MA 530 AUGUST 2018

1. If $\Omega \subset \mathbb{C}$ is open, $f \in \mathcal{O}(\Omega)$, and $u = \operatorname{Re} f$, $v = \operatorname{Im} f$, prove that

$$\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = |f'|^2.$$

Proof. Set

$$A = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

Since $f' = u_x + iu_y$, we see

$$|f'|^2 = u_x^2 + u_y^2. \tag{1}$$

Now utilizing the Cauchy-Riemann equations, (1) becomes

$$|f'|^2 = u_x v_y - u_y v_x \tag{2}$$

and we recognize the expression on the right hand side of (2) is precisely $\det(A)$. $\square$

2. Compute the following integral (the path of integration is orientated counterclockwise):

$$\int_{|z|=1/2} \frac{e^z}{z^5 - z^3 + z^2}\, dz.$$

Proof. Notice

$$z^5 - z^3 + z^2 = z^2(z^3 - z + 1).$$

Now we claim that all roots of $z^3 - z + 1$ lie outside $D(0, 1/2)$. So for $|z| = 1/2$ we have

$$\begin{aligned}
|(z^3 - z + 1) - (1 - z)| &= |z^3| \\
&= 8^{-1} \\
&< |1 - 2^{-1}| \\
&= |1 - |z|| \\
&\leq |1 - z|
\end{aligned}$$

Since $1 - z$ has no roots inside of $D(0, 1/2)$, by Rouché's theorem we conclude that $z^3 - z + 1$ also has no roots inside of $D(0, 1/2)$. Therefore if we set

$$f(z) = \frac{e^z}{z^3 - z + 1}$$

then f is holomorphic in $D(0, 1/2)$. Hence by Cauchy's integral formula we have

$$\begin{aligned}
\int_{|z|=1/2} \frac{e^z}{z^5 - z^3 + z^2}\, dz &= \int_{|z|=1/2} \frac{f(z)}{z^2}\, dz \\
&= 2\pi i f'(0) \\
&= 2\pi i \left(\frac{(z^3 - z + 1)e^z - e^z(3z^2 - 1)}{(z^3 - z + 1)^2} \right) \Bigg|_{z=0} \\
&= 4\pi i
\end{aligned}$$

□

3. Given that $\phi : [0, \infty) \to \mathbb{C}$ is a bounded continuous function, prove that

$$H(z) = \int_0^\infty \frac{\phi(t)}{t^2 + z}\, dt$$

defines a function holomorphic on $\mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}$.

Proof. Set

$$\Omega = \mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}$$

and let $\gamma \subset \Omega$ denote a closed contour. Now note that Ω is simply connected domain and $(t^2 + z)^{-1}$ is holomorphic in Ω. So by Cauchy's theorem

$$\int_\gamma \frac{1}{t^2 + z}\, dz = 0 \quad \text{for all } z \in \Omega.$$

Hence

$$\begin{aligned}
\int_\gamma H(z)\, dz &= \int_\gamma \left(\int_0^\infty \frac{\phi(t)}{t^2 + z}\, dt \right) dz \\
&= \int_0^\infty \phi(t) \left(\int_\gamma \frac{1}{t^2 + z}\, dz \right) dt \\
&= \int_0^\infty 0\, dt \\
&= 0
\end{aligned}$$

Let us justify the interchange of integrals in the second line above. Clearly $\phi(t)(t^2 + z)^{-1}$ is continuous on Ω, so it is measurable. Since ϕ is bounded, there is $M > 0$ such that $|\phi(t)| < M$ for all $0 \leq t < \infty$. Let L denote the length of γ. Since γ is compact, there is $z^* \in \gamma$ such that

$$\sup_{z \in \gamma} \frac{1}{|t^2 + z|} = \frac{1}{|t^2 + z^*|}.$$

Hence

$$\begin{aligned}
\left| \int_0^\infty \left(\int_\gamma \left| \frac{\phi(t)}{t^2 + z} \right| dz \right) dt \right| &\leq \int_0^\infty \left| \int_\gamma \left| \frac{\phi(t)}{t^2 + z} \right| dz \right| dt \\
&\leq \int_0^\infty \frac{LM}{|t^2 + z^*|}\, dt \\
&< \infty
\end{aligned}$$

Notice the fact that $z^* \in \Omega$ was used to conclude that the last integral is finite. Hence by the Fubini-Tonelli theorem, the interchange is legitimate and so by Morera's theorem $H \in \mathcal{O}(\Omega)$. $\square$

4. Show that a function $g \in \mathcal{O}(\mathbb{C})$ is $2\pi i$-periodic (i.e., $g(z+2\pi i) = g(z)$) if and only if there is an $h \in \mathcal{O}(\mathbb{C}\setminus\{0\})$ such that $g(z) = h(e^z)$.

Proof. ($\Longleftarrow$): This is obvious since e^z is $2\pi i$ periodic.
($\Longrightarrow$): Set $\Omega = \mathbb{C}\setminus\{0\}$ and $f(z) = e^z$. Fix $w \in \Omega$ and find $z \in \mathbb{C}$ such that $f(z) = w$. Then

$$f^{-1}(\{w\}) = \{z + 2\pi i n : n \in \mathbb{Z}\}.$$

Hence g has the same value at all elements of $f^{-1}(\{w\})$ since g is $2\pi i$-periodic. Thus if we define $h(w) = g(z)$, where z is any element of $f^{-1}(\{w\})$, then h is well-defined.
For $w \in \Omega$, choose any $z \in f^{-1}(\{w\})$. Since $f'(z) \neq 0$, then there is an open set $V \subset \mathbb{C}$ such that

(a) $z \in V$,
(b) f is one-to-one in V,
(c) $W = f(V)$ is open,
(d) if $F : W \to V$ is defined by $F(f(z)) = F(w) = z$, then $F \in \mathcal{O}(W)$.

