Introduction to Action Principle

and its applications

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1D Lagrangian & Hamiltonian Recap

- Let M be a manifold with metric g.
- The 1D time-dependent Lagrangian is a smooth function:
- $$\mathscr{L} : \mathbb{R} \times TM \to \mathbb{R}$$
- And, the 1D time-dependent Hamiltonian is a smooth function:
- $$\mathscr{H} : \mathbb{R} \times T^*M \to \mathbb{R}$$
- A classical example of above Lagrangian:
- $$\mathscr{L}(t, x, v) = \frac{1}{2} g_x(v, v) - V(t, x)$$

Generalized Lagrangian

- Let M, N be two manifolds of dimension m, n respectively. We denote the configuration space as:
- $$C = M \times N$$
- and, the space of *canonical velocity* as:
- $$V = \bigcup_{(x,q)\in C} L(T_xM, T_qN), \quad (\dim(V) = m + n + mn)$$

Bundle Structure Recap

- A C^∞ fiber bundle is the structure that consists of:
  - three C^∞ manifolds
    - E, called the *total space* of the bundle
    - M, called the *base space* of the bundle
    - F, called the *standard fiber* of the bundle
  - a surjective C^∞ map $\pi : E \to M$, called the *projection*
  - an open covering $\mathscr{U}$ of M, and for each $U \in \mathscr{U}$ a C^∞ map $\varphi : \pi^{-1}U \to F$ such that the map $(\pi, \varphi) : \pi^{-1}U \to U \times F$ is a diffeomorphism; the map (π, φ) is called a *bundle chart* on E over U.
- A differentiable section $\sigma : M \to E$ of the bundle satisfies that $\pi \circ \sigma = \mathrm{Id}$.

Bundle Structure Recap Continued

- Let $h : N \to M$ be a C^∞ map. The pullback of a bundle $F \to E \xrightarrow{\pi} M$ is:
- $$h^*E = \{(p, \xi) \in N \times E \,|\, h(p) = \pi(\xi)\} = \bigcup_{p \in M} \{p\} \times \pi^{-1}(h(p))$$
- A classical example of the pullback function is $h = \pi_{A,B}$, which is projection function from A to B.

Generalized Lagrangian Picture

- Lets denote $\wedge_k M = \cup_{x \in M} \wedge_k (T_x M)$, where $\wedge_k$ is the space of totally antisymmetric k-linear forms on $T_x M$.
- We define the Lagrangian $\mathscr{L}$ as a differentiable section of the pullback bundle:
- $$\pi^*_{V,M} \wedge_m M$$

Generalized Lagrangian Picture Continued

- We define the Lagrangian $\mathscr{L}$ as a differentiable section of the pullback bundle:
- $$\pi^*_{V,M} \wedge_m M = \bigcup_{v \in V} \{v\} \times \pi^{-1}_{\wedge_m M, M}(\pi_{V,M}(v))$$
- And, with $u : M \to N$, we see that $du\,\big|_x : T_x M \to T_{u(x)} N$. So $du(x)$ is a section of $(V, \pi_{V,M})$.

Generalized Action

- Now, we have: $du(x) : M \to V, \mathscr{L} : V \to \pi^*_{V,M} \wedge_m M$
- With $\mathscr{L}$ and $du(x)$ in mind, we could define the action of u as:
- $$\mathscr{A}_{\mathscr{L}}[u; D] = \int_D \mathscr{L} \circ du$$
- where D is any open sets with compact closure in M.

