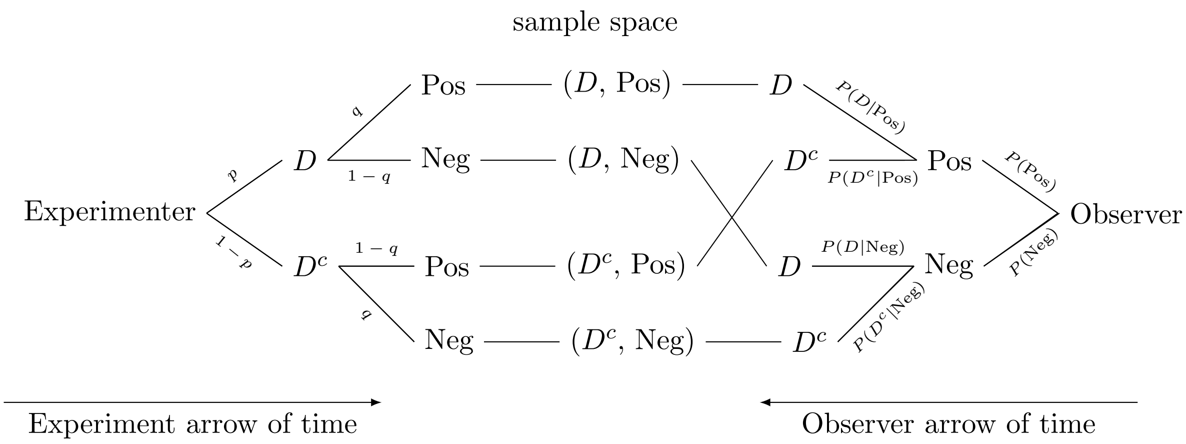
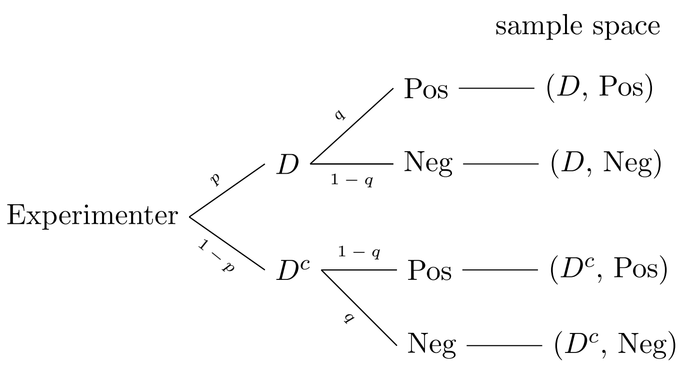
Example: Suppose there is a disease. Let D denote the event that one has the disease. Let D^c denote the event that one does not have the disease. Consider there is a test that gives positive (Pos) and negative (Neg) results. Furthermore, $P(D) = p \in (0,1)$ denotes the probability that a randomly selected person has the disease, and $q \in (0,1)$ denotes the probability that the test identifies the disease correctly.



In this setting, we want $P(\text{Pos}|D^c)$ and $P(\text{Neg}|D)$ to be small, as they correspond to the probabilities of incorrectly identifying the disease using the test. On the other hand, $P(\text{Pos}|D)$ and $P(\text{Neg}|D^c)$ should be large as they correspond to the probabilities of correctly identifying the disease using the test. For the observer, one expects $P(D|\text{Pos})$ and $P(D^c|\text{Neg})$ to be large, and $P(D^c|\text{Pos})$ and $P(D|\text{Neg})$ to be small.

Example: Consider again the disease example. $P(D) = p$, $P(D^c) = 1-p$, and $P(\text{test is correct}) = q$.



Then we can calculate the followings:

$$P(\text{Pos}) = P(D, \text{Pos}) + P(D^c, \text{Pos}) = p \cdot q + (1-p) \cdot (1-q)\,,$$

$$P(D|\text{Pos}) = \frac{P(D, \text{Pos})}{P(\text{Pos})} = \frac{pq}{pq + (1-p)(1-q)}\,,$$

$$P(D^c|\text{Pos}) = \frac{P(D^c, \text{Pos})}{P(\text{Pos})} = \frac{(1-p)(1-q)}{pq + (1-p)(1-q)}\,.$$

Similarly, one can calculate

$$P(\text{Neg}) = p \cdot (1-q) + q \cdot (1-p)\,,$$

$$P(D^c|\text{Neg}) = \frac{q(1-p)}{q(1-p) + p(1-q)}\,, \qquad P(D|\text{Neg}) = \frac{p(1-q)}{q(1-p) + p(1-q)}\,.$$

In this setting, we would like to have

$$P(D|\text{Pos}) = \frac{pq}{pq + (1-p)(1-q)}\,, \qquad \text{and} \qquad P(D^c|\text{Neg}) = \frac{(1-p)q}{(1-p)q + p(1-q)}$$

to be large.

