Problems

Physics 535 - General Relativity
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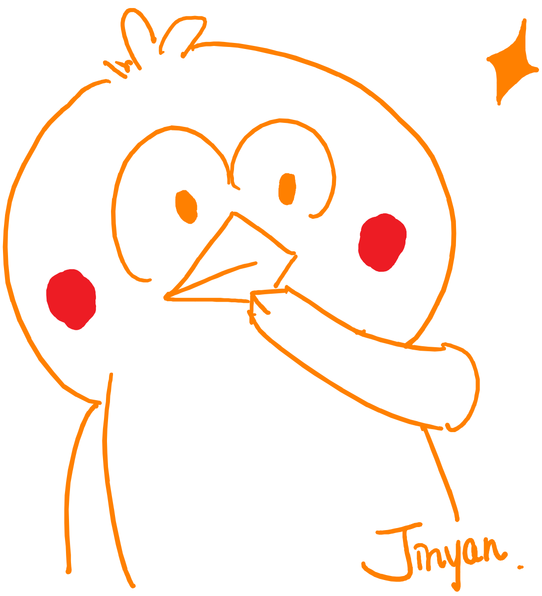

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Contents

Consider a metric

$$ds^2 = \frac{dr^2}{1 - 2\mu/r} + r^2(d\theta^2 + \sin^2(\theta)\, d\phi^2)\,,$$

The area of a sphere of radius R is then given by

$$\int \sqrt{\det(g)}\, dx_1\, dx_2\, \cdots dx_n = \int_0^\pi d\theta \int_0^{2\pi} (R^2 \sin(\theta))\, d\phi = 4\pi R^2\,.$$

The radial distance between the sphere $r = 2\mu$ and the sphere $r = 3\mu$ is then given by

$$\int_{2\mu}^{3\mu} \frac{dr}{\sqrt{1 - 2\mu/r}} = \left(r\sqrt{1 - \frac{2\mu}{r}} + 2\mu \tanh^{-1}\left(\sqrt{1 - \frac{2\mu}{r}}\right)\right) |_{r=2\mu}^{r=3\mu} = (\sqrt{3} + \ln(2 + \sqrt{3}))\mu$$

The volume of a sphere, characterized by $r > 2\mu$, of radius $r = R$ is given by

$$V = \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_{2\mu}^R dr\, \sqrt{\det(g)} = \int_0^\pi \sin(\theta) \int_0^{2\pi} d\phi \int_{2\mu}^R \frac{r^{5/2}}{\sqrt{r - 2\mu}}\,.$$

The worldline of a particle is described by the parametric equations in some Lorentz frame

$$t(\lambda) = a\sinh(\lambda/a)\,, \qquad x(\lambda) = a\cosh(\lambda/a)\,, \qquad y(\lambda) = z(\lambda) = 0\,.$$

The particle's four-velocity is given by

$$v^\mu = \frac{dx^\mu}{d\tau} = \left(\cosh(\lambda/a)\,\frac{d\lambda}{d\tau},\ \sinh(\lambda/a)\,\frac{d\lambda}{d\tau},\ 0,\ 0\right)\,.$$

To show that λ is the proper time along the worldline, we check that we have

$$-1 = g_{\mu\nu}v^\mu v^\nu = \left(-\cosh^2\left(\frac{\lambda}{a}\right) + \sin^2\left(\frac{\lambda}{a}\right)\right)\left(\frac{d\lambda}{d\tau}\right)^2\,,$$

from which we see here

$$\frac{d\lambda}{d\tau} = \pm 1\,,$$

taking $\lambda = \tau$ we see that the worldline is affinely parametrzied.

Here we can find the acceleration of the worldline

$$\alpha^\mu = \frac{dv^\mu}{d\tau} = \left(\frac{1}{a}\sinh\left(\frac{\lambda}{a}\right),\ \frac{1}{a}\cosh\left(\frac{\lambda}{a}\right),\ 0,\ 0\right)\,.$$

One thing to notice here is α^μ is orthogonal to v^μ, as we see here

$$0 = \frac{d}{d\tau}\left(g_{\mu\nu}v^\mu v^\nu\right) = 2g_{\mu\nu}\alpha^\mu v^\nu$$

as we have $g_{\mu\nu}v^\mu v^\nu$. Note further here α^μ is constant

$$|\alpha^\mu|^2 = \frac{1}{a}\,.$$

The usual velocity vector is given by

$$v_x = \frac{dx}{dt} = \frac{dx/d\tau}{dt/d\tau} = \tanh\left(\frac{\tau}{a}\right) = \frac{\sinh(\tau/a)}{\sqrt{\sinh^2(\tau/a)+1}} = \frac{t/a}{\sqrt{(t/a)^2+1}}$$

Consider a 2-space with metric

$$ds^2 = \frac{dr^2 + r^2\, d\theta^2}{r^2 - a^2} - \frac{r^2\, dr^2}{(r^2 - a^2)^2} = \frac{(dr^2 + r^2\, d\theta^2)(r^2 - a^2) - r^2\, dr^2}{(r^2 - a^2)^2} = -\frac{a^2\, dr^2}{(r^2 - a^2)^2} + \frac{r^2\, d\theta^2}{r^2 - a^2}\,.$$

For null geodesics, $ds^2 = 0$, thus we have

$$0 = -\frac{a^2\, dr^2}{(r^2 - a^2)^2} + \frac{r^2\, d\theta^2}{r^2 - a^2}\,,$$

or written in parameters

$$-\frac{a^2\, \dot{r}^2}{(r^2 - a^2)^2} + \frac{r^2 \dot{\theta}^2}{r^2 - a^2} = 0\,.$$

Note further that we have

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\theta}\,,$$

thus we have

$$a^2 \left(\frac{dr}{d\theta}\right)^2 + a^2 r^2 = r^4\,.$$

For finding the geodesics, we minimize the integral

$$\int \frac{1}{2} g_{\mu\nu} x^\mu x^\nu \, d\tau\,,$$

where we employ Euler-Lagrange equation to minimize

$$\mathscr{L} = \frac{1}{2}\left(\frac{-a^2 \dot{r}^2}{(r^2 - a^2)^2} + \frac{r^2 \dot{\theta}^2}{r^2 - a^2}\right)\,,$$

we get

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{\theta}} = \frac{d}{dt}\left(\frac{r^2 \dot{\theta}}{r^2 - a^2}\right) \coloneqq \frac{d}{dt}\, L = 0\,.$$

For timelike geodesic, we can write

$$1 = \frac{a^2\, \dot{r}^2}{(r^2 - a^2)} - \frac{r^2\, \dot{\theta}^2}{r^2 - a^2}$$