A Little More Concrete Example of Action

- Consider Φ be a section of the bundle $\wedge_m M \overset{\pi_{\wedge_m M, M}}{\rightarrow} M$. And, let $u : \mathbb{R}^m \to M$ be a smooth map. Then, we can define the action:
- $$\mathscr{A}_{\mathscr{L}}[u; D] = \int_D \Phi_{u(x)}(\frac{\partial u}{\partial x^1}(x), \cdots, \frac{\partial u}{\partial x^m}(x)))dx^1 \wedge \cdots \wedge dx^n$$
- We see that our previous example is also an example of above Φ with minor modification:
- $$\Phi(p)v = \frac{1}{2}g_p(v, v) - V(p)$$

Euler-Lagrange Equations

- The simple Euler-Lagrange equations:
- $$\frac{\partial \mathscr{L}}{\partial q^i}(t, q(t), \dot{q}(t)) - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}^i}(t, q(t), \dot{q}(t)) = 0$$

- The generalized Euler-Lagrange equation:
- $$\frac{\partial \mathscr{L}}{\partial q^k}(v) - \frac{\partial}{\partial x^l}\frac{\partial \mathscr{L}}{\partial v^k_l}(v) = 0$$

Canonical Momentum

- We define the $C-$vertical derivative of $\mathscr{L}$ at v_0, where $\pi_{V,C}(v_0) = (x, q)$, to be:
- $$\frac{\partial \mathscr{L}}{\partial v}(v_0)v_1 = \lim_{t \to 0} \frac{\mathscr{L}(v_0 + tv_1) - \mathscr{L}(v_0)}{t}$$
- for $v_1 \in \pi_{V,C}^{-1}(x, q) = L(T_xM, T_qN)$.
- If we consider the restriction $\mathscr{L} : L(T_xM, T_qN) \mapsto \wedge_m (T_xM)$, then:
- $$\frac{\partial \mathscr{L}}{\partial v}(v_0) \in L(L(T_xM, T_qN), \wedge_m (T_xM))$$

Canonical Momentum Continued

- If we consider the restriction $\mathscr{L} : L(T_xM, T_qN) \mapsto \wedge_m (T_xM)$, then:
- $$\frac{\partial \mathscr{L}}{\partial v}(v_0) \in L(L(T_xM, T_qN), \wedge_m (T_xM))$$
- We show that $L(T_qN, \wedge_{m-1} M) \cong L(L(T_xM, T_qN), \wedge_m M)$ by the isomorphism:
- $$(i\alpha)v_1(X_1, \cdots, X_m) = \sum_{i=1}^{m} (-1)^{i-1}\alpha(\dot{v}(X_i))(X_1, \cdots, \langle X_i \rangle, \cdots X_m), \quad \forall, X_1, \cdots, X_m \in T_xM$$
- where $\alpha \in L(T_qN, \wedge_{m-1} M), v_1 \in L(T_xM, T_qN)$.

Canonical Momentum Continued

- So, we have now recognize our $\dfrac{\partial \mathscr{L}}{\partial v}$ as element of $L(T_qN, \wedge_{m-1} M) \cong L(L(T_xM, T_qN), \wedge_m M)$. We make one final simplification by denoting:
- $$\wedge_{k,l}(M, N) = \bigcup_{(x,q)\in C} L(\wedge^k (T_xM), \wedge_l (T_qN)) = \bigcup_{(x,q)\in C} L(\wedge^l (T_qN), \wedge_k (T_xM))$$
- where $\wedge^k (V)$ is the totally antisymmetric $k-$fold tensor product of V with itself.

- Therefore, we finally have identified $\dfrac{\partial \mathscr{L}}{\partial v}$ to be a section of $\pi^*_{V,C} \wedge_{m-1,1} (M, N)$. And, we define the ***canonical momentum*** to be the section:
- $$p = \frac{\partial \mathscr{L}}{\partial v}$$

Euler-Lagrange Equations

- The generalized Euler-Lagrange equation:
- $$\frac{\partial \mathscr{L}}{\partial q^k}(v) - \frac{\partial}{\partial x^l}\frac{\partial \mathscr{L}}{\partial v_l^k}(v) = 0$$

