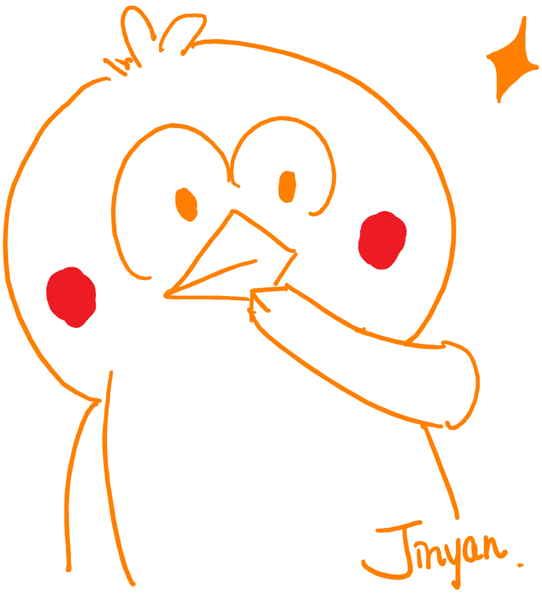
Homework 7

Physics 542 - Quantum Optics
Professor Alex Kuzmich



Jinyan Miao

Fall 2023

1

First we consider the case of $N = 2$ and $S = 1$, in which case we have (from Eq. (11a) on C. Genes and P. Berman, *Spin squeezing via atom-cavity field coupling*, Phys. Rev. A **68**, 043809 (2003),)

$$\langle \hat{S}_x^2 \rangle = \frac{1}{2} + \sum_{n=0}^{\infty} \left(\frac{1}{2} |c_{0,n}|^2 + c_{1,n} c_{-1,n} \right) , \tag{1.1}$$

where $c_{m,n}$ is the coefficient in the interaction representation of the state

$$|\psi(t)\rangle = \sum_{m=-N/2}^{N/2} \sum_{n=0}^{\infty} c_{m,k} e^{-i\omega(m+n)t} |m, n\rangle , \tag{1.2}$$

and here m labels the value of S_z and n labels the number of photons in the cavity field. Spin squeezing happens when we have

$$\xi_x = \frac{\sqrt{N\langle \hat{S}_x^2 \rangle}}{|\langle \mathbf{S} \rangle|} < 1 , \tag{1.3}$$

thus in this case we would like to have $\langle S_x^2 \rangle$ for spin squeezing, which is possible only when we have some $c_{1,n}c_{-1,n} < 0$ as $|c_{0,n}|$ is non-negative. From Eq. (21.57) from Berman's text *Principles of Laser Spectroscopy and Quantum Optics*, we know that

$$c_{1,n} c_{-1,n} \propto c_k c_{k+2} \tag{1.4}$$

where c_k are the initial state amplitudes for the field. Thus it is required to have $c_k c_{k+2} \neq 0$ in order for spin squeezing happening, thus requiring coherence between states differing by two. To generalize this effect to general N atoms system, it suffices to examine the form of $\hat{S}_x^2$ and the interaction Hamiltonian between the atoms and the field. The interaction Hamiltonian of the system is given by

$$\hat{H}_{\text{int}} = \hbar g \left(\hat{S}_+ \hat{a} + \hat{S}_- \hat{a}^\dagger \right) , \tag{1.5}$$

thus it is well-expected that the evolution of the wavefunction of the system, Eq. (1.2), satisfies $c_{m-1,k} \propto c_k$ and $c_{m+1,k} \propto c_{k+2}$, and thus Eq. (1.4) is again satisfied. Now from the form of $\hat{S}_x^2$, Eq. (21.41) from Berman's text *Principles of Laser Spectroscopy and Quantum Optics*,

$$\hat{S}_x^2 = \sum_{i,j=1}^{N} s_x^i s_x^j = \frac{N}{4} + \sum_{i,j\neq i}^{N} s_x^i s_x^j ,$$

it is expected that the only possible non-positive terms in $\langle S_x^2 \rangle$ are of the form $c_{m+1,n}c_{m-1,n}$. Following the argument as in the $N = 2$ case proves the result.

