③ Given that $\varphi : [0,\infty) \to \mathbb{C}$ is a bounded continuous function, prove that

$$H_z := \int_0^\infty \frac{\varphi(t)}{t^2+z}\,dt$$

defines a holomorphic function on $\mathbb{C}\setminus\{z\in\mathbb{R} : z\leq 0\}$.

Since φ is bounded, $\exists\, M>0 : \sup_{[0,\infty)} |\varphi| \leq M$.

Let $\varepsilon>0$ be arbitrary.

Let $a\in\mathbb{C}\setminus\mathbb{R}$, i.e., $\operatorname{Im} a\neq 0$. Choose $\delta\in\mathbb{R}$:

$$0<\delta<\min\left\{\frac{|\operatorname{Im} a|}{2},\ \varepsilon\left(\int_0^\infty \frac{3M}{|t^2+a|^2\cdot|\operatorname{Im} a|}\,dt\right)^{-1}\right\}.$$

Note that $\int_0^\infty \frac{1}{|t^2+a|^2}\,dt$ is convergent and nonzero since $a\in\mathbb{C}\setminus\{z\in\mathbb{R} : z\leq 0\}$.

For $z\in\mathbb{C} : |z-a|<\delta$, then

$$\left|\frac{H(z)-H(a)}{z-a}+\int_0^\infty \frac{\varphi(t)}{(t^2+a)^2}\,dt\right| = \left|\int_0^\infty \frac{\varphi(t)(z-a)}{(t^2+a)^2(t^2+z)}\,dt\right|$$

$$\leq \int_0^\infty \frac{M\cdot|z-a|}{|t^2+a|^2\cdot|t+z|}\,dt.$$

Since $0<\delta<\frac{|\operatorname{Im} a|}{2}$ and $|z-a|<\delta$, then

$$|t+z|\geq|\operatorname{Im}(t+z)| = |\operatorname{Im} z| > \frac{|\operatorname{Im} a|}{3}$$

Hence, $\left| \frac{H(z) - H(a)}{z - a} + \int_0^\infty \frac{\varphi(t)}{(t^2 + a)^2} dt \right| < \varepsilon$.

Now if a is real and positive choose $\delta \in \mathbb{R}$:

$0 < \delta < \min\left\{ \frac{a}{2}, \varepsilon \left(\int_0^\infty \frac{3M}{a(t^2+a)^2} dt \right)^{-1} \right\}$.

For $z \in \mathbb{C}$: $|z - a| < \delta$, we have

$$\left| \frac{H(z) - H(a)}{z - a} + \int_0^\infty \frac{\varphi(t)}{(t^2+a)^2} dt \right| \leq \int_0^\infty \frac{M \cdot |z - t|}{(t^2+a)^2 \cdot |t^2 + z|} dt.$$

Again we see that $|t^2 + z| \geq |\mathrm{Re}(t^2 + z)| > a/3$, so that

$$\int_0^\infty \frac{M \cdot |z - t|}{(t^2+a)^2 \cdot |t^2+z|} dt < \int_0^\infty \frac{3M\delta}{a(t^2+a)^2} dt < \varepsilon.$$

Therefore H is holomorphic on $\mathbb{C} \setminus \{ z \in \mathbb{R} : z \leq 0 \}$.