MA 530 JANUARY 2020

1. How many zeroes (counted with multiplicity) of the polynomial $10z^{10} + 25z^3 + 13z + 1$ fall outside the unit circle?

Proof. Let $f(z) = 10z^{10} + 25z^3 + 13z + 1$. Notice that for $|z| = 1$ we have

$$|f(z)| \geq |25|z|^3 - 10|z|^{10} - 13|z| - 1| = 1 > 0$$

thus f has no zeroes on the unit circle. So now for $|z| = 1$ we compute

$$\begin{aligned} |f(z) - 25z^3| &= |10z^{10} + 13z + 1| \\ &\leq 10|z|^{10} + 13|z| + 1 \\ &= 24 \\ &< 25 \\ &= |25z^3| \end{aligned}$$

Hence by Rouché's theorem we see that f has exactly 3 zeroes inside of the unit circle. Then since f has exactly 10 zeroes in $\mathbb{C}$ we conclude that f has 7 roots outside of the unit circle. $\square$

2. Let C denote the unit circle parameterized in the counterclockwise sense. Compute $\displaystyle\int_C \frac{z}{2z^2+5z+1}\,dz$.

Proof. First we factor

$$(2z^2+5z+1) = 2\left(z+\frac{5}{4}+\frac{\sqrt{17}}{4}\right)\left(z+\frac{5}{4}-\frac{\sqrt{17}}{4}\right).$$

So the roots are $(-5\pm\sqrt{17})/4$. Clearly

$$\frac{-5-\sqrt{17}}{4} \notin D_1(0) \quad \text{and} \quad \frac{-5+\sqrt{17}}{4} \in D_1(0).$$

Thus by the residue theorem

$$\begin{aligned}
\int_C \frac{z}{2z^2+5z+1}\,dz &= 2\pi i \operatorname{Res}\left(\frac{z}{2z^2+5z+1}, z = \frac{-5+\sqrt{17}}{4}\right) \\
&= 2\pi i\left(\frac{(-5+\sqrt{17})/4}{2(\sqrt{17}/2)}\right) \\
&= \frac{\pi i}{2}\left(\frac{-5+\sqrt{17}}{\sqrt{17}}\right)
\end{aligned}$$

□

3. Compute the residue of $\dfrac{e^{3z}}{1-\cos z}$ at $z=0$.

Proof. We have that

$$1-\cos(z) = 1 - \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^{2n}}{(2n)!}$$

and

$$e^{3z} = \sum_{n=0}^{\infty}\frac{3^n z^n}{n!}.$$

Hence

$$\frac{e^{3z}}{1-\cos(z)} = \frac{1}{z^2}\left(\frac{\sum_{n=0}^{\infty}\frac{3^n z^n}{n!}}{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{z^{2n-2}}{(2n)!}}\right) = \frac{1}{z^2}\left(\frac{1+3z+9z^2/2!+\dots}{1/2!-z^2/4!+z^4/6!-\dots}\right).$$

Write

$$F(z) = \frac{1+3z+9z^2/2!+\dots}{1/2!-z^2/4!+z^4/6!-\dots}.$$

We see that F is holomorphic in a neighborhood of 0, and thus can be expanded as a power series. So write

$$F(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

Then this power series expansion must satisfy the relation

$$1+3z+\frac{9z^2}{2!}+\dots = (a_0+a_1z+a_2z^2+\dots)\left(\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\dots\right).$$

Expanding the product on the right hand side, we see that $a_1 = 6$. Hence

$$\frac{e^{3z}}{1-\cos(z)} = \frac{F(z)}{z^2} = \frac{a_0}{z^2}+\frac{6}{z}+a_2+\dots$$

so we conclude

$$\operatorname{Res}\left(\frac{e^{3z}}{1-\cos(z)}, z=0\right) = 6.$$

$\square$

4. What is the image of the upper half plane under the mapping $L(z) = \dfrac{z-a}{z-b}$ where a and b are real numbers with $0 < a < b$? Explain.

