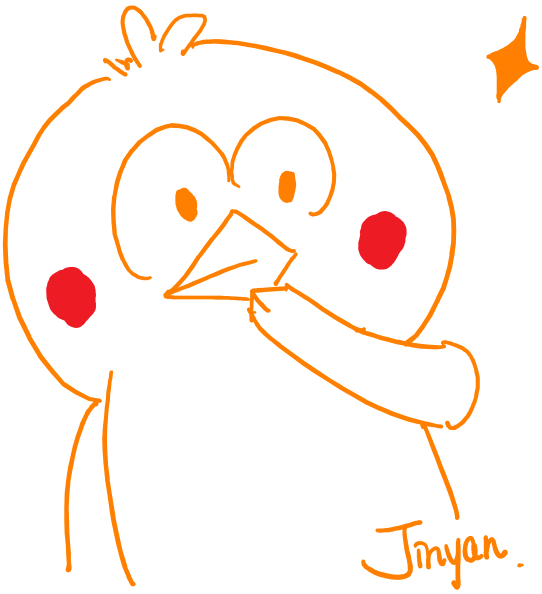
Homework 8

Physics 542 - Quantum Optics
Professor Alex Kuzmich



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First we compute the commutation relation, for obvious reason we drop the hat $\hat{()}$ on the operators $\hat{a}$ and $\hat{a}^\dagger$,

$$\begin{aligned}
[\alpha a^\dagger - a\alpha^*, \beta a^\dagger - a\beta^*] =& (\alpha a^\dagger - a\alpha^*)(\beta a^\dagger - a\beta^*) - (\beta a^\dagger - a\beta^*)(\alpha a^\dagger - a\alpha^*)\\
=& \alpha\beta a^\dagger a^\dagger - \alpha\beta^* a^\dagger a - \alpha^*\beta a a^\dagger + \alpha^*\beta^* aa - \alpha\beta a^\dagger a^\dagger + \alpha^*\beta a^\dagger a + \alpha\beta^* a a^\dagger - \alpha^*\beta^* aa\\
=& (\alpha^*\beta - \alpha\beta^*)a^\dagger a + (\alpha\beta^* - \alpha^*\beta)aa^\dagger\\
=& (\alpha^*\beta - \alpha\beta^*)a^\dagger a + (\alpha\beta^* - \alpha^*\beta)(1 + a^\dagger a)\\
=& \alpha\beta^* - \alpha^*\beta\,.
\end{aligned}$$

Now we can compute

$$\begin{aligned}
D(\alpha,\alpha^*)\, D(\beta,\beta^*) &= e^{\alpha a^\dagger - a\alpha^*} e^{\beta\alpha^\dagger - a\beta^*}\\
&= e^{\alpha a^\dagger - a\alpha^* + \beta a^\dagger - a\beta^*} e^{[\alpha a^\dagger - a\alpha^*, \beta a^\dagger - a\beta^*]/2}\\
&= e^{(\alpha+\beta)a^\dagger - a(\alpha^*\beta^*)} e^{[\alpha a^\dagger - a\alpha^*, \beta a^\dagger - a\beta^*]/2}\\
&= D(\alpha+\beta,\, \alpha^*+\beta^*)\, e^{(\alpha\beta^* - \alpha^*\beta)/2}\,.
\end{aligned}$$

Thus we can write

$$D(\alpha,\alpha^*)\, D(\beta,\beta^*)\, e^{(\alpha^*\beta - \alpha\beta^*)/2} = D(\alpha+\beta,\, \alpha^*+\beta^*)$$

We can consider $e^{(\alpha\beta^* - \alpha^*\beta)/2} = e^{i\Im(\alpha^*\beta)} \in \mathbb{C}$ ($\Im$ is the operator of taking the imaginary part), which has unitary magnitude as the phase of the coherent state. Mathematically,

$$D(\alpha,\alpha^*)\, D(\beta,\beta^*)\, |0\rangle \neq D(\alpha+\beta, \alpha^*+\beta^*)|0\rangle = |\alpha+\beta\rangle\,.$$

That is $D(\alpha,\alpha^*)\, D(\beta,\beta^*)$ generates a different coherent state than $D(\alpha+\beta, \alpha^*+\beta^*)$. Phase difference between coherent states can be measured, thus the phase $e^{i\Im(\alpha^*\beta)}$ does matter and is not purely a geometrical phase.