Here we consider some special cases:

1. Suppose the disease is highly prevalent, for instance, $P(D) = p = 0.9999$. That is, 9999 out of 10000 people on average have the disease. Suppose further that $P(\text{test is correct}) = q = 0.9999$. That is, the test is correct 9999 times out of 10000 times on average. For the test to be useful, we want to have both $P(D|\text{Pos})$ and $P(D^c|\text{Neg})$ to be large. In this case, we compute their numerical values
$$P(D|\text{Pos}) = \frac{pq}{pq + (1-p)(1-q)} = \frac{0.9999 \cdot 0.9999}{0.9999 \cdot 0.9999 + 0.0001 \cdot 0.0001} \approx 1\,.$$
However, we see that
$$P(D^c|\text{Neg}) = \frac{(1-p)q}{(1-p)q + p(1-q)} = \frac{0.0001 \cdot 0.9999}{0.0001 \cdot 0.9999 + 0.9999 \cdot 0.0001} = 0.5\,,$$
from which we see that a negative test result does not mean the person has no disease. The test is not very useful in this sense.

2. Now suppose we have a rare disease, for instance, $P(D) = p = 0.0001$. Suppose further that the test accuracy is $P(\text{test is correct}) = 0.9999 = q$. For the test to be useful, we again want to have both $P(D|\text{Pos})$ and $P(D^c|\text{Neg})$ to be large, but we again see that
$$P(D|\text{Pos}) = \frac{pq}{pq + (1-p)(1-q)} = \frac{0.0001 \cdot 0.9999}{0.0001 \cdot 0.9999 + 0.9999 \cdot 0.0001} = 0.5\,,$$
which is equivalent to a coin toss.

Example: Consider choosing 3 digits from the set $\{1, 2, 3, 4, 5\}$.

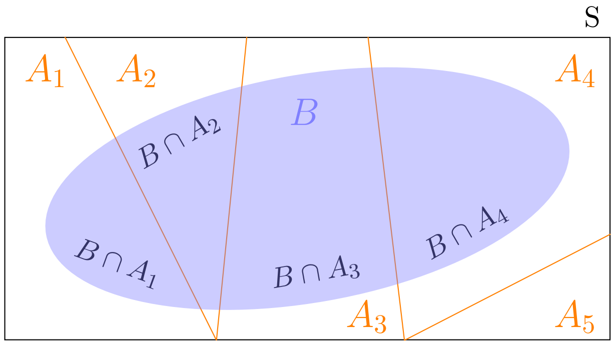
	order matters	order does not matter
with replacement	(1,2,2) is allowed but is different from (2,1,2)	(1,2,2) is allowed and is the same as (2,1,2)
without replacement	(1,2,2) is not allowed, (1,2,3) is different from (1,3,2)	(1,2,2) is not allowed, (1,2,3) is the same as (1,3,2)

Example: Consider choosing 2 digits without replacement from the set $\{1, 2, 3, 4\}$, order does not matter. All the options are $(1, 2)$, $(1, 3)$, $(1, 4)$, $(2, 3)$, $(2, 4)$, and $(3, 4)$. If order matters, more options are available, they are $(2, 1)$, $(3, 1)$, $(4, 1)$, $(3, 2)$, $(4, 2)$, and $(4, 3)$.

Example: Now consider again choosing 3 digits from the set of integers $\{1, 2, 3, 4, 5\}$. In this case, $n = 5$ and $k = 3$. We can compute
$${}^{n}\mathrm{P}_{k} = {}^{5}\mathrm{P}_{3} = \frac{5!}{(5-3)!} = 5 \cdot 4 \cdot 3 = 60\,, \qquad {}^{n}\mathrm{C}_{k} = {}^{5}\mathrm{C}_{3} = \frac{5!}{(5-3)!\,3!} = \frac{60}{3!} = 10\,,$$
$$n^k = 5^3 = 125\,, \qquad {}^{n+k-1}\mathrm{C}_{n-1} = {}^{7}\mathrm{C}_{4} = \frac{7!}{4!\,3!} = 35\,.$$

	order matters	order does not matter
with replacement	125	35
without replacement	60	10

Example: Given a coin with two sides H and T, $P(H) = P(\text{“Coin lands } H\text{”}) = p \in (0, 1)$. In the case where $p = 1/2$, the coin is fair, otherwise not a fair coin.


S
A_1
A_2
A_4
B
$B \cap A_2$
$B \cap A_1$
$B \cap A_3$
$B \cap A_4$
A_3
A_5