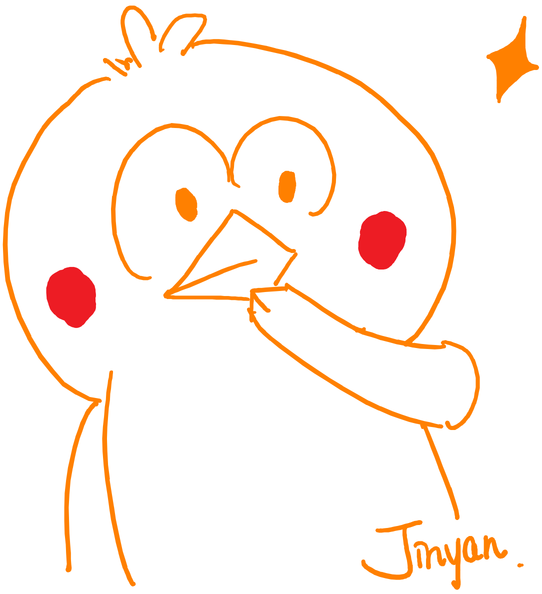
Homework 3

Physics 542 - Quantum Optics
Professor Alex Kuzmich



Jinyan Miao

Fall 2023

1

Straightforward algebra leads to

$$
\begin{aligned}
&-\gamma(\hat{\sigma}_0\hat{\rho} + \hat{\rho}\hat{\sigma}_0) + \gamma_2\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+ + 2\Gamma\hat{\sigma}_0\hat{\rho}\hat{\sigma}_0 \\
&\quad = \begin{bmatrix} 0 & 0 \\ 0 & -\gamma \end{bmatrix}\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} + \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & -\gamma \end{bmatrix} + \begin{bmatrix} 0 & \gamma_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2\Gamma \end{bmatrix}\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&\quad = \begin{bmatrix} 0 & 0 \\ -\gamma\rho_{21} & -\gamma\rho_{22} \end{bmatrix} + \begin{bmatrix} 0 & -\gamma\rho_{12} \\ 0 & -\gamma\rho_{22} \end{bmatrix} + \begin{bmatrix} \gamma_2\rho_{21} & \gamma_2\rho_{22} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2\Gamma\rho_{21} & 2\Gamma\rho_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&\quad = \begin{bmatrix} 0 & 0 \\ -\gamma\rho_{21} & -\gamma\rho_{22} \end{bmatrix} + \begin{bmatrix} 0 & -\gamma\rho_{12} \\ 0 & -\gamma\rho_{22} \end{bmatrix} + \begin{bmatrix} \gamma_2\rho_{22} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2\Gamma\rho_{22} \end{bmatrix} \\
&\quad = \begin{bmatrix} \gamma_2\rho_{22} & -\gamma\rho_{12} \\ -\gamma\rho_{21} & -2\gamma\rho_{22} + 2\Gamma\rho_{22} \end{bmatrix}.
\end{aligned}
$$

Comparing Eq. (3.31a) and (3.20a), the relaxation term in the $\dot{\rho}_{11}$ equation is $\gamma_2\rho_{22}$, which agrees with the $(1,1)$-entry in the matrix computed.

Comparing Eq. (3.31b) and (3.20b), the relaxation terms in the $\dot{\rho}_{22}$ equation is

$$-\gamma_2\rho_{22} = -2(\gamma - \Gamma)\rho_{22} = -2\gamma\rho_{22} + 2\Gamma\rho_{22}\,,$$

which agrees with the $(2,2)$-entry in the matrix computed.