Proof. Let U denote the upper half plane and let V denote lower half plane. For $z = x + iy$ with $y > 0$ we see

$$\begin{aligned} L(z) &= \frac{x+iy-a}{x+iy-b} \\ &= \frac{x-a+iy}{x-b+iy} \cdot \frac{x-b-iy}{x-b-iy} \\ &= \frac{x^2-(a+b)x+y^2+ab}{(x-b)^2+y^2} - i\frac{(b-a)y}{(x-b)^2+y^2} \end{aligned}$$

And since $b-a > 0$ and $y > 0$ we get that $L(z) \in V$ so $L(U) \subset V$. Now simple computations show that

$$L^{-1}(z) = \frac{bz-a}{z-1}.$$

For $z = x - iy$ with $y > 0$ we see

$$\begin{aligned} L^{-1}(z) &= \frac{bx-iby-a}{x-iy-1} \\ &= \frac{bx-a-iby}{x-1-iy} \cdot \frac{x-1+iy}{x-1+iy} \\ &= \frac{bx^2-bx-ax+a+by^2}{(x-1)^2+y^2} + i\frac{(b-a)y}{(x-1)^2+y^2} \end{aligned}$$

and since $b-a > 0$ and $y > 0$ we see $L^{-1}(z) \in U$ and so $L^{-1}(V) \subset U$. Since L is a bijection, $V \subset L(U)$ which proves that $L(U) = V$. $\square$

5. In this problem, you will compute the real integral $I = \displaystyle\int_0^\infty \frac{\ln(t)}{t^3+1}\,dt$ by integrating the complex values function $f(z) = \dfrac{\mathrm{Log}(z)}{z^3+1}$ (where Log denotes the principal branch of the complex log function) around a closed contour γ consisting of the directed line segment L_1 going from the origin to the point $R > 0$, followed by the circular arc C_R parametrized by Re^{it} as t ranges from zero to $2\pi/3$, then the directed line segment L_2 from $Re^{i2\pi/3}$ to the origin.

a) Express the integral of $f(z)$ along the line L_2 in terms of real integrals.
b) Show that the integral of $f(z)$ along the circular part C_R of the curve tends to zero as $R \to \infty$.
c) Compute the residue of f at the point $e^{i\pi/3}$.
d) Explain how to use a-c to get a formula for I. (Do not simplify.)

Proof. a) We can parametrize L_2 using the curve

$$r(t) = (R-t)e^{2\pi i/3}, \quad 0 \le t \le R.$$

Since $r'(t) = -e^{2\pi i/3}$ we get

$$\begin{aligned}
\int_{L_2} \frac{\mathrm{Log}(z)}{z^3+1}\,dz &= \int_0^R \frac{\mathrm{Log}((R-t)e^{2\pi i/3})}{((R-t)e^{2\pi i/3})^3+1}(-e^{2\pi i/3})\,dt \\
&= -e^{2\pi i/3}\int_0^R \frac{\ln(R-t)}{(R-t)^3+1}\,dt - \frac{2\pi i}{3}e^{2\pi i/3}\int_0^R \frac{dt}{(R-t)^3+1} \\
&= -e^{2\pi i/3}\int_0^R \frac{\ln(t)}{t^3+1}\,dt - \frac{2\pi}{3}e^{7\pi i/6}\int_0^R \frac{dt}{t^3+1}
\end{aligned}$$

b) A simple computation using the basic estimate yields

$$\begin{aligned}
\left|\int_{C_R} \frac{\mathrm{Log}(z)}{z^3+1}\,dz\right| &\le \frac{2\pi R}{3}\left(\sup_{z\in C_R}\left|\frac{\mathrm{Log}(z)}{z^3+1}\right|\right) \\
&= \frac{2\pi R}{3}\left(\sup_{z\in C_R}\left|\frac{\ln|z| + i\arg(z)}{z^3+1}\right|\right) \\
&\le \frac{2\pi R}{3}\left(\frac{3\ln(R)+2\pi i}{3(R^3-1)}\right) \\
&\to 0
\end{aligned}$$

