

Final Exam

Math 505 - Classical Field Theory
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Time Difference between Two Events

In frame K , at time t_A , a person M located at (x_A, y_A, z_A) shoots an arrow at a target located at (x_B, y_B, z_B) . The arrow hits the target at time t_B . An observer in the primed frame K' witnessed the event, and from his perspective, M discharged the arrow at time t'_A with the arrow reaching the target at time t'_B .

Here we will compute the minimum possible value $t'_B - t'_A$ in terms of space-time coordinates in the K frame.

Assuming that the two frames are moving relative to each other in the xx' -direction with speed v , from Lorentz transformation, we have

$$t'_B = \gamma(t_B - vx_B/c^2), \quad t'_A = \gamma(t_A - vx_A/c^2).$$

Thus we can write

$$\begin{aligned} t'_B - t'_A &= \frac{1}{\sqrt{1 - v^2/c^2}}(t_B - t_A) - \frac{v}{c^2} \frac{1}{\sqrt{1 - v^2/c^2}}(x_B - x_A) \\ &= \frac{c}{\sqrt{c^2 - v^2}}(t_B - t_A) - \frac{v/c}{\sqrt{c^2 - v^2}}(x_B - x_A). \end{aligned}$$

For simplicity, we denote $t_B - t_A = T$ and $x_B - x_A = X$. Differentiating with respect to v we obtain

$$\frac{d}{dv}(t'_B - t'_A) = \frac{Tv - X}{(c^2 - v^2)\sqrt{1 - v^2/c^2}},$$

which vanishes when we have

$$\begin{aligned} Tv - X &= 0 \\ v &= X/T = (x_B - x_A)/(t_B - t_A). \end{aligned}$$

That is, $t'_B - t'_A$ obtains its minimum when $v = (x_B - x_A)/(t_B - t_A)$, in which case we can write

$$\begin{aligned} t'_B - t'_A &= c \left(c^2 - \left(\frac{x_B - x_A}{t_B - t_A} \right)^2 \right)^{-1/2} (t_B - t_A) - \frac{v}{c} \left(c^2 - \left(\frac{x_B - x_A}{t_B - t_A} \right)^2 \right)^{-1/2} (x_B - x_A) \\ &= (c^2(t_B - t_A)^2 - (x_B - x_A)^2)^{1/2} c^{-1}, \end{aligned}$$

which is minimized when we have $x_B - x_A$ being maximized. That is, with the assumption that K' and K frame moving with respect to each other in the xx' -direction, if we want to minimize $t'_B - t'_A$, then the arrow should be moving in the x -direction in the K frame. In that case, the minimized time interval in K' frame is given by

$$t'_B - t'_A = (c^2(t_B - t_A)^2 - (x_B - x_A)^2)^{1/2} c^{-1}. \quad (1)$$

This result can be derived via spacetime interval. We can write

$$\begin{aligned} c^2(t_B - t_A)^2 - (x_B - x_A)^2 - (y_B - y_A)^2 - (z_B - z_A)^2 \\ = c^2(t'_B - t'_A)^2 - (x'_B - x'_A)^2 - (y'_B - y'_A)^2 - (z'_B - z'_A)^2. \end{aligned} \quad (2)$$

As we have timelike events, we also have

$$0 \leq c^2(t'_B - t'_A)^2 - (x'_B - x'_A)^2 - (y'_B - y'_A)^2 - (z'_B - z'_A)^2.$$

The LHS of Eq. (2) is fixed, thus $t'_B - t'_A$ is minimized when

$$(x'_B - x'_A)^2 + (y'_B - y'_A)^2 + (z'_B - z'_A)^2 = 0.$$

From Math 513 HW1¹, we can find a Lorentz frame K' such that the spatial coordinates of event A (arrow is shot by the person) coincide with the spatial coordinates of event B (arrow hits the target). In that case, we simply obtain

$$c^2(t'_B - t'_A)^2 = c^2(t_B - t_A)^2 - (x_B - x_A)^2 - (y_B - y_A)^2 - (z_B - z_A)^2.$$

Rearranging we have

$$t'_B - t'_A = \left(c^2(t_B - t_A)^2 - (x_B - x_A)^2 - (y_B - y_A)^2 - (z_B - z_A)^2 \right)^{-1/2} c^{-1},$$

which agrees with the result that we obtained above, except we have assumed the arrow moves in the x -direction when deriving Eq. (1).

