots, t_n\}$  be a partition of  $[a, b]$ , let  $f : [a, b] \rightarrow \mathbb{R}$  be a function, let n-tuple  $X^* = \{x_1^*, x_2^*, \dots, x_n^*\}$  be a subset of  $\mathbb{R}$  compatible with  $P$ . We can write  $\lim_{||P|| \rightarrow 0} R(f, P, X^*) = l$  for some  $l \in \mathbb{R}$  provided that for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all partitions  $P$  of  $[a, b]$  with  $||P|| < \delta$ , we have  $|R(f, P, X^*) - l| < \epsilon$ .

Let  $P = \{t_0, t_1, t_2, \dots, t_n\}$  be a partition of  $[a, b]$ , let  $f : [a, b] \rightarrow \mathbb{R}$  be a function, let n-tuple  $X^* = \{x_1^*, x_2^*, \dots, x_n^*\}$  be a subset of  $\mathbb{R}$  compatible with  $P$ .  $f$  is said to be **Riemann Integrable** on the interval  $[a, b]$  provided that  $\lim_{||P|| \rightarrow 0} R(f, P, X^*) = l$  for some  $l \in \mathbb{R}$ . If  $f$  is Riemann Integrable on  $[a, b]$ ,  $I_R := \lim_{||P|| \rightarrow 0} R(f, P, X^*)$  is called the **Riemann Integral** of  $f$  on  $[a, b]$ .

Fact: If a function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann Integrable on  $[a, b]$ , then  $f$  is bounded on  $[a, b]$ .

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function.

By Theorem,  $f$  is said to be **Integrable** on  $[a, b]$  provided that any one of the followings holds:

1.  $f$  is bounded on  $[a, b]$  and  $f$  is Darboux integrable on  $[a, b]$ .
2.  $f$  is bounded on  $[a, b]$  and  $f$  is S-integrable on  $[a, b]$ .
3.  $f$  is Riemann integrable on  $[a, b]$ .

If  $f$  is integrable on  $[a, b]$ ,  $\int_a^b f$  is called the **Integral** of  $f$  on  $[a, b]$ .

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function that is integrable on  $[a, b]$ . Given  $a < b$ ,  $\int_b^a f := -\int_a^b f$ .

The function  $\ln : (0, \infty) \rightarrow \mathbb{R} \quad x \mapsto \int_1^x \frac{1}{t}$  is called the **Natural Logarithm Function**.

Notation: For  $x \in \mathbb{R}$ ,  $0 < x < 1$  we write  $\ln(x) = \int_1^x \frac{1}{t} = -\int_x^1 \frac{1}{t}$ .

Fact: The function  $\gamma : (0, \infty) \rightarrow \mathbb{R} \quad t \mapsto \frac{1}{t}$  is continuous on  $(0, \infty)$ , hence integrable on  $(0, \infty)$ .

Fact: The Natural Logarithm function is strictly increasing and continuous on  $(0, \infty)$ .

Fact: The Natural Logarithm function is a bijection from  $(0, \infty)$  to  $\mathbb{R}$ .

Fact: Given  $\ln : (0, \infty) \rightarrow \mathbb{R} \quad x \mapsto \int_1^x \frac{1}{t}$ , we have  $\ln(1) = 0$ , and  $\ln((0, \infty)) = \mathbb{R}$ .

The inverse function of the Natural Logarithm function,  $\exp : \mathbb{R} \rightarrow (0, \infty) \quad \exp(t) = s \iff t = \ln(s)$ , is called the **Natural Exponential Function**.

Note: The Natural Logarithm function is a bijection from  $(0, \infty)$  to  $\mathbb{R}$ , hence invertible.

The assumptions,  $a, b \in \mathbb{R}$  with  $a < b$ ,  $n \in \mathbb{N}$ ,  $[a, b]$  is an interval, are now removed.

We continue using the topological space  $(\mathbb{R}, \mathcal{T}_{EUC})$ .

The term IWIMP refers to Interval With Infinitely Many Points.

Let  $A, B$  be subsets of  $\mathbb{R}$ , let  $f : A \rightarrow B$  be a function, let  $a \in A$ . We say that the limit of  $f$  at  $a$  when  $f$  approaches  $a$  from the right is  $l$ , denoted as  $\lim_{x \rightarrow a^+} f(x) = l$  for some  $l \in \mathbb{R}$ , provided that  $\forall \epsilon > 0, \exists \delta > 0$  s.t. if  $x \in \mathbb{R}$  satisfies  $0 < x - a < \delta$ , then  $|f(x) - l| < \epsilon$ .

Let  $A, B$  be subsets of  $\mathbb{R}$ , let  $f : A \rightarrow B$  be a function, let  $a \in A$ . We say that the limit of  $f$  at  $a$  when  $f$  approaches  $a$  from the left is  $m$ , denoted as  $\lim_{x \rightarrow a^-} f(x) = m$  for some  $m \in \mathbb{R}$ , provided that  $\forall \epsilon > 0, \exists \delta > 0$  s.t. if  $x \in \mathbb{R}$  satisfies  $0 < a - x < \delta$ , then  $|f(x) - m| < \epsilon$ .

