Examples:

1. Coin Toss

   (a). Experiment: Toss a coin once.
   All possible outcomes are contained in the set $\{H, T\}$, with H representing obtaining a head and T representing obtaining a tail. The total number of outcomes is 2.

   (b). Experiment: Toss the coin two times.
   All possible outcomes are contained in the set $\{HH, HT, TH, TT\}$. Number of outcomes is 4.

   (c). Experiment: Toss the coin 10 times.
   All possible outcomes are contained in the set $\{HHH \cdots H,\ HTHH \cdots H,\ \cdots\}$. Number of outcomes is $2^{10} = 1024$.

2. Die Rolls

   (a). Roll a six-sided die once. Outcomes are in the set $\{1, 2, 3, 4, 5, 6\}$.

   (b). Roll the die two times. Outcomes are in the set

$$\left\{\begin{matrix} (1,1), & (1,2), & (1,3), & \cdots, & (1,6), \\ (2,1), & (2,2), & (2,3), & \cdots, & (2,6), \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (6,1), & (6,2), & (6,3), & \cdots, & (6,6) \end{matrix}\right\}.$$

3. Consider the following experiment: In step 1, we toss a coin; In step 2, we roll a 6-sided die. All possible outcomes are contained in the set

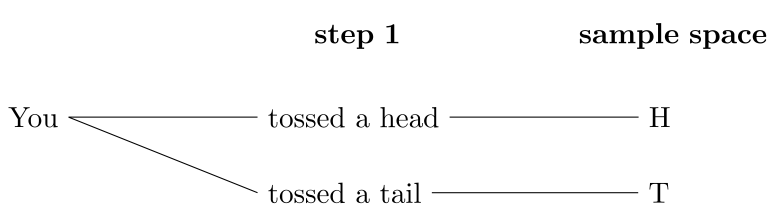
$$\left\{\begin{matrix} (H,1), & (H,2), & (H,3), & (H,4), & (H,5), & (H,6), \\ (T,1), & (T,2), & (T,3), & (T,4), & (T,5), & (T,6) \end{matrix}\right\}.$$

4. Consider the following experiment: Pick a random student from campus, and ask if they have walked more than 3000 steps today. All possible outcomes of this experiment are contained in the set {“Yes”, “No”}.

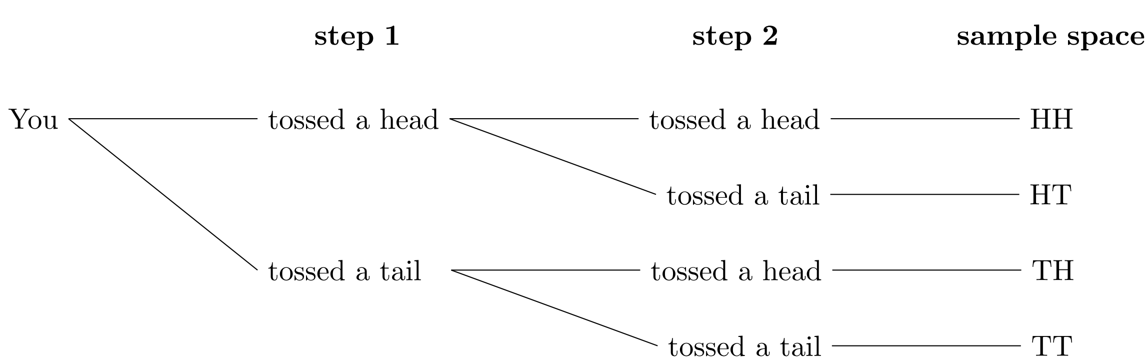
5. Consider the following experiment: Pick a student from campus and measure their height. All possible outcomes are contained in the set of positive real numbers, that is $(0, \infty)$.

