The Application of Floquet Theory in Quantum Optics

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We present a review of the application of Floquet theory in addressing problems in quantum optics. The theoretical framework of applying Floquet theory in studying temporally periodic Hamiltonian is introduced, and parallels between Floquet theory and Bloch theory are drawn. Then we explore how the Floquet theory is used to address two-level system problems describing an atom interacting with an external field, in both the semi-classical approach and the Jaynes-Cummings model. With numerical simulations, we analyze the efficiency advantages of the Floquet theory-based approach. Finally, we examine the utilization of Floquet engineering in investigating optical lattices.

I. INTRODUCTION

When addressing two-level system problems describing an atom interacting with an external field, the widely employed rotating-wave approximation (RWA) is effective only under the condition where the Rabi frequency is much smaller than the transition frequency. In the cases where RWA is inapplicable, a common approach is to employ the Floquet theory [1–3]. This approach entails expressing the evolution of a system as a Fourier series with respect to the oscillation frequency [4, 5]. The Floquet treatment of the two-level system problem under strong driving was initially treated by Shirley. Beyond its application to simple two-level systems in quantum optics, Floquet theory offers a powerful platform for dealing with the time-dependent Schrödinger equation by leveraging any time-periodicity within the Hamiltonian. Its utility extends broadly across various time-dependent scenarios, encompassing areas such as photonic Floquet topological insulators [6], quantum information processing [7], heat engines and laser cooling [8], quantum optimal control [9, 10], and time crystals [11, 12]. Furthermore, the combination of Floquet theory with its spatially analog, Bloch theory, provides an efficient and rigorous approach to studying optical lattices, which is particularly pivotal in researching the control of cold atom systems [5, 13, 14].

In this project, we will introduce the Floquet theory in describing temporally periodic systems, drawing its parallels with the Bloch theory, and apply it to solve two-level system problems, in both the semi-classical approach and the Jaynes-Cummings model. We will have a closer look at the solutions by solving the two-level system numerically, and discuss why the approach involving the Floquet theory is more efficient. Lastly, we will look at the application of Floquet engineering in studying optical lattices, based on the work by Kilian Sandholzer et al. [13].

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II. A REVIEW ON THE TWO-LEVEL SYSTEM

For simplicity, we will work in units where $\hbar = 1$. To motivate the discussion of Floquet theory applied in quantum optics, we start with a brief review of a two-level system describing an atom interacting with an oscillating electric field,

$$\mathbf{E}(t) = (E_0 e^{-i\omega t} + E_0^* e^{i\omega t})\hat{z} \tag{1}$$

with angular frequency ω. The total system Hamiltonian can then be written as

$$H = H_\mathrm{a} + H_\mathrm{int}\,, \tag{2}$$

where

$$H_\mathrm{a} = \frac{\omega_0}{2}\,\sigma_z \tag{3}$$

is the bare Hamiltonian of the atom, and

$$H_\mathrm{int} = (\Omega e^{-i\omega t} + \widetilde{\Omega} e^{i\omega t})|e\rangle\langle g| + (\mathrm{h.c.}) \tag{4}$$

is the interaction Hamiltonian, with Ω being the Rabi frequency of the two level system, $\widetilde{\Omega}$ being the associated counter-rotating frequency, and (h.c.) meaning the Hermitian conjugate terms. In the interaction picture $\widetilde{H} = U^\dagger H U$, where $U = \exp(-iH_\mathrm{a}t)$, we can write

$$\widetilde{H} = \left(\Omega e^{i\delta t} + \widetilde{\Omega} e^{i(\omega+\omega_0)t}\right)|e\rangle\langle g| + (\mathrm{h.c.})\,, \tag{5}$$

with $\delta = \omega - \omega_0$. Whenever $|\widetilde{\Omega}/(\omega_0+\omega)| \ll 1$ and $|\delta/(\omega_0+\omega)| \ll 1$, the rapidly oscillating terms do not contribute much since they average to zero in a very short period of time. In other words, the contribution from these rapidly varying terms would be negligibly small compared with the slowly varying terms if we take a coarse-grain time average over a time interval much greater than $1/(\omega_0+\omega)$. We therefore employ the rotating wave approximation (RWA), that is neglecting the counter-rotating terms in the Hamiltonian, whenever we have $|\widetilde{\Omega}/(\omega_0+\omega)| \ll 1$ and $|\delta/(\omega_0+\omega)| \ll 1$. Under RWA, the Hamiltonian in the original frame reads

$$H_\mathrm{RWA} = H_\mathrm{a} + \left(\Omega e^{-i\omega t}|e\rangle\langle g| + \Omega^* e^{i\omega t}|g\rangle\langle e|\right)\,. \tag{6}$$

The Hamiltonian described by Eq. (6) has been extensively studied, and there have been numerous researches demonstrating the Rabi oscillation characterized by Ω [15–17].

