Lecture 23 Linear fractional transformations (LFTs)

Hwk 6 due Thurs, 3/10

$$L(z) = \frac{az+b}{cz+d} = w \qquad D = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \neq 0$$

Solve for z:

($D=0$ means rows lin dependent $\Rightarrow$ num $L(z) = (\text{const})$ denom $L(z)$ or worse: denom $\equiv 0$.)

$$z = \frac{dw-b}{-cw+a} = L^{-1}(w)$$

$$\det\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \det \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\mathbb{A}} = D \neq 0$$

$$\mathbb{A}^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Cool thing: LFT's form a group under composition.

$L \sim \mathbb{A}$ $\qquad id(z) = z = \dfrac{1\cdot z + 0}{0\cdot z + 1} \sim \mathbb{I}$

$L^{-1} \sim \mathbb{A}^{-1}$

$$\left.\begin{array}{l} L_1 \sim \mathbb{A}_1 \\ L_2 \sim \mathbb{A}_2 \end{array}\right\} \quad L_1 \circ L_2 \sim \mathbb{A}_1 \mathbb{A}_2 \leftarrow \text{matrix mult}$$

Fact: LFTs isomorphic to grp of 2x2 complex matrices with $\det \neq 0$.

Fact: LFTs: $\left\{\begin{array}{l}\text{lines,} \\ \text{circles}\end{array}\right\} \circlearrowleft$

Lemma: LFTs are generated by 4 basic maps

1) $z \mapsto rz$, $r \in \mathbb{R}^+$ (stretch)

2) $z \mapsto e^{i\theta} z$ (rotation)

3) $z \mapsto z+b$ (translation)

4) $z \mapsto \frac{1}{z}$ (inversion)

Pf: Case $c=0$: $L(z) = \underbrace{A}_{re^{i\theta}} z + B$ 1, 2, 3 all we need.

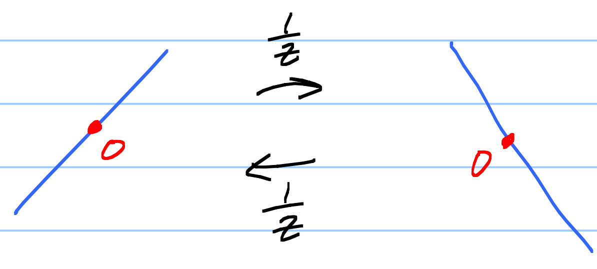
Case $c \neq 0$: $\dfrac{az+b}{cz+d} = \dfrac{a}{c} + \dfrac{(b-\frac{ad}{c})}{\underset{\uparrow\, c=re^{i\theta}}{c}z+d}$ $\leftarrow Re^{i\psi}$

Need 7 basic maps to get L!

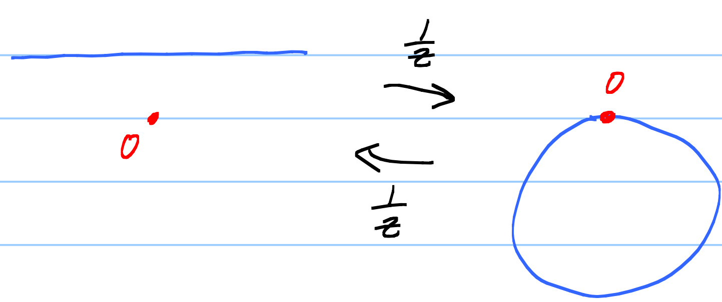
Cor: LFTs : $\left\{\begin{matrix}\text{lines,} \\ \text{circles}\end{matrix}\right\} \circlearrowleft$

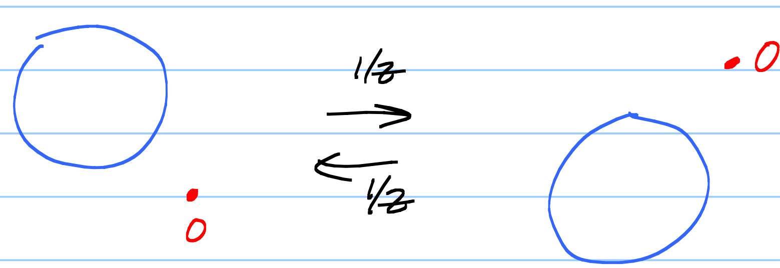
Pf: 1, 2, 3 : $\{\text{lines}\} \circlearrowleft$, $\{\text{circles}\} \circlearrowleft$

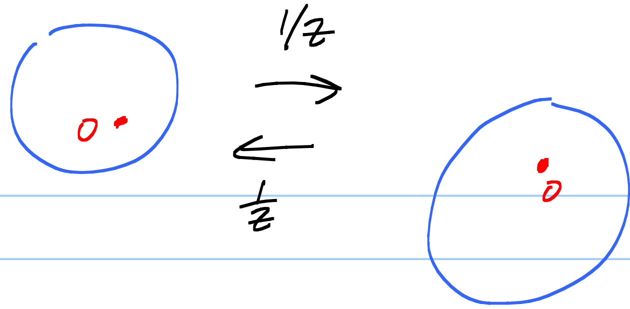
4 mixes them up:



$0 \mapsto \infty$
$\infty \mapsto 0$

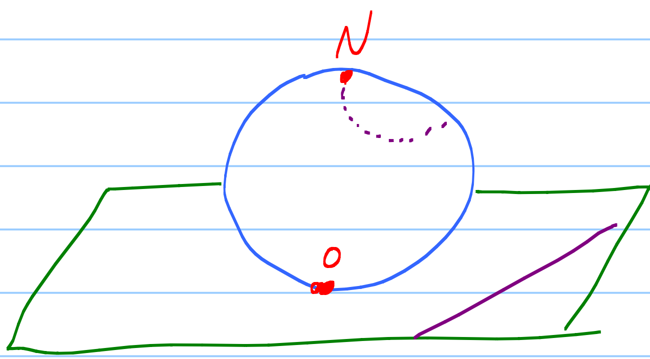






Pf: HS analytic geom.

Fun fact: On Riemann sphere $\left\{\begin{matrix}\text{lines,}\\ \text{circles}\end{matrix}\right\} \sim$ slices



LFTs : $\hat{\mathbb{C}}$ 1-1 onto

$$\lim_{z\to\infty} \frac{az+b}{cz+d} = \frac{a}{c} \quad (c \neq 0)$$

$$\begin{cases} L(\infty) = \frac{a}{c} \\ L(-\frac{d}{c}) = \infty \end{cases}$$

Lemma: An LFT that fixes 3 distinct pts must be the identity.

Pf: $L(z_j) = z_j \qquad j = 1,2,3$

$$\frac{az_j + b}{cz_j + d} = z_j$$

$$cz_j^2 + (d-a)z_j - b = 0 \Leftarrow$$ Quad eqn with 3 roots! All coeff must $= 0$.

So $c = 0$, $\underset{a=d}{d-a=0}$, $b = 0$

$D = ad \neq 0$, So $a \neq 0, d \neq 0$.

$$L(z) = \frac{az+0}{0\cdot z + d} = \left(\frac{a}{d}\right) z$$

Cor: If two LFTs agree at 3 pts, they are the same.

3 pts determine circle

2 pts (plus the pt at ∞) determine a line.

Useful LFTs

$$\frac{z-a}{z-b} \qquad a \mapsto 0, \quad b \mapsto \infty$$

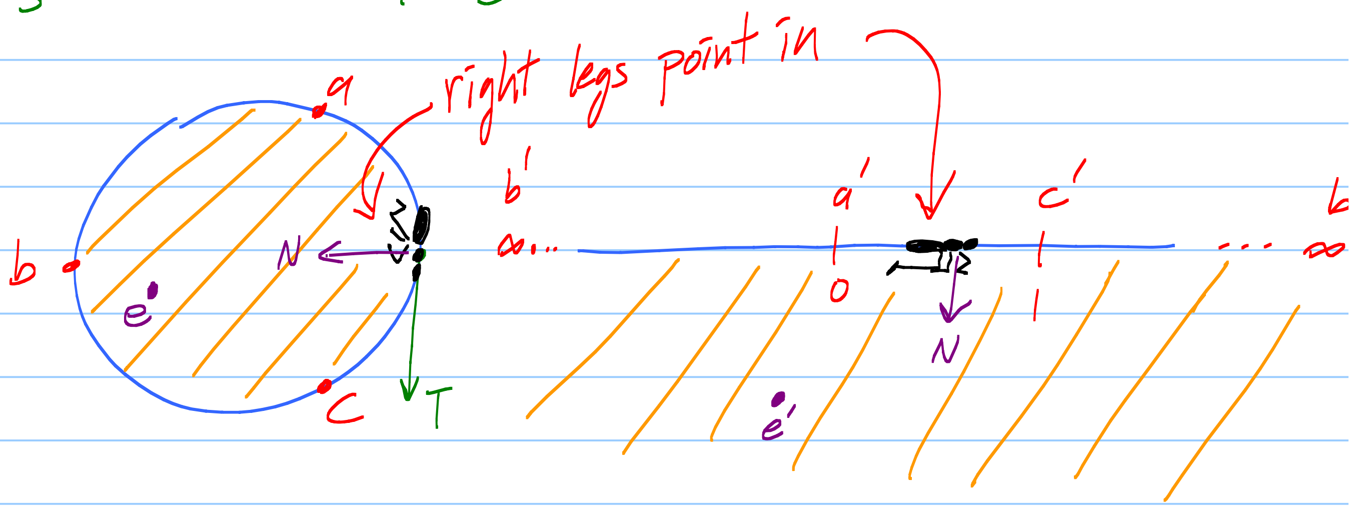
$$\left(\frac{c-b}{c-a}\right)\cdot\frac{z-a}{z-b} \qquad a \mapsto 0, \quad b \mapsto \infty, \quad c \mapsto 1$$

Prob: Find an LFT: $z_1 \mapsto w_1$, $z_2 \mapsto w_2$, $z_3 \mapsto w_3$ in finite $\mathbb{C}$

$$\begin{array}{ccccc} & L_1 & & L_2 & \\ z_1 & \mapsto & 0 & \leftarrow\!\dashv & w_1 \\ z_2 & \mapsto & \infty & \leftarrow\!\dashv & w_2 \\ z_3 & \mapsto & 1 & \leftarrow\!\dashv & w_3 \end{array}$$

$L_2^{-1} \circ L_1$ solves prob.

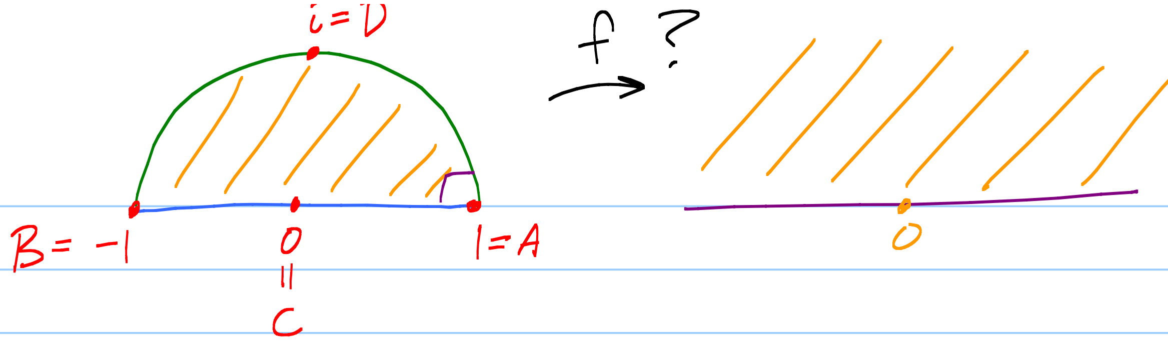
EX:



Bug test: Use conformality to see which side goes to what side.

Check one pt test:

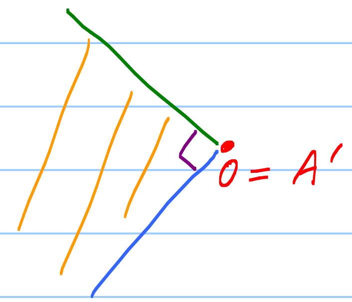
EX:



Trick: blow a corner to ∞.

$$L(z) = \frac{z-1}{z-(-1)}$$

Hmmm.



Quadrant!

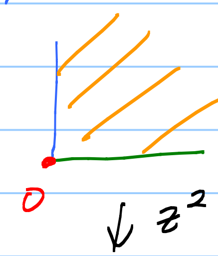
$L(1) = 0$

$L(-1) = \infty$

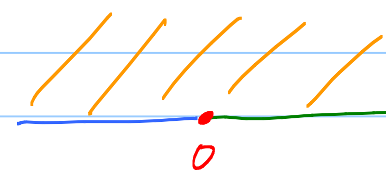
$L(i) = \frac{i-1}{i+1} = i$

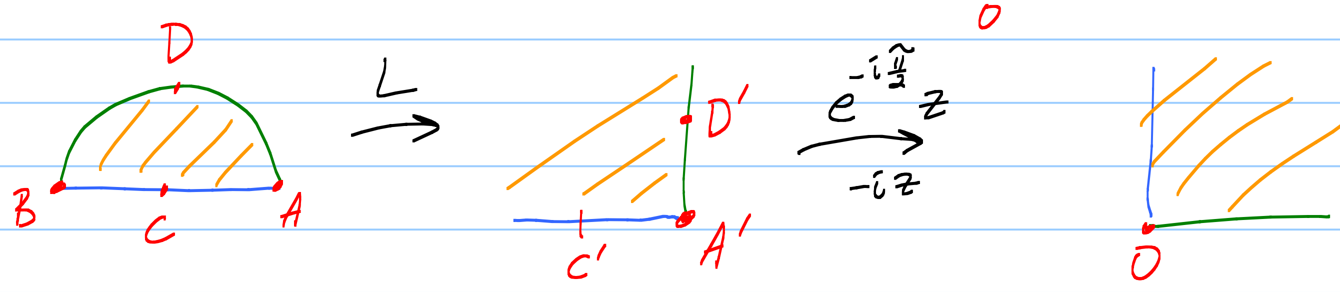
$L(0) = \frac{0-1}{0+1} = -1$

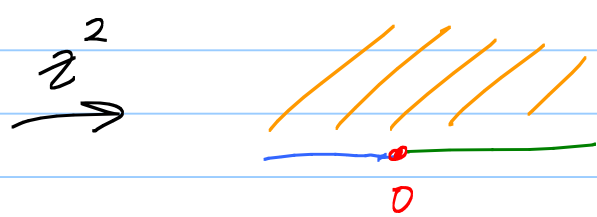
rotate



Square

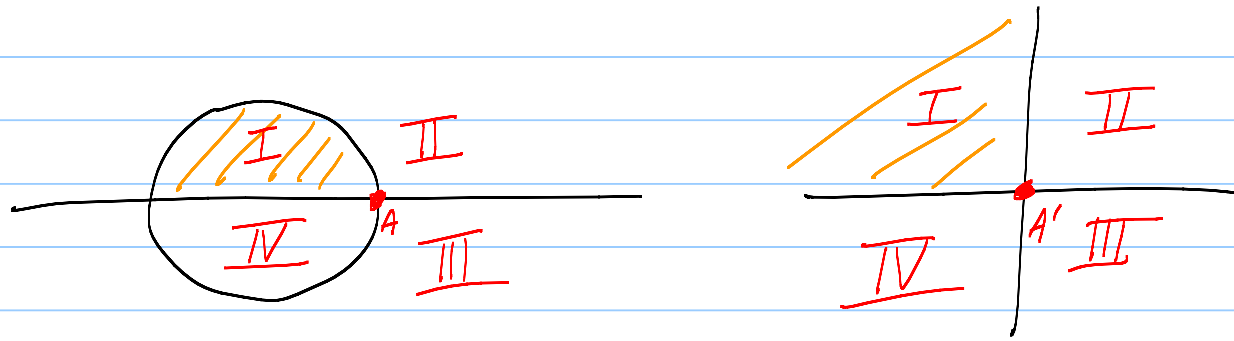






$$f(z) = \left[-i\left(\frac{z-1}{z+1}\right) \right]^2 = -\left(\frac{z-1}{z+1}\right)^2$$

