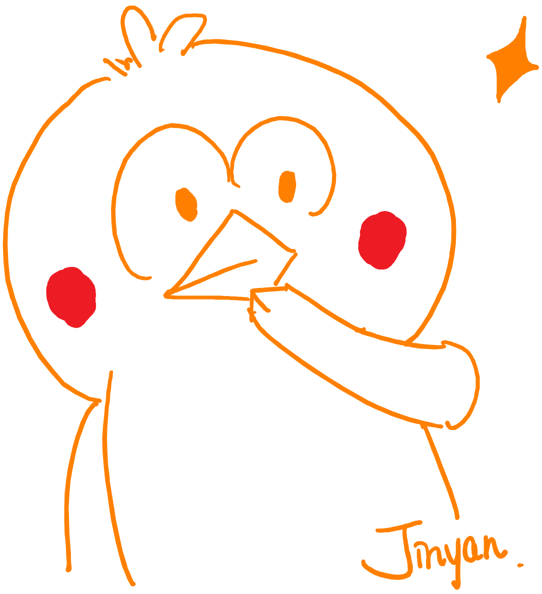
Homework 4

Physics 542 - Quantum Optics
Professor Alex Kuzmich



Jinyan Miao

Fall 2023

1

Here we suppose that the right eigenstate of the creation operator exists, that is, we write

$$\hat{a}^\dagger|m\rangle = m|m\rangle\,,$$

for some $m \in \mathbb{C}$. We can write $|m\rangle$ as superposition of the usual basis,

$$|m\rangle = \sum_{n=0}^{\infty} c_n|n\rangle\,.$$

Then we must have

$$\hat{a}^\dagger|m\rangle = \sum_{n=0}^{\infty} c_n\,\hat{a}^\dagger|n\rangle = \sum_{n=0}^{\infty} c_n\sqrt{n+1}\,|n+1\rangle\,.$$

On the other hand,

$$\hat{a}^\dagger|m\rangle = m|m\rangle = \sum_{n=0}^{\infty} c_n m|n\rangle = \sum_{n=0}^{\infty} c_n\sqrt{n+1}\,|n+1\rangle\,.$$

As $\langle n_1|n_2\rangle = 0$ for $n_1 \neq n_2$, thus we conclude that we have $c_0 = 0$, and

$$c_{n+1} = \frac{c_n\sqrt{n+1}}{m}\,.$$

As $c_0 = 0$, we thus require $c_n = 0$ for all $n \in \mathbb{N}$, so we have $|m\rangle = 0$, such a state does not exist.

2

For coherent state $|\beta\rangle$, we first compute

$$C_N(\lambda) = \mathrm{tr}(\hat{\rho} e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}}) = \langle \beta | e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}} | \beta \rangle = e^{\lambda^* \hat{a}} \langle \beta | e^{-\lambda^* \hat{a}} | \beta \rangle = \langle e^{\lambda^* \beta} \beta | e^{-\lambda^* \beta} \beta \rangle = e^{\lambda \beta^* - \lambda^* \beta} . \tag{2.1}$$

Now we compute

$$\begin{aligned}
W(\alpha) &= \frac{1}{\pi^2} \int e^{\lambda^* \alpha - \lambda \alpha^*} \, C_N(\lambda) \, e^{-|\lambda|^2/2} \, d^2\lambda \\
&= \frac{1}{\pi^2} \int e^{\lambda^* \alpha - \lambda \alpha^*} \, e^{\lambda \beta^* - \lambda^* \beta} \, e^{-|\lambda|^2/2} \, d^2\lambda \\
&= \frac{1}{\pi^2} \int e^{\lambda^* \alpha - \lambda \alpha^* + \lambda \beta^* - \lambda^* \beta - |\lambda|^2/2} \, d^2\lambda \\
&= \frac{1}{\pi^2} \int e^{\lambda^* (\alpha - \beta) - \lambda (\alpha^* - \beta^*) - |\lambda|^2/2} \, d^2\lambda \\
&= \frac{1}{\pi^2} \int_0^\infty \int_0^{2\pi} \exp(rse^{-i(\theta-\phi)} - rse^{i(\theta-\phi)} - r^2/2) \, r \, d\theta \, dr \\
&= \frac{1}{\pi^2} \int_0^\infty \int_0^{2\pi} e^{-2irs \sin(\theta - \phi)} \, d\theta \, e^{-r^2/2} \, r \, dr \\
&= \frac{2}{\pi} \int_0^\infty J_0(2rs) \, e^{-r^2/2} \, r \, dr \\
&= \frac{2}{\pi} \, e^{-2s^2} \\
&= \frac{2}{\pi} e^{-2|\alpha - \beta|^2} ,
\end{aligned}$$

where we have abbreviated $\lambda = re^{i\theta}$ and $\alpha - \beta = se^{i\phi}$ and the last integral is computed via Eq. (4) in Erdelyi's *Tables of Integral Transforms* vol II, p. 13.