III. THE FLOQUET THEORY

A. Quasi-energies and Floquet Modes

The Floquet's Theorem asserts that a collection of time-dependent differential equations, where the coefficients vary periodically with respect to time, will yield solutions that exhibit identical periodicity. This is a temporal analog of Bloch's theorem in spatial contexts, with solutions described in quasi-energies rather than quasi-momenta. We will first introduce the Floquet modes and quasi-energies, then we will compare them to their spatial analog from the Bloch's Theorem.

We first consider the time-dependent Schrodinger equation

$$i\frac{d}{dt}|\psi(t)\rangle = H(t)\,|\psi(t)\rangle\,, \tag{7}$$

for a periodic Hamiltonian

$$H(t) = H(t+nT) \tag{8}$$

for all $n \in \mathbb{Z}$. Via the Floquet Theorem, a solution $|\psi_\alpha(t)\rangle$ to Eq. (7) can be written as,

$$|\psi_\alpha(t)\rangle = e^{-i\epsilon_\alpha t}|\phi_\alpha(t)\rangle\,, \tag{9}$$

where $|\phi_\alpha(t)\rangle = |\phi_\alpha(t+nT)\rangle$ is called a Floquet mode, the time-independent quantity ϵ_α is called the associated quasi-energies, uniquely defined up to multiples of $\omega = 2\pi/T$, and $|\psi_\alpha(t)\rangle$ is called a Floquet state. A general solution to Eq. (7) can be written as a linear combination of the Floquet state $|\phi_\alpha(t)\rangle$. Now we define

$$G(t) = H(t) - i\frac{d}{dt}\,, \tag{10}$$

and combining Eq. (9) with Eq. (7), we obtain the eigenvalue problem that we are interested in solving,

$$G(t)\,|\phi_\alpha(t)\rangle = \epsilon_\alpha|\phi_\alpha(t)\rangle\,, \tag{11}$$

with G called the quasi-energy operator. That is, to find the Floquet modes and quasi-energies, we solve Eq. (10) numerically, or even analytically.

B. Comparison with the Bloch Theory

To gain more insights into the definition of Floquet modes and quasi-energies, we first have a review of the spatial analog of the Floquet theory, the Bloch theory. In this case, we consider the time-independent Schrodinger equation for a Hamiltonian with a spatially periodic potential $V(x) = V(x+a)$, where a is the lattice spacing. Bloch's Theorem suggests that the solution to the time-independent Schrodinger's equation has the form

$$|\psi(x,q)\rangle = u(x,q)\cdot e^{iqx}\,, \tag{12}$$

where $u(x,q)$ is called the Bloch function and has the same periodicity as the potential, $\psi(x,q)$ is called the Bloch wave, and q is called quasi-momentum or lattice momentum. We see immediately that Eq. (9) from the Floquet theory is of a form closely related to Eq. (12) from the Bloch theory, hence the name of quasi-energy for ϵ_α, and Floquet modes for $|\phi_\alpha(t)\rangle$. It seems that, in either Floquet theory or Bloch theory, we merely transform the problem from finding the unknown states $|\psi(t)\rangle$ or $|\psi(x,q)\rangle$ to finding the unknown states $|\phi_\alpha(t)\rangle$ or $u(x,q)$, but the crucial advantage is the periodicity in $|\phi_\alpha(t)\rangle$ and $u(x,q)$, which allows for Fourier transformation on those functionals. In Bloch theory, we can expand

$$u(x,q) = \sum_l c_l(q)\,e^{iklx}\,, \tag{13}$$

where $q \bmod 2k = 2\pi/a$, and thus we can write

$$\psi(x,q) = \sum_l c_l(q)\,e^{i(2kl+q)x}\,, \tag{14}$$

which allows us to rewrite the corresponding Hamiltonian in the basis of plane waves. Similarly, for the Floquet theory model, we can expand

$$|\phi_\alpha(t)\rangle = \sum_n e^{-i\omega nt}|\alpha,n\rangle\,, \tag{15}$$

and decompose the Hamiltonian

$$H(t) = \sum_n e^{i\omega nt}H_n\,, \tag{16}$$

where we define the Fourier components $|\alpha,n\rangle$ and H_n via the usual way,

$$|\alpha,n\rangle = \frac{1}{T}\int_0^T dt\,e^{i\omega nt}|\phi_\alpha(t)\rangle\,, \tag{17}$$