¹The proof is attached at the end of the text

Acceleration of a Charged Particle

In the context of special relativity, under the influence of electric and magnetic fields, the equation of motion for a particle with a charge q is

$$\frac{d\vec{p}}{dt} = \vec{F} = q \left(\vec{E} + \vec{v} \times \vec{B} \right) ,$$

and the energy obeys

$$\frac{dE}{dt} = \vec{F} \cdot \vec{v} = q \vec{E} \cdot \vec{v} .$$

We will show that we have

$$\frac{d\vec{v}}{dt} = \frac{q}{m_0} \sqrt{1 - \frac{v^2}{c^2}} \left(\vec{E} + \vec{v} \times \vec{B} - \frac{\vec{v}}{c^2} (\vec{v} \cdot \vec{E}) \right) ,$$

where m_0 is the rest mass of the particle.

From class result, we have

$$\frac{d}{dt}(m\vec{v}) = \frac{d}{dt} \left(\frac{m_0 \vec{v}}{\sqrt{1 - v^2/c^2}} \right) = \frac{d\vec{p}}{dt} = q \left(\vec{E} + \vec{v} \times \vec{B} \right) .$$

Thus we have

$$\frac{dm}{dt} \vec{v} + \frac{d\vec{v}}{dt} m = q(\vec{E} + \vec{v} \times \vec{B}) .$$

Rearranging we obtain

$$\frac{d\vec{v}}{dt} = \frac{q}{m_0} \left(1 - \frac{v^2}{c^2} \right)^{1/2} \left((\vec{E} + \vec{v} \times \vec{B}) - \frac{\vec{v}}{q} \frac{dm}{dt} \right) .$$

Thus we are left to evaluate

$$\frac{\vec{v}}{q} \frac{dm}{dt} = \frac{\vec{v}}{qc^2} \frac{dE}{dt} = \frac{\vec{v}}{qc^2} (q \vec{E} \cdot \vec{v}) = \frac{\vec{v}}{c^2} \vec{E} \cdot \vec{v} .$$

Combining we obtain

$$\frac{d\vec{v}}{dt} = \frac{q}{m_0} \left(1 - \frac{v^2}{c^2} \right)^{1/2} \left((\vec{E} + \vec{v} \times \vec{B}) - \frac{\vec{v}}{c^2} (\vec{v} \cdot \vec{E}) \right)$$

as expected.

Explosion in Special Relativity

An object with rest mass m_0 fragments into two pieces with rest masses m_A and m_B respectively. We will determine the relative velocity between the two fragments.

We select an inertial frame where the initial velocity of the object is zero. That is, before the explosion, the object is at rest, thus we have

$$p^\mu = \begin{bmatrix} m_0 c \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

After the explosion, we have

$$p^\mu = \begin{bmatrix} \gamma_A m_A c \\ \gamma_A m_B v_{A,x} \\ \gamma_A m_B v_{A,y} \\ \gamma_A m_B v_{A,z} \end{bmatrix} + \begin{bmatrix} \gamma_B m_B c \\ \gamma_B m_B v_{B,x} \\ \gamma_B m_B v_{B,y} \\ \gamma_B m_B v_{B,z} \end{bmatrix}.$$

Thus by conservation, we have,

$$\begin{bmatrix} m_0 c \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma_A m_A c \\ \gamma_A m_B v_{A,x} \\ \gamma_A m_B v_{A,y} \\ \gamma_A m_B v_{A,z} \end{bmatrix} + \begin{bmatrix} \gamma_B m_B c \\ \gamma_B m_B v_{B,x} \\ \gamma_B m_B v_{B,y} \\ \gamma_B m_B v_{B,z} \end{bmatrix},$$

where we have

$$\gamma_B = \frac{1}{\sqrt{1 - v_B^2/c^2}}, \quad \gamma_A = \frac{1}{\sqrt{1 - v_A^2/c^2}}.$$