Let  $A \subseteq \mathbb{R}$  be an interval. If  $A$  has a minimal element  $a$ , then  $a$  is called a **End Point** of  $A$ . If  $A$  has a maximal element  $b$ , then  $b$  is called a **End Point** of  $A$ .

Let  $A \subseteq \mathbb{R}$  be an IWIMP, let  $B$  be a subset of  $\mathbb{R}$ , let  $a \in A$ , let  $f : A \rightarrow B$  be a function. If  $a$  is not an end point of  $A$ , then  $f$  is **Differentiable** at  $a$  provided that there exists  $l \in \mathbb{R}$  such that  $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = l$ . If  $a$  is the minimal element of  $A$ , then  $f$  is **Differentiable** at  $a$  provided that there exists  $l \in \mathbb{R}$  such that  $\lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h} = l$ . If  $a$  is the maximal element of  $A$ , then  $f$  is **Differentiable** at  $a$  provided that there exists  $l \in \mathbb{R}$  such that  $\lim_{h \rightarrow 0^-} \frac{f(a+h) - f(a)}{h} = l$ . If  $f$  is **Differentiable** at  $a$ , then  $f'(a) := l$  is called the **Derivative** of  $f$  at  $a$ .

Note: Continuity of a function does not imply differentiability of the function.

Let  $A \subseteq \mathbb{R}$  be an IWIMP, let  $B$  be a subset of  $\mathbb{R}$ , let  $a \in A$ , let  $f : A \rightarrow B$  be a function.  $f$  is **Differentiable** on  $A$  provided that  $\forall a \in A$ ,  $f$  is differentiable at  $a$ . If  $f$  is differentiable on  $A$ , then the function  $f' : A \rightarrow \mathbb{R} \quad a \mapsto f'(a)$  is called the **Derivative** of  $f$ . If the function  $f$  is differentiable on its domain, then the function is said to be **Differentiable**.

Let  $A \subseteq \mathbb{R}$  be an IWIMP, let  $B$  be a subset of  $\mathbb{R}$ , let  $f : A \rightarrow B$  be a function, let  $a, b \in A$ .  $a$  is called a **Local Maximum** of  $f$  provided that  $\exists \delta > 0$  s.t.  $\forall x \in B_\delta(a) \cap A$ , we have  $f(a) \geq f(x)$ .  $b$  is called a **Local Minimum** of  $f$  provided that  $\exists \lambda > 0$  s.t.  $\forall x \in B_\lambda(b) \cap A$ , we have  $f(b) \leq f(x)$ .

Let  $A \subseteq \mathbb{R}$  be an IWIMP, let  $B$  be a subset of  $\mathbb{R}$ , let  $f : A \rightarrow B$  be a function, let  $a \in A$ .  $a$  is called a **Critical Point** of  $f$  on  $A$  provided that either one of the followings holds:

1.  $f$  is not differentiable at  $a$ .
2.  $f$  is differentiable at  $a$  and  $f'(a) = 0$ .

If  $a$  is a critical point of  $f$  on  $A$ , then  $f(a)$  is called a **Critical Value** of  $f$  on  $A$ .

Fact: Let  $A \subseteq \mathbb{R}$  be an IWIMP, let  $B$  be a subset of  $\mathbb{R}$ , let  $f : A \rightarrow B$  be a function. If  $f$  has a local maximum or a local minimum at  $a \in A$ , and if  $f$  is differentiable at  $a$ , then  $f'(a) = 0$ .

Let  $r \in \mathbb{R}$ , let  $x \in \mathbb{R}_{>0}$ ,  $x^r := \exp(r \cdot \ln(x))$  is called  $x$  **raised to the power of  $n$** , in this case,  $x$  is called the **Base**,  $n$  is called the **Exponent** or the **Power**.

## Sequences

Let  $(X, \mathcal{T})$  be a topological space, a function  $seq : \mathbb{N} \rightarrow X \quad n \mapsto x_n$  is called a **Sequence** in  $(X, \mathcal{T})$ .  
 Notation: A sequence  $seq : \mathbb{N} \rightarrow X \quad n \mapsto x_n$  in  $(X, \mathcal{T})$  can be denoted as  $n \mapsto x_n$  or  $(x_n)$ .

Let  $(X, \mathcal{T})$  be a topological space, let  $(x_n)$  be a sequence in  $(X, \mathcal{T})$ . We say  $(x_n)$  **Converges** to  $l \in X$  provided that  $\forall U \in \mathcal{T}$ , if  $l \in U$ , then  $\exists N \in \mathbb{N}$  s.t.  $\forall n \in \mathbb{N}$  with  $n \geq N$ , we have  $x_n \in U$ . If  $(x_n)$  converges to  $l \in X$ ,  $(x_n)$  is called a **Convergent Sequence**, and we write  $\lim_{n \rightarrow \infty} x_n = l$ .