Hence

$$\lim_{R\to\infty}\int_{C_R} \frac{\mathrm{Log}(z)}{z^3+1}\,dz = 0.$$

c) Recall if f and g are holomorphic and z_0 is a simple zero of $g(z)$ then

$$\mathrm{Res}(f/g, z = z_0) = \frac{f(z_0)}{g'(z_0)}.$$

Since $e^{\pi i/3}$ is a simple zero of z^3+1, using this formula we get

$$\mathrm{Res}\left(\frac{\mathrm{Log}(z)}{z^3+1}, z = e^{\pi i/3}\right) = \frac{\mathrm{Log}(e^{\pi i/3})}{3e^{2\pi i/3}} = \frac{\pi i}{9}e^{-2\pi i/3} = \frac{\pi}{9}e^{-\pi i/6}$$

d) We have that

$$\int_\gamma \frac{\text{Log}(z)}{z^3+1}\,dz = \int_0^R \frac{\ln(x)}{x^3+1}\,dx + \int_{C_R} \frac{\text{Log}(z)}{z^3+1}\,dz + \int_{L_2} \frac{\text{Log}(z)}{z^3+1}\,dz.$$

The integral on the left hand side can be evaluated by multiplying our answer from c) by $2\pi i$. Then sending $R \to \infty$ on the right hand side, the first integral becomes I. The second integral goes to zero by our computations in b). And the final integral become the two real integrals we found in a). Technically, we are allowed to stop here, but for fun, we will keep going. So set

$$I_1 = \int_0^R \frac{\ln(t)}{t^3+1}\,dt \quad \text{and} \quad I_2 = \int_0^R \frac{dt}{t^3+1}.$$

When $R \to \infty$, we see $I_1 \to I$. Now we consider I_2. Since $t^3+1 = (t+1)(t^2-t+1)$, then after applying partial fractions we have

$$I_2 = \frac{1}{3}\left(\int_0^R \frac{dt}{t+1} - \int_0^R \frac{t-2}{t^2-t+1}\,dt\right).$$

The first integral is easy, we get

$$\int_0^R \frac{dt}{t+1} = \ln(R+1).$$

The second is slightly more difficult. Notice $t-2 = (2t-1)/2 - 3/2$ thus

$$\int_0^R \frac{t-2}{t^2-t+1}\,dt = \frac{1}{2}\int_0^R \frac{2t-1}{t^2-1+1}\,dt - \frac{3}{2}\int_0^R \frac{dt}{t^2-t+1}.$$

The first integral now is simple, observe

$$\frac{1}{2}\int_0^R \frac{2t-1}{t^2-t+1}\,dt = \frac{1}{2}\ln(R^2-R+1).$$

To evaluate the second integral, note that

$$t^2-t+1 = \left(t-\frac{1}{2}\right)^2 + \frac{3}{4}.$$

Thus if we make the substitution $u = (2t-1)/\sqrt{3}$ we get

$$\frac{3}{2}\int_0^R \frac{dt}{t^2-t+1} = \sqrt{3}\int_{-1/\sqrt{3}}^{(2R-1)/\sqrt{3}} \frac{du}{u^2+1} = \sqrt{3}\left(\tan^{-1}\left(\frac{2R-1}{\sqrt{3}}\right) + \frac{\pi}{6}\right)$$

Thus

$$I_2 = \frac{1}{3}\ln(R+1) - \frac{1}{6}\ln(R^2-R+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2R-1}{\sqrt{3}}\right) + \frac{\pi}{6\sqrt{3}}.$$

Since

$$\lim_{R\to\infty} \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2R-1}{\sqrt{3}}\right) = \frac{\pi}{2\sqrt{3}}$$

and

$$\lim_{R\to\infty}\left(\frac{1}{3}\ln(R+1) - \frac{1}{6}\ln(R^2-R+1)\right) = \lim_{R\to\infty}\ln\left(\frac{\sqrt[3]{R+1}}{\sqrt[6]{R^2-R+1}}\right) = 0$$

then

$$I_2 \to \frac{2\pi}{3\sqrt{3}}.$$

Now sending $R \to \infty$ we get

$$\frac{2\pi^2}{9} e^{\pi i/3} = (1 - e^{2\pi i/3}) I - \frac{4\pi^2}{9\sqrt{3}} e^{7\pi i/6}.$$