Now we consider the number state $|n\rangle$. We first compute

$$\begin{aligned}
C_N(\lambda) = \langle n|e^{\lambda\hat{a}^\dagger} e^{-\lambda^*\hat{a}}|n\rangle &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\langle n|\frac{\lambda^i(\hat{a}^\dagger)^i}{i!}\frac{(-1)^j(\lambda^*)^j\hat{a}^j}{j!}|n\rangle \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j\lambda^i(\lambda^*)^j}{i!\,j!}\langle n|(\hat{a}^\dagger)^i\hat{a}^j|n\rangle \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^j\lambda^i(\lambda^*)^j}{i!\,j!}\hat{a}^i\langle n|\hat{a}^j|n\rangle \\
&= \sum_{i=0}^{n}\frac{(-1)^i|\lambda|^{2i}}{i!\,i!}\frac{n!}{(n-i)!} \\
&= L_n(|\lambda|^2)\,.
\end{aligned}$$

Then we compute

$$\begin{aligned}
W(\alpha) &= \frac{1}{\pi^2}\int e^{\lambda^*\alpha-\lambda\alpha^*}L_n(|\lambda|^2)\,e^{-|\lambda|^2/2}\,d^2\lambda \\
&= \frac{1}{\pi^2}\int e^{\lambda^*\alpha-\lambda\alpha^*-|\lambda|^2/2}L_n(|\lambda|^2)\,d^2\lambda \\
&= \frac{1}{\pi^2}\int_0^{\infty}\int_0^{2\pi} L_n(r^2)\,\exp(rse^{-i(\theta-\phi)}-rse^{i(\theta-\phi)}-r^2/2)\,r\,d\theta\,dr \\
&= \frac{1}{\pi^2}\int_0^{\infty}\int_0^{2\pi} e^{-2irs\sin(\theta-\phi)}\,d\theta\,L_n(r^2)e^{-r^2/2}\,r\,dr \\
&= \frac{2}{\pi}\int_0^{\infty} J_0(2rs)\,L_n(r^2)\,e^{-r^2/2}\,r\,dr \\
&= (-1)^n\frac{2}{\pi}\,e^{-2s^2}\,L_n(4s^2) \\
&= (-1)^n\frac{2}{\pi}\,e^{-2|\alpha|^2}\,L_n(4|\alpha|^2)\,,
\end{aligned}$$

where we have abbreviated $\lambda = re^{i\theta}$ and $\alpha = se^{i\phi}$ and the last integral is computed via Eq. (4) in Erdelyi's *Tables of Integral Transforms* vol II, p. 13.

3

Here we consider the superposition of coherent states $|\pm\beta\rangle$,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\beta\rangle + |-\beta\rangle\right) .$$

(a) Here we see that we have

$$\langle\Psi|\Psi\rangle = \frac{1}{2}\left(\langle\beta|\beta\rangle + \langle-\beta|-\beta\rangle + \langle-\beta|\beta\rangle + \langle\beta|-\beta\rangle\right) = \frac{1}{2}\left(2 + 2e^{-2|\beta|^2}\right) .$$

For large enough $|\beta|^2 \gg 1$, $e^{-2|\beta|^2} \sim 0$, it follows that $\langle\Psi|\Psi\rangle = 1$.

(b) Here we simply compute

$$\langle n|\Psi\rangle = \frac{1}{\sqrt{2}}\left(\langle n|\beta\rangle - \langle n|-\beta\rangle\right) = \frac{e^{-|\beta|^2/2}}{\sqrt{2}}\left(\frac{\beta^n}{\sqrt{n!}} + \frac{(-\beta)^n}{\sqrt{n!}}\right) = \frac{e^{-|\beta|^2/2}}{\sqrt{2}}\frac{\beta^n}{\sqrt{n!}}\left(1 + (-1)^n\right)$$

Thus we have

$$\mathbb{P}(n) = \frac{e^{-|\beta|^2}|\beta|^{2n}}{2(n!)}\left(1 + (-1)^n\right)^2$$