Note: In  $(\mathbb{R}, \mathcal{T}_{ind})$ , all sequences converge and they converge to every real number.

Note: In  $(\mathbb{R}, \mathcal{T}_{dis})$ , only those sequences that are eventually constant converge.

Fact: If a sequence in a Hausdorff topological space converges, then the limit of the sequence is unique, that is, the sequence converges to a unique element in that Hausdorff topological space.

A sequence  $(x_n)$  in  $(\mathbb{R}, \mathcal{T}_{EUC})$  is called a **Sequence of Real Numbers in the Euclidean Topology**. We say the sequence  $(x_n)$  **Converges** to some  $l \in \mathbb{R}$  provided that for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$  with  $n \geq N$ , we have  $|x_n - l| < \epsilon$ .

Note: Not all sequence of real numbers in the Euclidean topology converges.

Let  $(X, \mathcal{T})$  be a topological space, let  $\mathcal{A} : \mathbb{N} \rightarrow X \quad n \mapsto a_n$  be a sequence in  $(X, \mathcal{T})$ , let  $\mathcal{J} : \mathbb{N} \rightarrow \mathbb{N} \quad k \mapsto j_k$  be a strictly increasing function, let  $\circ$  denote function composition. The function  $\mathcal{A} \circ \mathcal{J} : \mathbb{N} \rightarrow X \quad n \mapsto a_{j_n}$  is called a **Subsequence** of the sequence  $\mathcal{A}$ .

Let  $(t_m)$  be a sequence of real numbers in the Euclidean topology.

$(t_m)$  is said to be **Monotonic Increasing** provided that  $\forall a, b \in \mathbb{N}$  with  $a > b$ , we have  $t_a \geq t_b$ .

$(t_m)$  is said to be **Monotonic Decreasing** provided that  $\forall a, b \in \mathbb{N}$  with  $a > b$ , we have  $t_a \leq t_b$ .

Let  $(t_m)$  be a sequence of real numbers in the Euclidean topology.  $(t_m)$  is said to be **Monotonic** provided that  $(t_m)$  is either monotonic increasing or monotonic decreasing.

Let  $(X, d)$  be a metric space, let  $(x_n)$  be a sequence in  $(X, \mathcal{T}_d)$ .  $(x_n)$  is called a **Cauchy Sequence** in  $X$  provided that  $\forall \epsilon > 0$ ,  $\exists N \in \mathbb{N}$  s.t.  $\forall n, m \in \mathbb{N}$  with  $n > N$  and  $m > N$ , we have  $d(x_n, x_m) < \epsilon$ .

Let  $(X, d)$  be a metric space. The metric space  $(X, d)$  is said to be **Cauchy Complete** provided that every Cauchy sequence in  $(X, \mathcal{T}_d)$  converges to some  $l \in X$ .

Note: Cauchy Complete is not a topological invariant.

Fact:  $(\mathbb{R}, d_{EUC})$  is Cauchy complete.

Fact:  $([0, 1], d_{EUC})$  is Cauchy complete.

## Polynomials

Consider using the topological space  $(\mathbb{R}, \mathcal{T}_{EUC})$  from now on.

The term IWIMP refers to Interval With Infinitely Many Points.

A **Polynomial Function of degree  $n$** , denoted as  $P : \mathbb{R} \rightarrow \mathbb{R}$ , is a function from  $\mathbb{R}$  to  $\mathbb{R}$  of the form  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ , with  $a_n, a_{n-1}, \dots, a_1, a_0 \in \mathbb{R}$ ,  $n \in \mathbb{N}$ , and  $a_n \neq 0$ .

Let  $A \subseteq \mathbb{R}$  be an IWIMP, let  $B$  be a subset of  $\mathbb{R}$ , let  $a \in A$ , let  $n \in \mathbb{N}$ , let  $f : A \rightarrow B$  be a function that is  $n$ -times differentiable at  $a$ . The function  $P_{n,a} : \mathbb{R} \rightarrow \mathbb{R} \quad x \mapsto \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k$  is called the **Taylor Polynomial** of degree  $n$ , centered at  $a$ , associated to  $f$ .

Note: By definition, given  $n \in \mathbb{N}$ ,  $a \in A$ ,  $P_{n,a}$  is unique and  $n$ -times differentiable.

Let  $A \subseteq \mathbb{R}$  be an IWIMP, let  $B$  be a subset of  $\mathbb{R}$ , let  $a \in A$ , let  $n \in \mathbb{N}$ , let  $f : A \rightarrow B$  be a function that is  $n$ -times differentiable at  $a$ , let  $P_{n,a} : \mathbb{R} \rightarrow \mathbb{R}$  be the Taylor polynomial of degree  $n$ , centered at  $a$ , associated to  $f$ .  $R_{n,a} : \mathbb{R} \rightarrow \mathbb{R} \quad x \mapsto f(x) - P_{n,a}(x)$  is called the **Remainder Term** of  $P_{n,a}$ .