Canonical Force

- We want to define a $N-$horizontal derivative of $\mathscr{L}$. This requires a choice of connection A on TN with associated covariant derivative D. Let also assume it to be symmetric:
- $$D_Y Z - D_Z Y = [Y, Z]$$
- for Y, Z are vectors fields on N.
- Now, consider a curve $\gamma : (-1,1) \to N$ through q by $\gamma(0) = q$ with tangent vector $\dot{\gamma}(0) = Q$. We can define a curve $\gamma_{TN}^{\#Q'}(-1,1) \to TN$ where $\gamma_{TN}^{\#Q'}(0) = Q' \in T_q N.$ such that:
- $$\pi_{TN,N} \circ \gamma_{TN}^{\#Q'} = \gamma$$
- This is called the *horizontal lift* of γ to TN through Q'. And, the tangent vector $\dot{\gamma}_{TN}^{\#Q'}(0)$ is called the *horizontal lift* to TN through Q' of the vector $Q \in T_q N$, and it is denoted by $Q_{TN}^{\#Q'}$

Canonical Force Continued

- Again, let $\gamma : (-1,1) \to N$ be a curve in N through $q = \gamma(0)$ with $Q = \dot{\gamma}(0)$. We now define a lift to V.
- We can define a curve $\gamma_V^{\#v} : (-1,1) \to V$ through $v = \gamma_V^{\#v}(0)$ by requiring that:
  - $\pi_{V,N} \circ \gamma_V^{\#v} = \gamma$
  - For some fixed $\pi_{V,M}(v) = x \in M$, we have $\pi_{V,M} \circ \gamma_V^{\#v} = x$
  - For each $X \in T_x M$, we must have: $\gamma_V^{\#v} \cdot X = \gamma_{TN}^{\#v \cdot X}$
- With these tools, we define the $N-$horizontal derivative of $\mathscr{L}$ at v relative to the connection A be the element
- $$D\mathscr{L}(v) \in L(T_q N, \wedge_m (T_x M)) \text{ where } (x, q) = \pi_{V,C}(v)$$
- $$D\mathscr{L}(v) \cdot Q = \frac{d}{dt}\Big|_{t=0} \mathscr{L}(\gamma_V^{\#v}(t)), \forall Q \in T_q N$$
- where γ is any curve in N through q with $\dot{\gamma}(0) = Q$.

Canonical Force Continued

- With these tools, we define the $N-$horizontal derivative of $\mathscr{L}$ at v relative to the connection A be the element
- $$D\mathscr{L}(v) \in L(T_qN, \wedge_m (T_xM)) \text{ where } (x, q) = \pi_{V,C}(v)$$
- $$D\mathscr{L}(v) \cdot Q = \frac{d}{dt}|_{t=0} \mathscr{L}(\gamma_V^{\#v}(t)), \forall Q \in T_qN$$
- where γ is any curve in N through q with $\dot{\gamma}(0) = Q$.
- We see from above that $DL(v) \in \wedge_{m,1} (M, N)$, and DL is a differential section of the bundle $\pi^*_{V,C} \wedge_{m,1} (M, N)$. And, we define the *canonical force* f to be the section:
- $$f = DL$$

Application of Generalized Euler-Lagrange Equations

- *Harmonic Functions*
  - Lets take our manifold $M = \mathbb{R}^m, N = \mathbb{R}$, and we define
  - $$\mathscr{L}(\mathbf{x}, q, v_1, \cdots, v_m)(X^1, \cdots, X^m)\cdots = \frac{1}{2}\sum_{i=1}^{m} v_i^2 \omega(X^i, \cdots, X^m)$$
  - where ω is the volume form.
  - Then, the action is:
  - $$\mathscr{A}_{\mathscr{L}}[u; D] = \int_D \mathscr{L}(\mathbf{x}, u(\mathbf{x}), du(\mathbf{x})) = \frac{1}{2}\int_D \sum_{i=1}^{m} (\frac{\partial u}{\partial \mathbf{x}^i})^2 d\mathbf{x} = \frac{1}{2}\int_D ||\nabla u||^2 d\mathbf{x}$$

