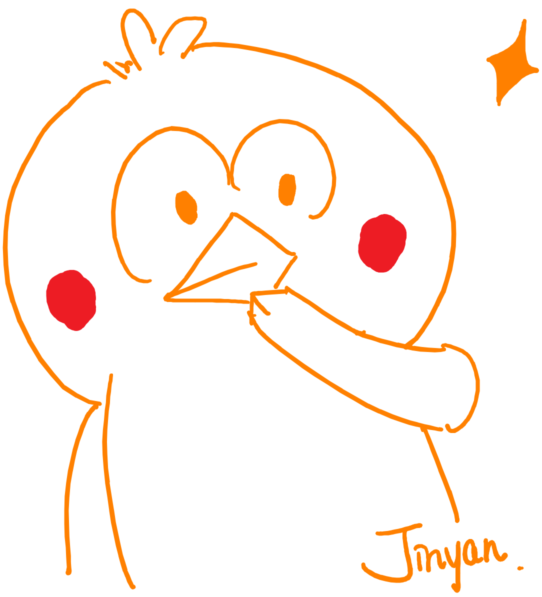
Homework 5

Physics 542 - Quantum Optics
Professor Alex Kuzmich



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(a) First we normalize $|\phi\rangle \sim \hat{a}|\psi\rangle$.

$$|\phi\rangle = C\hat{a}|\psi\rangle\,, \qquad \text{thus } \langle\phi|\phi\rangle = |C|^2\langle\psi|\hat{a}^\dagger\hat{a}|\psi\rangle = |C|^2\langle\hat{n}\rangle = 1\,,$$

rearranging we obtain

$$C = \frac{1}{\sqrt{\bar{n}}}\,.$$

Thus we write

$$|\phi\rangle = \frac{\hat{a}}{\sqrt{\bar{n}}}|\psi\rangle\,.$$

Now we can compute

$$\begin{aligned}
\langle\phi|\hat{n}|\psi\rangle &= \frac{1}{\bar{n}}\langle\psi|\hat{a}^\dagger\hat{n}\hat{a}|\psi\rangle \\
&= \frac{1}{\bar{n}}\langle\psi|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|\psi\rangle \\
&= \frac{1}{\bar{n}}\langle\psi|\hat{a}^\dagger(\hat{a}\hat{a}^\dagger - 1)\hat{a}|\psi\rangle \\
&= \frac{1}{\bar{n}}\Big(\langle\psi|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|\psi\rangle - \langle\psi|\hat{a}^\dagger\hat{a}|\psi\rangle\Big) \\
&= \frac{1}{\bar{n}}\left(\langle\psi|\hat{n}^2|\psi\rangle - \langle\psi|\hat{n}|\psi\rangle\right) \\
&= \frac{1}{\bar{n}}\left(\langle\hat{n}^2\rangle - \bar{n}\right) \\
&= \frac{\langle\hat{n}^2\rangle}{\langle\hat{n}\rangle} - 1\,.
\end{aligned}$$

(b) Now we consider the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |10\rangle\right)\,.$$

We first calculate

$$\langle\psi|\hat{n}|\psi\rangle = \frac{1}{2}\left(\langle 0|\hat{n}|0\rangle + \langle 10|\hat{n}|10\rangle\right) = 5\,.$$

Next we consider

$$|\phi\rangle = \frac{\hat{a}}{\sqrt{\bar{n}}}|\psi\rangle = \frac{\sqrt{10}}{\sqrt{5}\sqrt{2}}|9\rangle = |9\rangle\,.$$

Thus we have

$$\langle\phi|\hat{n}|\phi\rangle = 9\,.$$

Here we calculate

$$\langle\psi|\hat{n}^2|\psi\rangle = \frac{1}{2}\left(\langle 0|\hat{n}^2|0\rangle + \langle 10|\hat{n}^2|10\rangle\right) = \frac{100}{2} = 50\,.$$

Thus we obtain

$$\frac{\langle\hat{n}^2\rangle_{|\psi\rangle}}{\langle\hat{n}\rangle_{|\psi\rangle}} - 1 = 9 = \langle\phi|\hat{n}|\phi\rangle$$

as expected. The result that we find in this example is consistent with what we found in part (a). This result makes sense as $\hat{a}|0\rangle$ eliminate uncertainty on the measurement of $\hat{n}$ caused by the vacuum state $|0\rangle$.