and

$$H_n = \frac{1}{T}\int_0^T dt\,e^{i\omega nt}H(t)\,. \tag{18}$$

This enables us to transform the open boundary condition problem characterized by Eq. (7) into a periodic boundary condition problem. In this sense, one can define the Floquet Hamiltonian H_{F} to be the periodic-time-averaged of the quasi-energy operator, which has a matrix representation in the basis of the Fourier components, with entry $\langle\alpha,m|H_{\mathrm{F}}|\beta,n\rangle$ given by

$$\frac{1}{T}\int_0^T dt\,\langle\alpha,m|H_{\mathrm{F}}|\beta,n\rangle = \langle\alpha|H_{m-n}|\beta\rangle + \delta_{m,n}\delta_{\alpha,\beta}m\omega\,. \tag{19}$$

C. Time Evolution Operator

Furthermore, we can consider the propagator $U(T+t,t)$ for the time-dependent Schrodinger equation Eq. (7), defined by

$$U(T+t,t)|\psi(t)\rangle = |\psi(T+t)\rangle \,. \tag{20}$$

Using the periodicity of $|\Phi_\alpha(t)\rangle$, we can write

$$U(nT+t,t)\,|\phi_\alpha(t)\rangle = e^{-i\epsilon_\alpha T}|\phi_\alpha(t)\rangle = \eta_\alpha|\phi_\alpha(t)\rangle \,, \tag{21}$$

which suggests that the Floquet modes are eigenmodes of the multiple-period propagator, and we can therefore find the Floquet modes and quasienergies $\epsilon_\alpha = -\arg(\eta_\alpha)/T$ by numerically calculating $U(nT+t,t)$ and diagonalizing it. This process can be first done to $U(T,0)$ to find $|\phi_\alpha(0)\rangle$ and the quasi-energies, then by employing Eq. (9), (19) and (20), one can directly evaluate $|\phi_\alpha(t)\rangle$, $|\psi_\alpha(t)\rangle$ and the general solution $|\psi(t)\rangle$ as a superposition of $|\psi_\alpha(t)\rangle$ at any other time t, with coefficients c_α determined by the initial wavefunction

$$|\psi(0)\rangle = \sum_\alpha c_\alpha|\psi_\alpha(0)\rangle \,. \tag{22}$$

This method is utilized by the Python library QuTiP to numerically compute the solution to the time-dependent Schrodinger equation [18], and is used to compute the evolution of a state presented in Section IV.

D. The Extended Hilbert Space

Finally, we discuss a method of obtaining the quasi-energies by extending the Hilbert space. Using Eq. (15), it can be easily seen that

$$(\epsilon_\alpha + n\omega)|\alpha,n\rangle = \sum_k H_{n-k}|\alpha,k\rangle \,, \tag{23}$$

where H_l for $l\in\mathbb{Z}$ is again the Fourier components of the Hamiltonian H. We see that we have created an over-complete problem as ϵ_α are only defined up to multiples of ω, and Eq. (23) can be recast into

$$\mathbf{M}\mathbf{v}_\alpha = \epsilon_\alpha \mathbf{v}_\alpha \tag{24}$$

where $\mathbf{M}$ is a matrix

$$\mathbf{M} = \begin{pmatrix} \ddots & H_{-1} & H_{-2} & \ddots \\ H_1 & H_0 - n\omega & H_{-1} & H_{-2} \\ H_2 & H_1 & H_0-(n+1)\hbar\omega & H_{-1} \\ \ddots & H_2 & H_1 & \ddots \end{pmatrix} , \tag{25}$$

and $\mathbf{v}$ is a vector

$$\mathbf{v}_\alpha = \begin{pmatrix} \vdots \\ |\alpha,n\rangle \\ |\alpha,n+1\rangle \\ \vdots \end{pmatrix} . \tag{26}$$

Note that $\mathbf{M}$ consists of blocks of $d\times d$ entries where d is the dimension of the Hilbert space of H, and likewise, $|\alpha,n\rangle$ is d-dimensional.

A common time-dependent Hamiltonian, such as the one defined by Eq. (2) describing a qubit interacting with an external field, has the form

$$H(t) = H_0 + Ve^{i\omega t} + V^\dagger e^{-i\omega t} \,, \tag{27}$$

in which case $\mathbf{M}$ takes the form

$$\mathbf{M} = \begin{pmatrix} \ddots & V & 0 & 0 & \ddots \\ V^\dagger & H_0+\omega & V & 0 & 0 \\ 0 & V^\dagger & H_0 & V & 0 \\ 0 & 0 & V^\dagger & H_0-\omega & V \\ \ddots & 0 & 0 & V^\dagger & \ddots \end{pmatrix} , \tag{28}$$

which is similar to a tight-binding Hamiltonian with the nearest-neighbor hopping.