Hence

$$\begin{aligned}
I &= \frac{2\pi^2}{9} \left(\frac{e^{\pi i/3} + \frac{2}{\sqrt{3}} e^{7\pi i/6}}{1 - e^{2\pi i/3}} \right) \\
&= -\frac{\pi^2}{9i} \left(\frac{2i}{e^{\pi i/3} - e^{-\pi i/3}} \right) \left(1 + \frac{2}{\sqrt{3}} e^{\pi i/6} \right) \\
&= -\frac{\pi^2}{9i} \left(\frac{1}{\sin(\pi/3)} \right) \left(\frac{i}{\sqrt{3}} \right) \\
&= -\frac{2\pi^2}{27}
\end{aligned}$$

$\square$

6. Find a one-to-one conformal mapping of the open unit disc onto the first quadrant of the complex plane.

Proof. Recall
$$L(z) = \frac{z-i}{z+i}$$
is a one-to-one conformal mapping of the upper half plane onto the unit disc. So if f denotes the principal branch of the square root then our desired map is $(f \circ L^{-1})(z)$. $\square$

7. Suppose that $u(x, y)$ is harmonic on a disc $\{z : |z| < R\}$ where $R > 1$. Let M denote the maximum of $|u|$ on the unit circle. Prove that there is a real constant $C > 0$ that does not depend on u such that

$$\left| \frac{\partial u}{\partial x}(0, 0) \right| \leq CM.$$

Proof. Recall the Poisson kernel is given by

$$P(z, t) = \frac{1}{2\pi} \Re \left(\frac{e^{it} + z}{e^{it} - z} \right).$$

Since $P(z, t)$ is the real part of a holomorphic function, it is harmonic and therefore infinitely differentiable. Let us regard $P(z, t)$ instead as a function x, y, and t, so instead we write $P(x, y, t)$. Then we have the relation

$$u(x, y) = \int_{-\pi}^{\pi} P(x, y, t) u(\cos(t), \sin(t)) \, dt \quad \text{for all } x^2 + y^2 < 1$$

and so

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial}{\partial x} \int_{-\pi}^{\pi} P(x, y, t) u(\cos(t), \sin(t)) \, dt.$$

Now since P and u are infinitely differentiable, by the Leibniz rule we may move the partial derivative inside of the integral to get

$$\frac{\partial u}{\partial x}(x, y) = \int_{-\pi}^{\pi} P_x(x, y, t) u(\cos(t), \sin(t)) \, dt.$$

Let us choose

$$C = \|P_x(0, 0, t)\|_1 = \int_{-\pi}^{\pi} P_x(0, 0, t) \, dt.$$

So

$$\begin{aligned} \left| \frac{\partial u}{\partial x}(0, 0) \right| &\leq \int_{-\pi}^{\pi} |P_x(0, 0, t)| |u(\cos(t), \sin(t))| \, dt \\ &\leq \int_{-\pi}^{\pi} M |P_x(0, 0, t)| \, dt \\ &= CM \end{aligned}$$

This completes the proof. $\square$

8. Suppose that $f(z)$ is an entire function such that there exist positive constants C and $R_0 > 1$ such that

$$|f(z)| \le C \ln |z| \quad \text{if } |z| > R_0.$$

Prove that f must be constant.

Proof. First we show that $f(z)$ has a removable singularity at ∞. This is equivalent to showing that $f(1/z)$ has a removable singularity at 0. So we have

$$\begin{aligned} \lim_{z\to 0} \left| z f\left(\frac{1}{z}\right)\right| &\le \lim_{z\to 0} C|z| \ln \left|\frac{1}{z}\right| \\ &= C \lim_{x\to\infty} \frac{\ln(x)}{x} \\ &= 0 \end{aligned}$$

Hence

$$\lim_{z\to 0} z f(z^{-1}) = 0$$

so the singularity at ∞ is removable. So there is $R > 0$ such that $|f(z)|$ is bounded for all $|z| > R$. Without loss of generality, we may assume that $R > R_0$. So by the maximum modulus principle, f is bounded on $\overline{D(0, R)}$ and since the singularity at ∞ is removable it is bounded on $\mathbb{C} \setminus \overline{D(0, R)}$. Hence f is bounded on $\mathbb{C}$ and so constant by Liouville's theorem. $\square$