2

Here we calculate

$$\hat{a}^\dagger|\alpha\rangle\langle\alpha| = \hat{a}^\dagger \sum_{n,m} e^{-|\alpha|^2} \frac{\alpha^n(\alpha^*)^m}{\sqrt{n!m!}} |n\rangle\langle m| = \sum_{n,m} e^{-|\alpha|^2} \frac{a^n(a^*)^m}{\sqrt{n!m!}} (n+1)^{1/2}|n+1\rangle\langle m| \,,$$

On the other hand,

$$\begin{aligned}
&\left(\alpha^* + \frac{\partial}{\partial\alpha}\right)|\alpha\rangle\langle\alpha| \\
&\qquad = \alpha^* \sum_{n,m} e^{-|\alpha|^2} \frac{\alpha^n(\alpha^*)^m}{\sqrt{n!m!}} |n\rangle\langle m| - \alpha^* \sum_{n,m} e^{-|\alpha|^2} \frac{\alpha^n(\alpha^*)^m}{\sqrt{n!m!}} |n\rangle\langle m| + \sum_{n,m} e^{-|\alpha|^2} \frac{n\alpha^{n-1}(\alpha^*)^m}{\sqrt{n!m!}} |n\rangle\langle m| \\
&\qquad = \sum_{n,m} e^{-|\alpha|^2} \frac{n\alpha^{n-1}(\alpha^*)^m}{\sqrt{n!m!}} |n\rangle\langle m| \\
&\qquad = \sum_{n,m} e^{-|\alpha|^2} \frac{\alpha^{n-1}(\alpha^*)^m}{\sqrt{(n-1)!m!}} \sqrt{n}|n\rangle\langle m| \\
&\qquad = \sum_{n,m} e^{-|\alpha|^2} \frac{\alpha^n(\alpha^*)^m}{\sqrt{n!m!}} (n+1)^{1/2}|n+1\rangle\langle m| \\
&\qquad = \hat{a}^\dagger|\alpha\rangle\langle\alpha| \,.
\end{aligned}$$

For the second identity, it suffices to check it with a basis state $|n\rangle$, where we can write

$$
\begin{aligned}
|\alpha\rangle\langle\alpha|\hat{a}|n\rangle &= \sum_{k,m} e^{-|\alpha|^2} \frac{\alpha^k (\alpha^*)^m}{\sqrt{k!m!}} |k\rangle\langle m|\hat{a}|n\rangle \\
&= \sum_{k,m} e^{-|\alpha|^2} \frac{\alpha^k (\alpha^*)^m}{\sqrt{k!m!}} |k\rangle \sqrt{n} \langle m|n-1\rangle \\
&= \sum_{k} e^{-|\alpha|^2} \frac{\alpha^k (\alpha^*)^{(n-1)}}{\sqrt{k!(n-1)!}} |k\rangle \sqrt{n}\,.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\left(\alpha^* + \frac{\partial}{\partial\alpha}\right) |\alpha\rangle\langle\alpha|n\rangle \\
&\quad = \alpha^* \sum_{k,m} e^{-|\alpha|^2} \frac{\alpha^k (\alpha^*)^m}{\sqrt{k!m!}} |k\rangle\langle m|n\rangle - \alpha \sum_{k,m} e^{-|\alpha|^2} \frac{\alpha^k (\alpha^*)^m}{\sqrt{k!m!}} |k\rangle\langle m|n\rangle + \sum_{k,m} e^{-|\alpha|^2} \frac{m\alpha^k (\alpha^*)^{m-1}}{\sqrt{k!m!}} |k\rangle\langle m|n\rangle \\
&\quad = \sum_{k,m} e^{-|\alpha|^2} \frac{m\alpha^k (\alpha^*)^{m-1}}{\sqrt{k!m!}} |k\rangle\langle m|n\rangle \\
&\quad = \sum_{k} e^{-|\alpha|^2} \frac{n\alpha^k (\alpha^*)^{n-1}}{\sqrt{k!n!}} |k\rangle \\
&\quad = \sum_{k} e^{-|\alpha|^2} \frac{\alpha^k (\alpha^*)^{n-1}}{\sqrt{k!(n-1)!}} |k\rangle \sqrt{n} \\
&\quad = |\alpha\rangle\langle\alpha|\hat{a}|n\rangle\,,
\end{aligned}
$$