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First we derive the identity

$$
\begin{aligned}
e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} &= \left(1+\alpha\hat{A}+\frac{\alpha^2\hat{A}^2}{2!}+\frac{\alpha^3\hat{A}^3}{3!}+\cdots\right)\hat{B}\left(1-\alpha\hat{A}+\frac{\alpha^2\hat{A}^2}{2!}-\frac{\alpha^3\hat{A}^3}{3!}+\cdots\right)\\
&= \hat{B}+\alpha(\hat{A}\hat{B}-\hat{B}\hat{A})+\frac{\alpha^2}{2!}\left(\hat{A}[\hat{A},\hat{B}]-[\hat{A},\hat{B}]\hat{A}\right)+\cdots\\
&= \hat{B}+\alpha[\hat{A},\hat{B}]+\frac{\alpha^2}{2!}[\hat{A},[\hat{A},\hat{B}]]+\cdots .
\end{aligned}
$$

Now using (6.11) and (6.12), we obtain

$$
\hat{a}_2=\hat{U}^\dagger\hat{a}_0\hat{U}=e^{-i\pi\hat{J}_1/2}\hat{a}_0e^{i\pi\hat{J}_1/2}=\hat{a}_0+\left(-i\frac{\pi}{2}\right)[\hat{J}_1,\hat{a}_0]+\frac{1}{2!}\left(\frac{-i\pi}{2}\right)^2[\hat{J}_1,[\hat{J}_1,\hat{a}_0]]+\cdots
$$

where we have abbreviated $\hat{J}_1=(\hat{a}_0^\dagger\hat{a}_1+\hat{a}_0\hat{a}_1^\dagger)/2$. Now we evaluate

$$
\begin{aligned}
[\hat{J}_1,\hat{a}_0] &= \left(\hat{a}_0^\dagger\hat{a}_1+\hat{a}_0\hat{a}_1^\dagger\right)\hat{a}_0/2-\hat{a}_0\left(\hat{a}_0^\dagger\hat{a}_1+\hat{a}_0\hat{a}_1^\dagger\right)/2\\
&= \left(\hat{a}_0^\dagger\hat{a}_1\hat{a}_0-\hat{a}_0\hat{a}_0^\dagger\hat{a}_1+\hat{a}_0\hat{a}_1^\dagger\hat{a}_0-\hat{a}_0\hat{a}_0\hat{a}_1^\dagger\right)/2\\
&= \left(\hat{a}_0^\dagger\hat{a}_0\hat{a}_1-\hat{a}_0\hat{a}_0^\dagger\hat{a}_1+\hat{a}_0\hat{a}_0\hat{a}_1^\dagger-\hat{a}_0\hat{a}_0\hat{a}_1^\dagger\right)/2\\
&= \left(\hat{a}_0^\dagger\hat{a}_0\hat{a}_1-\hat{a}_0\hat{a}_0^\dagger\hat{a}_1\right)/2\\
&= \left([\hat{a}_0^\dagger,\hat{a}_0]\hat{a}_1\right)/2\\
&= -\hat{a}_1/2 .
\end{aligned}
$$

Similarly, one is able to obtain

$$
[\hat{J}_1,\hat{a}_1]=-\hat{a}_0/2 .
$$

Thus combining we obtain

$$
\begin{aligned}
\hat{a}_2 &= \hat{a}_0+i\frac{\pi}{4}\hat{a}_1+\frac{1}{4}\frac{\pi^2}{4}[\hat{J}_1,\hat{a}_1]+\cdots\\
&= \hat{a}_0+i\frac{\pi}{4}\hat{a}_1-\frac{\pi^2}{16}\hat{a}_0-i\frac{\pi^3}{64}\hat{a}_1^3+\cdots\\
&= \cos\left(\frac{\pi}{4}\right)\hat{a}_0+i\sin\left(\frac{\pi}{4}\right)\hat{a}_1\\
&= \frac{1}{\sqrt{2}}\left(\hat{a}_0+i\hat{a}_1\right)
\end{aligned}
$$

Now we perform similar calculation

$$
\begin{aligned}
\hat{a}_3 = \hat{U}^\dagger \hat{a}_1 \hat{U} &= \hat{a}_1 + \left(-i\frac{\pi}{2}\right)[\hat{J}_1, \hat{a}_1] + \frac{1}{2!}\left(\frac{-i\pi}{2}\right)^2 [\hat{J}_1, [\hat{J}_1, \hat{a}_1]] + \cdots \\
&= \cos\left(\frac{\pi}{4}\right)\hat{a}_1 + i\sin\left(\frac{\pi}{4}\right)\hat{a}_0 \\
&= \frac{1}{\sqrt{2}}\left(i\hat{a}_0 + \hat{a}_1\right) .
\end{aligned}
$$