Examples:

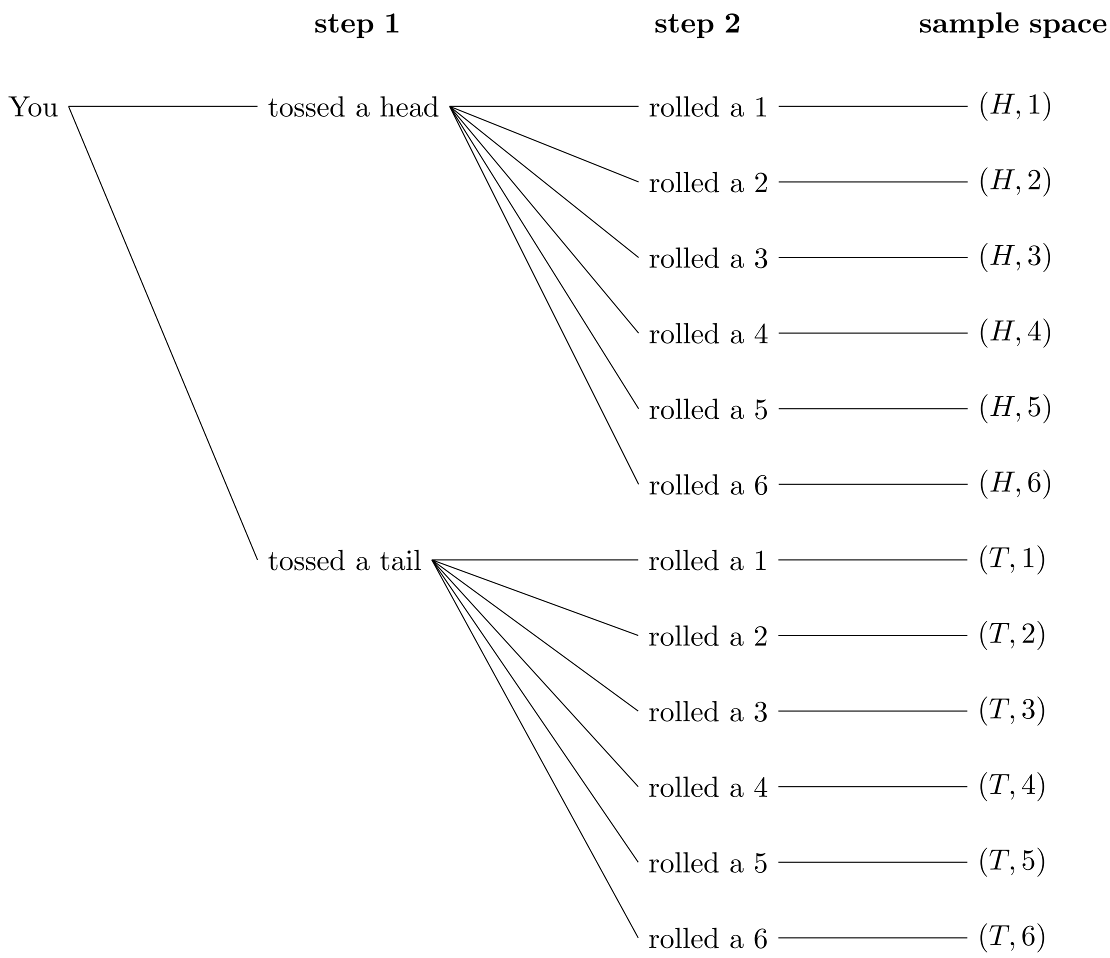
1. Consider the experiment that we toss the coin only once.



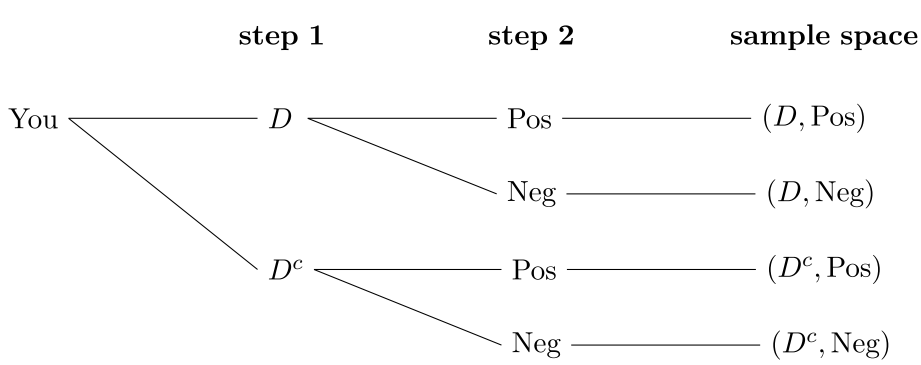
2. Consider the experiment that we toss the coin two times.



3. Consider the experiment: In the first step, we toss a coin; In the second step, we roll a 6-sided die.



4. Consider the following experiment: First choose a person, ask if they have the disease (result denoted as D or D^c), then administer the test to the person, getting positive (Pos) or negative (Neg) testing result.