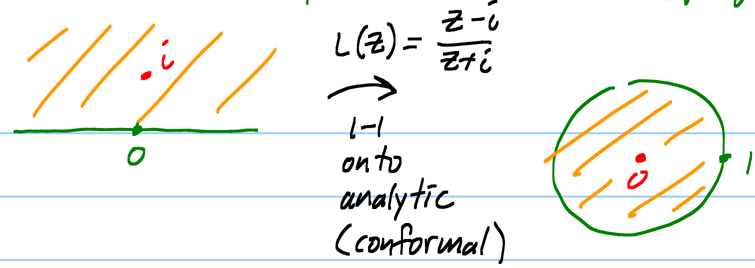
Big picture



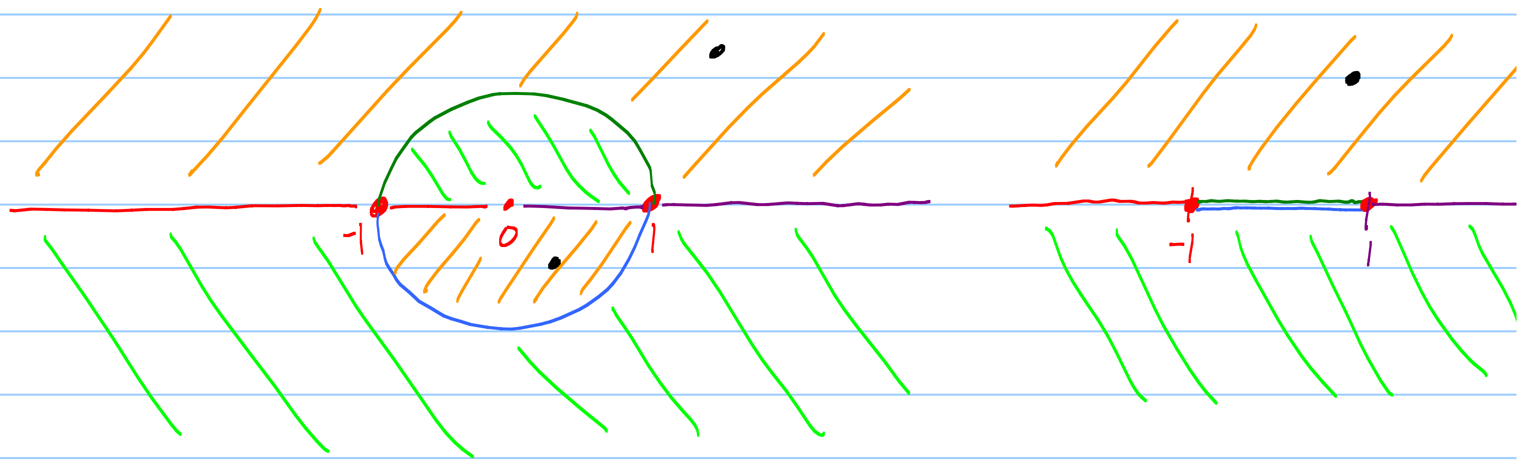
Lecture 24 LFTs, Jukovsky map, conformal mapping

HWK 6 due Thurs

Favorite LFT :



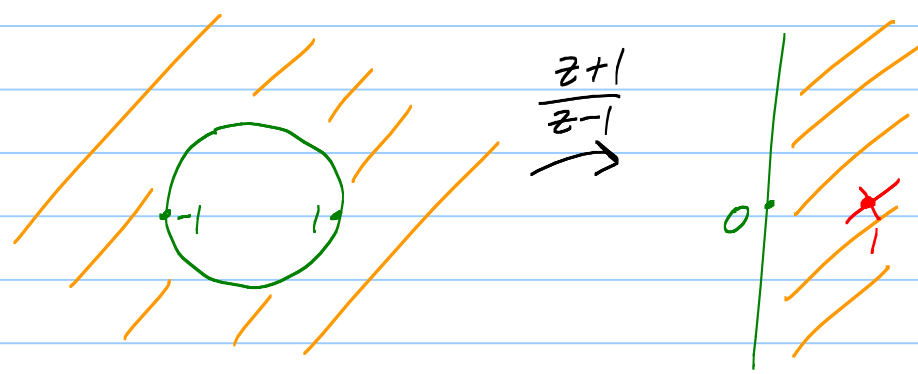
Jukovsky map $\quad J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

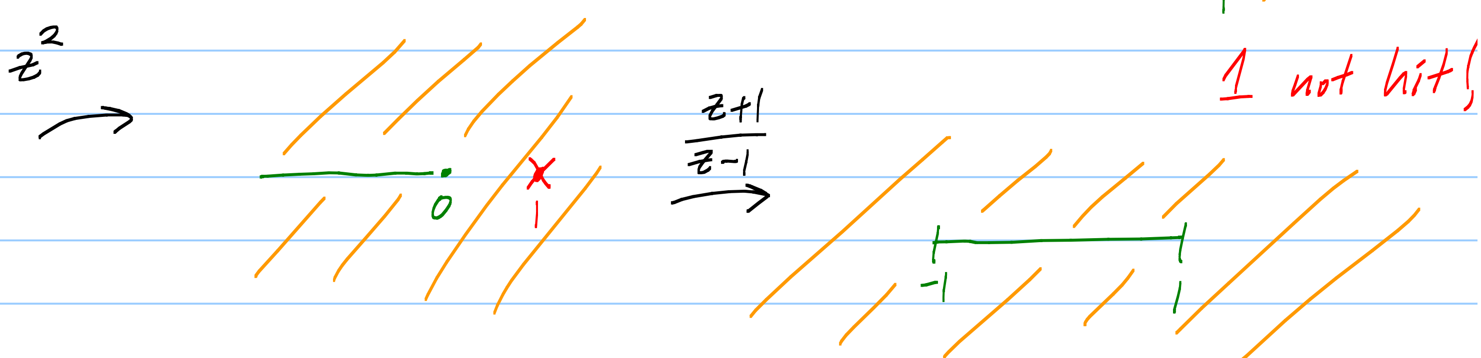


$J'(\pm 1) = 0 \qquad J : -1 \longmapsto -1$ with mult two

$\qquad\qquad\qquad\quad 1 \longmapsto 1$ " " "

Cook up $J(z)$:

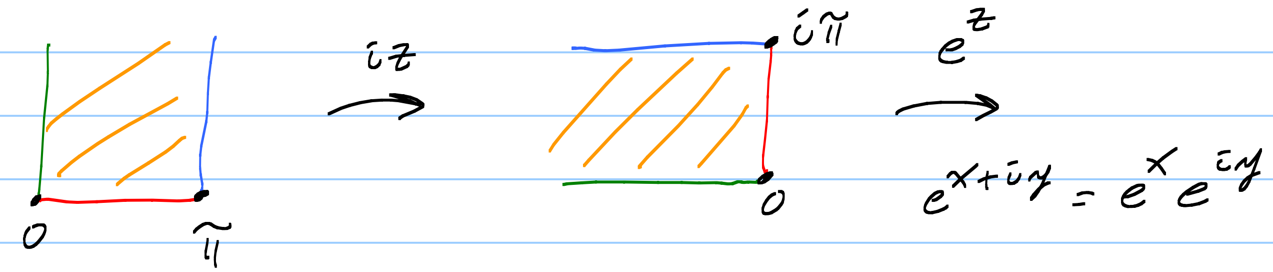


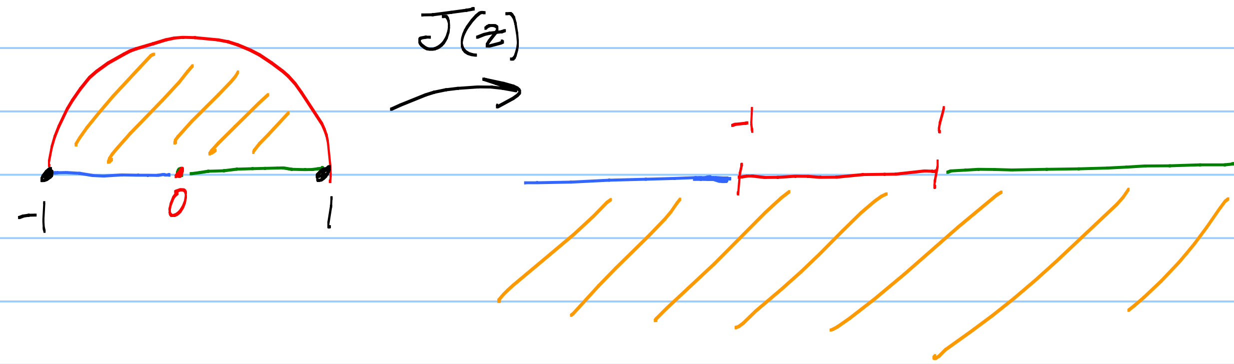


Composition $= J(z)$!

Complex cosine

$$\cos z = \frac{e^{iz}+e^{-iz}}{2} = J(e^{iz})$$

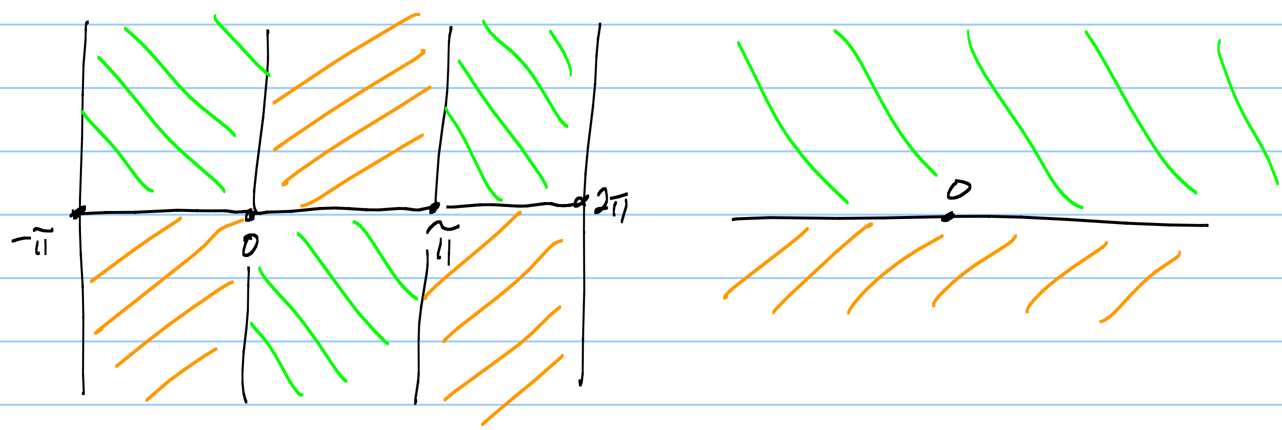




Composition $= \cos z$

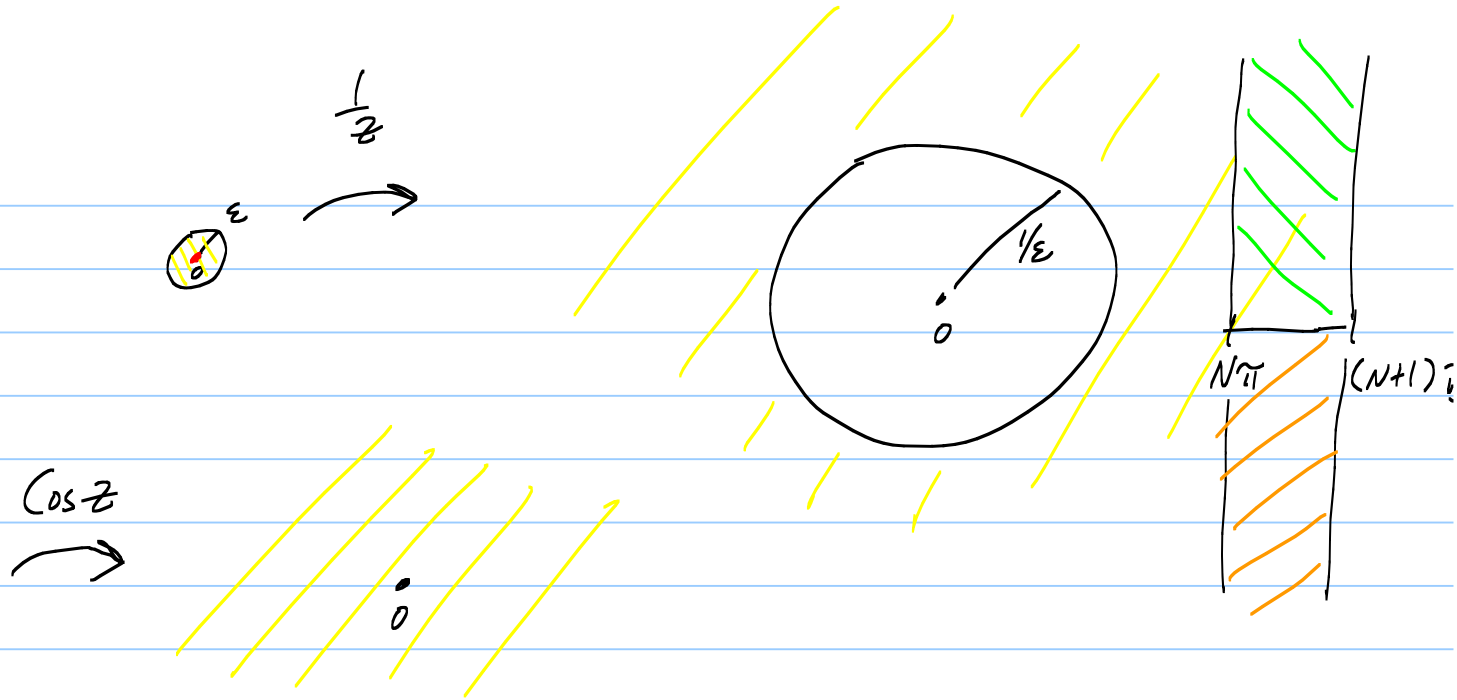
$\cos(-z) = \cos z$

$\cos(z+\pi) = -\cos z$



See $\cos \frac{1}{z}$ maps $\mathbb{C} \to \mathbb{C}$ infinite to one, missing nothing.

$\cos z$ has an "essential singularity at ∞," meaning that $\cos \frac{1}{z}$ has an essential sing at 0.



$f(z) = \cos\frac{1}{z}$ $\quad$ $f(D_\varepsilon(0) - \{0\})$ is dense in $\mathbb{C}$, no matter how small $\varepsilon > 0$

Branches of $\cos^{-1} w$

$$w = \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

Solve for z: Mult by $2e^{iz}$: $\quad 2e^{iz} w = \left(e^{iz}\right)^2 + 1$

$$0 = \left(e^{iz}\right)^2 - 2w\left(e^{iz}\right) + 1$$

pick correct $\sqrt{\ }$

$$e^{iz} = \frac{2w + \sqrt{4w^2 - 4}}{2} = w + \sqrt{w^2 - 1}$$

$$iz = \log\left(w + \sqrt{w^2 - 1}\right)$$

$$z = \frac{1}{i}\log\left(w + \sqrt{w^2-1}\right) \leftarrow \cos^{-1} w$$

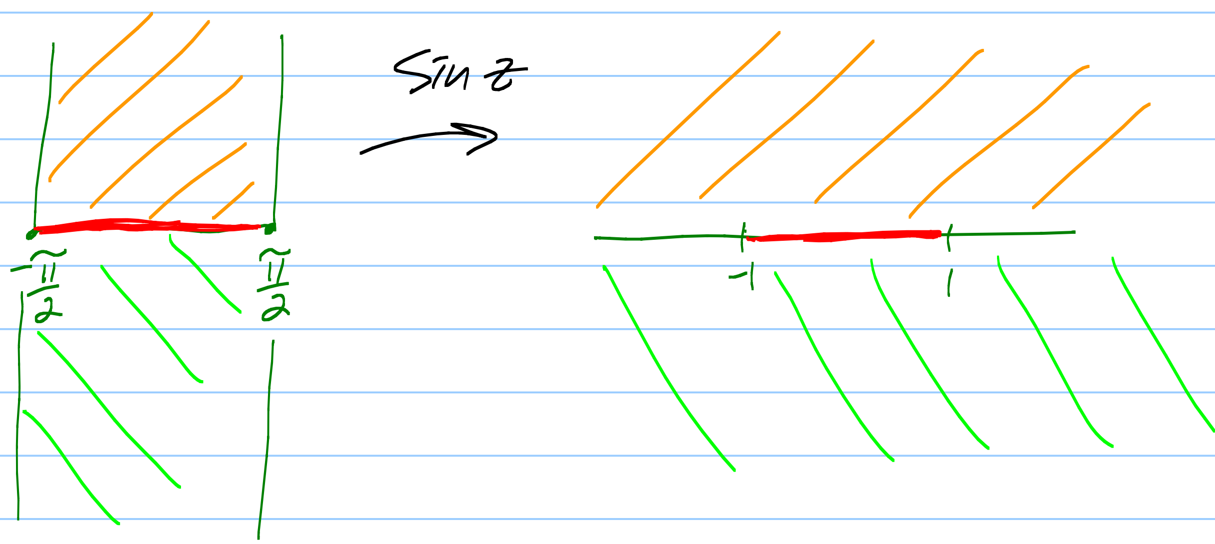
Liouville: e^z and its inverse plus algebraic functions generate the "elemtary fcns".

