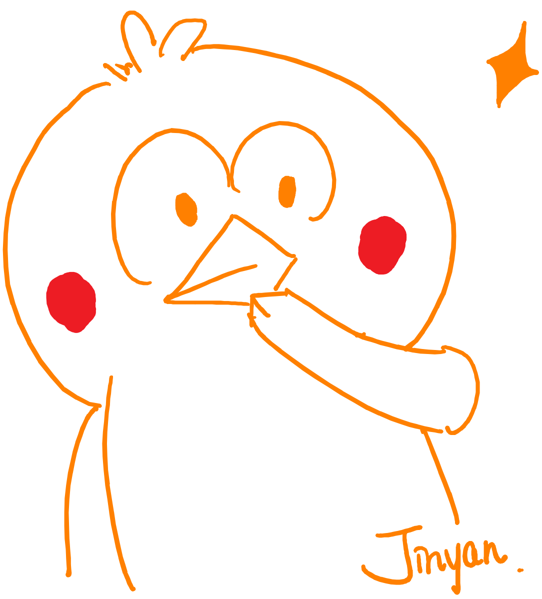
Homework 9

Physics 542 - Quantum Optics
Professor Alex Kuzmich



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1

Here we consider

$$S(N) = |A(N,t)|^2$$

where we have

$$A(N,t) = \sum_{j=1}^{N} e^{-i\omega t + i\phi_j}$$

with $\phi_j \in \mathbb{R}$ being random phase. Here we will compute, assume taking the time average,

$$\langle A \rangle = \left\langle e^{-i\omega t} \sum_{j=1}^{N} e^{i\phi_j} \right\rangle = 0 \cdot \sum_{j=1}^{N} e^{i\phi_j} = 0\,.$$

$$\langle S \rangle = \left\langle \sum_{j=1}^{N} e^{i\omega t - i\phi_j} \sum_{k=1}^{N} e^{-i\omega t + i\phi_k} \right\rangle = \left\langle \sum_{j,k=1}^{N} e^{i(\phi_k - \phi_j)} \right\rangle = \sum_{j,k=1}^{N} e^{i(\phi_k - \phi_j)} = \sum_{j \le k}^{N} 2\cos(\phi_j - \phi_k)\,.$$

$$\langle S^2 \rangle = \left\langle \left(\sum_{j,k=1}^{N} e^{i(\phi_k - \phi_j)} \right) \left(\sum_{j',k'=1}^{N} e^{i(\phi_{k'} - \phi_{j'})} \right) \right\rangle = \left(\sum_{j \le k}^{N} 2\cos(\phi_j - \phi_k) \right) \left(\sum_{j' \le k'}^{N} 2\cos(\phi_{j'} - \phi_{k'}) \right)\,.$$

$$(\Delta S)^2 = \langle S^2 \rangle - \langle S \rangle^2 = 0\,.$$

Now compute, assuming taking the ensemble average (over ϕ_j),

$$\langle A \rangle = \left\langle e^{-i\omega t} \sum_{j=1}^{N} e^{i\phi_j} \right\rangle = e^{-i\omega t} \cdot 0 = 0\,.$$

$$\langle S \rangle = \left\langle \sum_{j=1}^{N} e^{i\omega t - i\phi_j} \sum_{k=1}^{N} e^{-i\omega t + i\phi_k} \right\rangle = \left\langle \sum_{j,k=1}^{N} e^{i(\phi_k - \phi_j)} \right\rangle = N\,.$$

$$\langle S^2 \rangle = \left\langle \left(\sum_{j,k=1}^{N} e^{i(\phi_k - \phi_j)} \right) \left(\sum_{j',k'=1}^{N} e^{i(\phi_{k'} - \phi_{j'})} \right) \right\rangle = \left\langle \sum_{j,k,j',k'=1}^{N} e^{i(\phi_k - \phi_j + \phi_{k'} - \phi_{j'})} \right\rangle = 2N^2 - N\,.$$

