Continuous Random Variables

Suppose X is a random variable with cdf $F_X(x)$, that is $F_X(x) = P(X \leq x)$. We say X is a continuous random variable provided that $F_X : \mathbb{R} \to \mathbb{R}$ is a continuous function.

A probability density function for a continuous random variable is a function $f_X : \mathbb{R} \to \mathbb{R}$ satisfying

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\, dt\,.$$

We note that:

1. f_X need not be a continuous function for F_X to be continuous.
2. $P(a \leq X \leq b) = P(X \text{ takes values in } [a, b]) = \int_a^b f_X(x)\, dx$.
3. $P(X = a) = P(a \leq x \leq a) = \int_a^a f_X(x)\, dx = 0$. That is, the probability that X takes any fixed number equals to zero.
4. Via the Fundamental Theorem of Calculus, the integration and differentiation are inverse operations, that is

$$\frac{d}{dx}\left(\int_a^x f(u)\, du\right) = f(x)\,,$$

from which we obtain the following theorem:

Theorem 0.1
If the cdf is differentiable, $F'(x) = f_X(x)$. That is,

$$\frac{d}{dx}\int_{-\infty}^{x} f_X(t)\, dt = f_X(x)\,.$$

5. From (4), to calculate the pdf of X, we can first calculate the cdf $F_X(x)$ of X, then if $F_X(x)$ is dfferentiable, $f_X(x) = F'(x)$.
6. We also note that the Leibniz Rule for differentiating an integral gives

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(t)\, dt\right) = b'(x)\cdot f(b(x)) - a'(x)\cdot f(a(x))\,.$$

Theorem 0.2
Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f_X(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x)\, dx = 1$. Then $f(x)$ is a pdf of some random variable. On the other hand, if $f_X(x)$ is a pdf of X, then $f_X(x)$ satisfies $f_X(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Example Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2(1-x) & x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$

We would like to find a constant c such that $f_X = c \cdot f$ defines a pdf of a ramdom variable. First we notice that $f(x) = x^2(1-x) \geq 0$ whenever $x \in (0, 1)$. Now we compute

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^1 x^2(1-x)\, dx = \int_0^1 x^2 - x^3\, dx = \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^1 = \frac{1}{12}\,.$$

Thus if $c = 12$, $f_X = 12 \cdot f$, then f_X is a pdf of a random variable with the correct normalization.

$$f_X(x) = \begin{cases} 12 \cdot x^2(1-x) & x \in (0,1) \\ 0 & \text{otherwise} \end{cases}.$$

In this case, we can calculate

$$\begin{aligned} F_X(x) = \int_{-\infty}^{x} f_X(u)\, du &= \int_{-\infty}^{x} 12 \cdot u^2(1-u)\, du \\ &= 12 \cdot \lim_{c \to -\infty} \int_c^x u^2(1-u)\, du \\ &= 12 \cdot \lim_{c \to -\infty} \left(\left(\frac{x^3}{3} - \frac{x^4}{4}\right) - \left(\frac{c^3}{3} - \frac{c^4}{4}\right)\right) \\ &= 12\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\,. \end{aligned}$$

Here we observer that $F_X'(x) = F_X(x)$.

Example: Consider X being a random variable with pdf F_X and cdf F_X, and let $Y = 4X^5$ be a new random variable. To calculate the pdf of Y, we first note that in general $f_Y(x) \neq 4(f(x))^5$. For $y \in \mathbb{R}$,

$$F_Y(y) = P(Y \leq y) = P(4X^5 \leq y) = P(X^5 \leq y/4) = P(X \leq (y/4)^{1/5}) = F_X((y/4)^{1/5})\,.$$

Now, using chain rule we can compute

$$f_Y(y) = \frac{d}{dy}\left(F_Y(y)\right) = \frac{d}{dy}\left(F_X((y/4)^{1/5})\right) = F_X'\left((y/4)^{1/5}\right) \cdot (y/4)^{-4/5}/20\,.$$

With f_X defined by the last example, we obtain

$$F_X'\left(\left(\frac{y}{4}\right)^{1/5}\right) = f_x\left(\left(\frac{y}{4}\right)^{1/5}\right) = 12 \cdot \left(\frac{y}{4}\right)^{2/5} \cdot \left(1 - \left(\frac{y}{4}\right)^{1/5}\right)\,,$$

and thus

$$f_Y(y) = \begin{cases} \frac{3}{5} \cdot \left(\frac{y}{4}\right)^{-2/5} \cdot \left(1 - \left(\frac{y}{4}\right)^{1/5}\right) & y \in (0,1) \\ 0 & \text{otherwise} \end{cases}\,.$$

Parameters Attached to Continuous Variables

Suppose X is a continuous random variable with pdf $f_X(x)$.

1. The expected value of X is given by
$$E(X) = \mu_X = \int_{-\infty}^{\infty} x\, f_X(x)\, dx\,.$$

2. If $Y = h(X)$, then
$$E(Y) = E(h(X)) = \int_{-\infty}^{\infty} h(x) \cdot f_X(x)\, dx = \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy\,.$$

3. The variance of X is defined by
$$\sigma_X^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu_X)^2 \cdot f_X(x)\, dx\,.$$
Note that $V(X) = E(X^2) - E(X)$ still holds.

4. The moment generating function is defined by
$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x)\, dx\,.$$

5. For $\alpha \in (0,1)$, x_α is the α^{th} critical value for X is x_α satisfies $P(X \geq x_\alpha) = \alpha$, which holds iff $1 - P(X \leq x_\alpha) = \alpha$, iff $P(X \leq x_\alpha) = (1-\alpha)$, iff $F_X(x_\alpha) = 1 - \alpha$.

6. For $p \in (0,1)$, n_p is called the p^{th} percentile provided that $P(X \leq n_p) = p$, which is equivalent to $F_X(n_p) = p$. We note that α^{th} critical value is the same as the $(1-\alpha)^{\text{th}}$ percentile.