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First it is straightforward to write

$$|2\rangle_a|2\rangle_b = \frac{(\hat{a}^\dagger)^2}{\sqrt{2}}|0\rangle_a \frac{(\hat{a}^\dagger)^2}{\sqrt{2}}|0\rangle_b = \frac{(\hat{a}^\dagger)^2(\hat{b}^\dagger)^2}{2}|0\rangle_a|0\rangle_b\,.$$

We denote the operator for the first and second beam splittersa as $\hat{U}_1$ and $\hat{U}_2$, respectively. Then after the first beam splitter, we should have

$$\begin{aligned}
\hat{U}_1|\mathrm{in}\rangle &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}(\hat{a}^\dagger + i\hat{b}^\dagger)\right)^2\left(\frac{1}{\sqrt{2}}(i\hat{a}^\dagger + \hat{b}^\dagger)\right)^2|0\rangle_a|0\rangle_b \\
&= \frac{1}{8}(\hat{a}^\dagger + i\hat{b}^\dagger)^2(i\hat{a}^\dagger + \hat{b}^\dagger)^2|0\rangle_a|0\rangle_b \\
&= \frac{1}{8}\left(-(\hat{a}^\dagger)^4 - 2(\hat{a}^\dagger)^2(\hat{b}^\dagger)^2 - (\hat{b}^\dagger)^4\right)|0\rangle_a|0\rangle_b \\
&= -\frac{1}{8}\left(\sqrt{4!}|4\rangle_a|0\rangle_b + 4|2\rangle_a|2\rangle_b + \sqrt{4!}|0\rangle_a|4\rangle_b\right)\,.
\end{aligned}$$

Now we apply the mirror operator $\hat{U}_m$,

$$\hat{U}_m\hat{U}_1|\mathrm{in}\rangle = -\frac{1}{8}\left(\sqrt{4!}\left(|4\rangle_a|0\rangle_b + e^{i4\theta}|0\rangle_a|4\rangle_b\right) + 4e^{i2\theta}|2\rangle_a|2\rangle_b\right)\,.$$

In the next calculation we will show that

$$\begin{aligned}
|\mathrm{out}\rangle = &+\left(\frac{\sqrt{4!}e^{i2\theta}}{16} - \frac{\sqrt{4!}(1+e^{i4\theta})}{32}\right)(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b) \\
&- \frac{i\sqrt{6}(1-e^{i4\theta})}{8}(|3\rangle_a|1\rangle_b - |1\rangle_a|3\rangle_b) + \frac{3(1+e^{i4\theta}) + 2e^{i2\theta}}{8}|2\rangle_a|2\rangle_b\,.
\end{aligned}$$

We have the second beam splitter, represented by operator $\hat{U}_2$, here we abbreviate $\hat{U}_2\hat{U}_m\hat{U}_1 = \hat{U}$, then we can write