$$(\Delta S)^2 = \langle S^2 \rangle - \langle S \rangle^2 = 2(N^2 - N)\,.$$

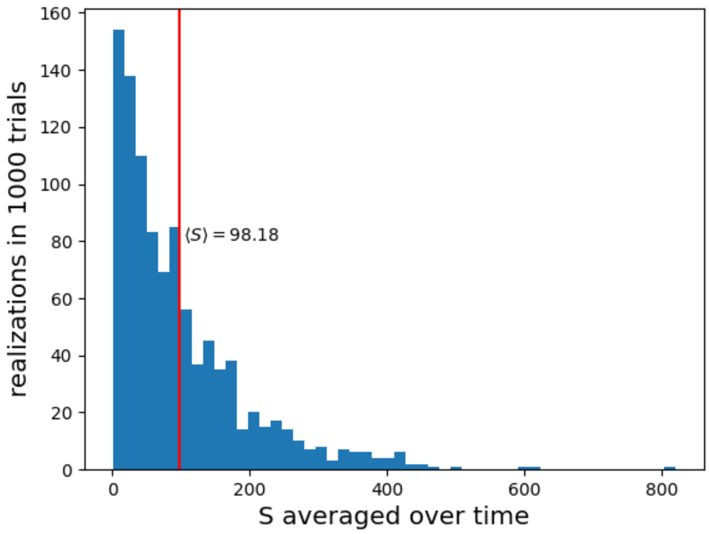
Quantities are computed with the help of Eq. (13.61) and (13.64) on Berman's text.

2

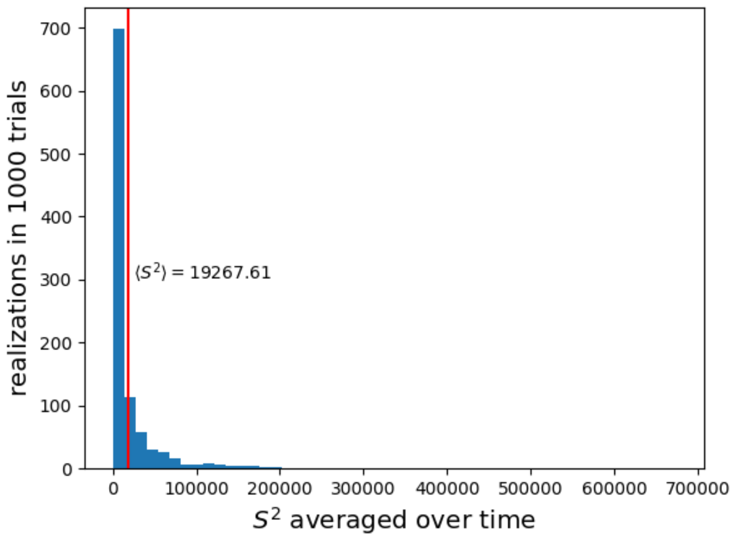
We evaluate

$$S = \left| \sum_{j=1}^{100} e^{-i\omega t + i\phi_j} \right|$$

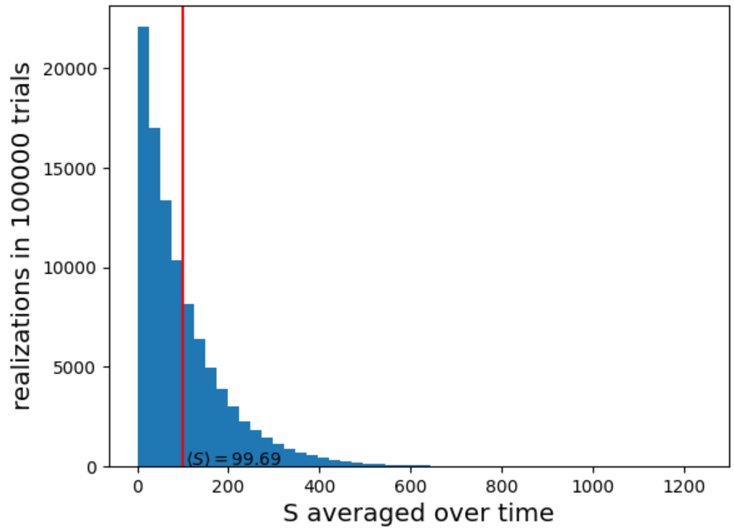
with random phases ϕ_j picked via random number generator. The three S's that we obtain are $S = 3.59$, $S = 114.65$, and $S = 94.02$. Then we repeat the calculation of S for a 1000 times and take the average, $\langle S \rangle = 98.18$ in this case, the result is shown on next page.

