5. Suppose that $f(z)$ is a continuous complex valued function on a disc such that the integral $\int_\gamma f(z)\,dz$ is equal to zero for every contour γ that is the boundary of a square in the disc. Prove that f must be analytic.

Because we can patch together squares to make rectangles, e.g. first use squares to make a rectangle having side-length ratio $m:n$ where m, n are positive integers, then shrink|grow those rectangles to make rectangles having side ratio $p:q$ for p, q rational, then using density of the rationals to approximate irrational ratios, we can conclude that

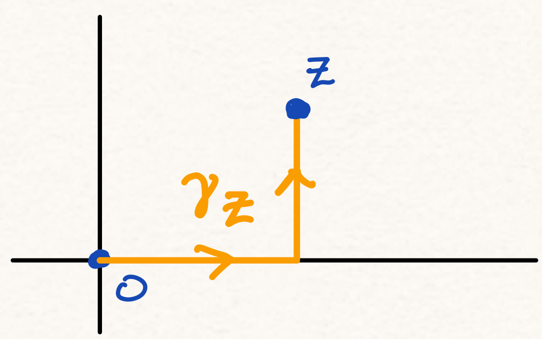
$$\left(\int_{\text{squares}} = 0 \right) \Rightarrow \left(\int_{\text{rectangles}} = 0 \right)$$

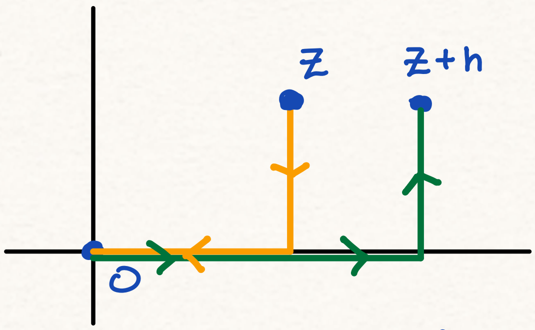
Now, given z in our disk, (WLOG we'll work with $D_1(0)$, and can scale as shown in Nick's submission if needed) define

$$F(z) \equiv \int_{\gamma_z} f(w)\,dw \;, \text{ where}$$

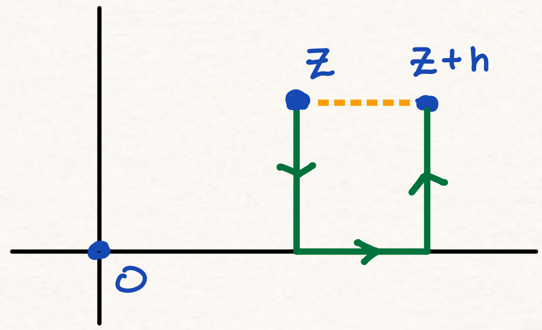
γ_z is the two line segments parallel to the axes, connecting $0, z$ as shown:

Taking h real, positive, we compute $\partial/\partial x\, F(z)$.

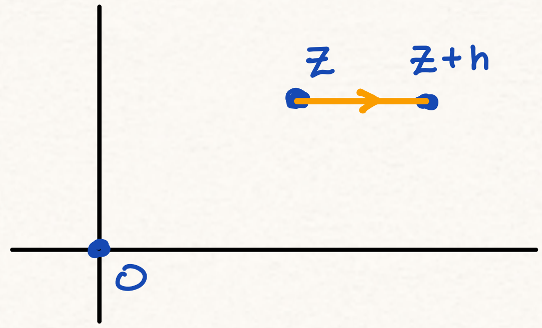




$=$



$\Big(F(z+h) - F(z) \Big) =$



Shown above we see $F(z+h) - F(z)$, which our "lemma" about the rectangles lets us make the last equality.

Then
$$\partial_x F(z) = \lim_{h \to 0} \frac{1}{h}\Big(F(z+h) - F(z) \Big)$$
$$= \lim_{h} \frac{1}{h} \int_0^h f(z+t)\,dt$$
$$= f(z) \qquad \text{(Fund. thm. calc.)}$$

We repeat this process now in the y direction,
$$\partial_y F(z) = \lim_{h \to 0} \frac{1}{h}\Big(F(z+ih) - F(z) \Big)$$
$$= \lim_{h} \left(\frac{1}{h} \int_0^h f(z + \underline{it})\,dt \right)$$
$$= if(z). \qquad \text{(chain rule!)}$$

So both $\partial_x F$ and $\partial_y F$ exist, and are cont. since they are $f(z)$ and $i f(z)$, resp., and f is assumed continuous. Moreover,

$$\partial_x F = \frac{1}{i} \partial_y F,$$ which is actually the Cauchy-Riemann condition for F, quickly we check it:

$$F = u + iv$$

$$\partial_x F = u_x + i v_x$$

$$\partial_y F = u_y + i v_y, \quad \frac{1}{i} \partial_y F = v_y + i u_y.$$

Thus $F(z)$ has continuous partial derivatives, those partials satisfy C-R, and so F is analytic! Then $f(z) = \partial_x F(z) = F'(z)$ (note $F'(z) \neq \partial_y F$, rather $F'(z) = \frac{1}{i} \partial_y F$),

so $f(z)$ is derivative of analytic

$\Rightarrow$ f analytic.