Bell Lempert August 2018

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1 Problem 4

4. Show that a function $g \in \mathcal{O}(\mathbb{C})$ is $2\pi i$–periodic (i.e., $g(z + 2\pi i) = g(z)$) if and only if there is an $h \in \mathcal{O}(\mathbb{C} \setminus \{0\})$ such that $g(z) = h(e^z)$.

What was written on the board but needs some clarifying.

Proof. ($\implies$) Suppose that g is $2\pi i$-periodic. Consider $f(z) = e^z$. Now, $f'(z) \neq 0$ for any z and we may find an open subset $\Omega \subset \mathbb{C}$ such that $f(z)$ is 1-to-1 on Ω. Pick $w \in \mathbb{C}\backslash\{0\}$ such that $f(z) = w$ and define $h(w) = g(z)$. Notice, $W = f(\Omega)$ is an open set by the open mapping theorem. By the Super Inverse Function theorem we may find $F : W \to \Omega$ defined by $F(f(z)) = F(e^z) = z$ for $F \in \mathcal{O}(W)$. Fix $w' \in h(w') = (g \circ F)(w') \in \mathcal{O}(W) \implies h \in \mathcal{O}(w) \implies h \in \mathbb{C}\backslash\{0\}$.

($\impliedby$) Suppose $g(z) = h(e^z)$. Then we have that :

$$g(z + 2\pi i) = h(e^{z+2\pi i}) = h(e^z e^{2\pi i}) = h(e^z) = g(z)$$

$\square$