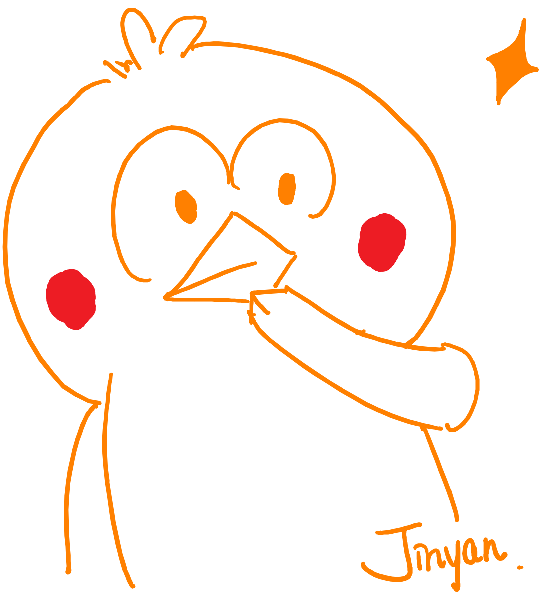
Homework 1

Physics 513 - Quantum Field Theory
Professor Ratindranath Akhoury



Jinyan Miao

Fall 2023

1

For pure rotation about the xy-axes by an angle θ, we can write

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(x) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

when θ is small or infinitesimal, denoted as $\delta\theta$, we can rewrite

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \left(\mathbb{I} + \delta\theta \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

with small angle approximation $\sin(\delta\theta) = \delta\theta$, and $\cos(\delta\theta) = 1$. From which we obtain one of the rotation generator

$$L_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For pure rotation about the xz-axes by an angle θ, we can write

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & 0 & \sin(\theta) \\ 0 & 0 & 1 & 0 \\ 0 & -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

when θ is small or infinitesimal, denoted as $\delta\theta$, we can rewrite

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \left(\mathbb{I} + \delta\theta \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

with small angle approximation $\sin(\delta\theta) = \delta\theta$, and $\cos(\delta\theta) = 1$. From which we obtain one of the rotation generator

$$L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

For pure rotation about the xz-axes by an angle θ, we can write

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

when θ is small or infinitesimal, denoted as $\delta\theta$, we can rewrite

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \left(\mathbb{I} + \delta\theta \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \right) \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

with small angle approximation $\sin(\delta\theta) = \delta\theta$, and $\cos(\delta\theta) = 1$. From which we obtain one of the rotation generator

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Now we consider a boost in x-direction, we can write

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

For infinitesimal boost with speed denoted as δv, we have $\gamma = 1/\sqrt{1-(\delta v)^2} \approx 1$. Thus we can rewrite

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \left(\mathbb{I} - \delta v \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

and thus we obtain the one of the boost generators

$$K_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we consider a boost in y-direction, we can write

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & -\gamma v & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma v & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

For infinitesimal boost with speed denoted as δv, we have $\gamma = 1/\sqrt{1-(\delta v)^2} \approx 1$. Thus we can rewrite

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \left(\mathbb{I} - \delta v \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

and thus we obtain the one of the boost generators

$$K_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we consider a boost in z-direction, we can write

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

For infinitesimal boost with speed denoted as δv, we have $\gamma = 1/\sqrt{1-(\delta v)^2} \approx 1$. Thus we can rewrite

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \left(\mathbb{I} - \delta v \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix},$$

and thus we obtain the one of the boost generators

$$K_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Here we have

$$K_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad K_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad L_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad L_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here we can check the transformation $[K_i, K_j] = K_iK_j - K_jK_i = -\epsilon_{ijk}L_k$.

$$[K_1, K_2] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = -L_3\,,$$

where we see here $\epsilon_{123} = 1$, and thus we have $[K_1, K_2] = -L_3$ as expected.

$$[K_2, K_1] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = L_3\,,$$

where we see here $\epsilon_{213} = -1$, and thus we have $[K_1, K_2] = L_3$ as expected.