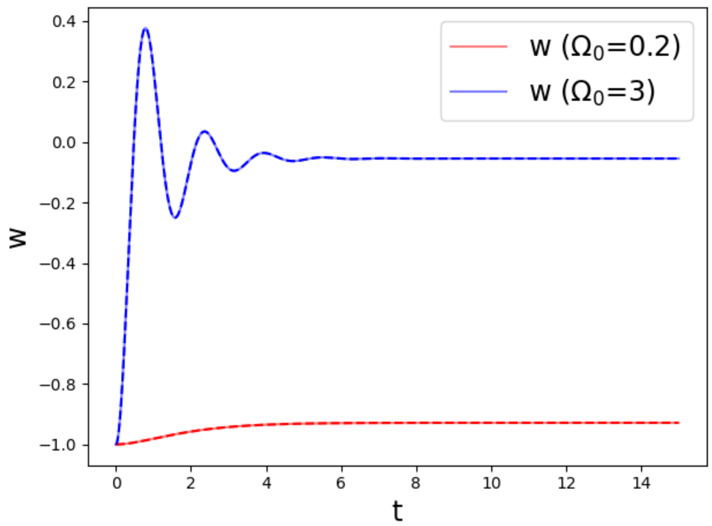
Comparing Eq. (3.31c) and (3.20c), the relaxation term in the $\dot{\rho}_{12}$ equation is $-\gamma\rho_{12}$, which agrees with the $(1,2)$-entry in the matrix computed.

Comparing Eq. (3.31d) and (3.20d), the relaxation term in the $\dot{\rho}_{21}$ equation is $-\gamma\rho_{21}$, which agrees with the $(2,1)$-entry in the matrix computed.

These complete the derivation.

2

Here we solve the Bloch equations numerically for $\gamma = \gamma_2/2 = 1/2$, $\delta = 0.1$, and $|\Omega_0| \in \{0.2, 3\}$. The solution of w is plot as a function of time, with initial condition $w(t=0) = \rho_{22} - \rho_{11} = -1$, that is the atom is initially in its ground state $\rho_{11} = 1$.



For comparison, the numerical solutions to the Bloch equations are plotted in solid curves, and the numerical solutions to Eq. (3.54) in *Berman* are plotted in dashed curves. We see here the solid curves agree perfectly with the dashed curves.

We also conclude that the Bloch vector approach its steady-state value monotonically in the case of $\Omega_0 = 0.2$ (plotted in red).

* *The script for numerical computations is attached on the next page.*

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import quad

# discretize time
delta_t = 0.00001
ts = np.linspace(0,15,int(20/delta_t))

# parameter settings
gamma = 1/2
gamma2 = 2*gamma
delta = 0.1
Omega0s = [0.2, 3]

# initial conditions
u_i = 0 # tilde_rho12 + tilde_rho21
v_i = 0 # 1j*(tilde_rho21 - tilde_rho12)
w_i = -1 # rho22 - rho11
m_i = 1 # rho22 + rho11

## verification
rho11_i = 1
rho22_i = 0
tilde_rho12_i = 0
tilde_rho21_i = 0

colors = ['red', 'blue']

for i in range(len(Omega0s)):
    Omega0 = Omega0s[i]
    chi = Omega0/2
    rho11_array = [rho11_i]
    rho22_array = [rho22_i]
    tilde_rho12_array = [tilde_rho12_i]
    tilde_rho21_array = [tilde_rho21_i]
    for t in ts[:-1]:
        rho11, rho22 = rho11_array[-1], rho22_array[-1]
        tilde_rho12, tilde_rho21 = tilde_rho12_array[-1], tilde_rho21_array[-1]
        dot_rho11 = (-1j*np.abs(chi)*tilde_rho21
                        + 1j*np.abs(chi)*tilde_rho12
                            + gamma2*rho22)
        dot_rho22 = (1j*np.abs(chi)*tilde_rho21
                        - 1j*np.abs(chi)*tilde_rho12
                            - gamma2*rho22)
        dot_tilde_rho12 = (1j*delta*tilde_rho12
                                - 1j*np.abs(chi)*(rho22-rho11)
                                    - gamma*tilde_rho12)
        dot_tilde_rho21 = (-1j*delta*tilde_rho21
                                + 1j*np.abs(chi)*(rho22-rho11)
                                    -gamma*tilde_rho21)
        rho11_array.append(rho11_array[-1]+delta_t*dot_rho11)
        rho22_array.append(rho22_array[-1]+delta_t*dot_rho22)
        tilde_rho12_array.append(tilde_rho12_array[-1]+delta_t*dot_tilde_rho12)
        tilde_rho21_array.append(tilde_rho21_array[-1]+delta_t*dot_tilde_rho21)
    plt.plot(ts, np.real([rho22_array[i]-rho11_array[i]
                                for i in range(len(rho22_array))]),
             linestyle='--', color=colors[i])
```