Application of Generalized Euler-Lagrange Equations

- *Harmonic Functions*
  - Then, the action is:
  - $$\mathscr{A}_{\mathscr{L}}[u;D] = \int_D \mathscr{L}(\mathbf{x}, u(\mathbf{x}), du(\mathbf{x})) = \frac{1}{2}\int_D \sum_{i=1}^{m} (\frac{\partial u}{\partial \mathbf{x}^i})^2 d\mathbf{x} = \frac{1}{2}\int_D ||\nabla u||^2 d\mathbf{x}$$
  - We find that our $\mathscr{L}$ does not change with respect to x and we obtain the relations:
  - $$\frac{\partial \mathscr{L}}{\partial x}(x, u(x), v_1, \cdots, v_m) = 0, \quad \frac{\partial \mathscr{L}}{\partial v_i}(x, u(x), v_1, \cdots, v_m) = v_i$$
  - Then, our function u satisfies the Euler-Lagrangian equation only when:
  - $$0 = \frac{\partial}{\partial x^l}\frac{\partial \mathscr{L}}{\partial v_l}(\mathbf{x}, u(\mathbf{x}), v_1, \cdots, v_m) - \frac{\partial \mathscr{L}}{\partial u}(\mathbf{x}, u(\mathbf{x}), v_1, \cdots, v_m) = \sum_{i=1}^{m} \frac{\partial^2 u}{(\partial x^i)^2}(\mathbf{x}) = \Delta u(\mathbf{x})$$

Background for Noether's Theorem

- Lets take a even more simplified setting:
  - Recap: $\mathscr{L}(v) \in \wedge_m (T_x M), \pi_{V,M}(v) = x$
  - We have seen that we could define something simple of the form:
  - $$\mathscr{L}(v) = \mathscr{L}^*(v)\omega$$
  - for $v \in V$ and $\mathscr{L}^* : V \to \mathbb{R}$.
  - Let us focus on the simplified $\mathscr{L}^*$ only.
  - We will then assume our $M = \mathbb{R}^m$ for the rest of the discussion.
  - We will also assume that L^* only depends on q, φ for $v = (x, q, \varphi)$ with $x \in M, q \in N, \varphi \in L(T_x M, T_q N)$. Essentially, this means that $\mathscr{L}^* : TQ^{\oplus m} \to \mathbb{R}$ where $TQ^{\oplus m}$ will contain elements of form $(q, v_1^1, \cdots, v_m^n)$.

Background for Noether's Theorem Continued

- We define the followings:
  - The $l-$th partial fiber derivative of $\mathscr{L}*$ is the map $(\mathbb{F}\mathscr{L}*)^l : TN^{\oplus m} \to T^*N$ given by:
  - $$(\mathbb{F}\mathscr{L}*)^l(q, \mathbf{v}_1, \cdots, \mathbf{v}_m)\mathbf{w} = \frac{d}{dt}|_{t=0}\mathscr{L}*(q, \mathbf{v}_1, \cdots, \mathbf{v}_l + t\mathbf{w}, \cdots, \mathbf{v}_m)$$
  - The total fiber derivative of $\mathscr{L}*$ is the map $\mathbb{F}\mathscr{L}* : TN^{\oplus m} \to T^*N^{\oplus m}$ given by:
  - $$(\mathbb{F}\mathscr{L}*)^l(q, \mathbf{v}_1, \cdots, \mathbf{v}_m)(\mathbf{w}_1, \cdots, \mathbf{w}_m) = (\mathbb{F}\mathscr{L}*)^l(q, \mathbf{v}_1, \cdots, \mathbf{v}_m)\mathbf{w}_l$$
  - We say that $\mathscr{L}*$ is *regular* if $\mathbb{F}\mathscr{L}*$ is a local diffeomorphism, and it is called *hyperregular* if $\mathbb{F}\mathscr{L}*$ is a global diffeomorphism.

