Bell Lempert August 2015

Jelena Mojsilovic

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1 Problem 1

If $\Omega \subset \mathbb{C}$ is open, $f \in \mathcal{O}(\Omega)$ and $u = \mathrm{Re} f, v = \mathrm{Im} f$ then

$$\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = |f'|^2.$$

Proof. Since $f \in \mathcal{O}(\Omega)$, then $f = u + iv$ is holomorphic so that u_x, u_y, v_x, v_y exist and the Cauchy-Reimann equations hold: $u_x = v_y$ and $u_y = -v_x$.

Thus $\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x v_y - u_y v_x = u_x^2 + v_x^2$.

Namely since f is holomorphic by Red box 1: $f' = u_x + iv_x \implies |f'| = \sqrt{u_x^2 + v_x^2} \implies |f'|^2 = u_x^2 + v_x^2$.

Thus $\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = |f'|^2$. $\square$