For the system described by Eq. (2) with $\widetilde{\Omega} = \Omega^*$, Eq. (27) captures the full picture of the system, and thus the numerical evaluation of the Eq. (28) can be very efficient. In general, to truncate the evaluation of Eq. (28), it is crucial to distinguish two regimes. In the weak driving regime ($\omega \gg \langle V\rangle$), only one block

$$\begin{pmatrix} H_0+\omega & V \\ V^\dagger & H_0 \end{pmatrix}$$

is relevant. When $\omega \sim \langle V\rangle$, which is the strong-driving limit, more blocks of matrix $\mathbf{M}$ shall be taken into account [5].

IV. APPLICATION IN STUDYING TWO-LEVEL SYSTEMS

Here we consider the two-level system described by the Hamiltonian defined in Eq. (2), (3), and (4). We first obtain the evolution of a state, for instance, $|\psi(0)\rangle = |g\rangle$, using the Floquet theory and the time evolution operator. The method is entailed in part C of Section II. With $\omega_0 = 10$, $\omega = 9$, and $\Omega = \widetilde{\Omega} = 1$, the computed occupation probability for states $|e\rangle$ and $|g\rangle$ are plotted using dashed curves in Fig. 1(a). Then we compute the evolution of the same state, $|\psi(0)\rangle = |g\rangle$, by numerically solving the time-dependent Schrodinger equation, Eq. (7), with the result computed with RWA plotted in dotted curves in Fig. 1(a), and the result computed without RWA plotted in solid curves. It is easy to see that all three methods agree quite well in this case as the conditions of using RWA are satisfied. We perform the same computations but with parameters $\omega_0 = 10$, $\omega = 9$, and $\Omega = \widetilde{\Omega} = 10$, with results depicted in Fig. 1(b). In this case, $|\widetilde{\Omega}/(\omega_0+\omega)| \ll 1$ does not hold and thus RWA shall not be employed. Indeed, we see that the result computed via the Floquet theory agrees with the result obtained by numerically solving Eq. (7) without RWA, and they disagree with the result computed with RWA. Lastly, Fig. 1(c) depicts the computations with $\omega_0 = 10$, $\omega = 3$, and $\Omega = \widetilde{\Omega} = 1$, where RWA is again not applicable.

We note that one advantage of employing the Floquet theory is that it requires less computational cost than numerically solving the time-dependent Schrodinger equation. This is because we can exploit the periodicity in the problem: To determine the evolution of a state $|\psi(t)\rangle$ at any time t, we only need to compute ϵ_α and the states $|\phi_\alpha(0)\rangle$ at $t = 0$, and the states

$$|\phi_\alpha(\tau + nT)\rangle = |\phi_\alpha(\tau)\rangle = e^{i\epsilon_\alpha \tau} U(\tau, 0)|\phi_\alpha(0)\rangle \tag{29}$$

in the Brillouin zone $\tau \in [0, T]$. Then $|\psi(t)\rangle$ can be computed via the superposition

$$|\psi(t)\rangle = \sum_\alpha c_\alpha e^{-i\epsilon_\alpha t}|\phi_\alpha(t)\rangle\,, \tag{30}$$

with the time-independent coefficients c_α determined by the initial state at time $t = 0$. Whereas numerically solving Eq. (7) requires fine time steps, and thus the computation cost for evaluating $|\psi(t)\rangle$ at a later time $t \gg T$ is much larger. Moreover, if there is no analytic solution available and without the application of Floquet theory, it is difficult to look up the evolution of the state at an arbitrary time t_0 without numerically solving the state at all time $t < t_0$, while the application of Floquet theory can be used to efficiently lookup the Floquet mode at an arbitrary time and thus constructing the state at an arbitrary time.

On the other hand, interesting observations can also be drawn from the quasi-energy of the system. Again, with $\omega = 9$ and $\omega_0 = 10$, the quasi-energies of the system as a function of Ω/ω can be computed, shown in Fig. 2. We observe that the quasi-energies cross at certain values of Ω/ω. These points are closely related to the coherent destruction of tunneling and have been shown to exhibit interesting phenomena [18–21].

