

Aug 2020, Q2

a) No, since for an essential singularity at a point $a \in \Omega$, say,

$\exists \rho > 0$ s.t. $\forall \varepsilon > 0$, where

$0 < \varepsilon < \rho$, $f(D_\varepsilon(a) \setminus \{a\})$ is dense in \mathbb{C} ,

but $f(D_\varepsilon(a) \setminus \{a\})$ cannot contain $D_2(0)$

\Rightarrow no pt. in $D_2(0)$ is contained $f(D_\varepsilon(a) \setminus \{a\})$ nor is a limit pt. in $f(D_\varepsilon(a) \setminus \{a\})$

This would contradict

$f(D_\varepsilon(a) \setminus \{a\})$ being dense in \mathbb{C} .

$$(b) \lim_{z \rightarrow \infty} f(z) = \infty$$

$$\Rightarrow \lim_{z \rightarrow \infty} \frac{1}{f(z)} = 0.$$

$$\Rightarrow \lim_{z \rightarrow 0} \frac{1}{f(\frac{1}{z})} = 0$$

So, $\frac{1}{f(\frac{1}{z})}$ has a

removable singularity at $z=0$

(Hardwavy?)

Let

$$F(z) = \begin{cases} \frac{1}{f(\frac{1}{z})} & , \quad z \neq 0 \\ 0 & , \quad z = 0 \end{cases}$$

$F(z)$ is analytic on ???

PTD

domain

(What, does $\frac{1}{z}$ map to Ω ?
It's the set $\{z \in \mathbb{C} \mid |z| < \frac{1}{2}\}$)

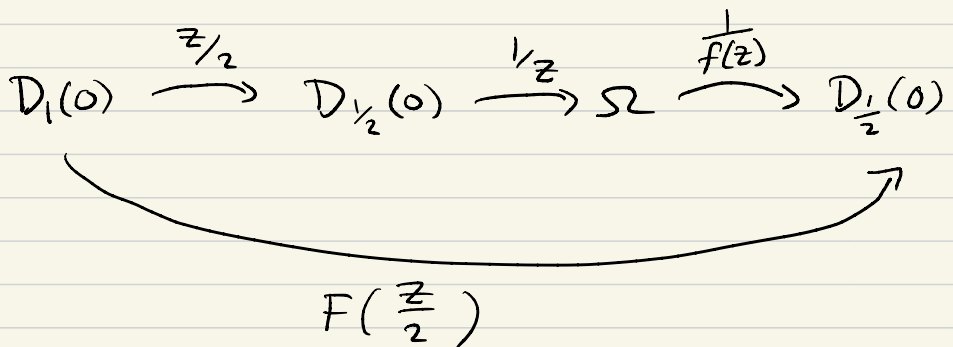
So, $F(z)$ is analytic on $D_{\frac{1}{2}}(0)$.

$\Rightarrow F(\frac{z}{2})$ is analytic on $D_1(0)$.

Note that $\max_{z \in D_1(0)} |F(\frac{z}{2})| \leq \frac{1}{2}$,

since f maps Ω to itself

and we've made it so



$\therefore G(z) = 2F\left(\frac{z}{2}\right)$ is analytic

on $D_1(0)$ and $G(D_1(0)) \subset D_1(0)$.

So, by Schwarz Lemma,

$$|G(z)| \leq |z|, \quad z \in D_1(0)$$

$$\Rightarrow |2F\left(\frac{z}{2}\right)| \leq |z| \quad \text{--- " ---}$$

$$\Rightarrow |F\left(\frac{z}{2}\right)| \leq \left|\frac{z}{2}\right|, \quad \text{--- " ---}$$

$$\Rightarrow \left| \frac{1}{f\left(\frac{z}{2}\right)} \right| \leq \left| \frac{z}{2} \right|, \quad \text{--- " ---}$$

$$\Rightarrow \left| \frac{1}{f(w)} \right| \leq \left| \frac{1}{w} \right|, \quad w \in \Omega$$

(let $\frac{z}{2} = w$)

$$\Rightarrow |w| \leq |f(w)|, \quad w \in \Omega$$