$$[K_1, K_3] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = L_2\,,$$

where we see here $\epsilon_{132} = -1$, and thus we have $[K_1, K_3] = L_2$ as expected.

$$[K_3, K_1] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = -L_2\,,$$

where we see here $\epsilon_{312} = 1$, and thus we have $[K_3, K_1] = -L_2$ as expected.

$$[K_2, K_3] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = -L_1\,,$$

where we see here $\epsilon_{231} = 1$, and thus we have $[K_2, K_3] = -L_1$ as expected.

$$[K_3, K_2] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = L_1\,,$$

where we see here $\epsilon_{321} = -1$, and thus we have $[K_3, K_2] = L_1$ as expected.

Next we will verify $[L_i, L_j] = L_iL_j - L_jL_i = \epsilon_{ijk}L_k$.

$$[L_1, L_2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = L_3\,,$$

as we have $\epsilon_{123} = 1$, we see that $[L_1, L_2] = L_3$ as expected.

$$[L_2, L_1] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = -L_3\,,$$

as we have $\epsilon_{213} = -1$, we see that $[L_2, L_1] = -L_3$ as expected.

$$[L_1, L_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = -L_2\,,$$

as we have $\epsilon_{132} = -1$, we see that $[L_1, L_3] = -L_2$ as expected.

$$[L_3, L_1] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = L_2\,,$$

as we have $\epsilon_{312} = 1$, we see that $[L_3, L_1] = L_2$ as expected.

$$[L_2, L_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = L_1\,,$$

as we have $\epsilon_{231} = 1$, we see that $[L_2, L_3] = L_1$ as expected.

$$[L_3, L_2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = -L_1\,,$$

as we have $\epsilon_{321} = -1$, we see that $[L_3, L_2] = L_1$ as expected.

Here we will check $[L_i, K_j] = L_iK_j - K_jL_i = \epsilon_{ijk}K_k$.

$$[L_1, K_2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = K_3\,,$$

where we see that $\epsilon_{123} = 1$, thus we have $[L_1, K_2] = K_3$ as expected.

$$[L_2, K_1] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = -K_3\,,$$

where we see that $\epsilon_{213} = -1$, thus we have $[L_2, K_1] = -K_3$ as expected.

$$[L_1, K_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = -K_2\,,$$

where we see that $\epsilon_{132} = -1$, thus we have $[L_1, K_3] = -K_2$ as expected.

$$[L_3, K_1] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = K_2\,,$$

where we see that $\epsilon_{312} = 1$, thus we have $[L_3, K_1] = K_2$ as expected.

$$[L_2, K_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = K_1\,,$$

where we see that $\epsilon_{231} = 1$, thus we have $[L_2, K_3] = K_1$ as expected.

$$[L_3, K_2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = -K_1\,,$$

where we see that $\epsilon_{321} = -1$, thus we have $[L_3, K_2] = -K_1$ as expected.

Now we have verified all commutation relations $[K_i, K_j] = -\epsilon_{ijk}L_k$, $[L_i, L_j] = \epsilon_{ijk}L_k$, and $[L_i, L_j] = \epsilon_{ijk}K_k$.

2

(a) Here we consider two events A and B, we can set up an inertial frame such that the spacial distance between A and B only has x-component. That is, let S denote such a frame, event A has coordinate $(x_0, x_1, 0, 0)$ in S, and event B has coordinate $(y_0, y_1, 0, 0)$ in S.