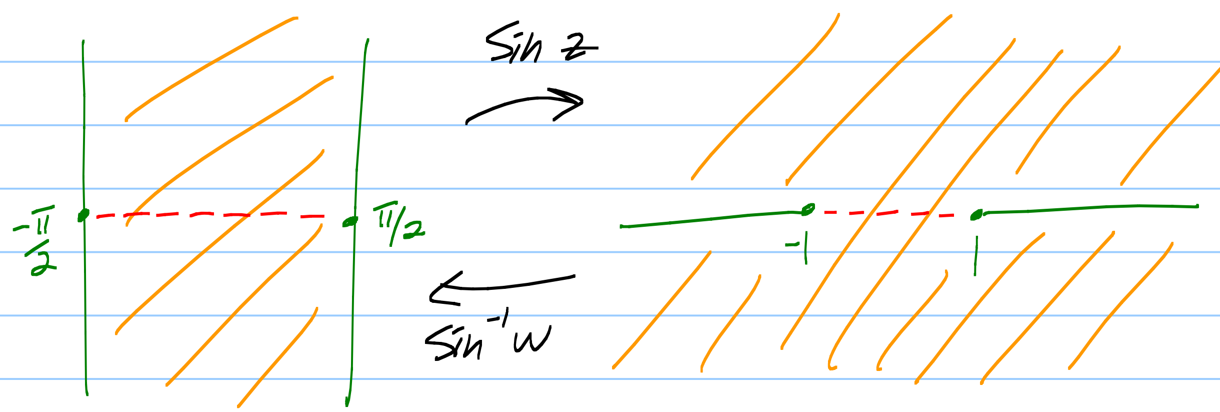
Thm: e^{x^2} does <u>not</u> have an antiderivative that is an elem fcn!

Alg fcns: $\quad P(z,w) = \sum_{n=0}^{N}\sum_{m=0}^{M} c_{nm} z^n w^m \equiv 0$

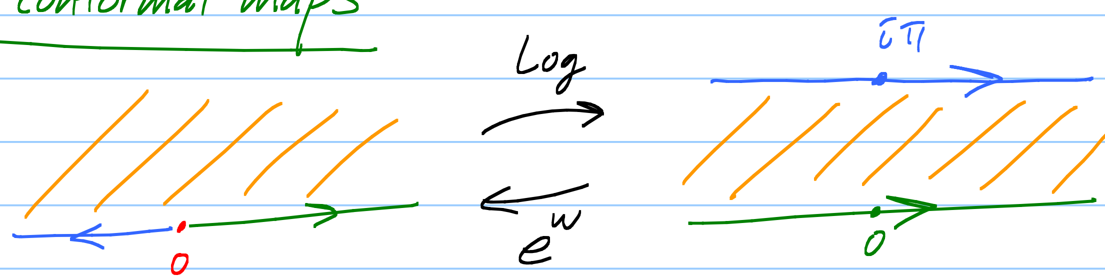
defines $w =$ fcn of z locally away from singular points.

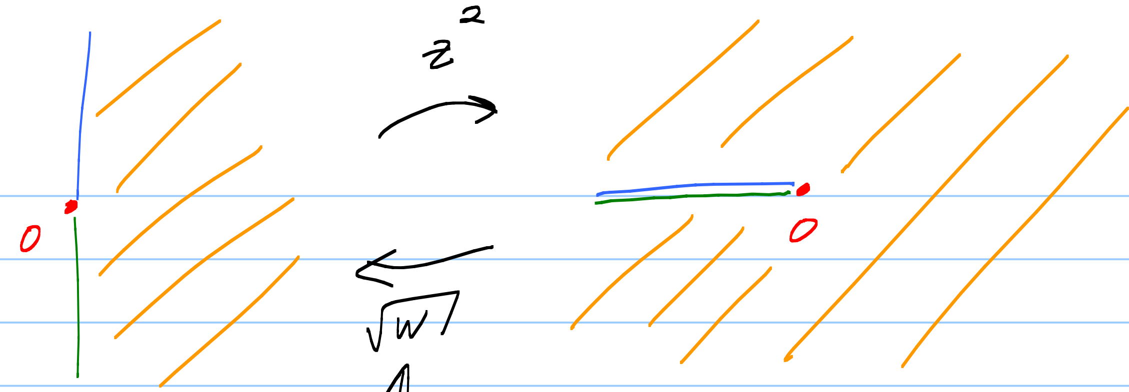
Remark $\quad \sin z = \cos(z - \frac{\pi}{2})$





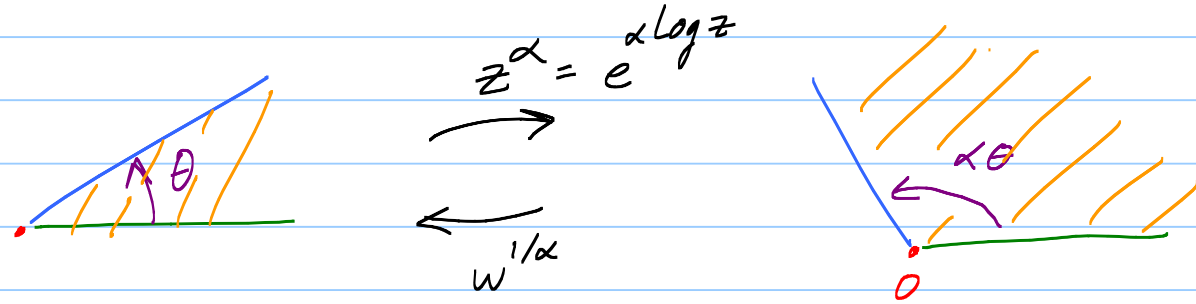
Other useful conformal maps

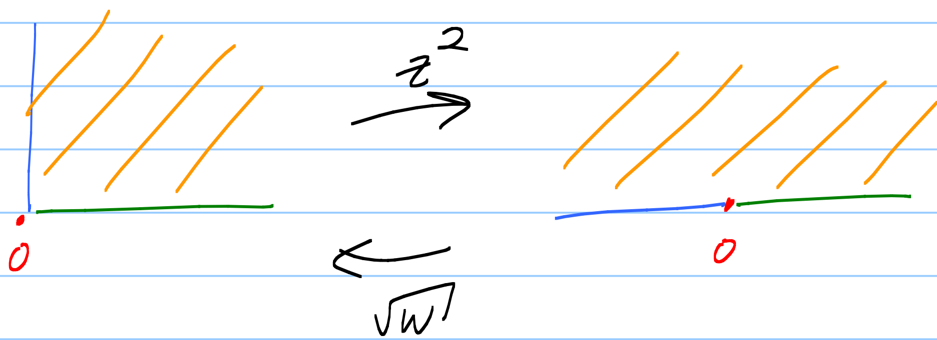


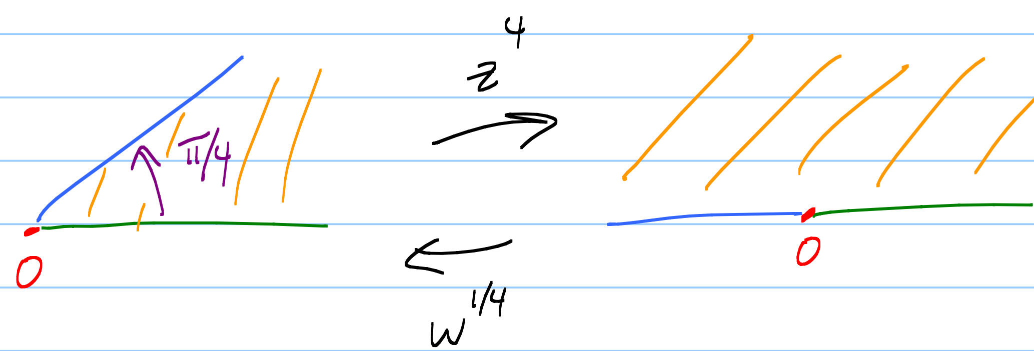


principal branch of $\sqrt{\ }$

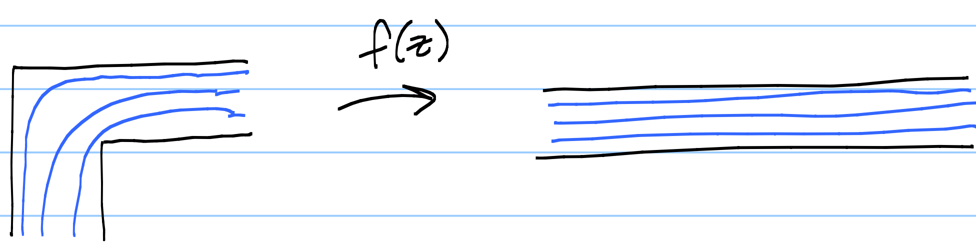
$$\sqrt{w} = e^{\frac{1}{2}\mathrm{Log}\, w} \leftarrow \text{principal branch of log}$$







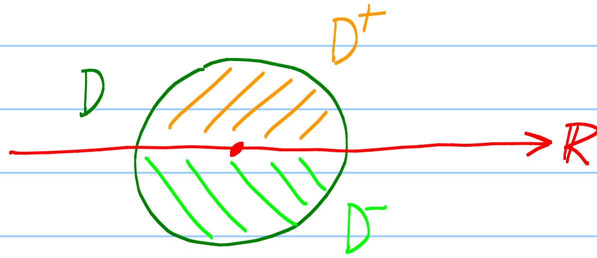
Engineers:



Lecture 25 Schwarz reflection

HWK 6 due Thurs in GS

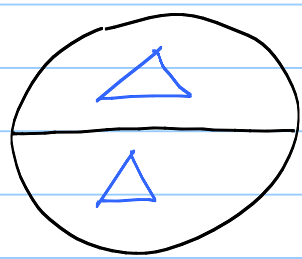
Lemma



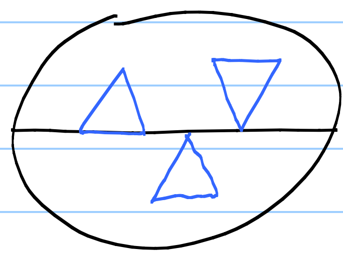
f continuous on D, analytic in D^+ and D^-.
open

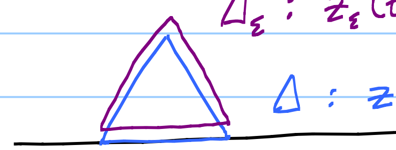
Then f analytic on D.

Pf:



$\int_\Delta f\,dz = 0$ when $\Delta \subset D^+$ or D^-.



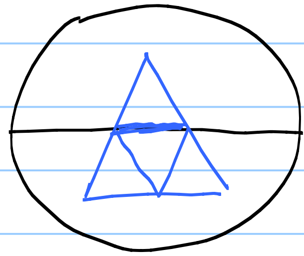


$\Delta_\varepsilon : z_\varepsilon(t) = z(t) + i\varepsilon$

$\Delta : z(t)$

$$\int_\Delta f\,dz = \lim_{\varepsilon \to 0^+} \underbrace{\int_{\Delta_\varepsilon} f\,dz}_{=0}$$

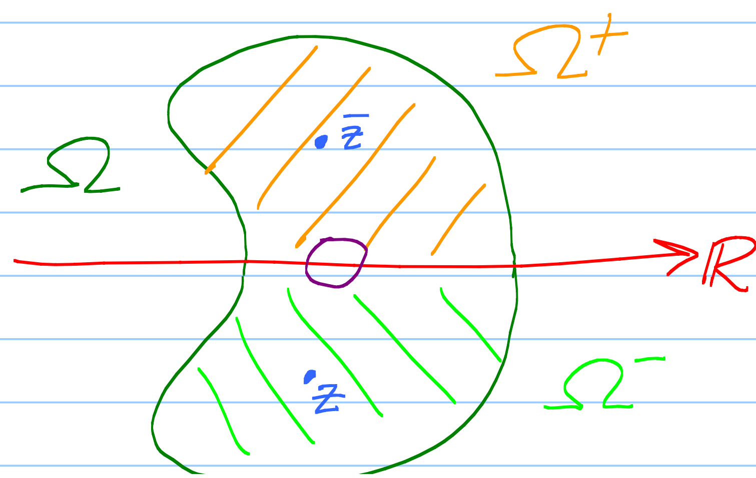
Goursat trick! $\rightarrow$



Aha $\int_\Delta f\,dz$ for any $\Delta \subset D$.

Morera's $\Rightarrow$ f analytic.

Schwarz reflection principle



Ω symmetric domain about real line

f continuous on Ω^+ up to $\mathbb{R}$.

) $f(x) \in \mathbb{R}$ for $x \in \Omega \cap \mathbb{R}$, and

) f is analytic in Ω^+.

Then
$$F(z) = \begin{cases} f(z) & \text{when } z \in \Omega^+ \\ f(x) & \text{when } x \in \Omega \cap \mathbb{R} \\ \overline{f(\bar{z})} & \text{when } z \in \Omega^- \end{cases}$$

is <u>analytic</u> on Ω.

<u>Pf</u>: HYP $\Rightarrow$ F continuous on Ω. ✓

F analytic on Ω^+. ✓

HWK 5 prob: $\overline{f(\bar{z})}$ is analytic on Ω^-. ✓

Lemma $\Rightarrow$ F analytic on Ω.

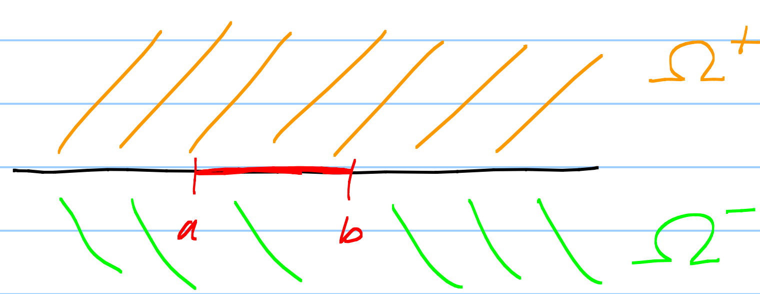
HWK 5 prob: $f(x+iy) = u(x,y) + iv(x,y) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$$\overline{f(x-iy)} = \underbrace{u(x,-y) - iv(x,-y)}_{\text{C-R eqns + Chain rule.}} = \underbrace{\overline{\sum_{n=0}^{\infty} a_n (\bar{z}-z_0)}}_{\text{Same R of C}}$$

✓

<u>Application of Schwarz</u> Suppose f is analytic on UHP and continuous up to $(a,b) \subset \mathbb{R}$ and <u>zero</u> there.

Then $f \equiv 0$ on UHP.

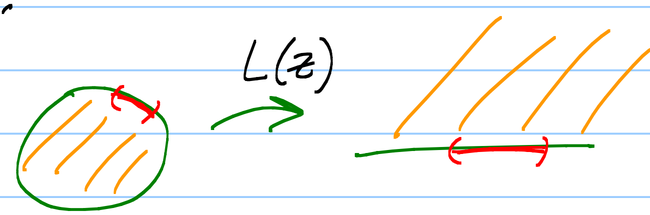


Schwarz $\Rightarrow$ f extends analytic to $\mathbb{C} - \left[(-\infty, a] \cup [b, \infty)\right]$

Every pt in (a,b) is a limit pt of $(a,b) \subset \Omega$.

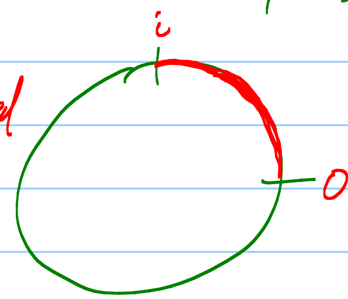
Identity thm $\Rightarrow$ $f \equiv 0$.

Example true for discs too.



Riemann mapping thm can be used to generalize.

<u>Fun exercise</u> f continuous on $\overline{D_1(0)}$, analytic in $D_1(0)$ and zero on red



Then $f \equiv 0$.

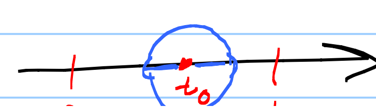
<u>Pf</u>: $F(z) = f(z) f(iz) f(-z) f(-iz)$ is cont on $\overline{D_1(0)}$, analytic in $D_1(0)$, $\equiv 0$ on $C_1(0)$.

Max princ $\Rightarrow$ $F \equiv 0$ on $D_1(0)$

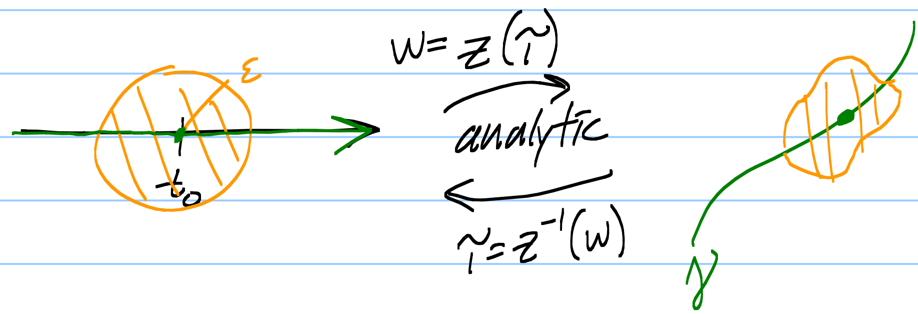
$\Rightarrow$ one of factors $\equiv 0$ $\Rightarrow$ $f \equiv 0$.