Other than the application in the semi-classical approach of describing the atom interacting with a driving field, the Floquet theory can also be applied in studying the Jaynes-Cummings model (JCM) [2, 3, 22]. Many works have focused on studying the JCM with RWA applied. With the counter-rotating terms included, a general treatment to study the JCM is again by applying the Floquet theory. In the Schrodinger picture, the total Hamiltonian of the JCM reads

$$H = H_0 + H_{\text{int}} \tag{31}$$

where

$$H_0 = \frac{\omega_0}{2}\sigma_z + \omega \hat{a}^\dagger \hat{a} \tag{32}$$

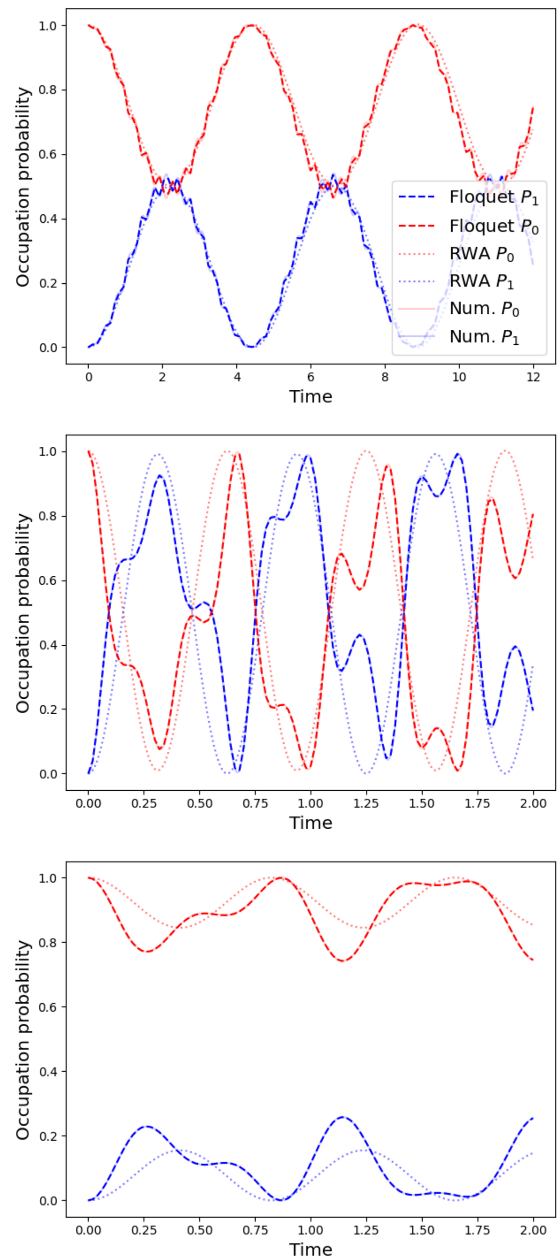

FIG. 1. The evolution of state $|\psi(0)\rangle = |g\rangle$, computed using three different methods: Using the time evolution operator with the Floquet theory, entailed in part C of Section II; Numerically solving Eq. (7) without RWA; Numerically solving Eq. (7) with RWA. Note $\omega_0 = 10$ in all three computations. Subplot (a) is computed with $\Omega = \widetilde{\Omega} = 1$ and $\omega = 9$, (b) is computed with $\Omega = \widetilde{\Omega} = 10$ and $\omega = 9$, and (c) is computed with $\Omega = \widetilde{\Omega} = 1$ and $\omega = 3$.

is the bare Hamiltonian of the atom and the field, and

$$H_{\text{int}} = g(\hat{a} + \hat{a}^\dagger)(\sigma_+ + \sigma_-) \tag{33}$$

is the interaction Hamiltonian. Turning to the interaction picture, a frame rotating with frequency ω_0, the

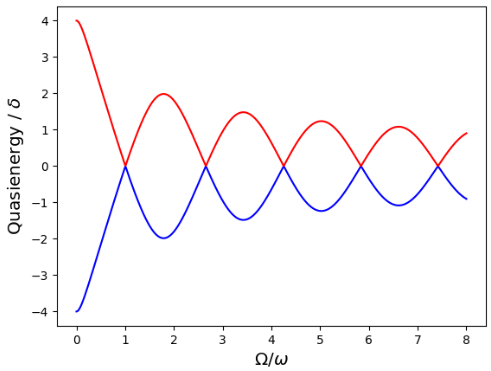

FIG. 2. Quasi-energy of the system as a function of Ω/ω with $\omega = 9$ and $\omega_0 = 10$.

