MA 530 JANUARY 2018

1. Suppose that $f(z)$ is analytic on the complex plane minus a single point z_0. Suppose further that f has a simple pole at z_0 and a removable singularity at infinity. Prove that

$$f(z) = \frac{A}{z - z_0} + B$$

where A and B are complex constants.

Proof. Let $g(z) = (z - z_0)f(z)$. Then g has a removable singularity at z_0, and hence can be extended to an entire function. Hence there are $\{c_n\} \subset \mathbb{C}$ such that

$$g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad \text{for all } z \in \mathbb{C}\,.$$

So

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^{n-1} \quad \text{for all } z \in \mathbb{C} \setminus \{z_0\}.$$

Thus

$$f(z) - \frac{c_0}{z - z_0} = \sum_{n=0}^{\infty} c_n (z - z_0)^{n-1} \quad \text{for all } z \in \mathbb{C} \setminus \{z_0\}. \tag{1}$$

The right hand side of (1) is a power series with an infinite radius of convergence. Thus the function on the left hand side of (1) can be extended to an entire function, which we will denote h. Now h also has a removable singularity at infinity, hence we know there are $M_1, R > 0$ such that

$$|h(z)| \leq M_1 \quad z \in D_R(0)^c. \tag{2}$$

Then since $|h(z)|$ is continuous on $\overline{D_R(0)}$, there is $M_2 > 0$ such that

$$|h(z)| \leq M_2 \quad \text{for all } z \in \overline{D_R(0)}.$$

Hence combining (2) and (3)

$$|h(z)| \leq M_1 + M_2 \quad \text{for all } z \in \mathbb{C}\,.$$

Since h is entire and bounded, by Liouville's theorem h is constant. So using the notation of (1) we conclude

$$f(z) - \frac{c_0}{z - z_0} = c_1 \quad \text{for all } z \in \mathbb{C} \setminus \{z_0\}$$

which is what we wished to show. $\square$

2. Let

$$f(z) = \frac{\log z}{(z^2+4)^2},$$

where log denotes a branch of the complex logarithm with branch cut along the negative imaginary axis that agree with the real logarithm ln on the positive real axis. For a radius $r > 0$, let C_r denote the half circle parametrized by $z(t) = re^{it}$ for $0 \leq t \leq \pi$, and for $a < b$, let $L[a,b]$ denote the line segment on the real line parametrized by $z(t) = t$ for $a \leq t \leq b$.

a) Assume that $r > 0$. Prove that $\int_{C_r} f(z)\,dz$ goes to zero as r goes to infinity and as r goes to zero.

b) Assume that $0 < \varepsilon < R$. Note that $\int_{L[\varepsilon,R]} f(z)\,dz = \int_\varepsilon^R \frac{\ln t}{(t^2+4)^2}\,dt$. Express $\int_{L[-R,-\varepsilon]} f(z)\,dz$ in terms of explicit real integrals.

c) Compute the residue of $f(z)$ at $2i$.

d) Finally, use the residue theorem, take limits, and take the real part to compute

$$I = \int_0^\infty \frac{\ln t}{(t^2+4)^2}\,dt.$$

Proof. a) First we deal with when $r \to \infty$. From the basic integral estimate we have

$$\begin{aligned}\left|\int_{C_r} f(z)\,dz\right| &\leq \pi r \left(\sup_{z\in C_r}\left|\frac{\log z}{(z^2+4)^2}\right|\right)\\ &= \pi r\left(\frac{\ln(r)+\pi}{(r^2-4)^2}\right)\\ &\to 0\end{aligned}$$

Using the same estimate we have

$$\left|\int_{C_r} f(z)\,dz\right| \leq \frac{\pi r(\ln(r)+\pi)}{(4-r^2)^2}.$$

And it is clear this goes to 0 as $r \to 0$.

b) Parametrize $L[-\varepsilon,-R]$ by $r(t) = -t$ for $\varepsilon \leq t \leq R$. Then since $r'(t) = -1$ we have

$$\begin{aligned}\int_{L[-\varepsilon,-R]} f(z)\,dz &= \int_\varepsilon^R -f(-t)\,dt\\ &= -\int_\varepsilon^R \frac{\log(-t)}{((-t)^2+4)^2}\,dt\\ &= -\int_\varepsilon^R \frac{\ln(t)+\pi i}{(t^2+4)^2}\,dt\\ &= -\int_\varepsilon^R \frac{\ln(t)}{(t^2+4)^2}\,dt - \pi i\int_\varepsilon^R \frac{dt}{(t^2+4)^2}\end{aligned}$$

But we need to reverse the orientation, hence

$$\int_{L[-R,-\varepsilon]} f(z)\,dz = \int_\varepsilon^R \frac{\ln(t)}{(t^2+4)^2}\,dz + \pi i\int_\varepsilon^R \frac{dt}{(t^2+4)^2}.$$

c) We compute

$$\begin{aligned}
\operatorname{Res}\left(\frac{\log z}{(z^2+4)^2}, z=2i\right) &= \lim_{z\to 2i}\left(\frac{\log z}{(z+2i)^2}\right)' \\
&= \lim_{z\to 2i}\frac{(z+2i)^2 z^{-1} - 2\log(z)(z+2i)}{(z+2i)^4} \\
&= \frac{8i - 8i\log(2i)}{256} \\
&= \frac{8i - 8i\ln(2) + 4\pi}{256} \\
&= \frac{\pi}{64} + i\frac{1-\ln(2)}{32}
\end{aligned}$$

d) Let γ denote the path which is formed by traversing C_ε, $L[\varepsilon, R]$, C_R, and $L_{[-R,-\varepsilon]}$ in the counterclockwise sense. Then we have

$$\int_\gamma f(z)\,dz = \int_{C_\varepsilon} f(z)\,dz + \int_{L[\varepsilon,R]} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{[-R,-\varepsilon]} f(z)\,dz.$$

The integral on the left hand side can be found by multiplying our answer in c) by $2\pi i$. Sending $R \to \infty$ and $\varepsilon \to 0$ causes the first and third integrals on the right hand side to tend to 0 by a) and so by b)

$$2\pi i\left(\frac{\pi}{64} + i\frac{1-\ln(2)}{32}\right) = 2\int_0^\infty \frac{\ln(t)}{(t^2+4)^2}\,dt + \pi i\int_0^R \frac{dt}{(t^2+4)^2}.$$

Taking the real part of both sides we get

$$\pi\frac{\ln(2)-1}{16} = 2I.$$

Hence

$$I = \pi\frac{\ln(2)-1}{32}.$$

$\square$

3. Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence of distinct points in the unit disc with no limit points in the disc. Prove that the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n z^n$ is *equal* to one.

Proof. Since $\{a_n\} \subset D_1(0)$, we have that $|a_n| < 1$. Hence

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq 1. \tag{3}$$

$\overline{D_1(0)}$ is compact, so there is a convergent subsequence of $\{a_n\}$. Since $\{a_n\}$ has no limit point *inside* the disc, all limit points of $\{a_n\}$ must lie on the boundary. So for fixed $\varepsilon \in (0,1)$, there will be infinitely many members of the sequence such that

$$1 - \varepsilon < |a_n|.$$

So let $\{a_{n_k}\}$ be a subsequence such that

$$1 - \varepsilon < |a_{n_k}| \quad \text{for all } k.$$

Hence

$$\sqrt[k]{1-\varepsilon} < \sqrt[k]{|a_{n_k}|}.$$

So

$$1 = \lim_{k\to\infty} \sqrt[k]{1-\varepsilon} \leq \limsup_{k\to\infty} \sqrt[k]{|a_{n_k}|} \leq \limsup_{n\to\infty} \sqrt[n]{|a_n|}. \tag{4}$$

From (3) and (4) we conclude that

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1$$

and so by the Cauchy-Hadamard theorem the radius of convergence is 1. $\square$

4. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{(z-n)^2}$ converges on the complex plane minus the positive integers to an analytic function with a double pole at each positive integer.

Proof. Fix $R > 0$. Then for $z \in D_R(0) \setminus \mathbb{N}$ we see

$$\sum_{n=1}^{\infty} \frac{1}{(z-n)^2} = \sum_{n \le R} \frac{1}{(z-n)^2} + \sum_{n > R} \frac{1}{(z-n)^2}. \tag{5}$$

The first sum on the right side of (5) converges because it is a finite sum. The second converges due to the comparison test. Thus by the Weierstrass M-test, $\sum \frac{1}{(z-n)^2}$ converges uniformly on $D_R(0) \setminus \mathbb{N}$ to an analytic function. Since $R > 0$ was arbitrary, we conclude that $\sum \frac{1}{(z-n)^2}$ converges to an analytic function on $\mathbb{C} \setminus \mathbb{N}$. Now for $k \ge 2$ and $m \in \mathbb{N}$, fix $R > m$. Then we know that convergence is uniform on $D_R(0)$ so we may interchange the sum and limit in the following computation to get

$$\lim_{z \to m} (z-m)^k \sum_{n=1}^{\infty} \frac{1}{(z-n)^2} = \sum_{n=1}^{\infty} \lim_{z \to m} \frac{(z-m)^k}{(z-n)^2} = \begin{cases} 1 & k = 2 \\ 0 & k \ge 3 \end{cases}.$$

Thus the poles at each of the natural numbers is of order 2. $\square$

5. Suppose that $f(z)$ is a continuous complex valued function on a disc such that the integral $\int_\gamma f(z)\,dz$ is equal to zero for every contour γ that is the boundary of a square in the disc. Prove that f must be analytic.

Proof. Let f be a continuous complex valued function on the disc $D_r(z')$. Define

$$g(z) = f\left(\frac{z-z'}{r}\right)$$

so g is defined on $D_1(0)$. For $z_0 \in D_1(0)$ let S be the square with 0 and z_0 being opposite vertices, then S is completely contained in $D_1(0)$. This follows from the triangle inequality and the convexity of $D_1(0)$. Define

$$G(z) = \int_0^{z_0} g(z)\,dz$$

where the integral is taken along the sides of S. Notice it does not matter which sides we pick since by assumption the integral of g over S is 0. Then for $h \in \mathbb{C}$ such that

$$z_0 + |h|e^{i\theta} \in D_1(0) \quad \text{for all } 0 \leq \theta \leq 2\pi$$

we see that

$$G(z_0+h) - G(z_0) = \int_{z_0}^{z_0+h} g(z)\,dz \tag{6}$$

where the integral in (6) is taken along the edges of the square whose opposite vertices are z_0 and z_0+h. Since g is continuous at z_0, there is a function $\psi(z)$ such that

$$\lim_{z\to z_0} \psi(z) = 0 \quad \text{and} \quad g(z) = g(z_0) + \psi(z).$$

Then rewriting (6) we have

$$G(z_0+h) - G(z_0) = \int_{z_0}^{z_0+h} g(z_0)\,dz + \int_{z_0}^{z_0+h} \psi(z)\,dz = hg(z_0) + \int_{z_0}^{z_0+h} \psi(z)\,dz. \tag{7}$$

Divide both sides of (7) by h and subtract $g(z_0)$ from both sides to get

$$\left|\frac{G(z_0+h)-G(z)}{h} - g(z_0)\right| = \left|\frac{1}{h}\int_{z_0}^{z_0+h} \psi(z)\,dz\right|. \tag{8}$$

Now we wish to estimate the right hand side of (8). The length of the path along the edges of the square whose opposite vertices are z_0 and z_0+h is $\sqrt{2}|h|$. Hence the basic integral estimate applied to (8) yields

$$\left|\frac{G(z_0+h)-G(z)}{h} - g(z_0)\right| \leq \sqrt{2} \sup_{z\in[z_0,z_0+h]} |\psi(z)|. \tag{9}$$

Take $\varepsilon > 0$. Since $\psi(z) \to 0$ as $z \to z_0$, there is $\delta > 0$ such that

$$z \in D_{2\delta}(z_0) \implies |\psi(z)| < \frac{\varepsilon}{\sqrt{2}}.$$

So if in (9) we require that $|h| < \delta$ we see

$$\left|\frac{G(z_0+h)-G(z)}{h} - g(z_0)\right| < \varepsilon.$$

Hence G is differentiable at z_0 and $G'(z_0) = g(z_0)$. But since $z_0 \in D_1(0)$ was arbitrary, we conclude that $G \in H(D_1(0))$ and $G'(z) = g(z)$. Thus $g \in H(D_1(0))$ which implies $f \in H(D_r(z'))$. $\square$