```


for i in range(len(Omega0s)):
    Omega0 = Omega0s[i]
    u_array = [u_i]
    v_array = [v_i]
    w_array = [w_i]
    for t in ts[:-1]:
        u, v, w = u_array[-1], v_array[-1], w_array[-1]
        dot_u = -delta*v - gamma*u
        dot_v = delta*u - np.abs(Omega0)*w - gamma*v
        dot_w = np.abs(Omega0)*v - gamma2*(w+1)
        u_array.append(u_array[-1]+delta_t*dot_u)
        v_array.append(v_array[-1]+delta_t*dot_v)
        w_array.append(w_array[-1]+delta_t*dot_w)
    plt.plot(ts, np.real(w_array), color=colors[i], alpha=0.5,
             label=r'w ($\Omega_0$='+str(Omega0)+')')


plt.ylabel(r'w', fontsize='xx-large')
plt.xlabel("t",fontsize='xx-large')
plt.legend(fontsize='xx-large')
plt.tight_layout()
plt.show()
```

3

Here we will compute

$$\langle \hat{M} \rangle = \langle \hat{n}(\hat{n}-1)(\hat{n}-2)\cdots(\hat{n}-r+1) \rangle$$

for the thermal state of one mode of light field. Via Eq. (2.145) in *Gerry & Knight*, the probability of fining the n photons in the field is given by

$$P_n = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \,,$$

and furthermore, via Eq. (2.138) we have

$$\hat{\rho} = \sum_{n=0}^{\infty} P_n \left| n \right\rangle \left\langle n \right| .$$

Then we con compute

$$\begin{aligned}
\langle \hat{M} \rangle = \mathrm{tr}(\hat{M}\hat{\rho}) &= \sum_{n=0}^{\infty} \langle n|\hat{M}\hat{\rho}|n\rangle \\
&= \sum_{n=0}^{\infty} \langle n|\hat{M} \sum_{m=0}^{\infty} P_m |m\rangle \left\langle m|n\right\rangle \\
&= \sum_{n=0}^{\infty} \langle n|\hat{M} P_n|n\rangle \\
&= \sum_{n=0}^{\infty} n(n-1)(n-2)\cdots(n-r+1)\, \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \\
&= \frac{1}{(1+\bar{n})} \frac{\bar{n}^r}{(1+\bar{n})^r} \sum_{n=0}^{\infty} n(n-1)(n-2)\cdots(n-r+1)\, \frac{\bar{n}^{n-r}}{(1+\bar{n})^{n-r}} \\
&= \frac{1}{\bar{n}} \left(\frac{\bar{n}}{1+\bar{n}}\right)^{r+1} \sum_{n=0}^{\infty} n(n-1)(n-2)\cdots(n-r+1) \left(\frac{\bar{n}}{1+\bar{n}}\right)^{n-r} .
\end{aligned}$$

Here we define

$$x = \frac{\bar{n}}{1+\bar{n}} \,,$$

then we can write

$$\begin{aligned}
\langle \hat{M} \rangle &= \frac{x^{r+1}}{\bar{n}} \sum_{n=0}^{\infty} n(n-1)(n-2)\cdots(n-r+1)\, x^{n-r} \\
&= \frac{x^{r+1}}{\bar{n}} \sum_{n=0}^{\infty} \frac{\partial^r}{\partial x^r} x^n \\
&= \frac{x^{r+1}}{\bar{n}} \frac{\partial^r}{\partial x^r} \left(\sum_{n=0}^{\infty} x^n \right) \\
&= \frac{x^{r+1}}{\bar{n}} \frac{\partial^r}{\partial x^r} \left(\frac{1}{1-x} \right) \\
&= \frac{x^{r+1}}{\bar{n}} \frac{r!}{(1-x)^{r+1}} \\