Hamiltonian in Eq. (25) reads

$$\widetilde{H} = \frac{\delta}{2}\sigma_z + g(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-) + g(\hat{a}\sigma_- e^{-2i\omega t} + \hat{a}\sigma_+ e^{2i\omega t}) , \tag{34}$$

where $\delta = \omega_0 - \omega$ is again the atom-light detuning. We notice that the Hamiltonian $\widetilde{H}$ is periodic with frequency 2ω, thus Floquet theory can be applied. We can decompose Eq. (28) into its Fourier components,

$$H = H_0 + H_1 e^{i2\omega t} + H_{-1} e^{-i2\omega t} , \tag{35}$$

where we have

$$H_0 = g(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-) + \delta\sigma_z/2 , \tag{36}$$
$$H_1 = g\hat{a}^\dagger\sigma_+ , \tag{37}$$
$$H_{-1} = g\hat{a}\sigma_- . \tag{38}$$

Via the Floquet theory, the long-time evolution of Eq. (29) can be described by an effective time-independent Floquet Hamiltonian,

$$H_{\mathrm{F,\,eff}} = \sum_{n=0}^{\infty} \frac{1}{(2\omega)^n} H^{(n)} \approx H^{(0)} + \frac{1}{2\omega}H^{(1)} + \frac{1}{4\omega^2}H^{(2)} , \tag{39}$$

with $H^{(0)} = H_0$,

$$H^{(1)} = [H_1, H_{-1}] = \frac{g^2}{2}\left((2\hat{n}+1)\sigma_z - \sigma_z^2\right) , \tag{40}$$

and

$$\begin{aligned} H^{(2)} &= \frac{1}{2}\left([[H_1, H_0], H_{-1}] + [[H_{-1}, H_0], H_1]\right) \\ &= -\frac{\delta g^2}{2}\left((2\hat{n}+1)\sigma_z - \sigma_z^2\right) - g^3\left(\hat{a}\sigma_+\hat{n} + \hat{n}\hat{a}^\dagger\sigma_0\right) . \end{aligned} \tag{36}$$

Combining and grouping terms, we obtain a high-frequency effective Hamiltonian,

$$H_{\mathrm{F,\,eff}} \approx \frac{\widetilde{\delta}_{\omega,\hat{n}}}{2}\sigma_z + (\hat{g}_{\omega,\hat{n}}\sigma_- + (\mathrm{h.c.})) + \frac{\delta_\omega}{2} , \tag{41}$$

with coefficients defined by

$$\widetilde{\delta}_{\omega,\hat{n}} = \left(1 - (2\hat{n}+1)\frac{g^2}{4\omega^2}\right)\delta + (2\hat{n}+1)\frac{g^2}{2\omega} , \tag{42}$$

$$\widetilde{g}_{\omega,\hat{n}} = \left(1 - \frac{g^2}{4\omega^2}\hat{n}\right) g , \tag{43}$$

$$\delta_\omega = -\frac{g^2}{2\omega} + \frac{g^2\delta}{4\omega^2} . \tag{44}$$

Comparing Eq. (35) with the Hamiltonian of JCM in RWA, one sees that the counter-rotating terms have effects on the effective detuning $\widetilde{\delta}_{\omega,\hat{n}}$ and the atom-light coupling $\widetilde{g}_{\omega,\hat{n}}$. A detailed analysis of Eq. (35) is given by Wang. et al. in [3].

More studies on the JCM with the application of the Floquet theory include JCM under monochromatic driving by Ermann et al. [22], the generation of light-matter entanglement, and the application of photon-resolved Floquet theory in quantum communication by Engelhardt et al. [2], and so on. These results demonstrate that the Floquet theory can be used to effectively study the two-level system, especially in the regime of strong driving, and strong matter-photon coupling.

V. OPTICAL LATTICES WITH FLOQUET ENGINEERING

One of the primary areas of research garnering significant attention lately is the Floquet engineering in optical lattices [13, 23–25]. This field combines Bloch theory, which describes spatial periodicity in potential, with Floquet theory, which characterizes temporal periodicity in potential, resulting in the emergence of Floquet-Bloch waves.

An intriguing effect brought by Floquet engineering that has captured people's interest is the shaking of optical lattices, which has been shown to manifest fascinating topological phenomena [26–30] and has become an important tool for controlling and manipulating cold atom systems [31–33]. In this section, we present the theoretical framework for describing optical lattice shaking.