2

Here we will show that $[\hat{S}_x, \hat{S}_y] = i\hat{S}_z$, and $\hat{S}_z\hat{S}_+ \neq \hat{S}_+/2$.
For cleanness we drop the operator hat in the notations.

$$S_x S_y = \sum_{j=1}^{N} s_x^j \sum_{i=1}^{N} s_y^i = \sum_{i,j=1}^{N} s_x^j s_y^i \,, \qquad S_y S_x = \sum_{i,j=1}^{N} s_y^j s_x^i \,.$$

Here we have that

$$\begin{aligned} s_x s_y &= -\frac{i}{4}\left(s_+ + s_-\right)\left(s_+ - s_-\right) \\ &= -\frac{i}{4}(s_+ s_+ - s_+ s_- + s_- s_+ - s_- s_-) \\ &= -\frac{i}{4}(|2\rangle\langle 1|2\rangle\langle 1| - |2\rangle\langle 1|1\rangle\langle 2| + |1\rangle\langle 2|2\rangle\langle 1| - |1\rangle\langle 2|1\rangle\langle 2|) \\ &= \frac{i}{4}\left(|2\rangle\langle 2| - |1\rangle\langle 1|\right) \,. \end{aligned}$$

$$\begin{aligned} s_y s_x &= -\frac{i}{4}(s_+ - s_-)\left(s_+ + s_-\right) \\ &= -\frac{i}{4}\left(s_+ s_+ + s_+ s_- - s_- s_+ - s_- s_-\right) \\ &= \frac{i}{4}\left(s_- s_+ - s_+ s_-\right) \\ &= \frac{i}{4}\left(|1\rangle\langle 1| - |2\rangle\langle 2|\right) \,. \end{aligned}$$

Thus we conclude

$$[s_y, s_y] = \frac{i}{2}\left(|2\rangle\langle 2| - |1\rangle\langle 1|\right) = i s_z \,.$$

As single-spin operators only operate on that single spin, thus $[s_x^j, s_y^i] = 0$ for $i \neq j$. It follows that we have

$$[S_x, S_y] = S_x S_y - S_y S_x = \sum_{i,j=1}^{N} s_x^j s_y^i - s_y^j s_x^i = \sum_{i=1}^{N} s_x^i s_y^i - s_y^i s_x^i = \sum_{i=1}^{N} [s_x^i, s_y^i] = \sum_{i=1}^{N} i s_z = i S_z$$

Now we see here

$$s_z s_+ = \frac{1}{2}\left(|2\rangle\langle 2| - |1\rangle\langle 1|\right)|2\rangle\langle 1| = \frac{1}{2}\left(|2\rangle\langle 1|\right) = \frac{1}{2} s_+ \,.$$

While on the other hand

$$\frac{1}{2}S_+ = \frac{1}{2}\sum_{i=1}^{N} s_+^i \,,$$

but

$$\begin{aligned} S_z S_+ = \sum_{i=1}^{N} s_z^i \sum_{j=1}^{N} s_+^j = \sum_{i,j=1}^{N} s_z^i s_+^j &= \sum_{i=1}^{N} s_z^i s_+^i + \sum_{i \neq j}^{N} s_z^i s_+^j \\ &= \left(\frac{1}{2} \sum_{i=1}^{N} s_+ \right) + \sum_{i \neq j}^{N} s_z^i s_+^j = \frac{1}{2} S_+ + \sum_{i \neq j}^{N} s_z^i s_+^j \neq \frac{1}{2} S_+ \end{aligned}$$

as the term $\sum_{i \neq j}^{N} s_z^i s_+^j$ does not vanish.

3

Here Eq. (21.45) from Berman's text *Principles of Laser Spectroscopy and Quantum Optics* reads

$$\xi_x = \frac{\sqrt{N}\sqrt{\langle S_x^2\rangle}}{|\langle \mathbf{S}\rangle|}$$

Again, from Eq. (11a) on C. Genes and P. Berman, *Spin squeezing via atom-cavity field coupling*, Phys. Rev. A **68**, 043809 (2003), we have that

$$(\Delta S_x)^2 = \frac{1}{2} + \sum_{n=0}^{\infty}\left(\frac{1}{2}|c_{0,n}|^2 + c_{1,n}c_{-1,n}\right) ,$$

with Eq. (21.57) from Berman's text *Principles of Laser Spectroscopy and Quantum Optics*, we compute $(\Delta S_x)^2$ up to the order of α^2,