&= \frac{r!}{\bar{n}} \left(\frac{x}{1-x} \right)^{r+1} .
\end{aligned}$$

Here we compute

$$\frac{x}{1-x} = \frac{\bar{n}}{1+\bar{n}} \left(1 - \frac{\bar{n}}{1+\bar{n}} \right)^{-1} = \frac{\bar{n}(1+\bar{n})}{1+\bar{n}} = \bar{n} \, .$$

Thus we have

$$\langle \hat{n}(\hat{n}-1)(\hat{n}-2)\cdots(\hat{n}-r+1) \rangle = \frac{r!\bar{n}^{r+1}}{\bar{n}} = r!\bar{n}^r = r!\langle \hat{n} \rangle^r ,$$

as expected.

4

Here we will compute

$$\langle \hat{M} \rangle = \langle \hat{n}(\hat{n}-1)(\hat{n}-2)\cdots(\hat{n}-r+1) \rangle = \langle \alpha | \hat{M} | \alpha \rangle$$

for a coherent state of one mode.

The number operator $\hat{n} = \hat{a}^\dagger \hat{a}$. For $r = 1$, we see that

$$\hat{M} = \hat{n} = \hat{a}^\dagger \hat{a}\,.$$

We claim that $\hat{M} = \hat{a}^{\dagger r} \hat{a}^r$ for all $r \in \mathbb{N}$. The base case for the claim has been proven, for the inductive case, we assume the claim is true for all $k < r$, and show that the claim is true for r. Here we write

$$\hat{n}(\hat{n}-1)(\hat{n}-2)\cdots(\hat{n}-r+1) = (\hat{a}^\dagger)^{r-1}\hat{a}^{r-1}(\hat{a}^\dagger \hat{a} - r + 1)\,.$$

Note that we have $[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a} = 1$, thus

$$\hat{a}^\dagger \hat{a} + 1 = \hat{a}\hat{a}^\dagger\,,$$

we apply this property $r-1$ times and obtain

$$\begin{aligned}
\hat{n}(\hat{n}-1)(\hat{n}-2)\cdots(\hat{n}-r+1) &= (\hat{a}^\dagger)^{r-1}\hat{a}^{r-1}(\hat{a}^\dagger \hat{a} - r + 1) \\
&= (\hat{a}^\dagger)^{r-1}\hat{a}^{r-1}\hat{a}^\dagger \hat{a} - (r-1)(\hat{a}^\dagger)^{r-1}\hat{a}^{r-1} \\
&= (\hat{a}^\dagger)^{r-1}\hat{a}^{r-2}(\hat{a}^\dagger \hat{a} + 1)\hat{a} - (r-1)(\hat{a}^\dagger)^{r-1}\hat{a}^{r-1} \\
&= (\hat{a}^\dagger)^{r-1}\hat{a}^{r-1} + (\hat{a}^\dagger)^{r-1}\hat{a}^{r-2}\hat{a}^\dagger \hat{a}^2 - (r-1)(\hat{a}^\dagger)^{r-1}\hat{a}^{r-1} \\
&= (\hat{a}^\dagger)^{r-1}\hat{a}^{r-2}\hat{a}^\dagger \hat{a}^2 - (r-2)(\hat{a}^\dagger)^{r-1}\hat{a}^{r-1} \\
&\ \ \vdots \\
&= \hat{a}^{\dagger r}\hat{a}^r - (r-r)(\hat{a}^\dagger)^{r-1}\hat{a}^{r-1} \\
&= \hat{a}^{\dagger r}\hat{a}^r\,.
\end{aligned}$$

Thus now we can compute

$$\langle \hat{M} \rangle = \langle \alpha | \hat{a}^{\dagger r} \hat{a}^r | \alpha \rangle = \langle \hat{a}^r \alpha | \hat{a}^r \alpha \rangle\,,$$

where we have

$$\hat{a}^r |\alpha\rangle = \alpha^r |\alpha\rangle\,,$$

thus combining we have

$$\langle \hat{n}(\hat{n}-1)(\hat{n}-2)\cdots(\hat{n}-r+1) \rangle = |\alpha|^{2r} \langle \alpha | \alpha \rangle = |\alpha|^{2r}\,.$$