In 1-dimensional crystal lattice, a common experimental method in implementing lattice shaking employs a piezo-electric actuator [5, 13], which moves the retro-reflecting mirror that defines the standing wave, and thus the potential of the optical lattice V is modulated by a

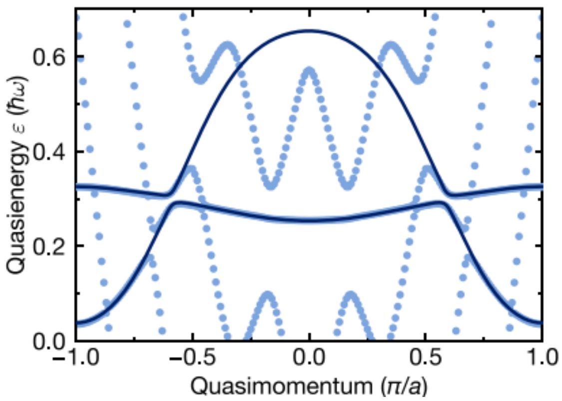

FIG. 3. Floquet-Bloch bands, taken from Fig. 7 in [13]. The dispersion of the effective two-band Hamiltonians (solid lines) is compared to the numerical exact solution of the Floquet-Bloch band structure (points, three lowest bands).

position and time $x(t)$. Another approach involves modulating the frequency of one of the two beams that form the standing wave that traps the atom [5]. Thus, in either case, the Hamiltonian describing the system takes the form

$$H_{\mathrm{lab}}(t) = \frac{\hat{p}^2}{2m} + V(\hat{x} - x(t)) \tag{45}$$

in the lab frame. It is possible to transform this Hamiltonian to the reference frame that is co-moving with the shaken lattice,

$$H_{\mathrm{cm}}(t) = \frac{\hat{p}^2}{2m} + V(\hat{x}) - F(t)\hat{x}\,, \tag{46}$$

where we have a time-periodic force $F(t)$ that shakes the lattice, and a spatial-periodic potential $V(\hat{x})$ responsible for trapping the atom. As Eq. (46) is periodic in space and time, Bloch and Floquet theory can be applied to study the system. As an example, the Floquet-Bloch bands in this type of system are shown in Fig. 3, attributed to the work by Kilian Sandholzer et al. on the *Floquet engineering of individual band gaps in an optical lattice using a two-tone drive* [13]. We note that the von Neumann–Wigner noncrossing rule leads to the formation of a band gap in quasimomentum when considering the single harmonic driving as the case depicted in Fig. 3 [34]. This results in the hybridization of the lowest band and the first excited band. In their work, the potential V in the lab frame takes the form

$$V = -V_X \cos^2\left(k_L\hat{x} - k_L x_0(\tau)\right)\,, \tag{47}$$

where the depth V_X and the phase $k_L x_0$ are controlled by varying the intensity of the laser and the position of the retroreflecting mirror. They use a piezoelectric actuator to obtain precise and fast control of the mirror position defining the phase of the lattice potential

$$k_L x_0(\tau) = \frac{2E_{\mathrm{rec}}}{\pi\hbar\omega}\left(K_\omega \cos(\omega\tau) + \frac{K_{l\omega}}{l}\cos(l\omega\tau + \phi)\right)\,, \tag{48}$$

where E_{rec} is the recoil energy, k_L is the wave vector of the lattice laser, and K_ω and $K_{l\omega}$ are the dimensionless driving strengths. Utilizing this setup, they experimentally and theoretically study two-frequency phase modulation to asymmetrically hybridize the lowest two bands of the one-dimensional lattice, and it is worth noticing that their theoretical framework in studying the model is based on that described by Eq. (28), turning a two-level-system problem into a tight-binding-model problem.

VI. SUMMARY

In this paper, we start by discussing the motivation for applying Floquet theory in describing two-level systems. We have demonstrated that the Floquet theory casts open boundary condition problems into periodic boundary condition problems by exploiting the periodicity in the Hamiltonian, which in turn would help us in efficiently computing the state of the two-level system in the semi-classical approach as demonstrated in Section IV. We have shown that the Floquet theory can turn a two-level problem into a tight-binding model in the extended Hilbert space. Moreover, Floquet theory can also be applied to study the Jaynes-Cummings model describing the two-level system, and we have investigated how the counter-rotating term plays a role in the evolution of the system in the Jaynes-Cummings model. Lastly, we have introduced Floquet engineering in optical lattice, based on the study by Sandholzer et al. on shaking optical lattice.

[1] J. H. Shirley, Phys. Rev. **138**, B979 (1965).

[2] G. Engelhardt, S. Choudhury, and W. V. Liu, Phys. Rev. Res. **6**, 013116 (2024).

[3] Y.-F. Wang H.-H. Yin, M.-Y. Yang, A.-C. Ji, and Q. Sun, Chin. Phys. B **30** 064204 (2021).