$$\begin{aligned}
\hat{U}_2\hat{U}_m\hat{U}_1|\text{in}\rangle &= -\frac{1}{8}\left(\sqrt{4!}\left(\frac{1}{4\sqrt{4!}}\left(\hat{a}^\dagger + i\hat{b}^\dagger\right)^4|0\rangle_a|0\rangle_b + \hat{U}e^{i4\theta}|0\rangle_a|4\rangle_b\right) + \hat{U}4e^{i2\theta}|2\rangle_a|2\rangle_b\right) \\
&= -\frac{1}{8}\left(\frac{1}{4}\left((\hat{a}^\dagger)^4 + i4(\hat{a}^\dagger)^3\hat{b}^\dagger - 6(\hat{a}^\dagger)^2(\hat{b}^\dagger)^2 - i4\hat{a}^\dagger(\hat{b}^\dagger)^3 + (\hat{b}^\dagger)^4\right)|0\rangle_a|0\rangle_b\right) + \hat{U}(\text{other terms}) \\
&= -\frac{1}{8}\frac{\sqrt{4!}}{4\sqrt{4!}}\left(\sqrt{4!}\left(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b\right) + i4\sqrt{3!}\left(|3\rangle_a|1\rangle_b - |1\rangle_a|3\rangle_b\right) - 12|2\rangle_a|2\rangle_b\right) \\
&\quad - \frac{e^{i4\theta}}{8}\frac{\sqrt{4!}}{4\sqrt{4!}}\left(\sqrt{4!}\left(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b\right) - i4\sqrt{3!}\left(|3\rangle_a|1\rangle_b - |1\rangle_a|3\rangle_b\right) - 12|2\rangle_a|2\rangle_b\right) \\
&\quad + \frac{e^{i2\theta}}{8}\frac{4}{8}\left(\sqrt{4!}\left(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b\right) + 4|2\rangle_a|2\rangle_b\right) \\
&= -\frac{\sqrt{4!}(1+e^{i4\theta})}{32}\left(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b\right) \\
&\quad - \frac{i4\sqrt{6}(1-e^{i4\theta})}{32}\left(|3\rangle_a|1\rangle_b - |1\rangle_a|3\rangle_b\right) \\
&\quad + \frac{12(1+e^{i4\theta})}{32}|2\rangle_a|2\rangle_b \\
&\quad + \frac{e^{2i\theta}}{16}\left(\sqrt{4!}\left(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b\right) + 4|2\rangle_a|2\rangle_b\right) \\
&= +\left(\frac{\sqrt{4!}e^{i2\theta}}{16} - \frac{\sqrt{4!}(1+e^{i4\theta})}{32}\right)\left(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b\right) && \text{(first term)} \\
&\quad - \frac{i\sqrt{6}(1-e^{i4\theta})}{8}\left(|3\rangle_a|1\rangle_b - |1\rangle_a|3\rangle_b\right) && \text{(second term)} \\
&\quad + \frac{3(1+e^{i4\theta}) + 2e^{i2\theta}}{8}|2\rangle_a|2\rangle_b\,. && \text{(third term)}
\end{aligned}$$

It is obvious that the parity operator $\hat{\Pi}_b$ defined by Eq. (11.3) does nothing to the first and third terms as the numbers of photons in b states are even, but flips the sign of the second term as number of photons in b state is odd, that is,

$$\begin{aligned}
\hat{\Pi}_b|\text{out}\rangle &= +\left(\frac{\sqrt{4!}e^{i2\theta}}{16} - \frac{\sqrt{4!}(1+e^{i4\theta})}{32}\right)\left(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b\right) \\
&\quad + \frac{i\sqrt{6}(1-e^{i4\theta})}{8}\left(|3\rangle_a|1\rangle_b - |1\rangle_a|3\rangle_b\right) \\
&\quad + \frac{3(1+e^{i4\theta}) + 2e^{i2\theta}}{8}|2\rangle_a|2\rangle_b\,.
\end{aligned}$$

Taking the inner product, we obtain

$$\langle\text{out}|\hat{\Pi}_b|\text{out}\rangle = 2\left|\frac{\sqrt{4!}e^{i2\theta}}{16} - \frac{\sqrt{4!}(1+e^{i4\theta})}{32}\right|^2 - 2\left|\frac{i\sqrt{6}(1-e^{i4\theta})}{8}\right|^2 + \left|\frac{3(1+e^{i4\theta}) + 2e^{i2\theta}}{8}\right|^2.$$

We simplify using Wolframe Mathematica, we obtain

$$\langle\hat{\Pi}_b\rangle = \langle\text{out}|\hat{\Pi}_b|\text{out}\rangle = \frac{1}{4}\left(1 + 3\cos(4\theta)\right)\,.$$

Lastly, we compute

$$\Delta\theta = \Delta\hat{\Pi}_b \cdot \left|\frac{\partial\langle\hat{\Pi}_b\rangle}{\partial\theta}\right|^{-1} = \sqrt{1-\langle\hat{\Pi}_b\rangle^2}\left|\frac{\partial\langle\hat{\Pi}_b\rangle}{\partial\theta}\right|^{-1} = \frac{\sqrt{1-(1+3\cos(4\theta))^2/16}}{3|\sin(4\theta)|}\,.$$

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After the first beam splitter, we have

$$\hat{U}_1|\text{in}\rangle = \frac{1}{\sqrt{2}}\left(|N\rangle_a|0\rangle_b + e^{i\phi_N}|0\rangle_a|N\rangle_b\right),$$

thus after the mirror, we have

$$|\text{hello}\rangle = \hat{U}_m\hat{U}_1|\text{in}\rangle = \frac{1}{\sqrt{2}}\left(|N\rangle_a|0\rangle_b + e^{i(\phi_N+N\theta)}|0\rangle_a|N\rangle_b\right)$$