Thus we have

$$m_0 = \gamma_A m_A + \gamma_B m_B, \quad m_B \gamma_B \vec{v}_B = -m_A \gamma_A \vec{v}_A,$$

From which we conclude that the fragments are moving antiparallel, in exactly the opposite direction. Solving the system (according the Physics 505 Class Notes) yields

$$v_A = c \left(1 - \frac{4m_0^2 m_A^2}{(m_0^2 + m_A^2 - m_B^2)^2} \right)^{1/2}, \quad v_B = c \left(1 - \frac{4m_0^2 m_B^2}{(m_0^2 + m_B^2 - m_A^2)^2} \right)^{1/2}.$$

Thus in this inertial frame, as the two are moving anti-parallel, the relative speed is

$$v_B - v_A = c \left(\left(1 - \frac{4m_0^2 m_B^2}{(m_0^2 + m_B^2 - m_A^2)^2} \right)^{1/2} - \left(1 - \frac{4m_0^2 m_A^2}{(m_0^2 + m_A^2 - m_B^2)^2} \right)^{1/2} \right)$$

We note that in other reference frame, the two fragments are not necessarily moving antiparallel to each other, the computation of the relative velocity between the two in other reference frames would then be more complicated.

Here's an outline of the computation of the relative velocity of the two fragments in other reference frames. Suppose in the current frame, the two fragments are moving

antiparallel to each other along the x -axis, and the initial object exploded at $(0, 0, 0, 0)$. This assumption does not alter any result derived above as translation and rotation of the axes in the current frame does not contradict with our assumption that the object is initially at rest in the current frame. In this case, we can write

$$x_A^\mu = (ct, v_A t, 0, 0), \quad x_B^\mu = (ct, -v_B t, 0, 0), \quad (3)$$

Computing the relative velocity of the two fragments in other frames involve Lorentz transforming Eq. (3) into other frames and taking the derivative with respect to new coordinate time.

Energy Flow

Consider a uniform electric field \vec{E} and a uniform magnetic field \vec{B} , the two fields are not parallel to each other. Consider further that there are two observers, one remains stationary in the current frame, and the other moves with a constant velocity \vec{v} perpendicular to both \vec{E} and \vec{B} . Both observers attempt to measure the energy current of the field. We will show that they measure the energy current to be in the same direction.

According to results from class, the energy current density as measured by the stationary observer has the direction

$$\vec{S} \propto (T^{10}, T^{20}, T^{30}) \propto \vec{E} \times \vec{B},$$

with \vec{E} and \vec{B} are measured in the current frame. WLOG, we set up the axes in the current frame such that $\vec{v} = (v, 0, 0)$. Thus \vec{B} and \vec{E} lie in the yz -plane. Now we transform to the primed frame that is moving with velocity \vec{v} with respect to the (unprimed) current frame. Then utilizing result from class, in the primed frame, we have

$$\begin{aligned} E'_x &= E_x = 0, & E'_y &= \gamma(E_y - vB_z), & E'_z &= \gamma(E_z + vB_y), \\ B'_x &= B_x = 0, & B'_y &= \gamma(B_y + vE_z/c^2), & B'_z &= \gamma(B_z - vE_y/c^2). \end{aligned}$$

In this frame, the energy current is thus

$$\vec{S}' \propto \vec{E}' \times \vec{B}'.$$

We are now left to show $\vec{E}' \times \vec{B}' \propto \vec{E} \times \vec{B}$. From direct computation, we obtain

$$\vec{E}' \times \vec{B}' = \gamma^2 \begin{bmatrix} (E_y - vB_z)(B_z - vE_y/c^2) - (E_z + vB_y)(B_y + vE_z/c^2) \\ 0 \\ 0 \end{bmatrix} \propto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Notice that on the other hand we have

$$\vec{E} \times \vec{B} \propto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

as both \vec{E} and \vec{B} are perpendicular to the x -axis. Thus we have shown that we have $\vec{E} \times \vec{B} \propto \vec{E}' \times \vec{B}'$, from which we conclude that $\vec{S}' \propto \vec{S}$, the energy current measured by the two observers have the same direction.