Examples

1. Consider the experiment of tossing a coin once. In this case, the sample space is $S = \{H, T\}$. The event space has the following elements:

$$\begin{aligned} \emptyset \quad &\to \text{Did not toss the coin,} \\ S \quad &\to \text{Tossed the coin,} \\ \{H\} &\to \text{Tossed the coin and the coin landed head,} \\ \{T\} &\to \text{Tossed the coin and the coin landed tail.} \end{aligned}$$

2. Now we consider a two-step experiment: In step 1, we toss a coin, with result H or T; In step 2, we roll a 6-sided die. The sample space is

$$0S = \begin{Bmatrix} (H,1), & (H,2), & (H,3), & (H,4), & (H,5), & (H,6), \\ (T,1), & (T,2), & (T,3), & (T,4), & (T,5), & (T,6) \end{Bmatrix}.$$

Here are some examples of events:

(a). The coin lands H, represented by the set $\{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6)\} \subseteq S$.

(b). The die rolls to 6, represented by $\{(H,6),(T,6)\}$.

(c). The coin lands H and the die rolls to an even number, represented by $\{(H,2),(H,4),(H,6)\}$.

(d). The die rolls to an odd number, represented by $\{(H,1),(H,3),(H,5),(T,1),(T,3),(T,5)\}$.

Example

Here we consider the experiment: Toss a coin followed by die roll. As discussed previously, the sample space is represented by the set

$$S = \begin{Bmatrix} (H,1), & (H,2), & (H,3), & (H,4), & (H,5), & (H,6), \\ (T,1), & (T,2), & (T,3), & (T,4), & (T,5), & (T,6) \end{Bmatrix}.$$

Now we consider some examples of events. Let E denote the event of coin lands H, then

$$E = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6)\}.$$

Let F denote the event of die rolls to an even number, then

$$F = \{(H,2),(H,4),(H,6),(T,2),(T,4),(T,6)\}.$$

Let G denote the event of die rolls to a 6, then

$$G = \{(H,6),(T,6)\}.$$

1. Notice that $E^c = \{(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$ represents the event of coin landing T. We also observe that when E^c happens, E does not happen.

2. We also observe that $G \nsubseteq E$, and $G \subseteq F$. That is, suppose A and B are events and $A \subseteq B$, then A happens implies B happens.

3. The set $E \cup G = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,6)\}$ represents the event of either E happening or G happening.

4. The set $E \cap F = \{(H,2),(H,4),(H,6)\}$ represents the event of both E happening and F happening.

Examples:

1. Suppose S is the sample space. The power set $\mathscr{P}(S)$, which is the set of all subsets of S, forms a sigma algebra about S. Furthermore, $\{\emptyset, S\}$ is also a sigma algebra about S, called the trivial sigma algebra.

2. Suppose the experiment of tossing a coin three times. The sample space is

$$S = \{HHH,,HHT,HTH,THH,TTH,HTT,THT,TTT\}.$$

   Here we see that $|S| = 8$, and $|\mathscr{P}(S)| = 2^{|S|} = 2^8 = 256$. One might be interested in the question: "Are there at least two H in three tosses?" This corresponds to the event $E = \{HHT,HTH,THH,HHH\}$. To assign "probability" to E, we only need to consider the sigma algebra generated by E, that is the set $\langle E \rangle = \{\emptyset, S, E, E^c\}$.

Example:

Now we consider the experiment consisting of tossing a coin two times. The sample space is $S = \{HH,HT,TH,TT\}$. Take the largest possible sigma algebra, that is the power set of S, denoted as $\mathscr{P}(S)$, which contains $|\mathscr{P}(S)| = 2^4 = 16$ elements. In this experiment, if the coin is a fair coin, we can assign the following probability function $P : \mathscr{P}(S) \to \mathbb{R}$:

event E	$P(E)$
$\emptyset$	0
$\{HH\}$	1/4
$\{HT\}$	1/4
$\{TT\}$	1/4
$\{TH\}$	1/4
$\{HH, HT\}$	1/4+1/4 = 1/2
$\vdots$	$\vdots$
$\{HH, HT, TH\}$	1/4+1/4+1/4 = 3/4
$\vdots$	$\vdots$
$\{HH, HT, TH, TT\}$	1

.