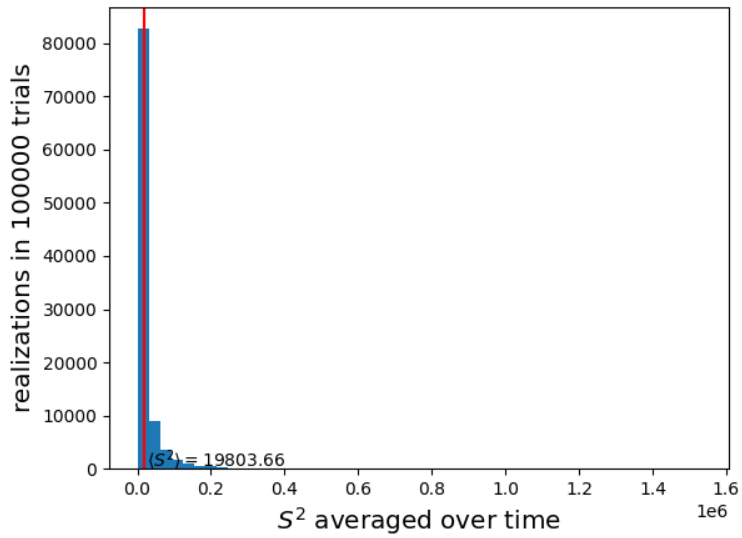


We do the same thing for S^2.



Now we evaluate them 100000 times, the numbers that we get here are very close to what we have calculated in problem 1.





3

First we consider

$$I(t) = \sum_{n=-\infty}^{\infty} \Theta(t + 0.1 - n) - \Theta(t - 0.1 - n)$$

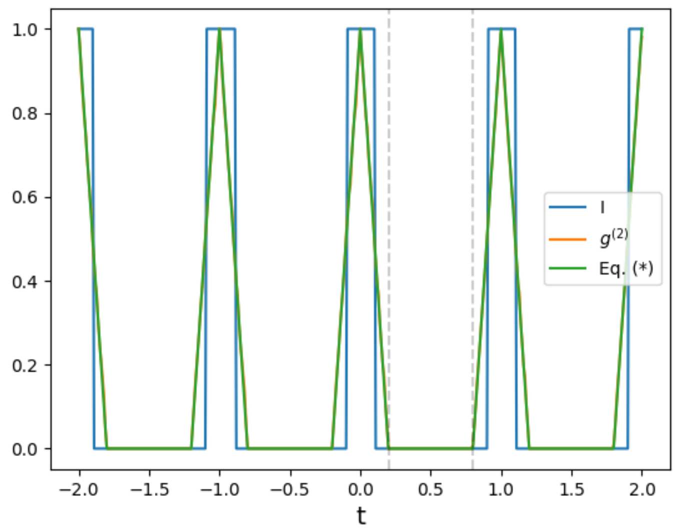
where Θ is the Heaviside step function. It is not hard to see that

$$\bar{I} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} I(t)\, dt = \lim_{T\to\infty} \frac{1}{T} \left(\lfloor T \rfloor \cdot 0.2 + \mu \right) = 0.2\,,$$

where $|\mu| < 1$. To compute $\langle I(t)\, I(t+\tau)\rangle$, as I is periodic with periods 1, it is not hard to see that $\langle I(t+n)\, I(t)\rangle = 0.2$ for all $n \in \mathbb{N}$ by symmetry. Furthermore, for $0.2 \leq \tau \leq 0.8$, we have either (1) $I(t+\tau) = 0$ with $I(t) = 1$, or (2) $I(t+\tau) = 1$ with $I(t) = 0$, thus their product must be $\langle I(t)\, I(t+\tau)\rangle = 0$ when $0.2 \leq \tau \leq 0.8$. Lastly, for $0 < \tau < 0.2$, by the periodicity of I, it is not hard to see that the time average of the product $\langle I(t)\, I(t+\tau)\rangle = 0$ decreases linearly from 0.2 to 0 as a function of τ. Then using periodicity and symmetry, we conclude

$$g^{(2)}(\tau) = \begin{cases} 1 - 5 \cdot |\tau - \mathrm{round}(\tau)| & |\tau - \mathrm{round}(\tau)| \leq 0.2 \\ 0 & |\tau - \mathrm{round}(\tau)| > 0.2 \end{cases}, \tag{*}$$

where round$(\cdot)$ is the function that rounds to the nearest integer. We calculate $g^{(2)}$ from the definition using I, and compare that with the result we obtain in Eq. (*).