(d) Here we calculate the Q function,

$$\begin{aligned}
Q(\alpha) &= \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle \\
&= \frac{1}{2\pi}\langle\alpha|\left(|\beta\rangle + |-\beta\rangle\right)\left(\langle\beta| + \langle-\beta|\right)|\alpha\rangle \\
&= \frac{1}{2\pi}\left(e^{-|\beta|^2/2-|\alpha|^2/2+\alpha^*\beta} + e^{-|\beta|^2/2-|\alpha|^2/2-\alpha^*\beta}\right)\left(e^{-|\alpha|^2/2-|\beta|^2/2+\beta^*\alpha} + e^{-|\alpha|^2/2-|\beta|^2/2-\beta^*\alpha}\right) \\
&= \frac{1}{2\pi}\left(e^{-|\beta|^2/2-|\alpha|^2/2}(e^{\alpha^*\beta} + e^{-\alpha^*\beta})\right)\left(e^{-|\alpha|^2/2-|\beta|^2/2}(e^{\beta^*\alpha} + e^{-\beta^*\alpha})\right) \\
&= \frac{1}{2\pi}e^{-|\alpha|^2-|\beta|^2}\left(e^{\alpha^*\beta} + e^{-\alpha^*\beta}\right)\left(e^{\beta^*\alpha} + e^{-\beta^*\alpha}\right) \\
&= \frac{1}{2\pi}e^{-|\alpha|^2-|\beta|^2}|e^{\beta^*\alpha} + e^{-\beta^*\alpha}|^2 .
\end{aligned}$$

Now we calculate

$$\begin{aligned}
C_N(\lambda) = \langle\Psi|e^{\lambda\hat{a}^\dagger}e^{-\lambda^*\hat{a}}|\Psi\rangle &= \frac{1}{2}\left(\langle\beta| + \langle-\beta|\right)e^{\lambda\hat{a}^\dagger}e^{-\lambda^*\hat{a}}\left(|\beta\rangle + |-\beta\rangle\right) \\
&= \frac{1}{2}\left(e^{\lambda\beta^*-\lambda^*\beta} + e^{-\lambda\beta^*-\lambda^*\beta-2|\beta|^2} + e^{\lambda\beta^*+\lambda^*\beta-2|\beta|^2} + e^{-\lambda\beta^*+\lambda^*\beta}\right) ,
\end{aligned}$$

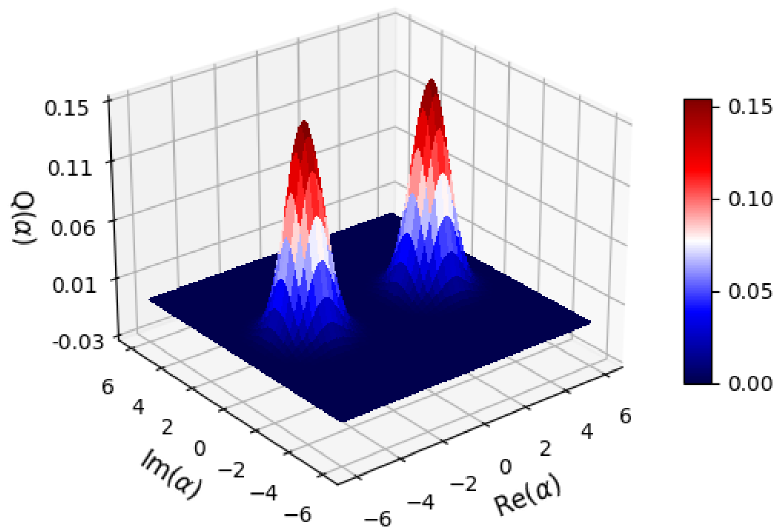
where the last equality comes from a similar computation as in Eq. (2.1) in this text. Now we can compute