Therefore F is a local holomorphic inverse of f. So fix $w' \in W$. Then $h(w') = (g \circ F)(w')$. Since $g \circ F \in \mathcal{O}(W)$ and $w' \in W$ was arbitrary, this shows $h \in \mathcal{O}(W)$. Then since $w \in \Omega$ was arbitrary, we conclude that $h \in \mathcal{O}(\Omega)$ which completes the proof. $\square$

5. Consider a function ϕ holomorphic on $\{z \in \mathbb{C} : |z| > r\}$, where $r \in (0,1)$. Suppose that there is a real number K and a natural number N such that $|\phi(z)| \leq K|z|^N$ for all z, and $|\phi(z)| \leq 1$ when $|z| = 1$. Prove that $|\phi(z)| \leq |z|^N$ when $|z| \geq 1$.

Proof. Notice

$$|\phi(z^{-1})| \leq K|z|^{-N} \quad \text{for all } |z| < r^{-1}. \tag{3}$$

Also since $|\phi(z)| \leq 1$ for $|z| = 1$, it follows that

$$|\phi(z^{-1})| \leq 1 \quad \text{for all } |z| = 1. \tag{4}$$

Since $\phi(z) \in \mathcal{O}(\mathbb{C} \setminus \overline{D_r(0)})$, we see that $\phi(z^{-1}) \in \mathcal{O}(D_{r^{-1}}(0) \setminus \{0\})$. We claim it can be extended to a function which is holomorphic on $D_{r^{-1}}(0)$. From (3) we get

$$|z^{N+1}\phi(z^{-1})| \leq K|z| \quad \text{for all } |z| < r^{-1}.$$

Hence sending $z \to 0$, we see that $z^{N+1}\phi(z^{-1}) \to 0$, which proves the singularity at 0 is removable. Thus we may regard $z^N\phi(z^{-1})$ as holomorphic on $D_{r^{-1}}(0)$. Since $r \in (0,1)$, we see that $\overline{D_1(0)} \subset D_{r^{-1}}(0)$. Hence by the maximum modulus principle, $|z^N\phi(z^{-1})|$ obtains its max on $\overline{D_1(0)}$ on the boundary. So by (4)

$$|z^N\phi(z^{-1})| \leq 1 \quad \text{for all } |z| \leq 1. \tag{5}$$

Now by sending $z \mapsto z^{-1}$ in (5) we get

$$|\phi(z)| \leq |z|^N \quad \text{for all } |z| \geq 1$$

which is exactly what we wished to show. □

6. Construct a biholomorphic map between $\{z \in \mathbb{C} : |z-1| < 1,\ |z-1/2| > 1/2\}$ and the unit disc. If the map is obtained as a composition of simpler maps, you need not write out explicitly the composition.

Proof. Let $\Omega \subset \mathbb{C}$ denote the region in the question and let $f_1(z) = z^{-1}$. Then f_1 transforms Ω onto the vertical strip

$$f_1(\Omega) = \left\{ z \in \mathbb{C} : \frac{1}{2} < \operatorname{Re} z < 1 \right\}.$$

Now let $f_2(z) = 2\pi i(z - 1/2)$. Then f_2 maps $f_1(\Omega)$ onto

$$f_2(f_1(\Omega)) = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < \pi\}.$$

Let $f_3(z) = e^z$. Then

$$f_3(f_2(f_1(\Omega))) = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

Finally recall that

$$L(Z) = \frac{z-i}{z+i}$$

maps the upper half plane onto the unit disc. So our desired biholomorphic map between Ω and the unit disc is $(L \circ f_3 \circ f_2 \circ f_1)(z)$. $\square$

7. In this problem $D_r = \{z : |z| < r\}$. Suppose $F, G \in \mathcal{O}(D_1)$, G is injective, $F(0) = G(0)$, and $F(D_1) \subset G(D_1)$. Prove that $F(D_r) \subset G(D_r)$ for all $r < 1$.

Proof. Take F and G as above and fix $0 < r < 1$. Since G is holomorphic and a bijection, it is invertible and G^{-1} is holomorphic. Define $g : D_1(0) \to D_1(0)$ by

$$g(z) = G^{-1}(F(z)).$$

Then it is clear that g is holomorphicc, $|g(z)| < 1$ for all $z \in D_1(0)$, and $g(0) = G^{-1}(F(0)) = 0$. Thus by the Schwarz lemma it follows that

$$|g(z)| \leq |z| \quad \text{for all } z \in D_1(0).$$

So if we select $|z| < r$ then $|g(z)| < r$. Thus it follows that

$$g(D_r(0)) \subset D_r(0).$$

So

$$G^{-1}(F(D_r(0))) \subset D_r(0).$$

So then using the fact G is a bijection we have

$$F(D_r(0)) \subset G(D_r(0))$$

which complete the proof. $\square$