$$\begin{aligned}
\text{SUM} := (\Delta S_x)^2 - \frac{1}{2} &\approx \frac{1}{2}|c_{0,0}|^2 + c_{1,0}c_{-1,0} + \frac{1}{2}|c_{0,1}|^2 + c_{1,1}c_{-1,1} \\
&= \frac{1}{2}\alpha^2\sin^2(\sqrt{2}gt) - \left(1 - \frac{\alpha^2}{2}\right)\left(\frac{\alpha^2}{3}(1 - \cos(\sqrt{6}gt)\right) + \frac{1}{2}\frac{\alpha^4}{3}\sin^2(\sqrt{6}gt) \\
&= \frac{\alpha^2}{2}\sin^2(\sqrt{2}gt) - \frac{\alpha^2}{3}\left(1 - \cos(\sqrt{6}gt)\right) + \frac{\alpha^4}{6}\left(1 - \cos(\sqrt{6}gt)\right) + \frac{\alpha^4}{6}\sin^2(\sqrt{6}gt) \\
&= \frac{\alpha^2}{2}\sin^2(\sqrt{2}gt) + \frac{\alpha^2}{3}\cos(\sqrt{6}gt) - \frac{\alpha^4}{6}\cos(\sqrt{6}gt) - \frac{\alpha^2}{3} + \frac{\alpha^4}{6} + \frac{\alpha^4}{6}\sin^2(\sqrt{6}gt) \\
&\approx \frac{\alpha^2}{2}\sin^2(\sqrt{2}gt) + \frac{\alpha^2}{3}\cos(\sqrt{6}gt) - \frac{\alpha^2}{3} \\
&= \alpha^2\left(\frac{1}{2}\sin^2(\sqrt{2}gt) + \left(\frac{1}{3}\cos(\sqrt{6}gt) - \frac{1}{3}\right)\right) \\
&= \alpha^2\left(\frac{1}{2}\sin^2(\sqrt{2}gt) - \frac{2}{3}\sin^2(\sqrt{6}gt/2)\right) .
\end{aligned}$$

Now via binomial approximation, we have

$$\Delta S_x = \left(\frac{1}{2} + \text{SUM}\right)^{1/2} = \frac{1}{\sqrt{2}}\left(1 + 2\text{SUM}\right)^{1/2} \approx \frac{1}{\sqrt{2}}\left(1 + \text{SUM}\right) .$$

Furthermore, we have

$$
\begin{aligned}
|\langle S\rangle| &= \sqrt{\langle S_x\rangle^2 + \langle S_y\rangle^2 + \langle S_z\rangle^2} \\
&= \left(2\alpha^2 \sin^2(\sqrt{2}gt) + \left(1 - \alpha^2 \sin^2(\sqrt{2}gt)\right)^2\right)^{1/2} \\
&= \left(2\alpha^2 \sin^2(\sqrt{2}gt) + 1 + \alpha^4 \sin^4(\sqrt{2}gt) - 2\alpha^2 \sin^2(\sqrt{2}gt)\right)^{1/2} \\
&= \left(1 + \alpha^4 \sin^4(\sqrt{2}gt)\right)^{1/2} \\
&\approx 1 + \frac{\alpha^4}{2} \sin^4(\sqrt{2}gt) \approx 1\,.
\end{aligned}
$$

Combining we can write

$$
\xi_x = \sqrt{2}\frac{\Delta S_x}{|\langle S\rangle|} \approx 1 + \mathrm{SUM} = 1 + \alpha^2 \left(\frac{1}{2}\sin^2(\sqrt{2}gt) - \frac{2}{3}\sin^2(\sqrt{6}gt/2)\right)\,,
$$

as expected. Here $\xi_{y'}$ can be computed similarly via $(\Delta S_y)^2$ given by Eq. (11b) on C. Genes and P. Berman's paper,

$$
(\Delta S_y)^2 = \langle S_y^2\rangle - \langle S_y\rangle^2 = \frac{1}{2} + \sum_{n=0}^{\infty}\left(\frac{1}{2}|c_{0,n}|^2 + c_{1,n}c_{-1,n}\right) - \langle S_y\rangle^2\,,
$$

thus giving

$$
\begin{aligned}
\xi_{y'} &\sim 1 + \alpha^2 \left(\frac{1}{2}\sin^2(\sqrt{2}gt) + \frac{2}{3}\sin^2(\sqrt{6}gt/2)\right) - \alpha^2 \sin^2(\sqrt{2}gt) \\
&= 1 + \alpha^2 \left(-\frac{1}{2}\sin^2(\sqrt{2}gt) + \frac{2}{3}\sin^2(\sqrt{6}gt/2)\right)
\end{aligned}
$$

as expected.

P3

$$(\Delta S_x)^2 = \frac{1}{2} + \sum_{n=0}^{\infty}\left(\frac{1}{2}|C_{0,n}|^2 + C_{1,n}C_{-1,n}\right)$$

$\boxed{\text{Eq. (11a) on paper}}$

$$= \frac{1}{2} + \sum_{n=0}^{1}\left(\frac{1}{2}|C_{0,n}|^2 + C_{1,n}C_{-1,n}\right) + \sim 0$$

$$(\Delta S_x)^2 - \frac{1}{2} = \frac{1}{2}|C_{0,0}|^2 + C_{1,0}C_{-1,0} + \frac{1}{2}|C_{0,1}|^2 + \cancelto{\sim 0}{C_{1,1}C_{-1,1}}$$

$$= \frac{1}{2}\alpha^2\sin^2(\sqrt{2}gt) - (1-\alpha^2/2)\frac{\alpha^2}{3}\left(1-\cos(\sqrt{6}gt)\right) + \frac{1}{2}\frac{\alpha^4}{3}\sin^2(\sqrt{6}gt)$$