Why Schwarz reflection is important:

"Real analytic" curve γ: $z(t) = \sum_{n=0}^{\infty} c_n (t - t_0)^n$



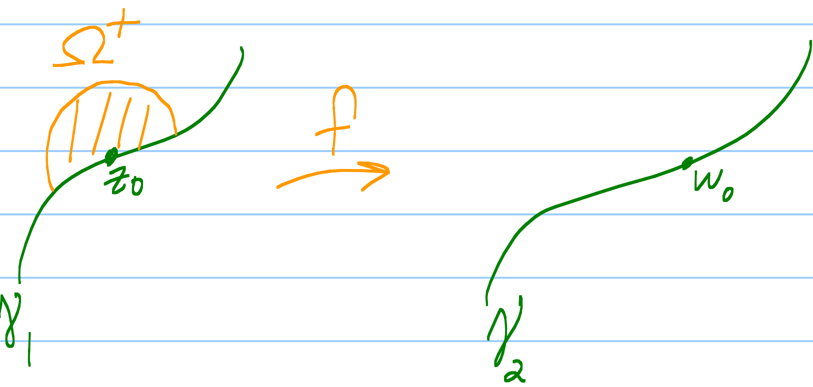
real conv power series $\Longrightarrow$ complex conv power series on $D_\varepsilon(t_0)$. Let $\tilde{\tau}$ be complex near t_0



Assume $z'(t_0) \neq 0$. Then $z'(\tilde{\tau})$ nonvanishing near t_0.

So can shrink ε so $z(\tilde{\tau})$ is 1-1 onto.

Cor:



real analytic curves

f analytic on Ω^+
and continuous up to γ_1.

If $f(\gamma_1) \subset \gamma_2$, then f extends to be analytic on a nbhd of z_0.

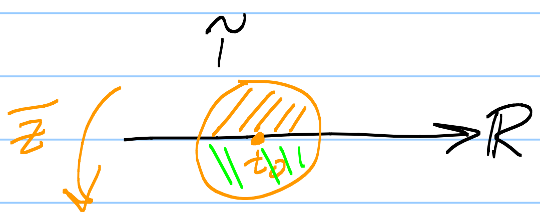
Schwarz reflection fcns:

Ex: For $\mathbb{R}$: $\quad R(z) = \bar{z}$ $\leftarrow$ antiholorphic (conjugate is holo)

R maps one side of $\mathbb{R}$ to other side and fixes $\mathbb{R}$.

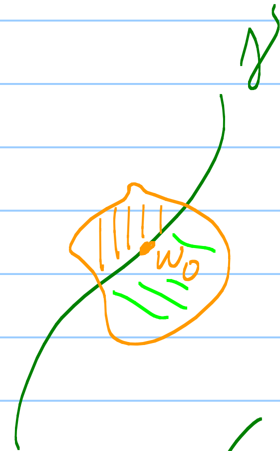
For $C_1(0)$: $\quad R(z) = 1/\bar{z}$

For any real analytic γ:



$w = z(\tau)$ 1-1 onto $\longrightarrow$

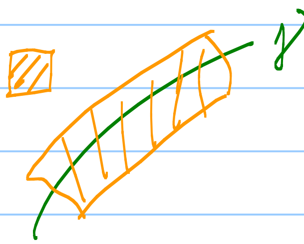
$\tau = z^{-1}(w)$ $\longleftarrow$



$$R(z) = z\left(\overline{z^{-1}(w)}\right)$$

Famous function: $S(z) = \overline{R(z)}$ is the Schwarz fcn.

It is analytic on



$S(z) = \bar{z}$ on γ.

For $\mathbb{R}$: $\quad S(z) = z$

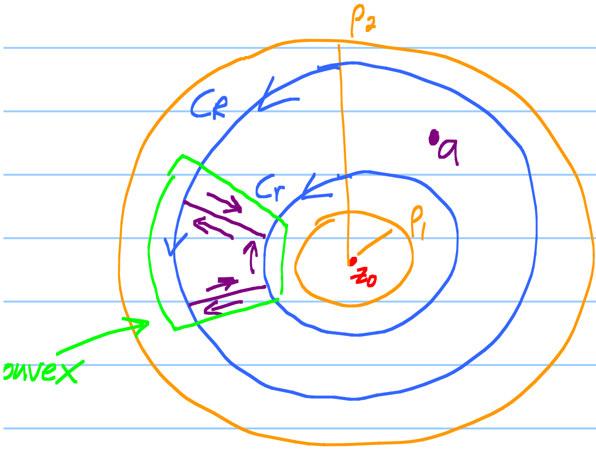
For $C_1(0)$: $\quad S(z) = 1/z$

Lecture 26 Laurent expansions

HWK 6 due tonight at 11:59 in GS

Midterm exam Fri, March 25

Find Practice problems for the midterm on the home page (not to be turned in). Feel free to discuss them on Piazza.

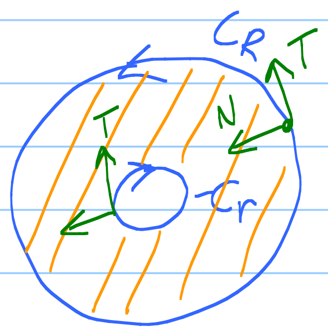


Suppose f is analytic on $A(\rho_1, \rho_2) = \{z : \rho_1 < |z - z_0| < \rho_2\}$

$$\rho_1 < r < R < \rho_2$$

Cauchy's theorem for an annulus: $$\left(\int_{C_R} - \int_{C_r}\right) f \, dz = 0$$

Pf: Add up "pieces of pie". Use Cauchy on convex. Cancel cuts.



$$\overset{\text{CCL}}{C_R} \cup \underbrace{(-C_r)}_{\text{CL}}$$

"standard orientation of the boundary"

(Inward pointing normal vector is to the left of the T vector.)

Cauchy integral formula: $$f(a) = \frac{1}{2\pi i}\left(\int_{C_R} - \int_{C_r}\right) \frac{f(z)}{z-a} \, dz$$

Pf: Apply the C. Thm to $F(z) = \dfrac{f(z) - f(a)}{z - a}$,

which has a removable sing at $z = a$.

$$0 = \left(\int_{C_R} - \int_{C_r}\right) \frac{f(z)}{z-a} - \frac{f(a)}{z-a}\, dz$$

Aha! $\left(\int_{C_R} - \int_{C_r}\right) \frac{f(a)}{z-a}\, dz = f(a)\left[\underbrace{\int_{C_R} \frac{1}{z-a}\, dz}_{2\pi i} - \underbrace{\int_{C_r} \frac{1}{z-a}\, dz}_{=0}\right]$

a inside C_R, outside C_r

Laurent series:

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{converging in } D_{\rho_2}(z_0),\ \text{absolutely on } D_R(z_0).} + \underbrace{\sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}}_{\text{converging in } \{z : |z-z_0| > \rho_1\},\ \text{unif in } \{z : |z-z_0| > r\}}$$

$$a_n = \frac{1}{2\pi i} \int_{C_{\tilde{r}}} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad \leftarrow n \in \mathbb{Z}$$

$\rho_1 < \tilde{r} < \rho_2$ Cauchy thm $\Rightarrow$ indep of choice of $\tilde{r}$

Theorem: f has an isolated singularity at z_0.

1) $a_{-n} = 0$ for $n = 1, 2, 3, \ldots \iff z_0$ is removable

2) Only finitely many $a_{-n} \neq 0$ $(n \in \mathbb{N})$

$\iff z_0$ is a pole

3) Infinitely many $a_{-n} \neq 0$ $(n \in \mathbb{N}) \iff z_0$ essential

Pf of expansion: Let $z_0 = 0$.

$$f(z) = \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{\underbrace{w-z}_{w\left(1-\frac{z}{w}\right)}}\,dw - \frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{\underbrace{w-z}_{-z\left(1-\frac{w}{z}\right)}}\,dw$$

$\left|\frac{z}{w}\right| < 1$ $\qquad$ $\left|\frac{w}{z}\right| < 1$

$$= \frac{1}{2\pi i}\int_{C_R} \frac{f(w)}{w}\left(1 + \frac{z}{w} + \left(\frac{z}{w}\right)^2 + \cdots \left(\frac{z}{w}\right)^N\right)dw$$

+ remainder term $\to 0$ unif on a shrunk disc.

$$= \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w^{n+1}}\,dw\right)}_{a_n} z^n$$

$$-\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{\underbrace{w-z}_{-z\left(1-\frac{w}{z}\right)}}\,dw = \frac{1}{2\pi i}\frac{1}{z}\int_{C_r}\frac{f(w)}{1-\frac{w}{z}}\,dw$$

$\left|\frac{w}{z}\right| < 1$

$$= \frac{1}{2\pi i}\cdot\frac{1}{z}\int_{C_r} f(w)\left[1 + \left(\frac{w}{z}\right) + \cdots + \left(\frac{w}{z}\right)^N + \underbrace{\frac{\left(\frac{w}{z}\right)^{N+1}}{1-\frac{w}{z}}}_{E_N\left(\frac{w}{z}\right)}\right]dw$$

$$= \sum_{n=1}^{N+1} \underbrace{\left(\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w^{-n+1}}\,dw\right)}_{a_{-n}}\frac{1}{z^n} + \mathcal{E}_N(z)$$

Finally $\quad |\mathcal{E}_N(z)| < \frac{1}{2\pi} \frac{1}{|z|} \cdot \underset{C_r}{\text{Max}} |f| \; \dfrac{\left(\frac{r}{|z|}\right)^{N+1}}{1 - \frac{r}{|z|}}$

$\to 0$ as $N \to \infty$.

Restrict z to smaller annulus to get unif estimate.

Notation: $\sum_{-\infty}^{\infty} a_n z^n$ converges means that

both $\sum_{0}^{\infty}$ and $\sum_{n=1}^{\infty} a_{-n} z^{-n}$ converge.

Remark: At an isolated sing, Laurent exp is valid on punctured disc.

Pf of Isolated sing thm:

(1): ($\Longrightarrow$) If $a_{-n} = 0 \;\; \forall\, n \in \mathbb{N}$, then
$f =$ power series on $D_R(z_0) - \{z_0\}$.
So z_0 removable.

($\Longleftarrow$) $\quad z_0$ removable. $\qquad (n \in \mathbb{N})$

$$a_{-n} = \frac{1}{2\pi i} \int_{C_{\tilde r}} \frac{f(w)}{(w - z_0)^{-n+1}}\, dw$$

$$= \frac{1}{2\pi i} \int_{C_{\tilde r}} \underbrace{f(w)\,(w - z_0)^{n-1}}_{\text{analytic inside}}\, dw = 0$$

Cauchy

✓

(2) Lemma: Laurent expansions are unique.

Why:

$$a_N = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z-z_0)^{N+1}}\, dz$$

$$= \frac{1}{2\pi i} \int_{C_r} \left[\sum_{-\infty}^{\infty} b_n (z-z_0)^n \right] \frac{1}{(z-z_0)^{N+1}}\, dz$$

$$= \sum_{-\infty}^{\infty} b_n \underbrace{\left(\frac{1}{2\pi i} \int_{C_r} (z-z_0)^{n-N-1}\, dz \right)}_{\begin{cases} =0 & \underbrace{n-N-1 \neq -1}_{n \neq N} \\ =1 & n=N \end{cases}}$$

$$= b_N \quad \checkmark$$

(2): ($\Rightarrow$) $f(z) = (\text{Princ Part}) + \text{power series}$.

So z_0 is a pole.

($\Leftarrow$) Princ part + power series is a Laurent exp. It is the unique Laurent exp.

(3): Left over case!

<u>Def^n</u>: At an isolated sing z_0

$$\text{Res}_{z_0} f = a_{-1} = \frac{1}{2\pi i} \int_{C_r} f(w)\, dw$$

<u>Meaning of residue</u>: f has an analytic antiderivative on $D_R(z_0) - \{z_0\} \iff \text{Res}_{z_0} f = 0$.

<u>Think</u>: Residue term leads to log terms.

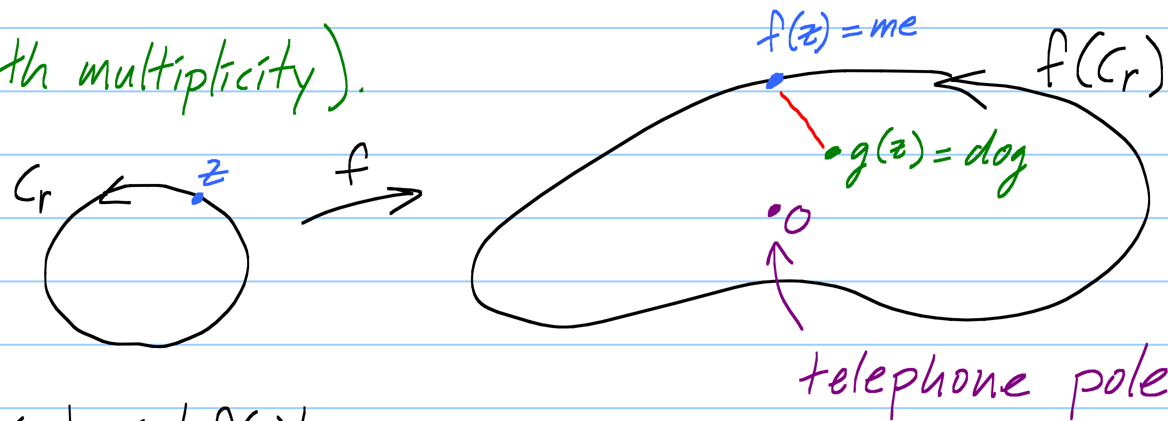
Lecture 27 Rouché's theorem, Hurwicz thm

Midterm exam a take-home exm
Out Thurs 2:00pm. Due Tues, 11:59pm
in Gradescope.

Rouché's thm for a disc (or "toy region"): Suppose $f(z)$ and $g(z)$ are analytic on $D_R(a)$ and $0 < r < R$. If

$|f(z) - g(z)| < |f(z)|$ on $C_r(a)$, then f and g have the same number of zeroes inside $C_r(a)$ (counted with multiplicity).

Meaning:



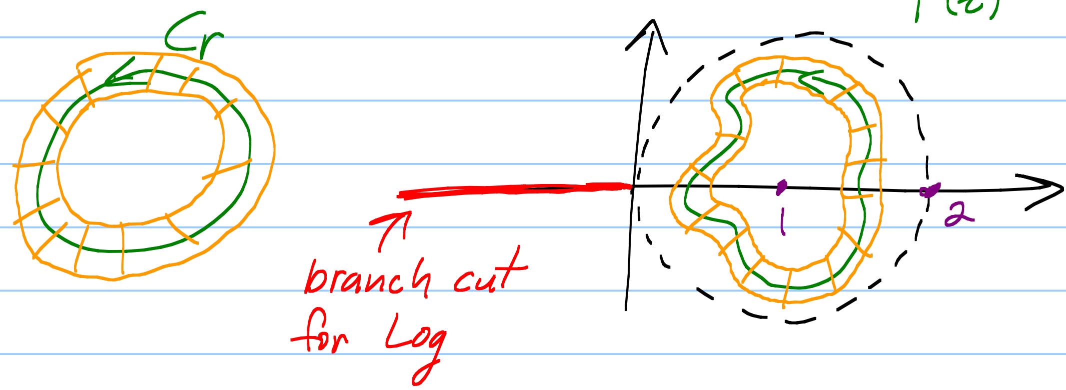
$$\underbrace{|f(z) - g(z)|}_{\text{leash length}} < \underbrace{|f(z)|}_{\text{dist me to pole}}$$

Arg Princ: #zeroes f inside C_r = # times $f(z)$ revolves around origin

= same # times dog goes around pole.

Note: Ineq $\Rightarrow$ f nonvanishing on C_r
$\Rightarrow$ g same

Pf: Divide ineq by $|f|$: $\quad \left|1 - \underbrace{\frac{g(z)}{f(z)}}_{F(z)}\right| < 1$ on C_r



$$\int_{C_r} \frac{F'}{F}\, dz = \int_{C_r} \frac{d}{dz} \operatorname{Log}(F(z))\, dz = 0$$

$$\frac{F'}{F} = \frac{g'}{g} - \frac{f'}{f} \quad \longleftarrow \text{"logarithm diff'tion"}$$

$$\text{So } 0 = \frac{1}{2\pi i} \int_{C_r} \frac{F'}{F}\, dz = \underbrace{\frac{1}{2\pi i}\int_{C_r} \frac{g'}{g}\, dz}_{\#\text{zeroes } g} - \underbrace{\frac{1}{2\pi i}\int_{C_r} \frac{f'}{f}\, dz}_{\#\text{ zeroes of } f}$$

Remark: Same proof extends to "toy regions" and allows zeroes and poles inside:

$$[\#\text{zeroes} - \#\text{poles}] \text{ of } f = \text{same \# for } g.$$

Hurwicz thm: Suppose f_n is a seq of analytic fcns on a domain Ω $f_n \twoheadrightarrow f$. ($\twoheadrightarrow$: unif on compacts)

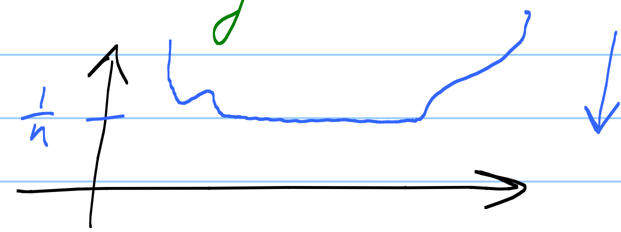
[Know f is analytic too.]