The gray dashed lines are $t = 0.2$ and $t = 0.8$, respectively. Here $g^{(2)}(\tau)$ does not approach 0 as $\tau \to \infty$ as I is a periodic function of time, which implies $I(t)$ and $I(t+\tau)$ are for sure correlated.

Here we consider

$$E^{+}(t)=\frac{e^{-i\omega t}}{\pi^{1/4}}e^{-t^2/2}\,.$$

Thus we can compute

$$E^{-}(t)=(E^{+})^{*}=\frac{e^{i\omega t}}{\pi^{1/4}}e^{-t^2/2}\,,$$

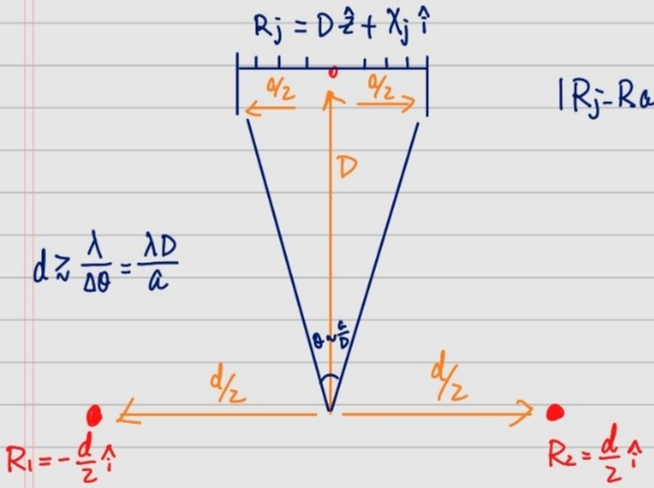
then we have

$$\begin{aligned}
g^{(1)}(\tau)=\frac{\langle E^{-}(t)\,E^{+}(t+\tau)\rangle}{\langle E^{-}(t)\,E^{+}(t)\rangle} &= \frac{\lim_{T\to\infty}\frac{1}{T\pi^{1/2}}\int_{-T/2}^{T/2}e^{-(t^2+(t+\tau)^2))/2}e^{-i\omega\tau}\,dt}{\lim_{T\to\infty}\frac{1}{T\pi^{1/2}}\int_{-T/2}^{T/2}e^{-t^2}\,dt}\\
&= \frac{e^{-i\omega\tau}\lim_{T\to\infty}\int_{-T/2}^{T/2}e^{-(t^2+(t+\tau)^2))/2}\,dt}{\lim_{T\to\infty}\int_{-T/2}^{T/2}e^{-t^2}\,dt}\\
&= \frac{e^{-i\omega\tau}}{\pi^{1/2}}\lim_{T\to\infty}\int_{-T/2}^{T/2}e^{-(t^2+(t+\tau)^2))/2}\,dt\\
&= e^{-i\omega\tau-\tau^2/4}\,.
\end{aligned}$$

While on the other hand,

$$\begin{aligned}
g^{(2)}(\tau) &= \frac{\langle E^{-}(t)\,E^{-}(t+\tau)\,E^{+}(t+\tau)\,E^{+}(t)\rangle}{\langle E^{-}(t)\,E^{+}(t)\rangle^2}\\
&= \frac{\langle e^{i\omega t-t^2/2}e^{i\omega(t+\tau)-(t+\tau)^2/2}e^{-i\omega t-t^2/2}e^{-i\omega(t+\tau)-(t+\tau)^2/2}\rangle}{\langle e^{-t^2}\rangle^2}\\
&= e^{-\tau^2}\frac{\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}e^{-2t^2-2\tau t}\,dt}{\lim_{T\to\infty}\frac{\pi}{T^2}}\\
&= \frac{\pi^{1/2}e^{\tau^2/2}e^{-\tau^2}}{2^{1/2}\pi}\frac{\lim_{T\to\infty}\frac{1}{T}}{\lim_{T\to\infty}\frac{1}{T^2}}\\
&= \frac{e^{-\tau^2/2}}{(2\pi)^{1/2}}\lim_{T\to\infty}T
\end{aligned}$$

which is an ill-defined quantity and obviously $g^{(2)}(0)=\infty$.