If events A and event B are spacelike related, then we can write

$$(y_0 - x_0)^2 - (y_1 - x_1)^2 < 0\,. \tag{2.1}$$

Here we will find an inertial frame S' in which the two event occur at the same time. Suppose S' is obtained by Loretnz boosting S in the x-direction with speed v. Then the coordinate of A in S' is given by

$$\begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma x_0 - \gamma v x_1 \\ -\gamma v x_0 + \gamma x_1 \\ 0 \\ 0 \end{bmatrix}, \tag{2.2}$$

where $\gamma = 1/\sqrt{1-v^2}$. Similarly, the coordinate of B in S' is given by

$$\begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma y_0 - \gamma v y_1 \\ -\gamma v y_0 + \gamma y_1 \\ 0 \\ 0 \end{bmatrix}. \tag{2.3}$$

If the two events have the same time coordinate in S', then we require v to satisfy

$$\begin{aligned} \gamma x_0 - \gamma v x_1 &= \gamma y_0 - \gamma v y_1\,, \\ x_0 - v x_1 &= y_0 - v y_1\,, \\ v y_1 - v x_1 &= y_0 - x_0\,, \\ v(y_1 - x_1) &= (y_0 - x_0)\,, \\ v &= (y_0 - x_0)/(y_1 - x_1)\,. \end{aligned}$$

Note from (2.1) we have that

$$-1 < \frac{y_0 - x_0}{y_1 - x_1} < 1\,, \tag{2.4}$$

and thus $v = (y_0 - x_0)/(y_1 - x_1)$ is well-defined. If $v = (y_0 - x_0)/(y_1 - x_1)$ is satisfied, event A and B have the same time coordinate in frame S'.

If one is able to find a Lorentz frame S' such that the spatial coordinates of events A and B coincide, then, denoting the coordinates of A as (x_0', x_1', x_2', x_3') and that of B as (y_0', y_1', y_2', y_3') in frame S', one can write

$$(y_0' - x_0')^2 - \sum_{i=1}^{3}(y_i' - x_i')^2 = (y_0' - x_0')^2 - 0 = (y_0' - x_0')^2 = (y_0 - x_0)^2 - (y_1 - x_1)^2 < 0\,,$$

but this is not possible as $(y_0' - x_0')^2$ is non-negative.

(b) Now suppose A and B are timelike related, then their coordinates in frame S satisfy the relation

$$(y_0 - x_0)^2 - (y_1 - x_1)^2 > 0\,,$$

rearranging we have

$$-1 < \frac{y_1 - x_1}{y_0 - x_0} < 1\,. \tag{2.5}$$

One immediately notice that S' obtained by a Lorentz boost in the x-direction with velocity v can never yield $x_0' = x_1'$ as Eq. (2.4) gives $v > 1$. However, one can find a frame S' by a Lorentz boost in the x-direction with velocity v such that events A and B have the same spatial coordinate components. Utilizing (2.2) and (2.3), we require

$$\begin{aligned} -\gamma v x_0 + \gamma x_1 &= -\gamma v y_0 + \gamma y_1\,, \\ v &= \frac{y_1 - x_1}{y_0 - x_0}\,. \end{aligned}$$

Such a velocity is possible as guaranteed by (2.5). Thus one can obtain an inertial frame in which events A and B have the same spatial coordinates, in the case where A and B are timelike related. In this case, it is not possible to obtain a frame S' in which A and B have the same time coordinate, as otherwise we would have

$$(y_0' - x_0')^2 - \sum_{i=1}^{3}(y_i' - x_i')^2 = -\sum_{i=1}^{3}(y_i' - x_i')^2 = (y_0 - x_0)^2 - (y_1 - x_1)^2 > 0\,,$$

which is not possible as $-\sum_{i=1}^{3}(y_i' - x_i')^2$ is non-positive.

3

Let $F_{\mu\nu}$ be an antisymmetric tensor. Here we can write

$$\begin{aligned}
(\partial_\nu F_\mu{}^\alpha) F^\nu{}_\alpha &= (\partial_\nu F_\mu{}^\alpha) F^{\nu\rho} g_{\rho\alpha} \\
&= (\partial_\nu F_\mu{}^\alpha g_{\rho\alpha}) F^{\nu\rho} \\
&= (\partial_\nu F_{\mu\rho}) F^{\nu\rho} \\
&= -(\partial_\nu F_{\mu\rho}) F^{\rho\nu} \\
&= -(\partial_\beta F_{\mu\alpha}) F^{\alpha\beta} \,,
\end{aligned}$$

where we have applied change of indices $\rho \to \alpha$ and $\nu \to \beta$ in the last step.