If all f_n's are nonvanishing on Ω, then

1) $f \equiv 0$ on Ω, or

2) f nonvanishing too.

Way false $\mathbb{R} \to \mathbb{R}$



$\underline{\text{EX}}$: $f_n(z) = z^n \twoheadrightarrow \equiv 0$ on $D_1(0)$.

$\underline{\text{Pf}}$ of Hurwicz: Suppose $f(z_0) = 0$. If $f \not\equiv 0$, then z_0 is an <u>isolated</u> zero. So $\exists\ \overline{D_\varepsilon(z_0)} \subset \Omega$ such that z_0 is the only zero of f in $\overline{D_\varepsilon(z_0)}$.

Let $m = \min\{|f(z)| ; z \in C_\varepsilon(z_0)\} > 0$ $|f(z(t))|$ on a compact $[0, 2\pi]$

$$\underbrace{|f_n(z) - f(z)| < m}_{\text{Get this for } n > N \text{ on compact } C_\varepsilon(z_0) \text{ by } \twoheadrightarrow.} \overset{\checkmark}{\leq} |f(z)| \quad \text{on } C_\varepsilon(z_0)$$

Aha! $\underset{\substack{\uparrow \\ \text{none!}}}{f_n}$ and $\underset{\substack{\uparrow \\ \text{at least one.}}}{f}$ have same # zeroes in $C_\varepsilon(z_0)$, $n > N$. $\lightning$.

Conclude that f must have been $\equiv 0$.

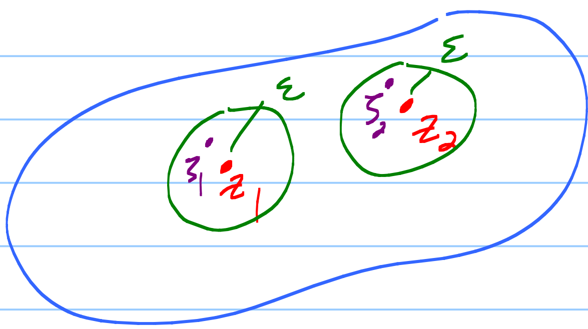
Corollary: Hurwicz #2. $f_n \rightrightarrows f$

If each f_n is one-to-one, then

1) $f \equiv \text{const}$, or

2) f is one-to-one too.

Pf: Redo pf of Hurwicz #1. Suppose $f(z_1) = f(z_2) = w_0$



If f not const, zeroes of $f(z) - w_0$ would be isolated.

Rouché argument $\Rightarrow$ for $n > \text{Max}(N_1, N_2)$,

$$\begin{pmatrix} \#\text{ zeroes of } f_n(z) - w_0 \\ \text{inside } C_\varepsilon(z_j) \end{pmatrix} = \underbrace{\begin{pmatrix} \#\text{ zeroes of } f(z) - w_0 \\ \text{inside } C_\varepsilon(z_j) \end{pmatrix}}_{\geq 1}$$

So $\exists\ \zeta_j \in C_\varepsilon(z_j)$ such that $f_n(\zeta_j) = w_0$, $j = 1, 2$, violating f_n 1-1. $\lightning$

Fun thing: Rouché's $\Rightarrow$ F. Thm. Alg

$$f(z) = a_N z^N \qquad g(z) = a_N z^N + \cdots + a_0$$

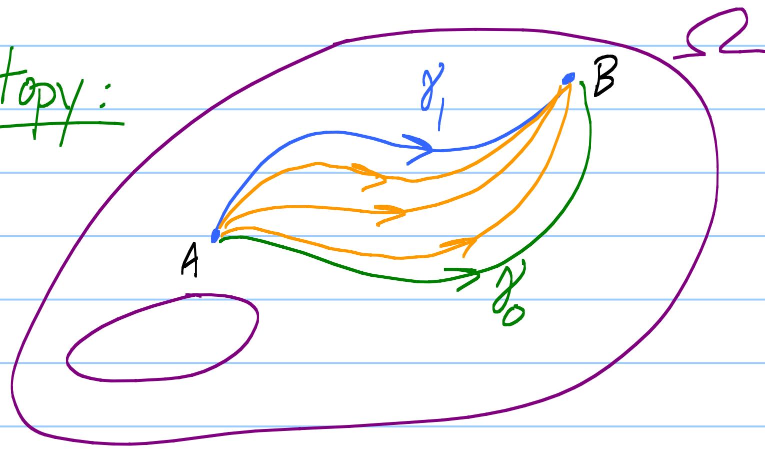
EX: How many zeroes does $\underbrace{z^5 + 3z^4 + \underbrace{8z^3}_{f(z)\text{ big one}} + z^2 + z + 1}_{g(z)}$ in unit disc?

$$|f(z) - g(z)| = |z^5 + 3z^4 + z^2 + z + 1| \le \underbrace{1+3+1+1+1}_{7} \le |8z^3|$$

on $|z| = 1$.

$g(z)$ has 3 zeroes in $D_1(0)$.

Homotopy:



$\gamma_j : z_j(t), \; 0 \le t \le 1$

$\gamma_0 \sim \gamma_1$: "γ_0 homotopic to γ_1 in Ω" means that there is a continuous $H : [0,1] \times [0,1] \to \Omega$ such that

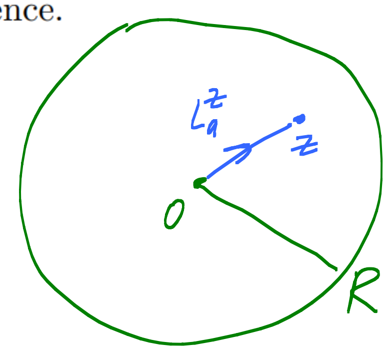
$$\begin{cases} H(0,t) = z_0(t) & 0 \le t \le 1 \\ H(1,t) = z_1(t) & 0 \le t \le 1 \end{cases}$$

$$\begin{cases} H(s,0) = A \\ H(s,1) = B \end{cases}$$

Think: $\gamma_s : z_s(t) = H(s,t) \leftarrow$ orange curves.

Practice problems, review

2. Prove that power series can be integrated term by term. To be precise, suppose that a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R > 0$ converges on the disc $D_R(0)$ to an analytic function $f(z)$. Prove that the power series $\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$ also has radius of convergence R and that this series converges to an analytic anti-derivative of $f(z)$ inside the circle of convergence.



$$F(z) = \int_{L_a^z} f(w)\,dw$$

$$= \sum_{0}^{\infty} a_n \underbrace{\int_{L_a^z} w^n\,dw}_{\frac{1}{n+1} z^{n+1}}$$

$R_F \geq R_f$

$R_F > R_f \implies$ power series for f converges on disc bigger than R_f.

Fact: Can diff and integrate Laurent expansions in same way, but the residue term gets in the way when you try to integrate.

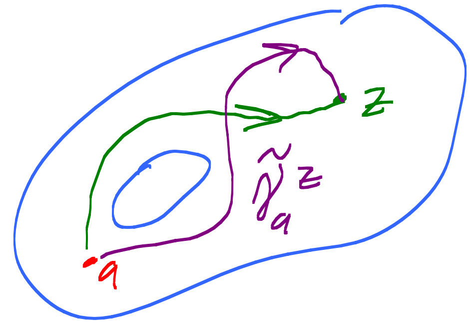
Lemma: f has an analytic antiderivative on a domain $\Omega \iff \int_\gamma f\,dz = 0$ for every closed contour γ in Ω.

Pf: $f = F'$. Then $\int_\gamma f\,dz = \int_\gamma F'\,dz = 0$, γ closed.
$\implies$

$\impliedby$ Suppose $\int_\gamma f\,dz = 0$, γ closed.

Try to define

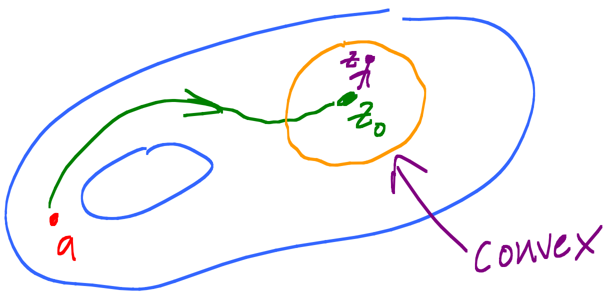
$$F(z) = \int_{\gamma_a^z} f(w)\,dw.$$



Claim: F well defined. ✓

$\gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z)$ is closed.

Trick:



$$\gamma_a^z = \gamma_a^{z_0} \cup L_{z_0}^z$$

$$\frac{d}{dz} F(z) = \frac{d}{dz}\left(\underbrace{\int_{\gamma_a^{z_0}} f\,dw}_{\text{const}} + \int_{L_{z_0}^z} f\,dw \right) = f(z)$$

Fact: Suppose f analytic on $D_r(a) - \{a\} =: \Omega$

f has an analytic antiderivative on Ω

$\iff \operatorname{Res}_a f = 0.$

Pf: $\int_{\gamma_a^z} \sum = \sum \int$

$$a_{-1} = \frac{1}{2\pi i} \int_{C_r} f(w)\,dw$$

3. Suppose that f and g are analytic in a neighborhood of a. If f has a simple zero at a, then

$$\operatorname{Res}_a \frac{g}{f} = \frac{g(a)}{f'(a)}.$$

Prove a similar formula in case f has a double zero at a, i.e., in case f is such that $f(a) = 0$, $f'(a) = 0$, but $f''(a) \neq 0$.

$$f(z) = \underbrace{a_0}_{=0} + \underbrace{a_1}_{=0}(z-a) + \underbrace{a_2}_{=\frac{f''(a)}{2!}}(z-a)^2 + \underbrace{a_3}_{=\frac{f'''(a)}{3!}}(z-a)^3 + \cdots$$

$$= (z-a)^2 \Big[\underbrace{a_2 + a_3(z-a) + \cdots}_{F(z)} \Big]$$

Note: $a_2 = F(a) = \dfrac{f''(a)}{2!}$, $a_3 = \dfrac{F'(a)}{1!} = \dfrac{f'''(a)}{3!}$

$$\frac{g(z)}{f(z)} = \frac{1}{(z-a)^2}\Big[\underbrace{\frac{g(z)}{F(z)}}_{A_0 + A_1(z-a) + \cdots}\Big]$$ analytic near a.

$$= \frac{A_0}{(z-a)^2} + \frac{A_1}{z-a} + \text{power series}$$

$$\operatorname{Res}_a \frac{g}{f} = A_1 = \frac{\frac{d}{dz}\left[\frac{g}{F}\right]\Big|_{z=a}}{1!} = \frac{g'(a)F(a) - g(a)F'(a)}{F(a)^2}$$

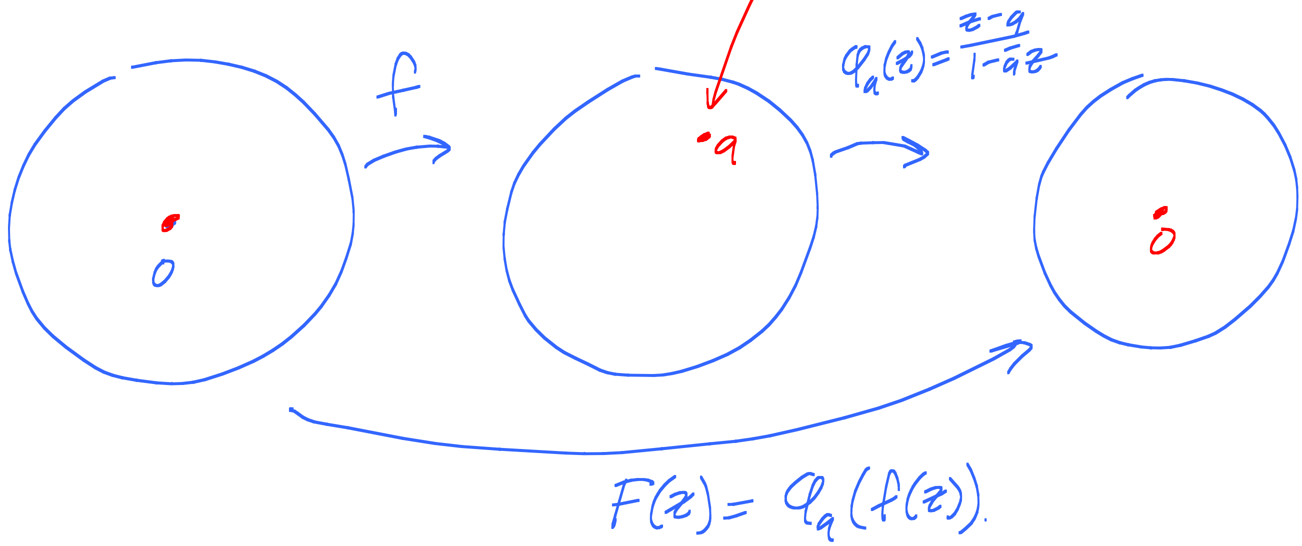
$$= \frac{g'(a)\dfrac{f''(a)}{2!} - g(a)\dfrac{f'''(a)}{3!}}{\left[\dfrac{f''(a)}{2!}\right]^2}$$

-ated formula:

$$\operatorname{Res}_a \frac{g}{f} = \lim_{z \to a} \frac{\frac{d}{dz}\left[(z-a)^2 \frac{g(z)}{f(z)}\right]}{1!}$$

5. Show that if f is an analytic mapping of the unit disk into itself, then $|f'(0)| \le 1$.

ouch! Can't use Schwarz



Schwarz: $|F'(0)| \le 1$

$$F'(z) = \varphi_a'(f(z))\, f'(z)$$

$$\underbrace{|F'(0)|}_{\le 1} = |\underbrace{\varphi_a'(a)}_{\frac{1}{1-|a|^2}}\, f'(0)| \le 1$$

Conclude $|f'(0)| \le 1 - |a|^2$

< 1 if $a \ne 0$.

Schwarz $\rightarrow$ (≤ 1 if $a = 0$.)

what if $|f'(0)| = 1$? Then $a = 0$. Schwarz part 2

$\Rightarrow$ $f(z) = \lambda z$, $|\lambda| = 1$.

6. Suppose that f is an analytic function on the unit disc such that $|f(z)| < 1$ for $|z| < 1$. Prove that if f has a zero of order n at the origin, then

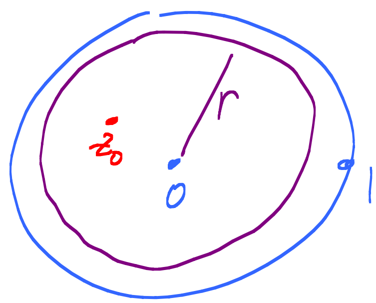
$$|f(z)| \le |z|^n$$

for $|z| < 1$. How big can $|f^{(n)}(0)|$ be?