[4] F. Campaioli, J. H. Cole, H. Hapuarachchi, *A Tutorial on Quantum Master Equations: Tips and tricks for quantum optics, quantum computing and beyond*, arXiv:2303.16449, 2023.

[5] K. Viebahn, *Introduction to Floquet theory*, Bounder School for Condensed Matter and Material Physics (2020).

[6] M. C. Rechtsman, J. M. Zeuner, Y. Plotnik, Y. Lumer, D. Podolsky, F. Dreisow, S. Nolte, M. Segev, and A. Sza-

meit, Nature **496**, 196 (2013).
[7] R. W. Bomantara and J. Gong, Phys. Rev. B **98**, 65421 (2018).
[8] S. Restrepo, J. Cerrillo, P. Strasberg, and G. Schaller, New J. Phys. **20**, 053063 (2018).
[9] B. Bartels and F. Mintert, Phys. Rev. A **88**, 052315 (2013).
[10] A. Castro, U. De Giovannini, S. A. Sato, H. Hubener, and A. Rubio, Phys. Rev. Res. **4**, 033213 (2022).
[11] D. V. Else, B. Bauer, and C. Nayak, Phys. Rev. Lett. **117**, 090402 (2016).
[12] K. Sacha and J. Zakrzewski, Rep. Prog. Phys. **81**, 016401 (2018).
[13] K. Sandholzer , A.-S. Walter, J. Minguzzi, Z. Zhu, K. Viebahn , and T. Esslinger, Phys. Rev. Res. **4**, 013056 (2022).
[14] B. M. Anderson, L. W. Clark, J. Crawford, A. Glatz, I. S. Aranson, P. Scherpelz, L. Feng, C. Chin, and K. Levin, Phys. Rev. Lett. **118**, 220401 (2017).
[15] Y. O. Dudin, L. Li, F. Bariani, and A. Kuzmich, Nat. Phys. **8**, 790 (2012).
[16] T. R. Gentile, B. J. Hughey, D. Kleppner, and T. W. Ducas, Phys. Rev. A **40**, 5103 (1989).
[17] K. Shandarova, C. E. Ruter, and D. Kip, Phys. Rev. Lett. **102**, 123905 (2009).
[18] P.D. Nation, J.R. Johansson, A.J.G. Pitchford, C. Granade, and A.L. Grimsmo, *QuTiP: Quantum Toolbox in Python. User Guide: Time Evolution and Quantum System Dynamics, Floquet Formalism*, 2017.
[19] F. Grossmann, T. Dittrich, P. Jung, and P. Hanggi, Phys. Rev. Lett. **67**, 4 (1991).
[20] Q. Miao and Y. Zheng, Sci. Rep. **6**, 28959 (2016).
[21] Y. Kayanuma and K. Saito, Phys. Rev. A **77**, 010101(R) (2008).
[22] L. Ermann, G. G. Carlo, A. D. Chepelianskii, and D. L. Shepelyansky, Phys. Rev. A **102**, 033729 (2020).
[23] S. Subhankar, P. Bienias, P. Titum, T.-C. Tsui, Y. Wang, A. V. Gorshkov, S. L. Rolston, and J. V. Porto, New J. Phys. **21**, 113058 (2019).
[24] S. Wu, W. Song, Z. Lin, C. Chen, S. Zhu, and T. Li, Opt. Express **30**, pp. 44983 (2022).
[25] K. Sandholzer, *Floquet Engineering of Ultracold Atoms in Optical Lattices*, Dr. sc. thesis, ETH Zurich, 2022.
[26] M. S. Rudner and N. H. Lindner, Nat. Rev. Phys. **2**, 229 (2020).
[27] T. Kitagawa, E. Berg, M. Rudner, and E. Demler, Phys. Rev. B **82**, 235114 (2010).
[28] J. Cayssol, B. Dóra, F. Simon, and R. Moessner, Phys. Status Solidi **7**, 101 (2013).
[29] Á. Gómez-León and G. Platero, Phys. Rev. Lett. **110**, 200403 (2013).
[30] T. Oka and S. Kitamura, Annu. Rev. Condens. Matter Phys. **10**, 387 (2019).
[31] N. R. Cooper, J. Dalibard, and I. B. Spielman, Rev. Mod. Phys. **91**, 015005 (2019).
[32] N. Goldman, J. C. Budich, and P. Zoller, Nat. Phys. **12**, 639 (2016).
[33] C. Weitenberg and J. Simonet, Nat. Phys. **17**, 1342 (2021).
[34] J. von Neumann and E. P. Wigner, Z. Phys. **30**, 467 (1929).