Examples of Continuous Random Variables

Uniform Distribution: We say that X has the uniform distribution on the interval $[A, B] \subseteq \mathbb{R}$ provided that the pdf is given by

$$f_X(x) = \begin{cases} \frac{1}{B-A} & x \in [A, B] \\ 0 & \text{otherwise} \end{cases} .$$

The cdf of the uniform distribution can be computed via the forllowing: For $x \leq A$,

$$F_X(x) = \int_{-\infty}^{x} f_X(u)\, du = 0 .$$

For $x \in [A, B]$,

$$F_X(x) = \int_A^x \frac{1}{B-A}\, du = \frac{x-A}{B-A} = \left(-\frac{A}{B-A}\right) + \left(\frac{1}{B-A}\right) x .$$

Furthermore, if $x \geq B$, $F_X(x) = 1$. Thus we conclude

$$F_X(x) = \begin{cases} 0 & x \leq A \\ \frac{x-A}{B-A} & x \in [A, B] \\ 1 & x \geq B \end{cases} .$$

Now we can calculate

$$E(X) = \frac{B+A}{2}, \qquad E(X^2) = \frac{B^2 + AB + A^2}{3}, \qquad V(X) = \frac{(B-A)^2}{12}, \qquad M_X(t) = \frac{e^{Bt} - e^{At}}{(B-A)t} .$$

Normal Distribution: A random variable Z has the standard normal distribution provided that the pdf of Z is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} ,$$

for $z \in \mathbb{R}$. To show that f_Z is a pdf, we notice that $f_Z(z) \geq 0$ for all z, the normalization is checked by computing

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = 1 .$$

To compute this integral, we notice that

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2/2}\, dx \cdot \int_{-\infty}^{\infty} e^{-y^2/2}\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\, dy\, dx ,$$

where we have made use of the Fubini's Theorem. Here we switch to the polar coordinate, $x = r\cos(\theta)$ and $y = r\sin(\theta)$, with

$$J = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}, \qquad \text{thus } \det(J) = r(\cos^2(\theta) + \sin^2(\theta)) = r .$$

Then we have

$$\begin{aligned} \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2/2}\, r\, dr\, d\theta &= \int_0^{2\pi} \left(\lim_{c\to\infty} \int_0^c re^{-r^2/2}\, dr\right) d\theta \\ &= \int_0^{2\pi} \left(\lim_{c\to\infty} \int_0^{c^2/2} e^{-u}\, du\right) d\theta \\ &= \int_0^{2\pi} \left(\lim_{c\to\infty} \left(1 - e^{-c^2/2}\right)\right) d\theta \\ &= \int_0^{2\pi} d\theta = 2\pi , \end{aligned}$$

where we have set $u = r^2/2$, thus $du = r\, dr$. It follows from here that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi} ,$$

and rearranging we obtain the desired result.

On the other hand, the cdf of Z has no closed form formula,

$$\Phi(z) = F_Z(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du\,.$$

To compute $\Phi(z)$, one needs numerical approximation.

Next we compute the expected value, the variance, and the mgf for Z.

1. Expected value. Here we compute

$$\mu_Z = E(Z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot z e^{-z^2/2}\, dz\,.$$

   We notice that $f(z) = ze^{-z^2/2}$ is an odd function, that is $f(z) = -f(-z)$, but we cannot use $\int_{-c}^{c} f(z)\,dz = 0$ to calculate $\int_{-\infty}^{\infty} f(z)\,dz$. Instead, we should write

$$\int_{-\infty}^{\infty} z\, f_Z(z)\, dz = \int_{-\infty}^{0} z\, f_Z(z)\, dz + \int_{0}^{\infty} z f_Z(z)\, dz\,.$$

   Here we have

$$\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} z e^{-z^2/2}\, dz = \lim_{c\to\infty} \int_{0}^{c} \frac{1}{\sqrt{2\pi}} z e^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \lim_{c\to\infty} \int_{0}^{c} z e^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}}\,.$$

   Similarly,

$$\int_{-\infty}^{0} z f_Z(z)\, dz = \frac{1}{\sqrt{2\pi}} \lim_{c\to-\infty} \left(\int_{c}^{c} z e^{-z^2/2}\, dz \right) = -\frac{1}{\sqrt{2\pi}}\,.$$

   Thus we conclude $E(Z) = 0$.

2. Now to calculate $V(Z)$, one fines that $E(Z^2) = 1$ and thus

$$V(Z) = E(Z^2) - (E(Z))^2 = 1\,.$$

3. Finally, we compute the mgf.

$$\begin{aligned}
M_Z(t) = E(e^{tZ}) &= \int_{-\infty}^{\infty} e^{tz} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}\, dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2+tz}\, dz \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2+tz+t^2/2-t^2/2}\, dz \\
&= \int_{-\infty}^{\infty} e^{t^2/2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-(z-t)^2/2}\, dz \\
&= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du \\
&= e^{t^2/2}\,,
\end{aligned}$$

   where we have set $u = z - t$ and thus $du = dz$.

Definition 0.1
Let Z has the standard normal distribution. X is called a normal distribution with parameters μ and σ^2, where μ is the mean and σ^2 is the variance, provided that $X = \sigma Z + \mu$, where $\sigma = \sqrt{\sigma^2}$. Such a normal distribution is denoted as $X \sim \mathrm{N}(\mu, \sigma^2)$

To calculate the pdf of X, we first need to calculate the cdf $F_X(x)$, then if $F_X(x)$ is differentiable, we obtain $f_X(x) = F_X'(x)$ from $F_X(x)$. For $x \in \mathbb{R}$, we see that

$$F_X(x) = P(X \le x) = P(\sigma Z + \mu \le x) = P(Z \le (x-\mu)/\sigma)\,.$$

Thus we see that

$$F_X(x) = F_Z((x-\mu)/\sigma)\,.$$

Using chain rule, we can write

$$f_X(x) = \frac{d}{dx}\left(F_X(x)\right) = \frac{d}{dx}\left(F_Z\left(\frac{x-\mu}{\sigma}\right)\right) = F_Z'\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{d}{dx}\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,,$$

for $x \in \mathbb{R}$. Now we want to calculate the followings

1. The cdf of X. Here we write

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-(u-\mu)^2/(2\sigma^2)}\, du = F_Z\left(\frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

   where $(X-\mu)/\sigma$ is the standardized normal distribution, or the z-score of X. Here we see that we can use the cdf of Z to calculate the cdf of X.

2. The mean of $X \sim \mathrm{N}(\mu, \sigma^2)$ is calculated via $X = \sigma Z + \mu$,

$$E(X) = E(\sigma Z + \mu) = \sigma E(Z) + \mu = \mu\,.$$

3. The variance of X is calculated similarly,

$$V(X) = V(\sigma Z + \mu) = \sigma^2 V(Z) = \sigma^2\,.$$

4. The mgf of X,

$$M_X(t) = M_{\sigma Z+\mu}(t) = e^{\mu t} \cdot M_Z(\sigma t) = e^{\mu t} \cdot e^{(\sigma t)^2/2} = e^{\mu t + (\sigma^2 t^2)/2}\,.$$

Example: Suppose John takes an exam, and he scores 72/100 on the exam. Without additional information, John has a C. John is then told that $\mu = 70$ and $\sigma = 2$ for this exam. That is, if the distribution is normal, 68% of the class is in the range $(68, 72)$; 95% of the class is in the range $(66, 74)$, and 99.7% of the class is in the range $(64, 76)$. Thus, only 18% of the class has done better than John, so John deserves a B. However, the fundamental problem here is that John is assuming the class grades are normally distributed, which is an incorrect assumption.

Gamma Distribution:

Definition 0.2
Let $\alpha > 0$, the function $\Gamma : \mathbb{R}^+ \to \mathbb{R}_{\geq 0}$ defined by

$$\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$

is called the Gamma function.

The Gamma function has the following properties:

1. $\Gamma(\alpha) > 0$ for all $\alpha > 0$, thus $\Gamma : \mathbb{R}^+ \to \mathbb{R}_{\geq 0}$.

2. $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, this is seen from integration by parts with $u = t^\alpha$, $dv = e^{-t}$ in the integral

$$\Gamma(\alpha+1) = \int_0^\infty t^{\alpha+1-1} e^{-t}\, dt = \int_0^\infty t^\alpha e^{-t}\, dt\,,$$

   with $du = \alpha t^{\alpha-1}\, dt$ and $v = -e^{-t}$, we obtain the desired result.

3. $\Gamma(1) = 1$, as we have

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1\,.$$

4. From (2) and (3), we obtain that for all $n \in \mathbb{N}$,

$$\Gamma(n+1) = n \cdot \Gamma(n) = n \cdot (n-1) \cdot \Gamma(n-1) = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 \cdot \Gamma(1) = n!\,,$$

   thus Γ interpolates the factorial function.

5. $\Gamma(1/2) = \sqrt{\pi}$, the proof of this fact is an easy exercise.

Definition 0.3
We say that T has the standard Gamma distribution with α being the shape parameter provided that the pdf of T is defined by

$$f_T(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-t} & t > 0 \\ 0 & \text{otherwise} \end{cases}.$$

We observe that

$$\int_{-\infty}^{\infty} f_T(t) = \int_0^{\infty} \frac{1}{\Gamma(\alpha)} \cdot t^{\alpha-1} \cdot e^{-t}\, dt = \frac{1}{\Gamma(\alpha)} \cdot \int_0^{\infty} t^{\alpha-1} e^{-t}\, dt = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1\,.$$

Now suppose we have $T \sim$Gamma$(\alpha, 1)$. We calculate $E(T)$, $V(T)$, and $M_T(t)$:

1. For the expectation value,

$$E(T) = \int_{-\infty}^{\infty} t \cdot f_T(t)\, dt = \int_0^{\infty} \frac{t}{\Gamma(\alpha)} \cdot t^{(\alpha-1)} e^{-t}\, dt = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \int_0^{\infty} \frac{t^{(\alpha+1)-1} \cdot e^{-t}}{\Gamma(\alpha+1)}\, dt = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha\,.$$

2. For the variance,

$$E(T^2) = \int_0^{\infty} \frac{t^2}{\Gamma(\alpha)} \cdot t^{\alpha-1} \cdot e^{-t}\, dt = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \int_0^{\infty} \frac{t^{(\alpha+2)-1)} \cdot e^{-t}}{\Gamma(\alpha+2)}\, dt = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} = \frac{(\alpha+1) \cdot \Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha(\alpha+1)$$

   Thus we have

$$V(T) = E(T^2) - (E(T))^2 = \alpha(\alpha-1) - \alpha^2 = \alpha\,.$$

3. Lastly we calculate $M_T(t)$,

$$\begin{aligned} M_T(s) = E(e^{sT}) = \int_0^{\infty} \frac{e^{st}}{\Gamma(\alpha)} \cdot t^{\alpha-1} e^{-t}\, dt &= \int_0^{\infty} \frac{t^{\alpha-1} e^{-t(1-s)}}{\Gamma(\alpha)}\, dt \\ &= \int_0^{\infty} \frac{1}{\Gamma(\alpha)} \cdot \left(\frac{u}{1-s}\right)^{\alpha-1} \cdot e^{-u} \frac{du}{1-s} \\ &= \left(\frac{1}{1-s}\right)^{\alpha} \int_0^{\infty} \frac{1}{\Gamma(\alpha)} \cdot u^{\alpha-1} e^{-u}\, du \\ &= \left(\frac{1}{1-s}\right)^{\alpha}, \end{aligned}$$

   where we have used the change of variables $u = t(1-s)$ thus $du/(1-s) = dt$.

Definition 0.4
We say X has the Gamma distribution with parameters α and β, where α denotes the shape parameter and β denotes the scale parameter, provided that $X = \beta T$ where T has the standard Gamma distribution with shape α.

Consider $X \sim$Gamma(α, β) and $T \sim$Gamma$(\alpha, 1)$. We would like to calculate the pdf of X. As usual, we first calculate the cdf then we differentiate the cdf.

$$F_X(x) = P(X \leq x) = P(\beta T \leq x) = P(T \leq x/\beta) = F_T(x/\beta)\,,$$

thus we can write

$$f_X(x) = \frac{d}{dx}(F_x(x)) = \frac{d}{dx}\left(F_T\left(\frac{x}{\beta}\right)\right) = F_T'\left(\frac{x}{\beta}\right) \cdot \frac{1}{\beta} = \begin{cases} \frac{1}{\Gamma(\alpha)} \cdot \frac{x^{\alpha-1}}{\beta^{\alpha}} \cdot e^{-x/\beta} & x > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Now we would like to calculate $E(X)$, $V(X)$ and $M_X(t)$.

1. For $E(X)$,

$$E(X) = E(\beta T) = \beta \cdot E(T) = \alpha\beta\,.$$

2. For $V(X)$,

$$V(X) = V(\beta T) = \beta^2 \cdot V(T)\,.$$

3. For $M_X(t)$,

$$M_X(t) = M_{\beta T}(t) = M_T(\beta t) = \left(\frac{1}{1-\beta t}\right)^{\alpha}.$$

Special Cases of the Gamma Distribution

a. **Exponential Distribution**: We say that X has the exponential distribution with λ being the rate parameter provided that X ~Gamma($\alpha = 1,\ \beta = 1/\lambda$). The pdf of X is defined by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

If X ~Exp(λ), here we can calculate:

(a) $E(X) = \alpha \cdot \beta = 1/\lambda$;
(b) $V(X) = \alpha\beta^2 = 1/\lambda^2$;
(c) $M_X(t) = 1/(1 - t/\lambda) = \lambda/(\lambda - t)$.

Note that the cdf of X ~Exp(λ) is given by

$$F_X(x) = \int_0^x \lambda \cdot e^{-\lambda u}\, du = \lambda \cdot \left.\frac{e^{-\lambda u}}{-\lambda}\right|_0^x = 1 - e^{\lambda x}\,.$$

The exponential distribution has the memoryless property. That is,

$$P(X > t + s | X > t) = \frac{P(X > t + s, X > t)}{P(X > t)} = \frac{P(X > t + s)}{P(X > t)} = \frac{1 - F_X(t + s)}{1 - F_X(t)} = \frac{e^{-(t+s)}}{e^{-t}} = e^{-s} = P(X > s)\,.$$

b. **Chi-Squared Distribution** We say that X has the Chi-squared distribution with parameter $\nu \in \mathbb{N}$ being the degrees of freedom provided that X ~Gamma($\alpha = \nu/2,\ \beta = 2$). Such a Chi-squared distribution is denoted by $X \sim \chi^2_\nu$. The pdf of X is defined by

$$f_X(x) = \begin{cases} \frac{1}{\Gamma(\nu/2)\cdot 2^{\nu/2}} x^{(\nu/2-1)} \cdot e^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Here we suppose $X \sim \chi^2_\nu$, it is not hard to compute

1. $E(X) = (\nu/2) \cdot 2 = \nu$;
2. $V(X) = (\nu/2) \cdot 2^2 = 2\nu$;
3. $M_X(t) = (1/(1 - 2t))^{\nu/2}$.

Theorem 0.5
Suppose Z ~N$(0, 1)$, then $X = Z^2$ has the distribution χ^2_1.

Example: Measuring the height of an object. Suppose the height is μ, and X be the single measurement of the height. It is reasonable to assume that X ~N(μ, σ^2). One is interested in modeling the error in the measurement, that is how X deviates from the true (unknown) height μ. The standardized deviation is

$$Z = \frac{X - \mu}{\sigma}\,,$$

and the squared standardized deviation of X from μ is thus

$$Z^2 = \left(\frac{X - \mu}{\sigma}\right)^2\,.$$

Other Continuous Random Variables

1. **Beta Distribution**: We say X has a Beta distribution on $(0, 1)$ with parameters $\alpha, \beta > 0$, denoted as X ~Beta(α, β), provided that its pdf is defined by

$$f_X(x) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\cdot\Gamma(\beta)} x^{\alpha-1} \cdot (1 - x)^{\beta-1} & x \in (0, 1) \\ 0 & \text{otherwise} \end{cases}.$$

Such a Beta distribution has

$$E(X) = \alpha/(\alpha + \beta)\,, \qquad \text{and } V(X) = \alpha\beta/((\alpha + \beta)^2 \cdot (\alpha + \beta + 1))\,.$$

2. **Cauchy Distribution**. We say X has the Cauchy distribution provided that its pdf is defined by

$$f_X(x) = \frac{1}{\pi}\left(\frac{1}{1 + x^2}\right)$$

for all $x \in \mathbb{R}$. The expectation value of the Cauchy distribution does not exist as the integral

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{1 + x^2}\, dx$$

does not converge.