which shows that

$$
|\alpha\rangle\langle\alpha|\,\hat{a} = \left(\alpha^* + \frac{\partial}{\partial\alpha}\right) |\alpha\rangle\langle\alpha|
$$

as $|n\rangle$ is arbitrary in the basis spanning the Hilbert space.

3

The state of the system is given by Eq. (4.120) on Gerry & Knight,

$$\begin{aligned}|\psi(t)\rangle = &\sum_{n=0}^{\infty} \left(C_e C_n \cos(\lambda t\sqrt{n+1}) - iC_g C_{n+1} \sin(\lambda t\sqrt{n+1})\right) |e\rangle|n\rangle \\ &+ \sum_{n=0}^{\infty} \left(-iC_e C_{n-1} \sin(\lambda t\sqrt{n}) + C_g C_n \cos(\lambda t\sqrt{n})\right) |g\rangle|n\rangle\end{aligned}$$

We abbreviate here

$$\begin{aligned}K_{e,n} &= C_e C_n \cos(\lambda t\sqrt{n+1}) - iC_g C_{n+1} \sin(\lambda t\sqrt{n+1})\,, \\ K_{g,n} &= -iC_e C_{n-1} \sin(\lambda t\sqrt{n}) + C_g C_n \cos(\lambda t\sqrt{n})\,.\end{aligned}$$

By Eq. (4.94) on Gerry & Knight, we have

$$\hat{d} = d|g\rangle\langle e| + d^*|e\rangle\langle g|\,.$$

Thus we have

$$\hat{d}|\psi(t)\rangle = \sum_{n=0}^{\infty} K_{e,n} d|g\rangle|n\rangle + K_{g,n} d^*|e\rangle|n\rangle\,.$$

Thus we have

$$\begin{aligned}\langle\psi(t)|\hat{d}|\psi(t)\rangle &= \left(\sum_{n=0}^{\infty} K_{e,n}^*\langle e|\langle n| + K_{g,n}^*\langle g|\langle n|\right)\left(\sum_{n=0}^{\infty} K_{e,n} d|g\rangle|n\rangle + K_{g,n} d^*|e\rangle|n\rangle\right) \\ &= \left(\sum_{n=0}^{\infty} K_{e,n}^*\langle e|\langle n| + K_{g,n}^*\langle g|\langle n|\right)\left(\sum_{n=0}^{\infty} K_{e,n} d|g\rangle|n\rangle + K_{g,n} d|e\rangle|n\rangle\right) \\ &= d\sum_{n=0}^{\infty} K_{e,n}^* K_{g,n} + K_{g,n}^* K_{e,n}\,. \qquad (3.1)\end{aligned}$$

In the case where the atom is initially in excited state and the field is initially at state $|n\rangle$, we have

$$C_n = 1\,, \qquad C_m = 0 \text{ for } m \neq n\,, \qquad C_e = 1\,, \qquad C_g = 0\,.$$

By comparing coefficients of terms in $K_{e,n}$ and $K_{g,n}$, we find that the products $K_{e,n}^* K_{g,n}$ and $K_{g,n}^* K_{e,n}$ always vanish in this case, thus $\langle\hat{d}\rangle = 0$.