End of the exam solution

The next two pages display supplement materials to the first problem in the exam.

2

(a) Here we consider two events A and B , we can set up an inertial frame such that the spacial distance between A and B only has x -component. That is, let S denote such a frame, event A has coordinate $(x_0, x_1, 0, 0)$ in S , and event B has coordinate $(y_0, y_1, 0, 0)$ in S .

If events A and event B are spacelike related, then we can write

$$(y_0 - x_0)^2 - (y_1 - x_1)^2 < 0. \quad (2.1)$$

Here we will find an inertial frame S' in which the two event occur at the same time. Suppose S' is obtained by Loretnz boosting S in the x -direction with speed v . Then the coordinate of A in S' is given by

$$\begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma x_0 - \gamma v x_1 \\ -\gamma v x_0 + \gamma x_1 \\ 0 \\ 0 \end{bmatrix}, \quad (2.2)$$

where $\gamma = 1/\sqrt{1-v^2}$. Similarly, the coordinate of B in S' is given by

$$\begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma y_0 - \gamma v y_1 \\ -\gamma v y_0 + \gamma y_1 \\ 0 \\ 0 \end{bmatrix}. \quad (2.3)$$

If the two events have the same time coordinate in S' , then we require v to satisfy

$$\begin{aligned} \gamma x_0 - \gamma v x_1 &= \gamma y_0 - \gamma v y_1, \\ x_0 - v x_1 &= y_0 - v y_1, \\ v y_1 - v x_1 &= y_0 - x_0, \\ v(y_1 - x_1) &= (y_0 - x_0), \\ v &= (y_0 - x_0)/(y_1 - x_1). \end{aligned}$$

Note from (2.1) we have that

$$-1 < \frac{y_0 - x_0}{y_1 - x_1} < 1, \quad (2.4)$$

and thus $v = (y_0 - x_0)/(y_1 - x_1)$ is well-defined. If $v = (y_0 - x_0)/(y_1 - x_1)$ is satisfied, event A and B have the same time coordinate in frame S' .

If one is able to find a Lorentz frame S' such that the spatial coordinates of events A and B coincide, then, denoting the coordinates of A as (x'_0, x'_1, x'_2, x'_3) and that of B as (y'_0, y'_1, y'_2, y'_3) in frame S' , one can write

$$(y'_0 - x'_0)^2 - \sum_{i=1}^3 (y'_i - x'_i)^2 = (y'_0 - x'_0)^2 - 0 = (y'_0 - x'_0)^2 = (y_0 - x_0)^2 - (y_1 - x_1)^2 < 0,$$

but this is not possible as $(y'_0 - x'_0)^2$ is non-negative.

(b) Now suppose A and B are timelike related, then their coordinates in frame S satisfy the relation

$$(y_0 - x_0)^2 - (y_1 - x_1)^2 > 0,$$

rearranging we have

$$-1 < \frac{y_1 - x_1}{y_0 - x_0} < 1. \quad (2.5)$$

One immediately notice that S' obtained by a Lorentz boost in the x -direction with velocity v can never yield $x'_0 = x'_1$ as Eq. (2.4) gives $v > 1$. However, one can find a frame S' by a Lorentz boost in the x -direction with velocity v such that events A and B have the same spatial coordinate components. Utilizing (2.2) and (2.3), we require

$$\begin{aligned} -\gamma v x_0 + \gamma x_1 &= -\gamma v y_0 + \gamma y_1, \\ v &= \frac{y_1 - x_1}{y_0 - x_0}. \end{aligned}$$

Such a velocity is possible as guaranteed by (2.5). Thus one can obtain an inertial frame in which events A and B have the same spatial coordinates, in the case where A and B are timelike related. In this case, it is not possible to obtain a frame S' in which A and B have the same time coordinate, as otherwise we would have

$$(y'_0 - x'_0)^2 - \sum_{i=1}^3 (y'_i - x'_i)^2 = - \sum_{i=1}^3 (y'_i - x'_i)^2 = (y_0 - x_0)^2 - (y_1 - x_1)^2 > 0,$$

which is not possible as $-\sum_{i=1}^3 (y'_i - x'_i)^2$ is non-positive.