In the case where the coin is not a fair coin, the probability function can be defined in the following way:

event E	$P(E)$
$\emptyset$	0
$\{HH\}$	1/3
$\{HT\}$	1/3
$\{TT\}$	1/3
$\{TH\}$	0
$\{HH, TH\}$	1/3+0 = 1/3
$\vdots$	$\vdots$
$\{HH, HT, TT\}$	1/3+1/3+1/3 = 1
$\vdots$	$\vdots$
$\{HH, HT, TH, TT\}$	1

.

Similarly, the following probability function is also allowed:

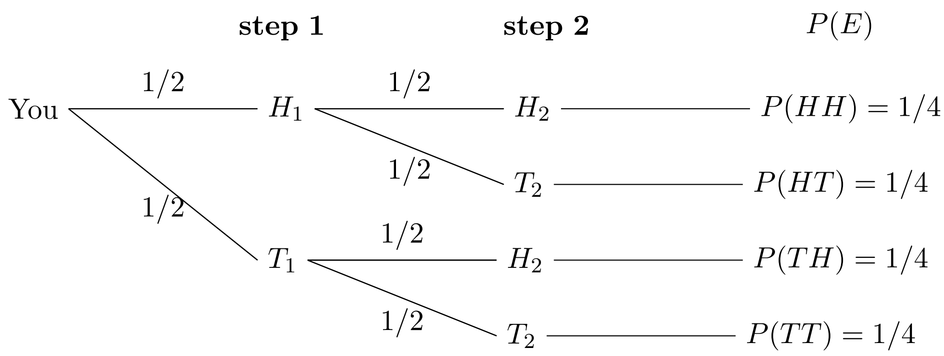
event E	$P(E)$
$\emptyset$	0
$\{HH\}$	1/8
$\{HT\}$	1/8
$\{TT\}$	3/8
$\{TH\}$	3/8
$\{HH, TH\}$	1/8+3/8 = 1/2
$\vdots$	$\vdots$
$\{HH, HT, TT\}$	1/8+1/8+3/8 = 5/8
$\vdots$	$\vdots$
$\{HH, HT, TH, TT\}$	1

.

Notice that the probabilities of events like $\{HH, TH\}$ and $\{HH, HT, TT\}$ are completely determined by the probabilities of the four simple events $\{HH\}$, $\{HT\}$, $\{TH\}$ and $\{TT\}$. While for event E, we note that $P(E)$ must be non-negative. We also observe that $P(S) = 1$, and as $S = S \cup \emptyset$, then $P(S) = P(S \cup \emptyset) = P(S) + P(\emptyset) = 1$, from which we deduce that we must have $P(\emptyset) = 0$. The tables shown above are called the distribution tables, and we see that there can be different distribution tables for the same sample space.

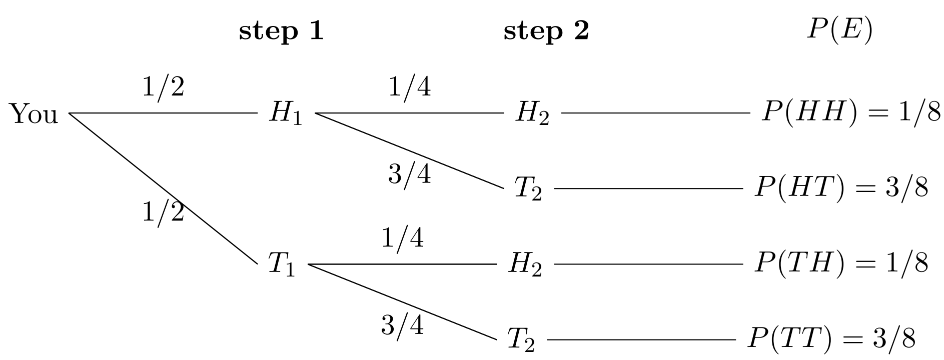
Examples:

1. Consider the experiment that we toss a fair coin two times.



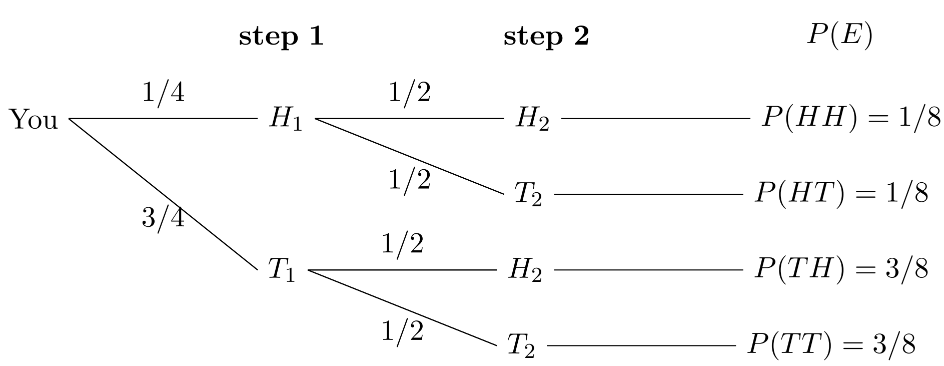
Event E	HH	HT	TH	TT
$P(E)$	1/4	1/4	1/4	1/4

2. Now consider first we flip a fair coin, then an unfair coin.
   The second coin has probability $P(H_2) = 1/4$ and $P(T_2) = 3/4$.



Event E	HH	HT	TH	TT
$P(E)$	1/8	3/8	1/8	3/8

3. Now consider first we flip an unfair coin, then a fair coin.
   The first coin has probability $P(H_1) = 1/4$ and $P(T_1) = 3/4$.

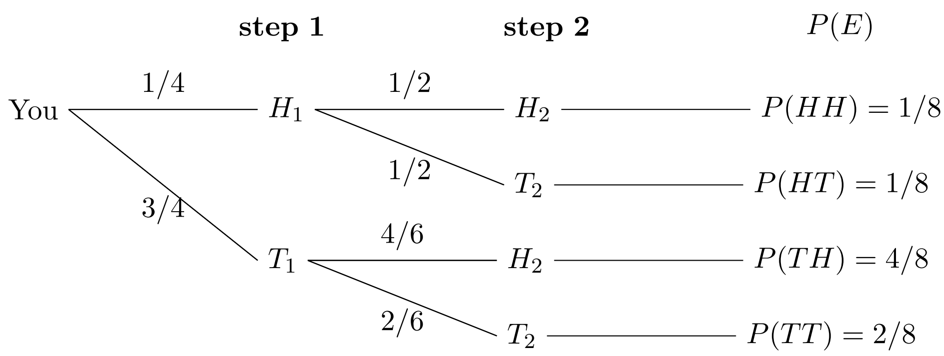


Event E	HH	HT	TH	TT
$P(E)$	1/8	1/8	3/8	3/8

4. One might ask if it is possible to construct the following distribution table:

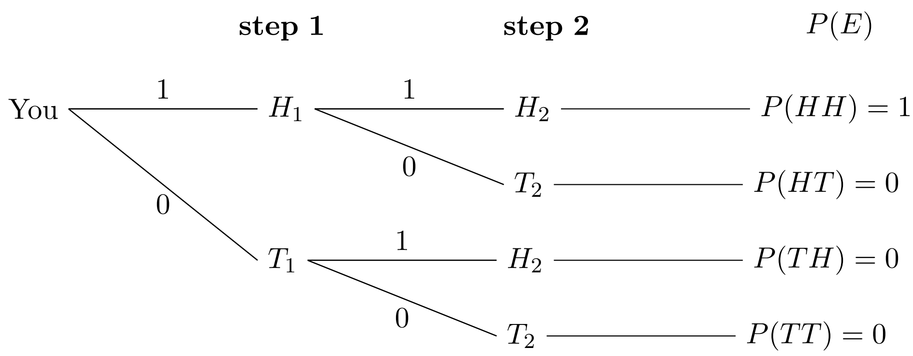
Event E	HH	HT	TH	TT
$P(E)$	1/8	1/8	4/8	2/8

   To construct such a distribution table, we consider the following experiment: In step 1, we toss a coin with $P(H_1) = 1/4$; In step 2, if we got H_1 in step 1, then we toss a coin with $P(H_2) = 1/2$, if we got T_1 in step 1, then we toss a coin with $P(H_2) = 4/6$.



We see that this experiment gives the desired distribution table.

5. Now suppose we toss a two-headed coin two times, that is, $P(H_1) = P(H_2) = 1$.



Event E	HH	HT	TH	TT
$P(E)$	1	0	0	0

We notice in the coin toss examples, the following diagram holds: