4. *Suppose* $\Omega \subset \mathbb{C}$ *is open,* $g \in \mathcal{O}(\Omega \setminus \{a\})$ *for some* $a \in \Omega$*, and* $\operatorname{Re} g \geq 0$ *everywhere. Prove that the singularity of* g *at* a *is removable.*

(Following Anna's solution) Because Ω is open, we can, without loss of generality, just consider some $D_r(a) \subset \Omega$, and assume Ω is this ball. If g is a constant on $\dot{D}_r(a)$, then by the RRST there is nothing to prove, so we assume g is not constant. g maps Ω to the right half-plane $\{Re(z) \geq 0\}$, but *because it is not constant*, the open-mapping theorem tells us that the image of $\dot{D}_r(a)$ under g must be an open subset of $\{Re(z) \geq 0\}$, and thus this image cannot contain any boundary points, i.e. those with $Re(z) = 0$. So $g(z)$ in fact maps to the interior set $\{Re(z) > 0\}$.

We map this plane to the unit disk by composing $z \mapsto iz$ and $z \mapsto \frac{z-i}{z+i}$, defining $L(z)$ to be this composition. We can in fact just write out

$$L(z) = \frac{iz - i}{iz + i} = \frac{z - 1}{z + 1}.$$

As L is analytic on $\{Re(z) > 0\}$, we know $L(g(z))$ is analytic as a map $\dot{D}_r(a) \to D_1(0)$, and thus $|L(g(z))|$ must be bounded on $\dot{D}_r(a)$, and we apply the RRST again to conclude there exists some $c \in \mathbb{C}$ such that $L(g(a)) := c$ provides an analytic extension of $L \circ g$ to all of $D_r(a)$. By continuity, certainly $c \in \overline{D_1(0)}$.

To conclude $g(a) = L^{-1}(c)$, we would like to be sure $c \neq 1 + 0i$, as $L^{-1}(z) = \frac{1+z}{1-z}$. Intuitively, $c = 1$ would mean $g(a) = \infty$, because $L(\infty) = 1$, so we know this won't happen. To be completely rigorous, we can again appeal to the open mapping theorem: If $c = 1$, then $L \circ g$ must map $D_r(a)$ to an open set about 1, which necessarily contains points $w : |w| > 1$. Because L, L^{-1} are biholomorphic (as maps $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$), the image of $D_r(a)$ containing such w is possible if and only if $g(z)$ mapped to some points in the half plane $\{Re(z) < 0\}$, contradicting our assumption. Thus $g(a) = L^{-1}(c)$ shows us g has a removable singularity at a.

Although we only needed to consider $c = 1$, you can see that this argument about $|w| > 1$ and the open mapping theorem would also prevent c being *any* point on the boundary $\{|w| = 1\}$, as those are the points with $Re[L^{-1}(w)] = 0$.