Following definition from the text,

$$\hat{J}_3 = (\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})/2\,, \qquad \hat{J}_2 = (\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})/(2i)\,, \qquad \hat{J}_0 = (\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})/2\,.$$

We see that obviously that $[\hat{J}_i, \hat{J}_0] = 0$ because, for instance,

$$\begin{aligned}
&\hat{J}_1(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}) - (\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})\hat{J}_1 \\
&\qquad = \hat{J}_1(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}) - (\hat{a}^\dagger(\hat{J}_1\hat{a} + \hat{b})) + \hat{b}^\dagger(\hat{J}_1\hat{b} + \hat{a}) \\
&\qquad = -\left(-\hat{b}^\dagger\hat{a} + \hat{a}^\dagger\hat{b} - \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{a}\right) = 0\,.
\end{aligned}$$

While on the other hand,

$$\begin{aligned}
[\hat{J}_1, \hat{J}_3] &= \frac{1}{2}\left(\hat{J}_1(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}) - (\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})\hat{J}_1\right) \\
&= \frac{1}{2}\left(\hat{J}_1(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}) - \left(\hat{a}^\dagger(\hat{J}_1\hat{a} + \hat{b}/2) - \hat{b}^\dagger(\hat{J}_1\hat{b} + \hat{a}/2)\right)\right) \\
&= -\frac{1}{2}(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a}) = -i\hat{J}_2\,,
\end{aligned}$$

and it can be computed that $[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}_k$. Thus $\hat{J}_1, \hat{J}_2, \hat{J}_3$ satisfy those identities of angular momentum operators. We further observe that we have $\hat{J}_0 - \hat{J}_3 = \hat{b}^\dagger\hat{b}$. Also note that $|\text{out}\rangle = e^{i\pi\hat{J}_1/2}|\text{hello}\rangle$ as shown in Problem 2. Thus it follows that we can write

$$\begin{aligned}
\langle\text{out}|\hat{\Pi}_b|\text{out}\rangle &= \langle\text{out}|e^{i\pi(\hat{J}_0-\hat{J}_3)}|\text{out}\rangle \\
&= \langle\text{hello}|e^{-i\pi\hat{J}_1/2}e^{i\pi\hat{J}_0}e^{-i\pi\hat{J}_3}e^{i\pi\hat{J}_1/2}|\text{hello}\rangle \\
&= \langle\text{hello}|e^{i\pi\hat{J}_0}e^{-i\pi\hat{J}_1/2}e^{-i\pi\hat{J}_3}e^{i\pi\hat{J}_1/2}|\text{hello}\rangle \\
&= \langle\text{hello}|e^{i\pi\hat{J}_0}e^{-i\pi\hat{J}_2}|\text{hello}\rangle \\
&= \frac{1}{2}\left({}_a\langle N|_b\langle 0| + e^{-i(\phi_N+N\theta)}{}_a\langle 0|_b\langle N|\right)e^{i\pi\hat{J}_0}\left(|0\rangle_a|N\rangle_b + (-1)^N e^{i(\phi_N+N\theta)}|N\rangle_a|0\rangle_b\right) \\
&= \frac{1}{2}\left({}_a\langle N|_b\langle 0| + e^{-i(\phi_N+N\theta)}{}_a\langle 0|_b\langle N|\right)e^{i\pi N/2}\left(|0\rangle_a|N\rangle_b + (-1)^N e^{i(\phi_N+N\theta)}|N\rangle_a|0\rangle_b\right) \\
&= \frac{e^{i\pi N/2}}{2}\left((-1)^N e^{i(\phi_N+N\theta)} + e^{-i(\phi_N+N\theta)}\right)
\end{aligned}$$

which gives $\pm\sin(\phi_N + N\theta)$ for odd N and $\pm\cos(\phi_N + N\theta)$ for even N. Now it is not hard to compute, for even N,

$$\Delta\theta = \sqrt{1 - \langle\hat{\Pi}_b\rangle} \left|\frac{\partial\langle\hat{\Pi}_b\rangle|}{\partial\phi}\right|^{-1} = \sin(\phi_N + N\theta)\sin^{-1}(\phi_N + N\theta)N^{-1} = \frac{1}{N}$$

and for odd N,

$$\Delta\theta = \cos(\phi_N + N\theta)\cos^{-1}(\phi_N + N\theta)N^{-1} = \frac{1}{N}\,,$$

as desired.