$$d \gtrsim \frac{\lambda}{\Delta\theta} = \frac{\lambda D}{a}$$

$$\begin{aligned}|R_j - R_a| &= \left(D^2 + (x_j \pm d/2)^2\right)^{1/2} \\ &= D\left(1 + \frac{(x_j \pm d/2)^2}{D^2}\right)^{1/2} \\ &\sim D\left(1 + \frac{(x_j \pm d/2)^2}{2D^2}\right) \\ &= D + \frac{(x_j \pm d/2)^2}{2D}\end{aligned}$$

Calculate $g^{(2)}(R_1, t\,;\, R_2, t) = \dfrac{\langle E_1^- E_2^- E_2^+ E_1^+ \rangle}{\langle E_1^- E_1^+ \rangle \langle E_2^- E_2^+ \rangle}$

at R_a

$$E_a^+(R_1, t) = \frac{1}{2} \sum_j E_0 e^{-i\omega t_j + i\phi_j(t - D/c)}$$

$$E_a^-(R_1, t) = \frac{1}{2} \sum_j E_0 e^{i\omega t_j - i\phi_j(t - D/c)}$$

With $t_j = t - |R_j - R_a|/c$

$r_j = |R_j - R_a|/c$

for simplicity. assume $c = 1$ $\Rightarrow$ $\omega \sim \frac{1}{\lambda}$

R_2 is $-$
R_1 is $+$

$$\Rightarrow r_j \sim D + \frac{(x_j \pm d/2)^2}{2D}$$

$$\langle E_a^- E_a^+ \rangle = \frac{E_0^2}{4} \left\langle \sum_j e^{i\omega t_j - i\phi_j(t-D/c)} \sum_j e^{-i\omega t_j + i\phi_j(t-D/c)} \right\rangle$$

$$= \frac{E_0^2}{4} \left\langle \sum_{j,k} e^{i\omega(t - r_j - t + r_k) - i(\phi_j(t-D/c) - \phi_k(t-D/c))} \right\rangle$$

$$= \frac{E_0^2}{4} \left\langle \sum_{j,k} e^{i\omega(r_k - r_j) - i(\phi_j(t-D/c) - \phi_k(t-D/c))} \right\rangle$$

$$= \frac{E_0^2 N}{4}$$

$$\langle E_1^- E_2^- E_2^+ E_1^+ \rangle$$

$$= \frac{E_0^4}{16} \left\langle \sum_j e^{-i\omega t_j^1 + i\phi_j^1(t-D/c)} \sum_{j'} e^{-i\omega t_{j'}^2 + i\phi_{j'}^2(t-D/c)} \right.$$

$$\left. \sum_k e^{i\omega t_k^1 - i\phi_k^1(t-D/c)} \sum_{k'} e^{i\omega t_{k'}^2 - i\phi_{k'}^2(t-D/c)} \right\rangle$$

$$= \frac{E_0^4}{16} \left\langle \sum_{j,j',k,k'} \exp\left(-i\omega(t - r_j^1 + t - r_{j'}^2 - t + r_k^1 - t + r_{k'}^2) + i(\phi_j^1 + \phi_{j'}^2 - \phi_k^1 - \phi_{k'}^2)\right) \right\rangle$$

$$= \frac{E_0^4}{16} \left\langle \sum_{j,j',k,k'} \exp\left(-i\omega(r_k^1 + r_{k'}^2 - r_j^1 - r_{j'}^2) + i(\phi_j^1 + \phi_{j'}^2 - \phi_k^1 - \phi_{k'}^2)\right) \right\rangle$$

count # of e^0

$j = j' = k = k' \Rightarrow N$

$j = k \neq j' = k' \Rightarrow N(N-1)$

$j = k' \neq j' = k \Rightarrow N(N-1)$

$$r_k^1 + r_{k'}^2 - r_j^1 - r_{j'}^2 = \frac{(x_k + d/2)^2}{2D} + \frac{(x_{k'} - d/2)^2}{2D} - \frac{(x_j + d/2)^2}{2D} - \frac{(x_{j'} - d/2)^2}{2D}$$

$$= (2D)^{-1}\left(x_k^2 - x_j^2 + x_{k'}^2 - x_{j'}^2 + d(x_k - x_{k'} - x_j + x_{j'})\right)$$