Know $f(z) = z^n \underbrace{F(z)}_{[a_n + a_{n+1} z + \dots]}$, $F(0) \neq 0$ and analytic

$F(0) = a_n = \dfrac{f^{(n)}(0)}{n!}$

$$F(z) = \frac{f(z)}{z^n}$$



$|z_0| < r < 1$

$$|F(z_0)| \le \max_{C_r} |F| = \max_{C_r} \frac{|f(z)|}{r^n} \le \frac{1}{r^n} \searrow 1 \text{ as } r \nearrow 1$$

Get $|F(z_0)| \le 1$; $|f(z_0)| \le |z_0|^n$ ✓

Also get $\Big|\underbrace{F(0)}_{\frac{f^{(n)}(0)}{n!}}\Big| \le 1$. So $|f^{(n)}(0)| \le n!$

equality? $f(z) = \lambda z^n$

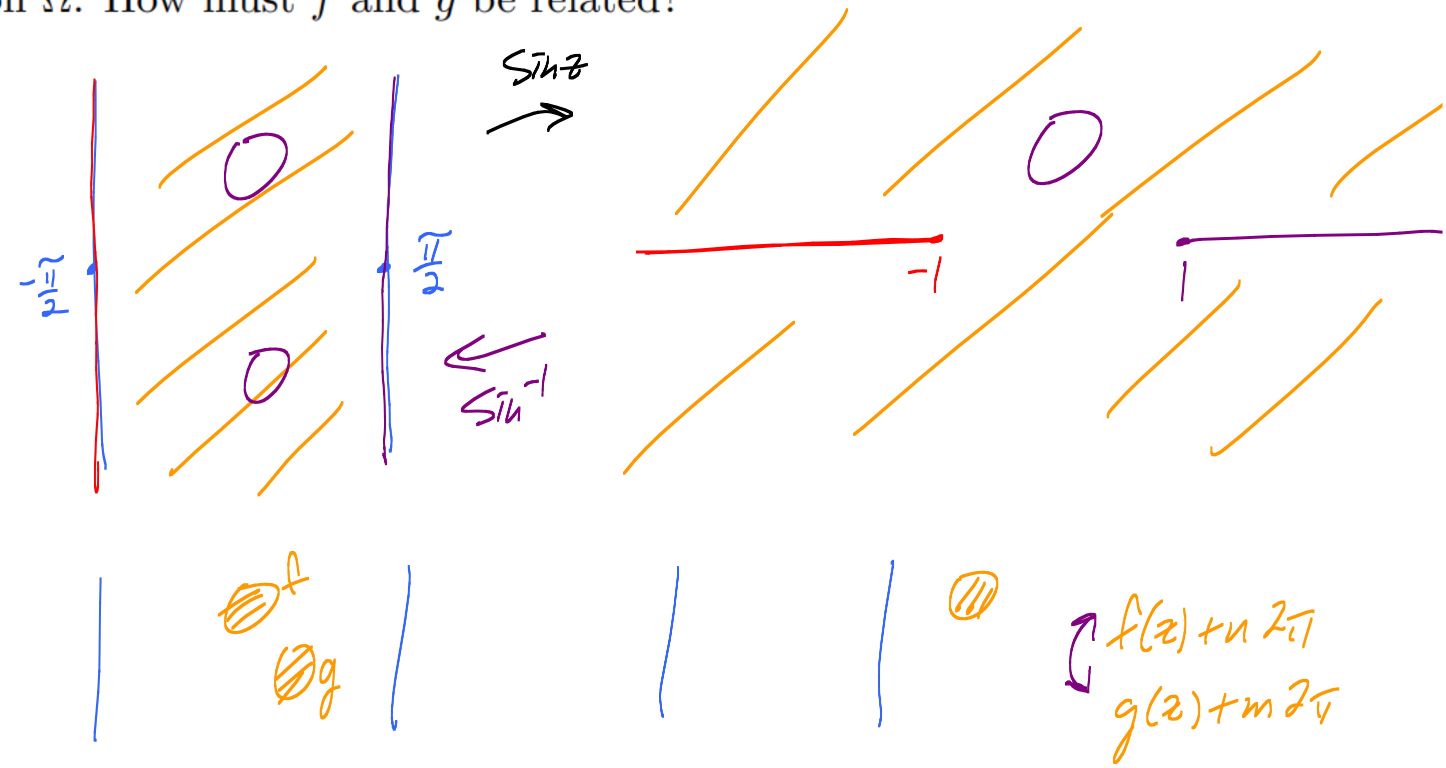
8. Suppose f and g are analytic functions on a domain Ω and that

$f(z)+2n\pi$

$$\sin(f(z)) \equiv \sin(g(z))$$

open mapping thm

on Ω. How must f and g be related?



7. $|f(z)| \le c(1+|z|^N)$ for $z \in \mathbb{C}$. f entire.

Cauchy est: $$|f^n(0)| \le \frac{n! \max_{C_R(0)} |f|}{R^n} \le \frac{n!\,(1+R^N)}{R^n}$$

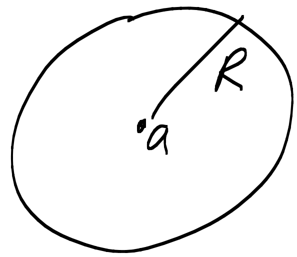
Aha! If $n > R$, see $|f^{(n)}(0)| \to 0$

as we let $R \to \infty$ in Cauchy est.

Power series: $f(z) = a_0 + a_1 z + \dots + a_N z^N + 0 + \dots$

might be zero.

Or:



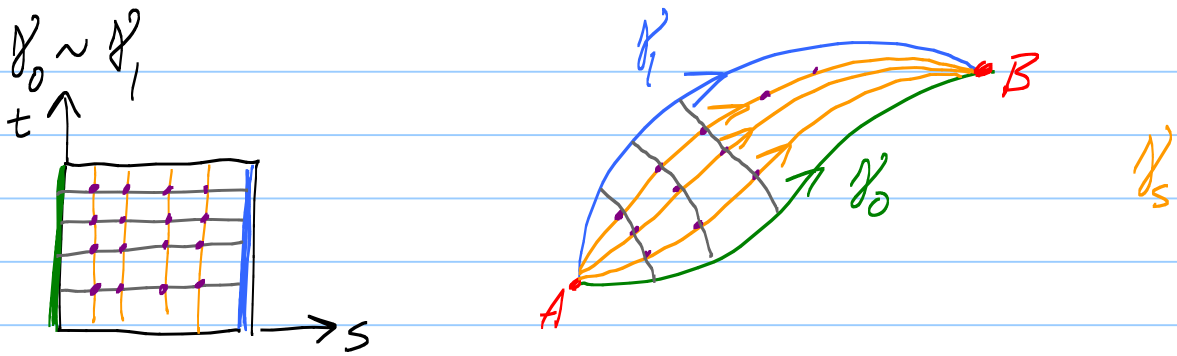
$f^{(n)}(a) \equiv 0$

when $n = N+1$

$f^{(N+1)}(z) \equiv 0 \implies f$ poly deg N or less

Lecture 28 Homotopy, simply connected domains, the Cauchy theorem

$\gamma_0 \sim \gamma_1$



Homotopy : $H : [0,1] \times [0,1] \longrightarrow \Omega$ continuous

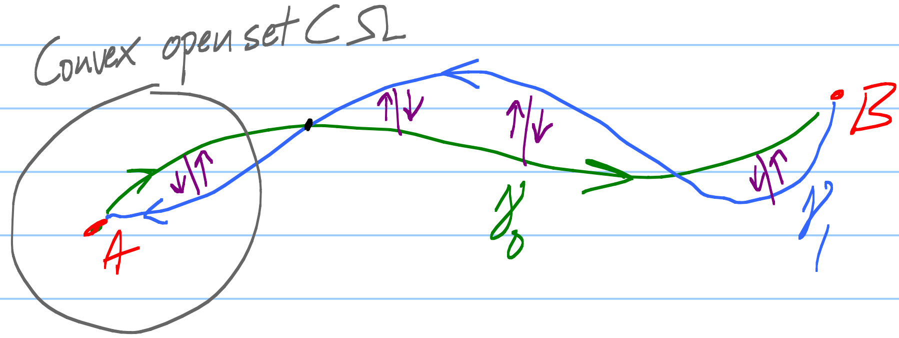
$$\begin{cases} H(0,t) = z_0(t) & : \gamma_0 \\ H(1,t) = z_1(t) & : \gamma_1 \end{cases}$$

$$\begin{cases} H(s,0) = A \\ H(s,1) = B \end{cases}$$

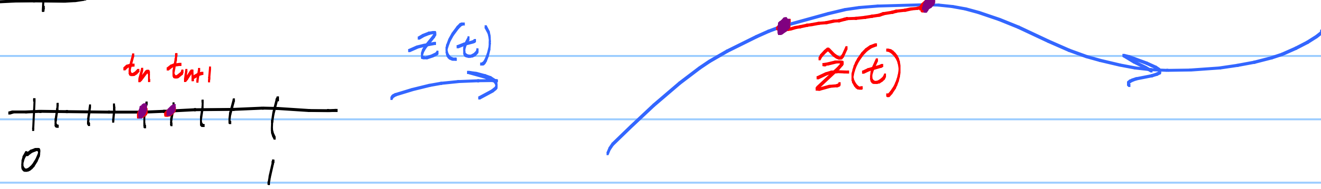
Think $\gamma_s : z_s(t) = H(s,t)$

Feeling: If γ_0 is "close" to γ_1 in Ω, then

$$\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz \qquad (f \text{ analytic on } \Omega)$$

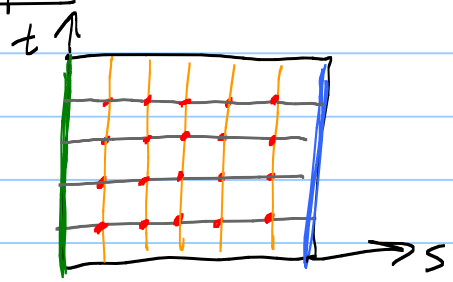


Step: Piecewise linearize a curve :

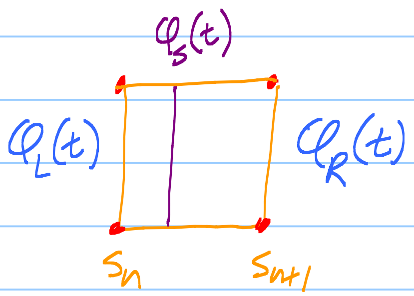


$$\tilde{z}(t) = z(t_n) + \frac{t-t_n}{t_{n+1}-t_n}\left[z(t_{n+1}) - z(t_n)\right]$$

Step 2: Do this in the vertical dir for homotopy.



Step 3: On little square



$$\varphi_s(t) = \varphi_L(t) + \frac{s-s_n}{s_{n+1}-s_n}\left[\varphi_R(t) - \varphi_L(t\right.$$

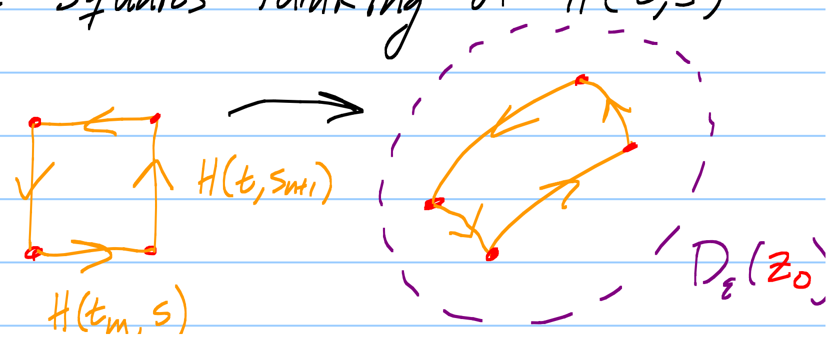
Check Get piecewise linear cont fcn on $[0,1]\times[0,1]$.

New homotopy, $H(t,s)$. Note: $\begin{cases} H(0,s) = A \\ H(1,s) = B \end{cases}$ ✓

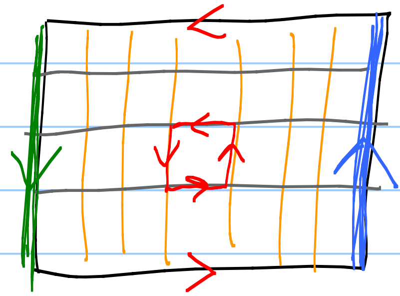
Thm: f analytic on Ω, $\gamma_0 \sim \gamma_1$ in Ω.

Then $\int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$. (Cauchy's Thm)

Pf: Add up $\int$ around little squares thinking of $H(t,s)$ as parametrizing curves:



Add up squares:



Use Cauchy on convex. ε comes from unif cont of cont fcn H on compact $[0,1]\times[0,1]$.

Cancel all the inner sides.

$$\int_{\text{Bottom}} = \int f(A)\underbrace{\left[\frac{d}{ds}A\right]}_{\equiv 0} ds = 0. \quad \text{Same for top}$$

Right side: $\int_{\gamma_1} f\, dz$.

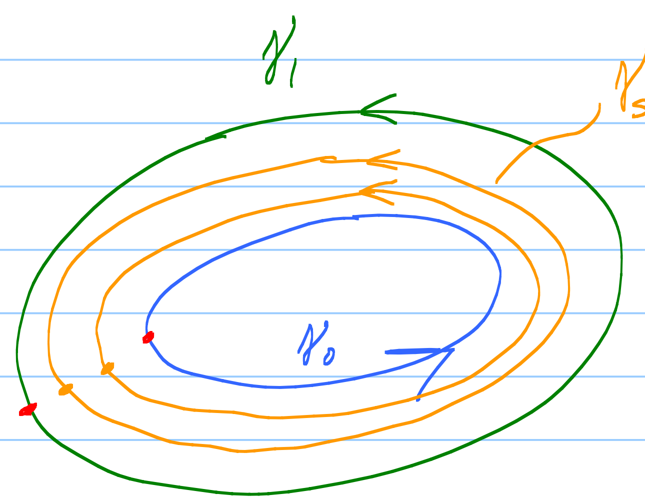
Left side: $-\int_{\gamma_0} f\, dz$.

Get $\int_{\gamma_0} = \int_{\gamma_1}$ ✓

Def^n: Two closed curves γ_0 and γ_1 in Ω are homotopic in Ω if there is a homotopy $H(t,s)$ such that

$$\begin{cases} H(t,0) = z_0(t) : \gamma_0 \\ H(t,1) = z_1(t) : \gamma_1 \end{cases}$$

$$H(0,s) = H(1,s)$$

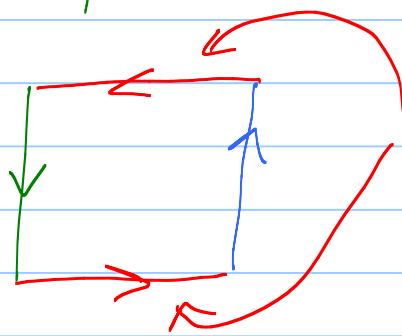


Think: $\gamma_s : z_s(t) = H(t,s)$.

Thm: f anal on Ω, $\gamma_0 \sim \gamma_1$ in Ω,

$$\text{then } \int_{\gamma_0} f\,dz = \int_{\gamma_1} f\,dz$$

Pf: Same pf, but



Top and bottom cancel

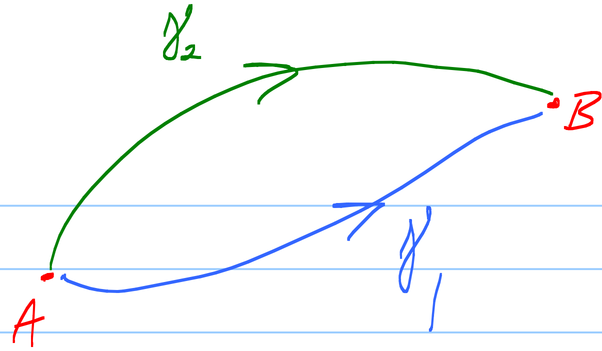
Def^n: A domain in $\mathbb{C}$ is simply connected if every closed curve in Ω is "homotopic to a point" in Ω. $[\,z(t) \equiv z_0 \in \Omega,\ 0 \le t \le 1\,]$

The Cauchy Thm: f analytic on a simply connected domain Ω, then $\int_\gamma f\,dz = 0$ for every closed curve in Ω. Also, for any two curves γ_1 and γ_2 that both start at A and stop at B,

$$\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz.$$

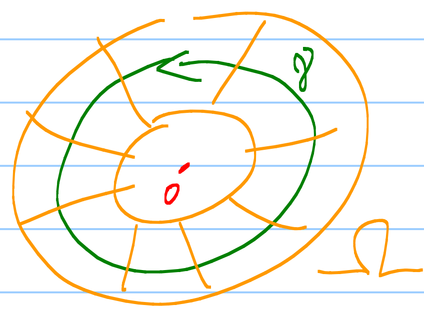
Pf: $\gamma \sim z_0$ $\qquad \int_\gamma f\,dz = \int_0^1 f(z_0) \underbrace{\frac{d}{dt} z_0}_{=0}\,dt = 0$

Part 2:



Closed curve $\gamma = \gamma_1 \cup -\gamma_2$. Use part 1.

EX:



$$\int_\gamma \frac{1}{z}\, dz = 2\pi i \leftarrow \text{Not zero}$$

Holes mess up Cauchy's thm.

Lecture 29 Consequences of the Cauchy Theorem

Midterm exam
due in GS
11:59 pm Tues

Cauchy theorem Suppose f is analytic on a simply connected domain Ω. Then

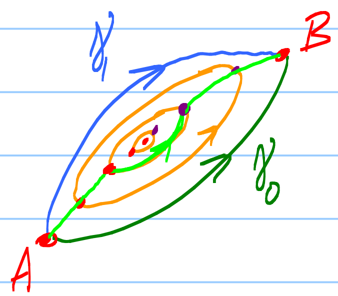
1) $\int_\gamma f\, dz = 0$, γ in Ω closed

2) $\int_{\gamma_A^B} f\, dz = \int_{\tilde{\gamma}_A^B} f\, dz \leftarrow \int_A^B f\, dz$

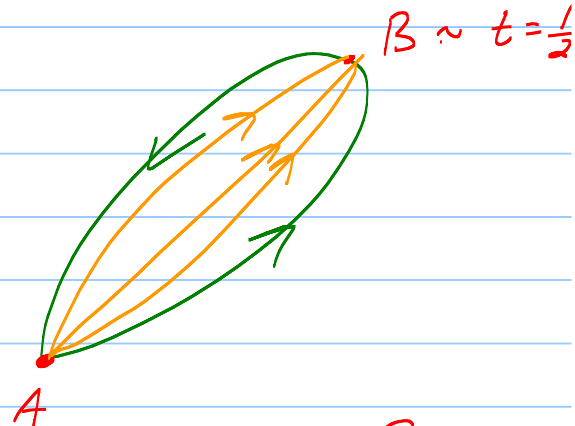
indep of path

Remark Ω simply connected $\Longleftrightarrow$ for any $A, B \in \Omega$
$\gamma_A^B \sim \tilde{\gamma}_A^B$.

