Bell Lempert August 2018

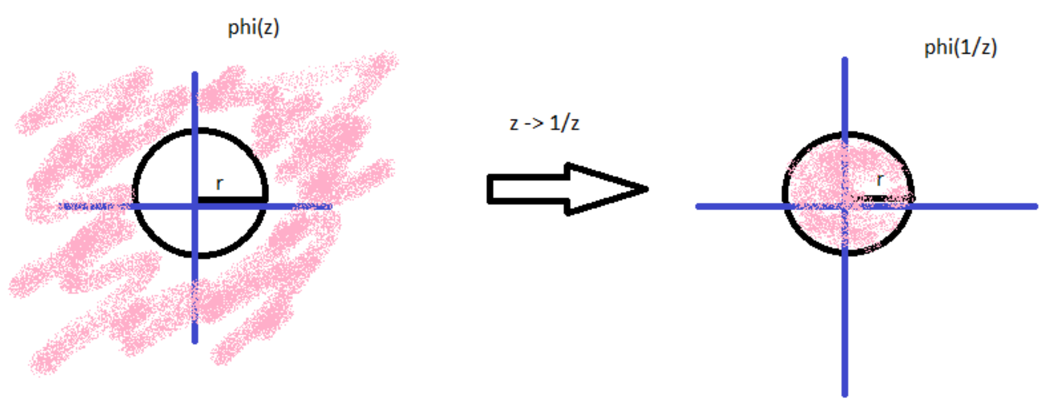
Jelena Mojsilovic

June 2022

1 Problem 5

5. Consider a function ϕ holomorphic on $\{z \in \mathbb{C} : |z| > r\}$, where $r \in (0, 1)$. Suppose that there are a real number K and a natural number N such that $|\phi(z)| \leq K|z|^N$ for all z, and $|\phi(z)| \leq 1$ when $|z| = 1$. Prove that $|\phi(z)| \leq |z|^N$ when $|z| \geq 1$.

Proof. Consider $\phi(\frac{1}{z})$ on $\{z \in \mathbb{C} : |z| < r\}$. Since $|\phi(z)| \leq K|z|^N$ on $\{z \in \mathbb{C} : |z| > r\}$, then $|\phi(\frac{1}{z})| \leq K|\frac{1}{z}|^N$ on $\{z \in \mathbb{C} : |z| < r\}$. Since $|\phi(z)| \leq 1$ when $|z = 1|$, then $|\phi(\frac{1}{z})| \leq 1$ when $|z = 1|$.



Notice, $\phi(z)$ is holomorphic on $\{z \in \mathbb{C} : |z| > r\}$ and hence $\phi(\frac{1}{z})$ is holomohpric on $\{z \in \mathbb{C} : |z| < r\}$. By the maximum modulus principle, the maximimum of $|\phi(\frac{1}{z})|$ is attained on the boundary $|z| = 1$. Thus $K|\frac{1}{z}|^N < 1$ and since $|z| \leq 1$, then $|\frac{1}{z}| \geq 1$ and it must be that $K \leq 1$. Therefore, $|\phi(\frac{1}{z})| \leq |\frac{1}{z}|^N$ on $|z| \leq 1$ implies that $|\phi(z)| \leq |z|^N$ when $|z| \geq 1$. $\square$