On the other hand, in the semiclassical treatment of the model, we have, from Eq. (2.91) in Berman,

$$c_1 = -i\sin(\Omega t/2)\,, \qquad c_2 = \cos(\Omega t/2)$$

for the atom initially in excited state with exact resonance with the field. Thus we can write

$$|\psi(t)\rangle_{\text{semiclass.}} = -i\sin(\Omega t/2)e^{-iE_g t/\hbar}|g\rangle + \cos(\Omega t/2)e^{-iE_e t/\hbar}|e\rangle\,.$$

The dipole measurement in this case reads

$$\begin{aligned}
\langle\hat{d}\rangle_{\text{semiclass.}} &= d\left(-i\cos(\Omega t/2)e^{iE_e t/\hbar}\sin(\Omega t/2)e^{-iE_g t/\hbar} + i\sin(\Omega t/2)e^{iE_g t/\hbar}\cos(\Omega t/2)e^{-iE_e t/\hbar}\right)\\
&= d\left(-ie^{i(E_e-E_g)t/\hbar}\cos(\Omega t/2)\sin(\Omega t/2) + ie^{i(E_g-E_e)t/\hbar}\sin(\Omega t/2)\cos(\Omega t/2)\right)\\
&= d\left((ie^{i(E_g-E_e)t/\hbar} - ie^{i(E_e-E_g)t/\hbar})\cos(\Omega t/2)\sin(\Omega t/2)\right)\\
&= d\sin((E_e-E_g)t/\hbar)\sin(\Omega t)\,,
\end{aligned}$$

which is non-zero for some $t > 0$, thus we see that $\langle\hat{d}\rangle = 0$ in the quantum approach is purely due to the entanglement between the atom state and the field state as seen from Eq. (3.1).

4

In the case of the field initially in a coherent state, with $n \geq 0$,

$$C_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} .$$

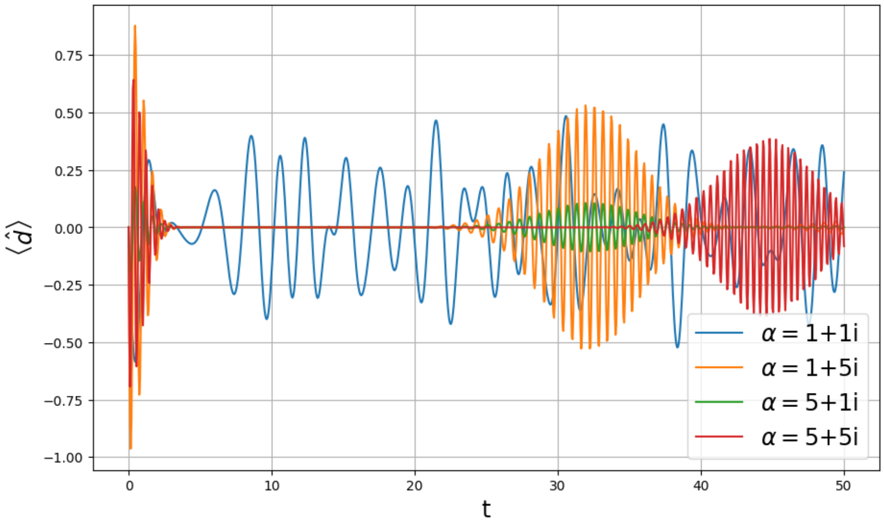
We simplify (3.1) in this text by consider the state of the atom initially in the excited state, as in the previous problems. In this case, $C_e = 1$ and $C_g = 0$, thus

$$K_{e,n} = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} \cos(\lambda t \sqrt{n+1}) , \qquad K_{g,n} = -ie^{-|\alpha|^2/2} \frac{\alpha^{n-1}}{\sqrt{(n-1)!}} \sin(\lambda t \sqrt{n}) .$$

In this case (4.1) becomes

$$\begin{aligned} \langle \hat{d} \rangle &= de^{-|\alpha|^2} \sum_{n=1}^{\infty} \frac{-2|\alpha|^{2(n-1)} \Im(\alpha)}{\sqrt{n!}\sqrt{(n-1)!}} \cos(\lambda t \sqrt{n+1}) \sin(\lambda t \sqrt{n}) \\ &= de^{-|\alpha|^2} \Im(\alpha) \sum_{n=0}^{\infty} \frac{-2|\alpha|^{2n}}{n! \sqrt{n+1}} \cos(\lambda t \sqrt{n+2}) \sin(\lambda t \sqrt{n+1}) \end{aligned}$$

In the case where $\alpha \in \mathbb{R}$, we see that $\langle \hat{d} \rangle$ vanishes, and we also observe that the sign of $\Re(\alpha)$ does not affect $\langle \hat{d} \rangle$, the sign of $\Im(\alpha)$ only flips the sign of $\langle \hat{d} \rangle$. Thus we make some plots for $\alpha \in \{1+1i, 5+1i, 1+5i, 5+5i\}$.



From the figure we see that $\langle \hat{d} \rangle$ also have the collapse and recover behavior similar to that of the Rabi oscillations of the atom. While in the case of the field being in Fock state, $\langle \hat{d} \rangle$ vanishes, and in the case of semiclassical treatment of the system, $\langle \hat{d} \rangle$ oscillates periodically.

5

Following the analysis in Section 4.5, the inversion for field initially in a thermal state is

$$W(t) = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \cos(2\lambda t\sqrt{n+1})\,.$$

Here we denote $\Omega(n) = 2\lambda\sqrt{n+1}$. The collapse time t_c is estimated from the time-frequency uncertainty relation

$$t_c\,(\Omega(\bar{n}+\Delta n) - \Omega(\bar{n}-\Delta n)) \sim 1\,.$$

For thermal state, Eq. (2.149) on Gerry & Knight gives

$$\Delta n = (\bar{n}+\bar{n}^2)^{1/2}\,.$$

Thus combining, we obtain

$$t_c \sim \left(2\lambda\left(\bar{n}+1+(\bar{n}+\bar{n}^2)^{1/2}\right)^{1/2} - 2\lambda\left(\bar{n}+1-(\bar{n}+\bar{n}^2)^{1/2}\right)^{1/2}\right)^{-1}\,,$$

which gives the relation between t_c and $\bar{n}$. In the limit $\bar{n} \gg 1$, we have

$$t_c \sim \left(2\lambda\,(2\bar{n})^{1/2}\right)^{-1} = \frac{1}{2\sqrt{2}\lambda\sqrt{\bar{n}}}$$

In the limit where $\bar{n} \ll 1$, we have

$$t_c \sim \left(2\lambda\left(1+\bar{n}^{1/2}\right)^{1/2} - 2\lambda\left(1-\bar{n}^{1/2}\right)^{1/2}\right)^{-1} \sim \left(2\lambda\left(1+\frac{\bar{n}^{1/2}}{2}\right) - 2\lambda\left(1-\frac{\bar{n}^{1/2}}{2}\right)\right)^{-1} = \frac{1}{2\lambda\sqrt{\bar{n}}}\,.$$

6

Here we have the initial state

$$|\psi\rangle = \widetilde{c}_{1,1}e^{i\omega t}e^{i\omega t/2}|1,1\rangle + \widetilde{c}_{2,0}e^{i\omega t}e^{i\omega t/2}|2,0\rangle + \widetilde{c}_{1,2}e^{i2\omega t}e^{i\omega t/2}|1,2\rangle + \widetilde{c}_{2,1}e^{i2\omega t}e^{i\omega t/2}|2,1\rangle\,.$$

Thus the probability for the atom in the excited state is

$$\mathbb{P} = ||\widetilde{c}_{2,0}||^2 + ||\widetilde{c}_{2,1}||^2\,.$$

From Eq. (15.40) in Berman, we can write

$$\widetilde{c}_{2,0} = -\frac{2ig_1}{\Omega_1}\sin\left(\frac{\Omega_1 t}{2}\right)\widetilde{c}_{1,1}(0) + \cos\left(\frac{\Omega_1 t}{2}\right)\widetilde{c}_{2,0}(0) = -\frac{ig}{\sqrt{2}|g|}\sin\left(|g|t\right)\,,$$

$$\widetilde{c}_{2,1} = -\frac{2ig_2}{\Omega_2}\sin\left(\frac{\Omega_2 t}{2}\right)\widetilde{c}_{1,2}(0) + \cos\left(\frac{\Omega_2 t}{2}\right)\widetilde{c}_{2,1}(0) = -\frac{ig}{\sqrt{2}|g|}\sin\left(\sqrt{2}|g|t\right)\,.$$

WLOG, we assume here $g = -i$, then we have

$$\widetilde{c}_{2,0} = -\frac{1}{\sqrt{2}}\sin(t)\,, \qquad \widetilde{c}_{2,0} = -\frac{1}{\sqrt{2}}\sin(\sqrt{2}t)\,.$$

Thus the probability in the excited state is

$$\mathbb{P} = \frac{1}{2}\left(\sin^2(t) + \sin^2(\sqrt{2}t)\right)$$

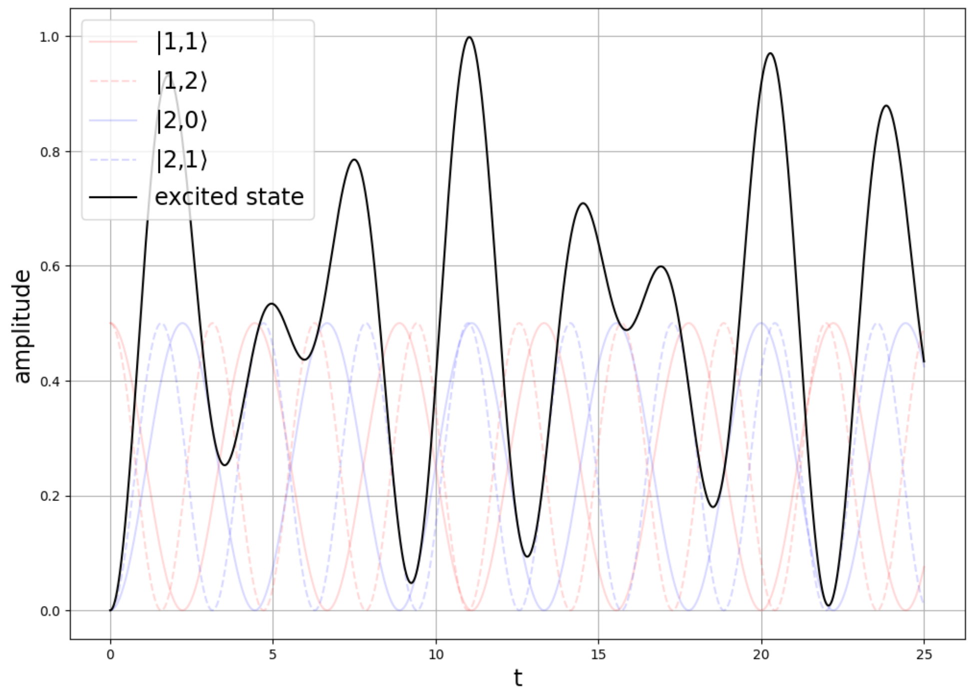
In order for $\mathbb{P} = 1$, we require

$$t = \frac{(2n-1)\pi}{2} = \frac{(2m-1)\pi}{2\sqrt{2}}$$

for integers $m, n \in \mathbb{Z}$, thus rearranging

$$\sqrt{2}(2n-1) = 2m-1\,,$$

while the RHS is always rational as $m \in \mathbb{Q}$, but the LHS is never a rational as $0 \neq (2n-1) \in \mathbb{Q}$ and $\sqrt{2} \notin \mathbb{Q}$, thus such a pair $(n,m) \in \mathbb{Z}^2$ does not exist. We conclude that $\mathbb{P} \neq 1$. Similar reasoning holds for $\mathbb{P} \neq 0$ except at $t = 0$. The plot for $\mathbb{P}$ is attached on next page, with $g = -i/\sqrt{2}$.



It seems like the atom reaches excited state at around $t = 11$, but in fact it is not.