Noether's Theorem

- **Theorem** (Noether): Let $\mathscr{L}^* : TN^{\oplus m} \to \mathbb{R}$ be a Lagrangian, and $(\varphi_s)_{s\in\mathbb{R}}$ is a 1-parameter group of diffeomorphisms of N that leaves $\mathscr{L}^*$ invariant. Then, the *Noether current* $\mathscr{J} = (\mathscr{J}^1, \cdots, \mathscr{J}^m) : TN^{\oplus m} \to \mathbb{R}^m$ given by:
- $$\mathscr{J}^l(q, \mathbf{v}_1, \cdots, \mathbf{v}_m) = (\mathbb{F}\mathscr{L}^*)^l(q, \mathbf{v}_1, \cdots, \mathbf{v}_m)(\frac{d}{ds}|_{s=0}\varphi_s(x))$$
- have zero divergence along the function $u : D \to N$, where $D \subset M = \mathbb{R}^m$, when u satisfies the Euler-Lagrange equation.

Proof of Noether's Theorem

- **Theorem** (Noether): Let $\mathscr{L}^* : TN^{\oplus m} \to \mathbb{R}$ be a Lagrangian, and $(\varphi_s)_{s\in\mathbb{R}}$ is a 1-parameter group of diffeomorphisms of N that leaves $\mathscr{L}^*$ invariant. Then, the *Noether current* $\mathscr{J} = (\mathscr{J}^1, \cdots, \mathscr{J}^m) : TN^{\oplus m} \to \mathbb{R}$ given by:
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- have zero divergence along the function $u : D \to N$, where $D \subset M = \mathbb{R}^m$, when u satisfies the Euler-Lagrange equation.
- Let $u_s = \varphi_s \circ u$. By invariance, we must have all u_s satisfy the Euler-Lagrange equation. And, we must have:
- $$\frac{\partial u_s}{\partial x^l}(x) = d(\varphi_s)_{u(x)}(\frac{\partial u}{\partial x^l}(x))$$
- Writing above in coordinate form, we have:
- $$(u_s(x), \nabla u_s(x)) = (q^1(s,x), \cdots, q^n(s,x), v_1^1(s,x), \cdots, v_m^n(s,x))$$
- And, the condition of invariance is simply:
- $$\mathscr{L}^*(u_s(x), \nabla u_s(x)) = \mathscr{L}^*(u(x), \nabla u(x))$$

Proof of Noether's Theorem

- **Theorem** (Noether): Let $\mathscr{L}^* : TN^{\oplus m} \to \mathbb{R}$ be a Lagrangian, and $(\varphi_s)_{s\in\mathbb{R}}$ is a 1-parameter group of diffeomorphisms of N that leaves $\mathscr{L}^*$ invariant. Then, the *Noether current* $\mathscr{J} = (\mathscr{J}^1, \cdots, \mathscr{J}^m) : TN^{\oplus m} \to \mathbb{R}$ given by:
- $$\mathscr{J}^l(q, \mathbf{v}_1, \cdots, \mathbf{v}_m) = (\mathbb{F}\mathscr{L}^*)^l(q, \mathbf{v}_1, \cdots, \mathbf{v}_m)(\frac{d}{ds}|_{s=0}\varphi_s(x))$$
- have zero divergence along the function $u : D \to N$, where $D \subset M = \mathbb{R}^m$, when u satisfies the Euler-Lagrange equation.
- And, the condition of invariance is simply:
- $$\mathscr{L}^*(u_s(x), \nabla u_s(x)) = \mathscr{L}^*(u(x), \nabla u(x))$$
- We take derivative against s on both side, and we find:
- $$\begin{aligned} 0 &= \sum_{k=1}^{n} \frac{\partial \mathscr{L}^*}{\partial q^k}(u_s(x), \nabla u_s(x))\frac{\partial q^k}{\partial s}(s,x) + \sum_{l=1}^{m}\sum_{k=1}^{n} \frac{\partial \mathscr{L}^*}{\partial v_l^k}(u_s(x), \nabla u_s(x))\frac{\partial u_l^k}{\partial s}(s,x) \\ &= \sum_{k=1}^{m}\sum_{l=1}^{n} \frac{\partial}{\partial x^l}\left(\frac{\partial \mathscr{L}^*}{\partial v_l^k}(u_s(x), \nabla u_s(x))\right)\frac{\partial q^k}{\partial s}(s,x) + \sum_{l=1}^{m}\sum_{k=1}^{n} \frac{\partial \mathscr{L}^*}{\partial v_l^k}(u_s(x), \nabla u_s(x))\frac{\partial}{\partial x^l}(\frac{\partial q^k}{\partial s}(s,x)) \\ &= \sum_{l=1}^{m} \frac{\partial}{\partial x^l}\left(\sum_{k=1}^{n} \frac{\partial \mathscr{L}^*}{\partial v_l^k}(u_s(x), \nabla u_s(x))\frac{\partial q^k}{\partial s}(s,x)\right) \end{aligned}$$