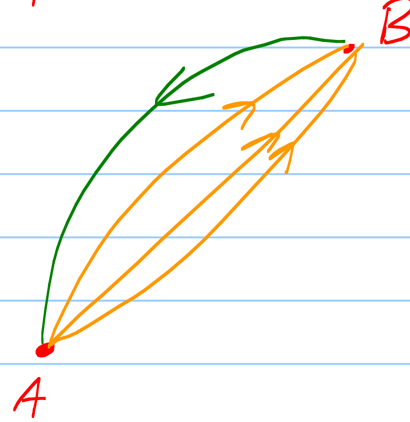
($\Rightarrow$)



($\Leftarrow$)

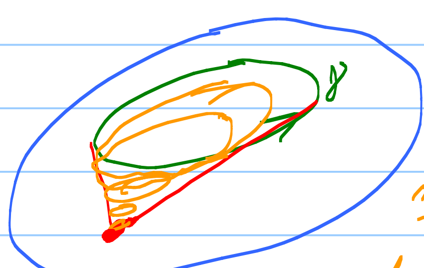


$\gamma = \gamma_0 \cup (-\gamma_1)$



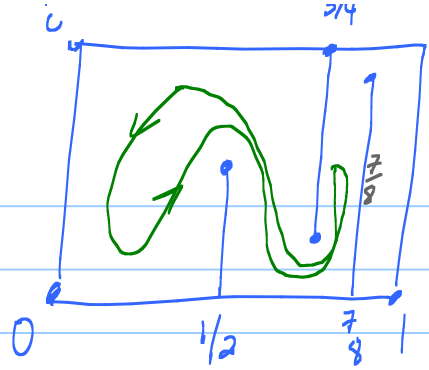
Examples of s.c. domains:

1) Convex domains are s.c.



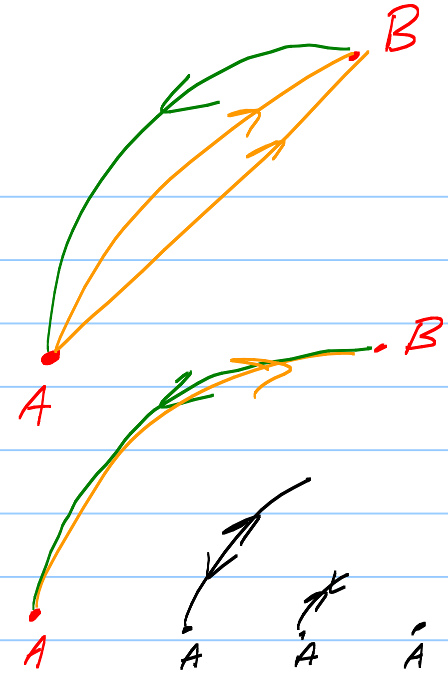
"tornado"
down to pt.

2)



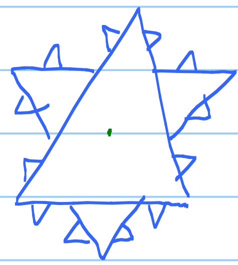
etc.

Spaghetti again.



Suck it in like spaghetti.

3) Serpinski snowflake



etc.

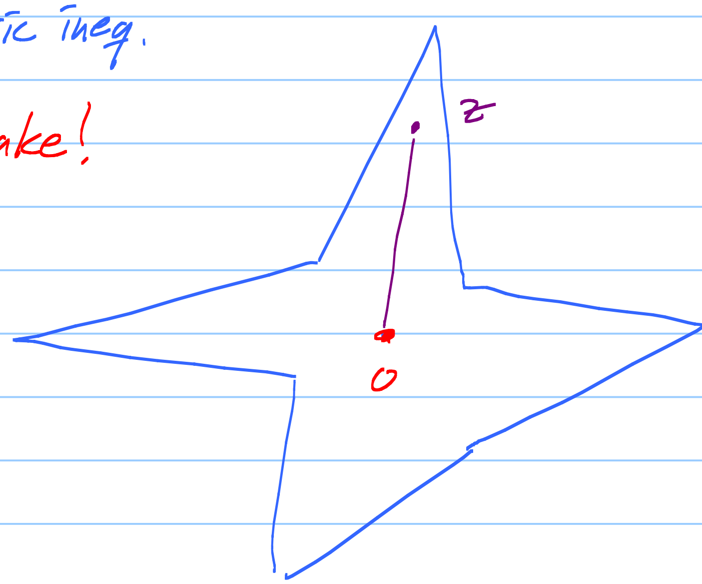
$A \leq \frac{L^2}{4\pi}$ Isoperimetric ineq.

bounded

$\to \infty$ in snowflake!

3) Starshaped domains

tornado method



Thm: Suppose f is analytic on a s.c. domain Ω.

A) $\exists$ analytic antiderivative F on Ω with $F' = f$.

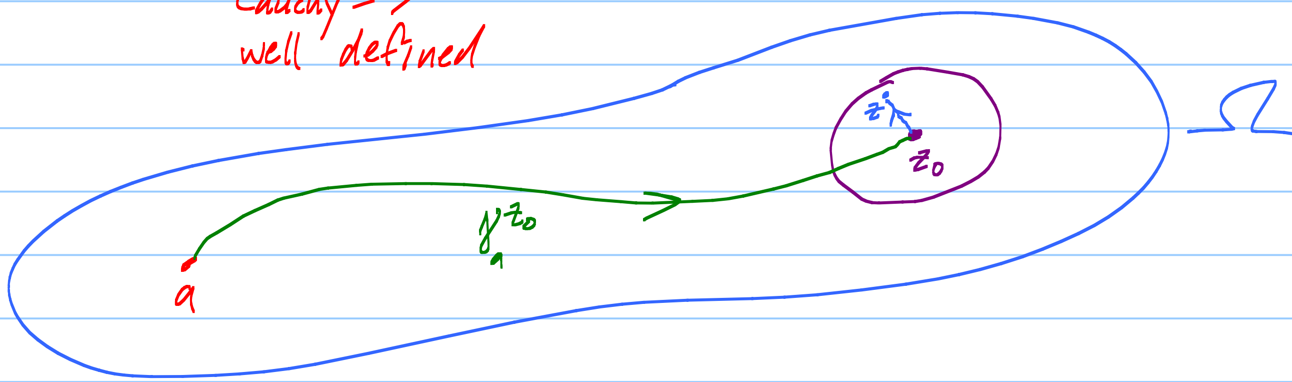
B) If f is nonvanishing on Ω, then $\exists$ analytic G on Ω with $f = e^G$. $\left[\frac{d}{dz} G = \frac{f'}{f}\right]$

And $\exists$ analytic H with $f = H^N$ $\quad \left[H = e^{G/N} \right]$

c) If u is harmonic on Ω, then u has a harmonic conjugate on Ω.

Pf: A) "Define" $\quad F(z) = \int_{\gamma_a^z} f(w)\,dw$

Cauchy $\Rightarrow$ well defined



$$F(z) = \underbrace{\int_{\gamma_a^{z_0}} f\,dw}_{\text{const.}} + \underbrace{\int_{L_{z_0}^{z}} f\,dw}_{\text{anti-deriv for } f \text{ on convex disc}}$$

So $F'(z) = f(z)$ on disc. $\leftarrow$ arb z_0 ✓

B) $$g(z) = \int_{\gamma_a^z} \frac{f'(w)}{f(w)}\,dw$$ $\leftarrow$ nonvanishing!

Get $\quad g'(z) = \dfrac{f'}{f}$

Trick: $\quad \dfrac{d}{dz}\left(\dfrac{e^g}{f}\right) \equiv 0$

So $\dfrac{e^g}{f} = \text{const.}$

$$f = k e^g = e^{\operatorname{Log} k} e^g = e^{\overbrace{(\operatorname{Log} k + g)}^{G}}$$

c) u harmonic. C-R Eqns $\Rightarrow$ $F = u_x - i u_y$ analytic

Get $\tilde{f}$ antideriv of $F = u_x - i u_y$

[Think: $f = u + iv \leftarrow$ analytic

$f' = \underbrace{u_x + i v_x}_{\text{red box \#1}} = u_x - i u_y$

CR Eqns.]

Now $F = \tilde{f}'$

$u_x - i u_y = \tilde{u}_x - i \tilde{u}_y \leftarrow$ $\nabla u \equiv \nabla \tilde{u}$

So $u = \tilde{u} + c$ on Ω.

Aha! $\tilde{f} = \tilde{u} + i\tilde{v} = (u - c) + i\tilde{v}$ is analytic.

So is $u + i\tilde{v}$ $\leftarrow$ harm conj

Thm: f analytic on a domain has a complex antiderivative $\Longleftrightarrow$ $\int_\gamma f\, dz = 0$ for all closed γ.

Pf: In review lecture.

Cor: f analytic on $D_r(z_0) - \{z_0\}$. [Isolated sing.]

f has an analytic antideriv on $D_r(z_0) - \{z_0\}$

$\Longleftrightarrow \operatorname{Res}_{z_0} f = 0.$

Pf: $(\Longrightarrow)$ $a_{-1} = \frac{1}{2\pi i} \int_{C_\rho(z_0)} \underbrace{f(w)}_{= F'(w)} dw = 0$

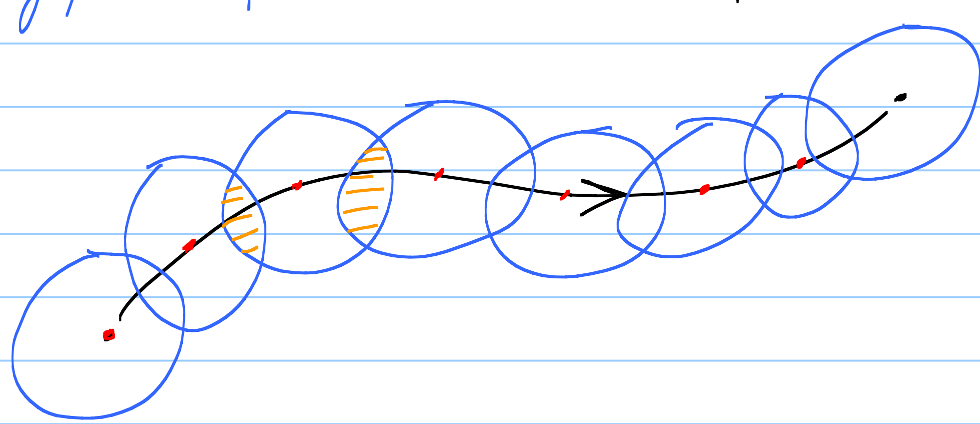
$(\Longleftarrow)$ Laurent exp converge unif on $\underbrace{\{\varepsilon < |z - z_0| < \rho < r}_{A_\varepsilon^\rho}$

Given closed γ in $D_r(z_0) - \{z_0\}$, $\exists$ A_ε^ρ containing γ.

$$\int_\gamma f \, dw = \int_\gamma \sum_{-\infty}^{\infty} a_n (w - z_0)^n dw$$

$$= \sum_{-\infty}^{\infty} a_n \int_\gamma \underbrace{(w - z_0)^n}_{= \text{deriv except } n=-1 \text{ case}} dw = \underbrace{a_{-1}}_{=0} \int_\gamma \frac{1}{w - z_0} dw$$

Hungry caterpillar idea γ simple curve.

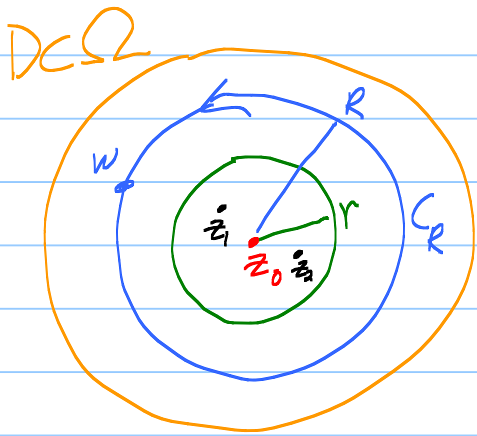


Simply connected caterpillar

Easy to see $\int_{WB} f \, dz$ well defined in worm. Use when γ_A^B is just cont

Lecture 30 Montel's theorem HWK 7 due Tues, 4/12

Arzela-Ascoli Thm : Baby Rudin p. 158
Ahlfors p. 219–227
Stein p. 225



Lemma Uniformly bounded family $\mathcal{F}$ of analytic fcns on a domain are uniformly equicontinuous on compact subsets.

$f \in \mathcal{F}$

$$\left| f(z_1) - f(z_2) \right| = \left| \frac{1}{2\pi i} \int_{C_R} f(w) \underbrace{\left[\frac{1}{w-z_1} - \frac{1}{w-z_2} \right]}_{\frac{z_1 - z_2}{(w-z_1)(w-z_2)}} dw \right|$$

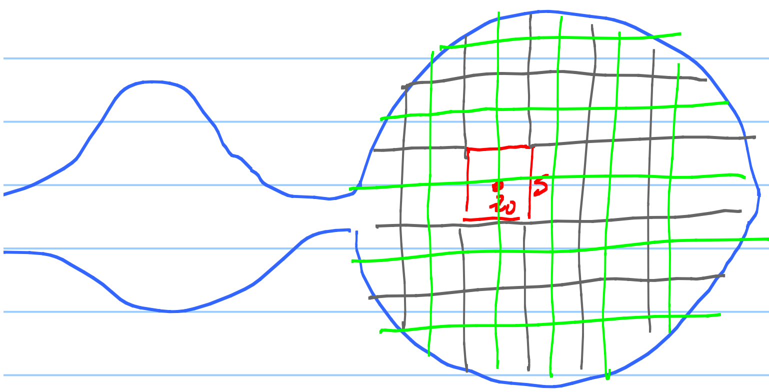
$$\leq |z_1 - z_2| \frac{1}{2\pi} \underbrace{\operatorname{Max}_{C_R} |f|}_{\text{indep of } f} \left(\frac{1}{(R-r)^2} \right) (2\pi R)$$

Finally, cover compact K by discs $D_r(z_0)$, $z_0 \in K$. Take finite subcover.

Montel's thm Suppose f_n are analytic on a domain Ω and are uniformly bounded on compact subsets of Ω. Then there is a subseq f_{n_k} that converges uniformly on compact subets (to analytic f).

Pf: Step 1: "Exhaust" Ω by compact sets:

$$K_1 \subset K_2 \subset K_3 \subset \cdots \subset \Omega, \quad \bigcup_j K_j = \Omega$$



K_1 = Square 1

K_2 = Union of squares side s inside $\Omega \cap D_{10s}(z_0)$

K_3 = Union of squares of side $s/2$ inside Ω $\cap D_{100s}(z_0)$

$\frac{s}{4}, \frac{s}{8}, \frac{s}{16}, \ldots \qquad 10s, 100s, 1000s, \ldots$

Fancier version: Domains $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega$

$K_1 = \overline{\Omega_1}, \ K_2 = \overline{\Omega_2}, \ \ldots$

$$\mathring{K}_j \subset K_{j+1}$$

Step 2: $z \in \mathbb{C}$; $\operatorname{Re} z, \operatorname{Im} z \in \mathbb{Q}$ is a countable dense subset of $\mathbb{C}$.

Fix K_j. Let $\{z_n\}_{n=1}^{\infty}$ be a countable dense subset of K_j from above.

Step 3: $f_1(z_1), f_2(z_1), f_3(z_1), \ldots$ is a bdd seq.

So $\exists$ conv subseq $f_{1,1}, f_{1,2}, f_{1,3} \ldots$

Now $f_{1,1}(z_2), f_{1,2}(z_2), f_{1,3}(z_2)$ is a bdd seq.

So $\exists$ conv subseq $f_{2,1}, f_{2,2}, f_{2,3}, \ldots$

$f_{2,1}(z_3), f_{2,2}(z_3), \ldots$ bdd seq

etc.

Aha! $\underbrace{f_{1,1}}_{F_1}, \underbrace{f_{2,2}}_{F_2}, \underbrace{f_{3,3}}_{F_3}, \ldots$

converges at all pts in dense subset of K_j.