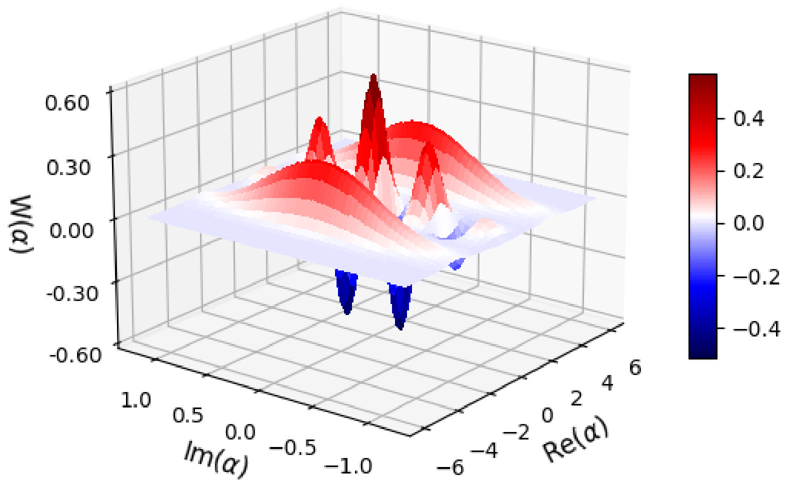
$$\begin{aligned}
W(\alpha) &= \frac{1}{\pi^2}\int e^{\lambda^*\alpha-\lambda\alpha^*}\,C_N(\lambda)\,e^{-|\lambda|^2/2}\,d^2\lambda \\
&= \frac{1}{2\pi^2}\int e^{\lambda^*\alpha-\lambda\alpha^*}\,\left(e^{\lambda\beta^*-\lambda^*\beta}+e^{-\lambda\beta^*-\lambda^*\beta-2|\beta|^2}+e^{\lambda\beta^*+\lambda^*\beta-2|\beta|^2}+e^{-\lambda\beta^*+\lambda^*\beta}\right)\,e^{-|\lambda|^2/2}\,d^2\lambda \\
&= \frac{1}{2}\left(\frac{2}{\pi}e^{-2|\alpha-\beta|^2}+\frac{2}{\pi}e^{-2|\alpha+\beta|^2}+e^{-2|\alpha|^2}\left(\frac{2}{\pi}e^{-2(\beta\alpha^*-\alpha\beta^*)}+\frac{2}{\pi}e^{-2(-\beta\alpha^*+\alpha\beta^*)}\right)\right) \\
&= \frac{1}{\pi}\left(e^{-2|\alpha-\beta|^2}+e^{-2|\alpha+\beta|^2}+e^{-2|\alpha|^2}\left(e^{-2(\beta\alpha^*-\alpha\beta^*)}+e^{-2(-\beta\alpha^*+\alpha\beta^*)}\right)\right) \\
&= \frac{1}{\pi}\left(e^{-2|\alpha-\beta|^2}+e^{-2|\alpha+\beta|^2}+2e^{-2|\alpha|^2}e^{\Re(-2(\beta\alpha^*-\alpha\beta^*))}\cos(-2\Im(\beta\alpha^*-\alpha\beta^*))\right) \\
&= \frac{1}{\pi}\left(e^{-2|\alpha-\beta|^2}+e^{-2|\alpha+\beta|^2}+2e^{-2|\alpha|^2}\cos(2|\beta\alpha^*-\alpha\beta^*|)\right)\,.
\end{aligned}$$

The plots of $W(\alpha)$ and $Q(\alpha)$ are attached on the next page. Clearly $|\Psi\rangle$ is not classical as $W(\alpha)$ has negative value in some regions as seen from the plots.

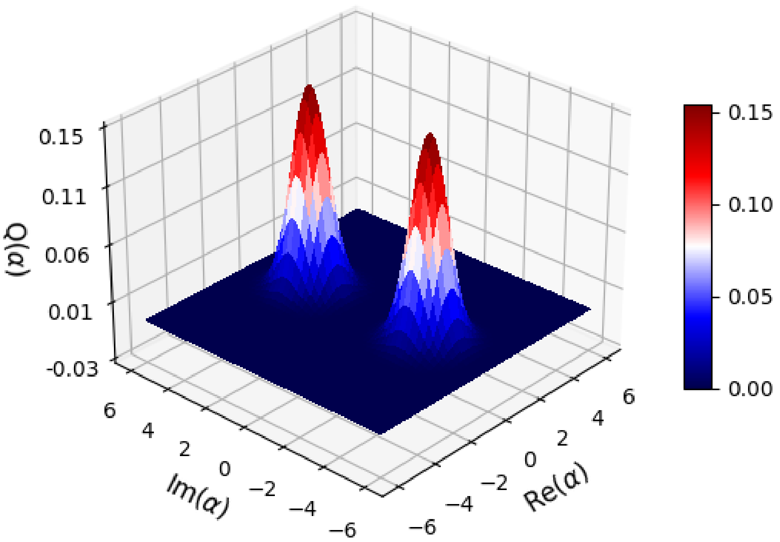
Here we attach the plot of $Q(\alpha)$ and $W(\alpha)$ for $\beta_1 = 3$ and $\beta_2 = 3i$, and $\beta_3 = 3(1+i)/2$.