Proof of Noether's Theorem

- **Theorem** (Noether): Let $\mathscr{L}* : TN^{\oplus m} \to \mathbb{R}$ be a Lagrangian, and $(\varphi_s)_{s\in\mathbb{R}}$ is a 1-parameter group of diffeomorphisms of N that leaves $\mathscr{L}*$ invariant. Then, the *Noether current* $\mathscr{J} = (\mathscr{J}^1, \cdots, \mathscr{J}^m) : TN^{\oplus m} \to \mathbb{R}$ given by:
- $$\mathscr{J}^l(q, \mathbf{v}_1, \cdots, \mathbf{v}_m) = (\mathbb{F}\mathscr{L}*)^l(q, \mathbf{v}_1, \cdots, \mathbf{v}_m)(\frac{d}{ds}|_{s=0}\varphi_s(x))$$
- have zero divergence along the function $u : D \to N$, where $D \subset M = \mathbb{R}^m$, when u satisfies the Euler-Lagrange equation.
- We have
- $$\sum_{l=1}^{m} \frac{\partial}{\partial x^l}\left(\sum_{k=1}^{n} \frac{\partial \mathscr{L}*}{\partial v_l^k}(u_s(x), \nabla u_s(x))\frac{\partial q^k}{\partial s}(s, x)\right) = 0$$
- Recall that:
- $$(\mathbb{F}\mathscr{L}*)^l(q, \mathbf{v}_1, \cdots, \mathbf{v}_m)\mathbf{w} = \frac{d}{dt}|_{t=0}\mathscr{L}*(q, \mathbf{v}_1, \cdots, \mathbf{v}_l + t\mathbf{w}, \cdots, \mathbf{v}_m)$$
- Hence, if we set $s = 0$, we already see that our theorem is proved. ∎

Application of Noether's Theorem

- *Harmonic Functions*
  - Again, lets visit our previous example $\mathscr{L}^* : \mathbb{R}^{\oplus m} \to \mathbb{R}$ (where $N = \mathbb{R} \implies TN \cong \mathbb{R}$) with $\mathscr{L}^*(u, v_1, \cdots, v_m) = \dfrac{1}{2}\sum_{i=1}^{m} (v_i)^2$.
  - We see that the 1-parameter group of translation $\tau_s : \mathbb{R} \to \mathbb{R}$ with $\tau_s(x) = x + s$ leaves the Lagrangian invariant.
  - Moreover, using our previous definition, it is easy to see that:
  - $$\begin{aligned} &\mathscr{J}(u, v_1, \cdots, v_m) \\ &= (\mathscr{J}^1, \cdots, \mathscr{J}^l)(u, v_1, \cdots, v_m) \\ &= ((\mathbb{F}\mathscr{L}^*)^1(u, v_1, \cdots, v_m), \cdots, (\mathbb{F}\mathscr{L}^*)(u, v_1, \cdots, v_m)) \\ &= (v_1, \cdots, v_m) \end{aligned}$$
  - So, this implies that $\mathscr{J}(u(x), \nabla u(x)) = \nabla u(x)$. By Noether's Theorem, it has zero divergence, which means
  - $$\nabla \cdot \nabla u(x) = \Delta u(x) = 0$$

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Thank You!