4

Consider a conserved current j^μ, and define the total charge as

$$Q = \int_M d^3x \, j^0(\vec{x}, t) \,. \tag{4.1}$$

where M is the region of interest.

(a) Here we write

$$\frac{dQ}{dt} = \frac{d}{dt} \int_M d^3x \, j^0(\vec{x}, t) \; = \int_M d^3x \, \frac{d}{dt} j^0(\vec{x}, t) \; = -\int_M d^3x \, \nabla \cdot \vec{j} \,,$$

where we have used the conservation law $\partial_\alpha j^\alpha(x) = 0$ in the last equality. Now applying Stokes' Theorem, with the assumption that M is large enough such that there is no net current flux on the surface ∂M, we can write

$$\frac{dQ}{dt} = -\int_M \nabla \cdot \vec{j} \, dV = -\int_{\partial M} \vec{j}(x) \cdot \hat{n} \, dS = 0 \,.$$

Thus we see that Q is independent of time.

(b) To see that Q is a Lorentz scalar, we redefine Q as

$$Q = \int_{M'} d^4x \, j^a(x) \, \partial_a \theta(n_\beta x^\beta) \tag{4.2}$$

where M' is the entire spacetime, θ is the Heaviside step function, and n_λ (is not a tensor, but for notation here, $n_\beta x^\beta$ indicate summing same indices) is defined by $n_1 = n_2 = n_3 = 0$, with $n_0 = 1$. To see that Q (4.2) agrees with (4.1), we simply expand

$$Q = \int_{M'} d^4x \, j^a(x) \, \partial_a \theta(n_\beta x^\beta) = \int_{M'} d^4x \, j^a(x) \, \partial_a \theta(t) = \int_{M'} d^4x \, j^0(x) \, \delta t \; = \int_M d^3x \, j^0(x) \,.$$

The effect of a Lorentz transformation on Q is then simply to change n,

$$n'_\beta = \Lambda^\gamma{}_\beta n_\gamma$$

for any Lorentz transformation $\Lambda^\gamma{}_\beta$. Then we can write

$$\begin{aligned} Q' - Q &= \int_{M'} d^4x \, j^a(x) \left(\partial_a \theta(n'_\beta x^\beta) - \partial_a \theta(n_\beta x^\beta) \right) \\ &= \int_{M'} d^4x \, j^a(x) \; \partial_a \left(\theta(n'_\beta x^\beta) - \theta(n_\beta x^\beta) \right) \,. \end{aligned} \tag{4.3}$$

From product rule, we see that, for reasonable function $k(x)$, we have

$$\int_{M'} d^4x\, \partial_a \left(j^a(x)\, k(x)\right) = \int_{M'} d^4x\, \left(k(x)\, \partial_a j^a(x) + j^a(x)\, \partial_a k(x)\right) = \int_{M'} d^4x\, j^a(x)\, \partial_a k(x)\,,$$

with the help of conservation law $\partial_a j^a = 0$ everywhere. And thus (4.3) becomes

$$\begin{aligned} Q' - Q &= \int_{M'} d^4x\, \partial_a \left(j^a(x) \left(\theta(n'_\beta x^\beta) - \theta(n_\beta x^\beta)\right)\right) \\ &= \int_{\partial M'} d^3x\; j^a(x) \left(\theta(n'_\beta x^\beta) - \theta(n_\beta x^\beta)\right)\,, \end{aligned} \tag{4.4}$$

where we have applied Stokes' Theorem. With the assumption that $j^a(x)$ vanishes for large enough $|x|$ at fixed t, and $\theta(n'_\beta x^\beta) - \theta(n_\beta x^\beta)$ vanishes for large enough $|t|$ with fixed x, we conclude that we have

$$Q - Q' = \int_{\partial M} d^3x\; j^a(x) \left(\theta(n'_\beta x^\beta) - \theta(n_\beta x^\beta)\right) = 0\,,$$

showing that Q is a Lorentz scalar.