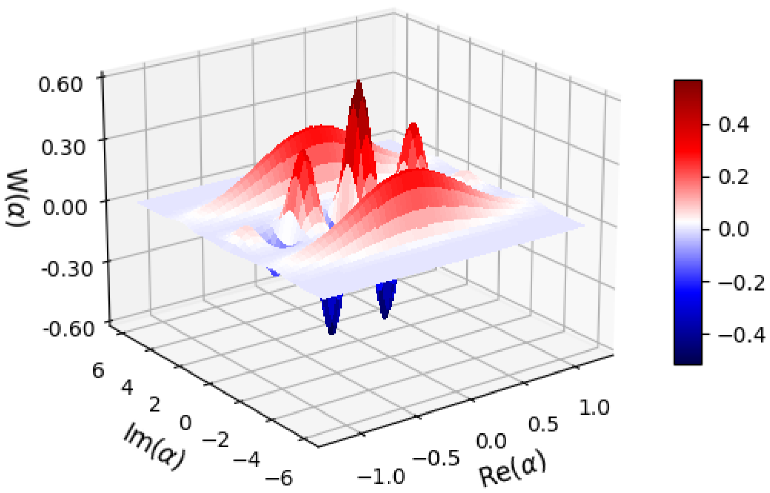
(1) $\beta_1 = 3$. Note that the $\Im(\alpha)$-axis is re-scaled for the $W(\alpha)$ plot.



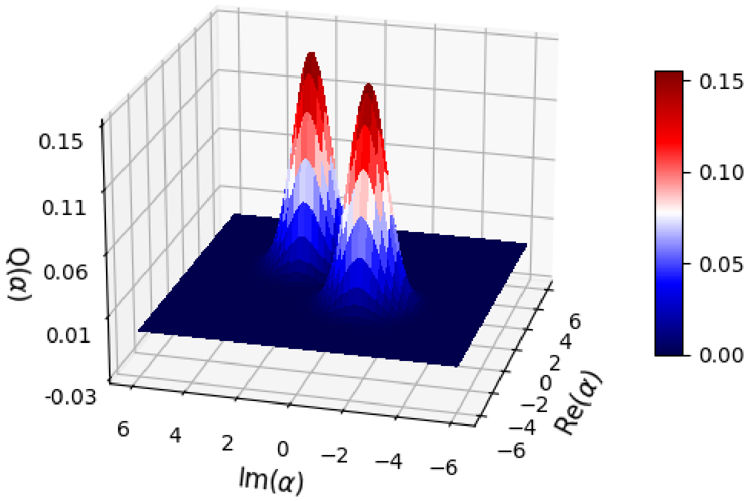


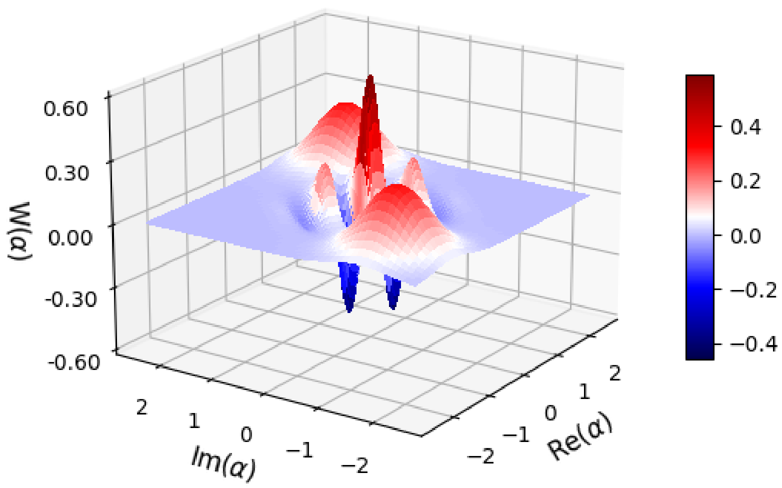
(2) $\beta_2 = 3i$. Note that the $\Re(\alpha)$-axis is re-scaled for the $W(\alpha)$ plot.





(3) $\beta_3 = 3(1+i)/2$.


0.15
0.11
0.06
0.01
-0.03
Q(α)
6
4
2
0
−2
−4
−6
Im(α)
Re(α)
0.15
0.10
0.05
0.00


0.60
0.30
0.00
-0.30
-0.60
W(α)
2
1
0
−1
−2
Im(α)
Re(α)
0.4
0.2
0.0
−0.2
−0.4