Step 4: F_n are uniformly Cauchy on K_j. $\varepsilon > 0$

$$|F_n(z) - F_m(z)| = |\underbrace{\overbrace{F_n(z) - F_n(z_j)}^{|\,|<\frac{\varepsilon}{3}}}_{} + \underbrace{F_n(z_j) - F_m(z_j)}_{} + \underbrace{\overbrace{F_m(z_j) - F_m(z)}^{|\,|<\frac{\varepsilon}{3}}}_{}|$$

Pick z_j close to z. equicont makes small

$F_n(z_j)$ conv $\Rightarrow$ unif Cauchy

Pick $z_0 \in K_j$. Get $D_\delta(z_0)$ from equicont. fact, $\frac{\varepsilon}{3}$, Shrink: $\frac{\delta}{2}$.

Pick z_j in $D_{\delta/2}(z_0)$. Get estimate on $D_{\delta/2}(z_0)$.

Cover K_j by discs. Take finite subcover. ✓

Step 5: F_n unif Cauchy on K_j. So it converges unif.

$\underbrace{F_{1,1}}_{F_1}, \underbrace{F_{1,2}}_{F_2}, \underbrace{F_{1,3}}_{F_3}, \dots$ conv unif on K_1

Subseq $F_{2,1}, F_{2,2}, F_{2,3}, \dots$ conv unif on K_2

Subseq $F_{3,1}, F_{3,2}, F_{3,3}, \dots$ conv unif on K_3

etc.

$F_{n,n}$ is a subseq of f_n converges uniformly on each K_j.!

__Step 6__ Finally, given compact $K \subset \Omega$, there is a K_j with $K \subset K_j$.

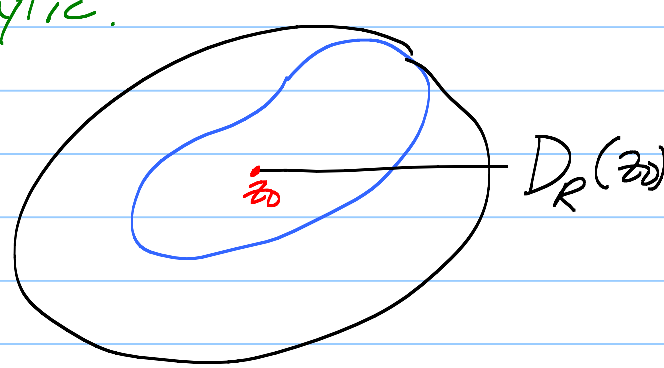
__Technical point.__ $\Omega = \mathbb{C}$. No $f : \mathbb{C} \xrightarrow[\text{onto}]{1-1} D_1(0)$ analytic, Liouville's.

If Ω is simply connected domain and $\Omega \neq \mathbb{C}$,

Riemann mapping thm $\Rightarrow$ $\exists\, f : \Omega \xrightarrow[\text{onto}]{1-1} D_1(0)$ analytic.

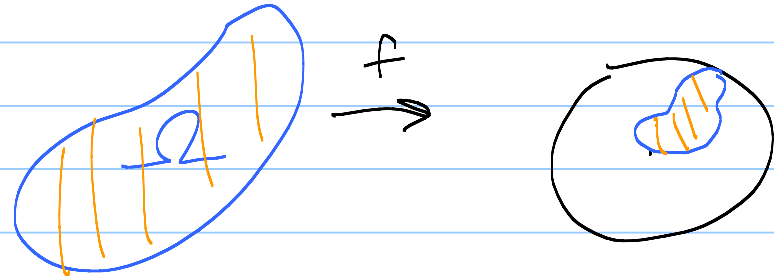
__Lemma__ Given Ω s.c. $\neq \mathbb{C}$, $\exists\, f : \Omega \to D_1(0)$ that is 1-1 and analytic.

Easy if Ω is bounded:



$$R = \max_{z \in \overline{\Omega}} |z - z_0|$$

$f(z) = \dfrac{z - z_0}{R}$ maps Ω 1-1 into $D_1(0)$, analytic.



What to do if Ω unbounded?

$\Omega \neq \mathbb{C}$. $\exists\ z_0 \in \mathbb{C} - \Omega$.

Hmmm. $z - z_0$ is nonvanishing on Ω.

$\exists$ analytic $g(z)$ on Ω with $g(z)^2 = z - z_0$

Show g is 1-1, and misses a disc.

ecture 31 Proof of the Riemann mapping theorem

HWK 7 due Thurs, 4/7

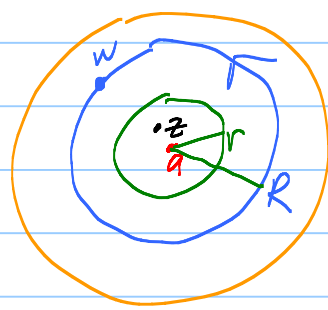
Thm: Ω simply connected domain $\Omega \neq \mathbb{C}$. There is an analytic (conformal) map $f:\Omega \to D_1(0)$ that is one-to-one and onto.

Lemma: If f_n analytic on Ω and $f_n \rightrightarrows f$, then $f_n^{(k)} \rightrightarrows f^{(k)}$ for each k.

Way false $\mathbb{R}\to\mathbb{R}$: $h_N(t) = \sum_{n=1}^{N} \frac{1}{2^n} \sin(2^n t) \leftarrow$ $h_N \to$ nowhere diff fcn

Montel's false $\mathbb{R}\to\mathbb{R}$: $\sin(2^n t)$

Pf of lemma



$D \subset \Omega$

$$\left| f_n^{(k)}(z) - f^{(k)}(z) \right| = \left| \frac{k!}{2\pi i} \int_{C_R} \frac{f_n(w)-f(w)}{(w-z)^{k+1}}\, dw \right|$$

$$\leq \frac{k!}{2\pi} \frac{\max_{C_R} |f_n - f|}{|R-r|^{k+1}} (2\pi R) \to 0$$

See unif conv on $D_r(a)$. Given $K \subset\subset \Omega$, cover by $D_r(z_0)$.

Lemma: $\exists\ f : \Omega \neq \mathbb{C}$, s.c. $\longrightarrow D_1(0)$ that is 1-1.

Pf: Easy if Ω bounded: $f(z) = \dfrac{z}{R}$ $\quad R = \max\limits_{z \in \Omega} |z|$.

Unbounded: $\Omega \neq \mathbb{C}$. So $\exists\ w_0 \in \mathbb{C} - \Omega$.

Now $z - w_0$ is nonvanishing on Ω s.c.

So $\exists$ analytic $g(z)$ such that $g(z)^2 = z - w_0$.

Claim: g is 1-1:

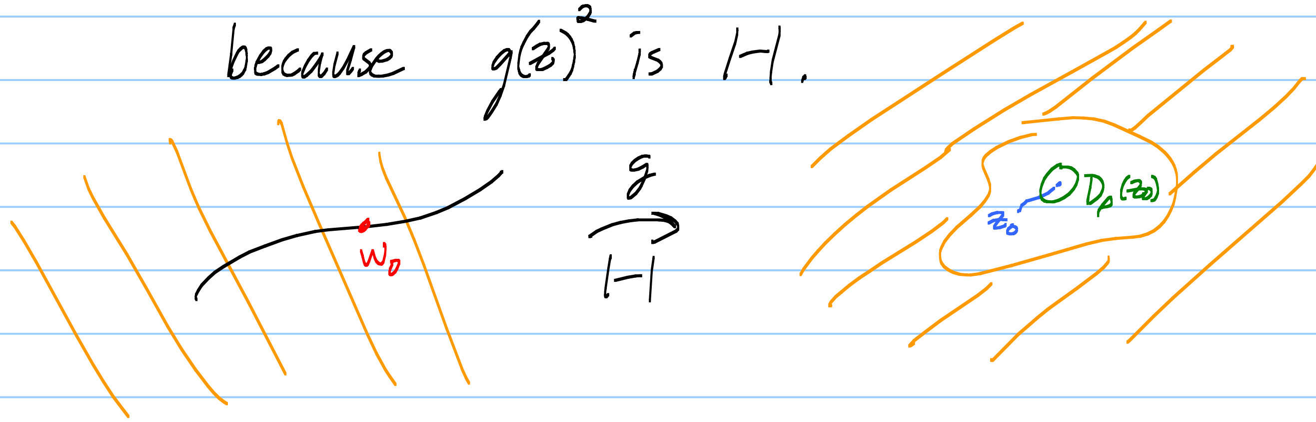
$$g(z_1) = g(z_2)$$
$$g(z_1)^2 = g(z_2)^2$$
$$z_1 - w_0 = z_2 - w_0 \quad \checkmark$$

Claim: $g(\Omega)$ misses a disc:

g not constant. Open mapping thm $\Longrightarrow \exists$ disc $D_\rho(z_0) \subset g(\Omega)$.

Aha! $-D_\rho(z_0) = \{-z : z \in D_\rho(z_0)\} \cap g(\Omega) = \emptyset$

because $g(z)^2$ is 1-1.



$$\xrightarrow[1-1]{} \quad \frac{\rho}{z - z_0}$$

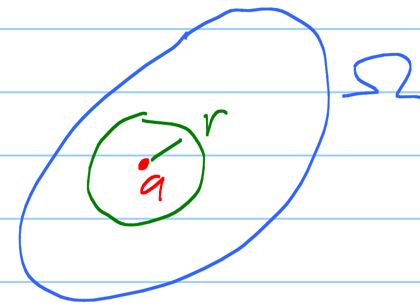
Pf of RMT: Let $\tilde{\mathcal{F}}$ = family of analytic fns $f:\Omega \to D_1(0)$ that are 1-1.

Lemma $\Rightarrow$ $\tilde{\mathcal{F}}$ not empty.

Pick $a \in \Omega$. Let $M = \sup\{|f'(a)| : f \in \tilde{\mathcal{F}}\}$

Step 1. Claim: $0 < M < \infty$

$\uparrow$ f 1-1 $\Rightarrow f'(a) \neq 0$.



Cauchy estimate: $|f'(a)| \leq \dfrac{1! \max_{C_r(a)} |f|}{r^1}$

$\leq \dfrac{1}{r}$. So $M \leq \dfrac{1}{r}$.

[Let r max out to see $|f'(a)| \leq \dfrac{1}{\text{dist}(a, b\Omega)}$.]

Step 2: Take seq $f_n \in \tilde{\mathcal{F}}$ such that $|f_n'(a)| \to M$ as $n \to \infty$. [$|f_n(z)| < 1 \ \forall n, z \in \Omega$.]

Montel's $\Rightarrow$ $\exists$ subseq $f_{n_k} \rightrightarrows F$ on Ω.

[F is the Riemann map!] $\uparrow$ analytic ✓

Step 3 Lemma $\Rightarrow$ $f_{n_k}'(a) \to F'(a)$

So $|F'(a)| = M$. $\leftarrow$ not zero

$F'(a)$ not zero $\Longrightarrow$ F not constant.

<u>Hurwicz</u> $\Longrightarrow$ F is 1-1 on Ω.

<u>Step 4</u> Claim: $F \in \tilde{\mathcal{F}}$. Note: $f_n : \Omega \to D_1(0)$

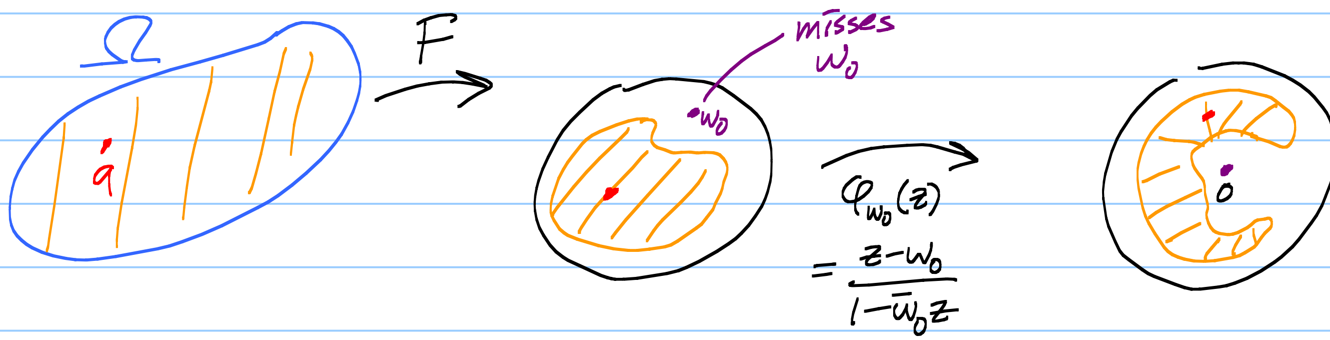
$|f_n(z)| < 1$

So $|F(z)| \le 1$.

Aha! If $|F(z_0)| = 1$ for some $z_0 \in \Omega$, Max Princ $\Longrightarrow$

$F \equiv$ const. ↯ So $F : \Omega \to D_1(0)$. Get $F \in \tilde{\mathcal{F}}$ ✓

<u>Step 5</u>: Claim: F is <u>onto</u> $D_1(0)$. Suppose not.



<u>Note</u>: $\varphi_{w_0} \circ F$ is non-vanishing, 1-1, analytic.

So $\exists$ analytic g on Ω with $g^2 = \varphi_{w_0} \circ F$.

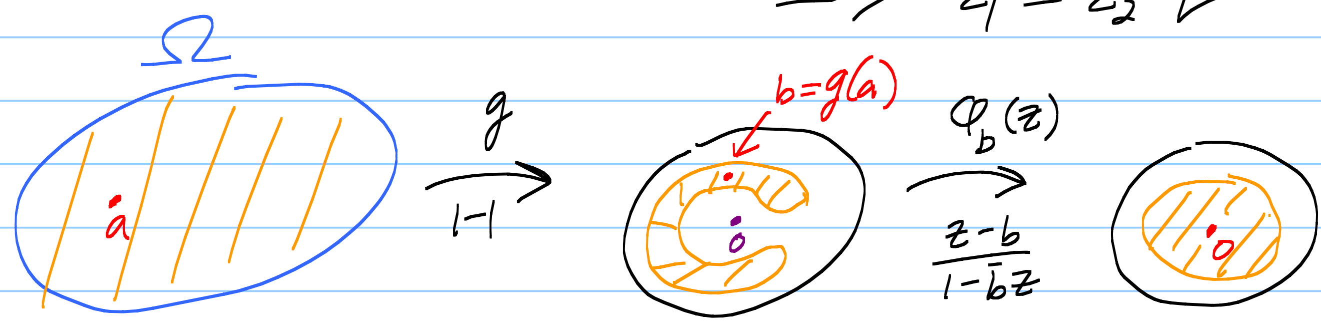
$\left[\, |g^2(z)| < 1 \Longrightarrow |g(z)| < 1. \right] \leftarrow g : \Omega \to D_1(0).$ ✓

<u>Claim</u>: $g \in \tilde{\mathcal{F}}$. Need g is 1-1.

<u>Why</u>: $g(z_1) = g(z_2) \Longrightarrow g(z_1)^2 = g(z_2)^2$

$\varphi_{w_0}(F(z_1)) = \varphi_{w_0}(F(z_2)) \leftarrow$ 1-1

$\Longrightarrow z_1 = z_2$ ✓



Claim: $\varphi_b \circ g \in \tilde{\mathcal{F}}$ ✓ and derivative at a is strictly bigger in modulus than M. ↯ So no such w_0.

Write $\tilde{F} = \varphi_b \circ g$.

Check that $F = G \circ \tilde{F}$

where $G = \varphi_{w_0}^{-1} \circ s \circ \varphi_b^{-1}$, $s(z) = z^2$

Note: $G : D_1(0) \to D_1(0)$

Cor to Schwarz: $|G'(0)| \leq 1 \leftarrow$ if $=1$, then $G(z) = \underbrace{\lambda z}_{\text{1-1}}$, $|\lambda| = 1$.

Aha! Since $\underbrace{G \text{ is not 1-1}}_{s(z) \text{ is not 1-1}}$, we must have $|G'(0)| < 1$.

Finally,

$$F'(a) = G'(\underbrace{\tilde{F}(a)}_{0})\, \tilde{F}'(a)$$

$$|F'(a)| = |G'(0)|\, |\tilde{F}'(a)|$$

M $\quad < 1 \quad$ must be $> M!$ $\lightning$

No such w_0. F is onto. Done.

Fact from proof: $f : \Omega \to D_1(0)$. $\quad f(a) = b \neq 0$.

Then $\left| (\varphi_b \circ f)'(a) \right| > |f'(a)|$.

$\left[\text{because} \quad |\varphi_b'(b)| = \frac{1}{1-|b|^2} > 1 \right]$

The Riemann map associated to $a \in \Omega$: $f_a : \Omega \to D_1(0)$

$f(a) = 0$

$f'(a) = M$

1-1, onto

analytic

$f_a : \lambda F$

$f_a'(a) = \lambda F'(a) \qquad$ Take $\lambda = \frac{\overline{F'(a)}}{|F'(a)|}$

Makes unique.