(sum)

$$= \frac{1}{2}\alpha^2\sin^2(\sqrt{2}gt) - \frac{\alpha^2}{3}\left(1-\cos(\sqrt{6}gt)\right) + \frac{\alpha^4}{6}\left(1-\cos(\sqrt{6}gt)\right) + \frac{1}{2}\frac{\alpha^4}{3}\sin^2(\sqrt{6}gt)$$

$$= \frac{1}{2}\alpha^2\sin^2(\sqrt{2}gt) + \frac{\alpha^2}{3}\cos(\sqrt{6}gt) - \cancelto{\sim 0}{\frac{\alpha^4}{6}\cos(\sqrt{6}gt)} - \frac{\alpha^2}{3} + \cancel{\frac{\alpha^4}{6}} + \cancelto{\sim 0}{\frac{\alpha^4}{6}\sin^2(\sqrt{6}gt)}$$

ignored term of order α^4

$$|\langle S\rangle| = \sqrt{\langle S_x\rangle^2 + \langle S_y\rangle^2 + \langle S_z\rangle^2}$$

$$= \left(2\alpha^2\sin^2(\sqrt{2}gt) + \left(1-\alpha^2\sin^2(\sqrt{2}gt)\right)^2\right)^{1/2}$$

$$= \left(\cancel{2\alpha^2\sin^2(\sqrt{2}gt)} + 1 + \alpha^4\sin^4(\sqrt{2}gt) - \cancel{2\alpha^2\sin^2(\sqrt{2}gt)}\right)^{1/2}$$

$$= \left(1+\alpha^4\sin^4(\sqrt{2}gt)\right)^{1/2}$$

$$\sim 1 + \cancelto{\sim 0}{\frac{\alpha^4}{2}\sin^4(\sqrt{2}gt)} \quad \Rightarrow \quad |\langle S\rangle| \sim 1$$

$$\Delta S_x = \left(\tfrac{1}{2} + \mathrm{SUM}\right)^{1/2} = \left(\tfrac{1}{2}\left(1 + 2\,\mathrm{SUM}\right)\right)^{1/2} = \tfrac{1}{\sqrt{2}}\left(1 + 2\,\mathrm{SUM}\right)^{1/2} \sim \tfrac{1}{\sqrt{2}}\left(1 + \mathrm{SUM}\right)$$

$\Rightarrow$ approximations leads to $\boxed{\xi_x = \sqrt{2}\,\dfrac{\Delta S_x}{|\langle S \rangle|} \sim 1 + \mathrm{SUM}}$

Now employ trig-identity

from SUM. $\dfrac{\alpha^2}{3}\left(\cos(\sqrt{6}gt) - 1\right) = \dfrac{\alpha^2}{3}\left(1 - 2\sin^2(\sqrt{6}gt/2) - 1\right)$

$$= -\alpha^2 \tfrac{2}{3}\sin^2(\sqrt{6}gt/2)$$

Combining all

$$\xi_x = \sqrt{2}\,\frac{\Delta S_x}{|\langle S \rangle|} \sim 1 + \mathrm{SUM} \sim 1 + \tfrac{1}{2}\alpha^2 \sin^2(\sqrt{2}gt) - \alpha^2 \tfrac{2}{3}\sin^2(\sqrt{6}gt/2)$$

$$\Rightarrow \boxed{\xi_x \sim 1 + \alpha^2\left(\tfrac{1}{2}\sin^2(\sqrt{2}gt) - \tfrac{2}{3}\sin^2(\sqrt{6}gt/2)\right)}$$

The calculation of $\xi_{y'}$ follows similarly using Eq. (11b)

$\sqrt{2}gt$ term

$$(\Delta S_y)^2 = \langle S_y^2 \rangle - \langle S_y \rangle^2 = \frac{1}{2} + \sum_{n=0}^{\infty}\left(\frac{1}{2}|C_{0,n}|^2 - C_{1,n}C_{-1,n}\right) - \langle S_y \rangle^2$$

minus $\sqrt{6}gt/2$ term

$$\Rightarrow \quad \xi_{y'} \sim 1 + \alpha^2\left(\tfrac{1}{2}\sin^2(\sqrt{2}gt) + \tfrac{2}{3}\sin^2(\sqrt{6}gt/2)\right) - \alpha^2\sin^2(\sqrt{2}gt)$$

$$\boxed{\xi_{y'} \sim 1 + \alpha^2\left(-\tfrac{1}{2}\sin^2(\sqrt{2}gt) + \tfrac{2}{3}\sin^2(\sqrt{6}gt/2)\right)}$$