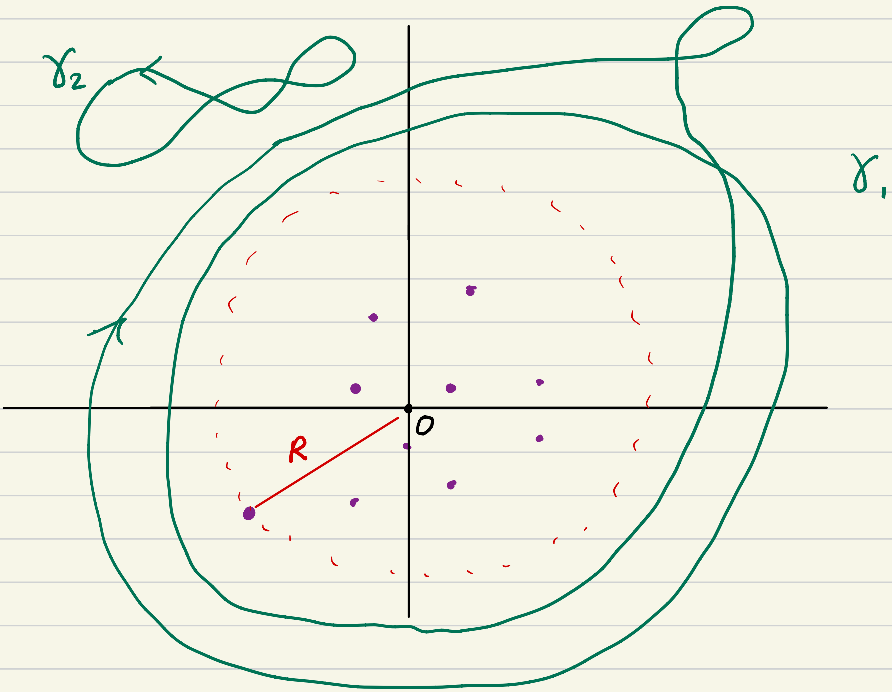
Aug 2020, Q 4



Let γ be some arbitrary closed curve lying in $\mathbb{C} \setminus \overline{D_R(0)}$

γ winds about each element of A the same no. of times (possibly not at all or in an anti-clockwise or clockwise direction)

i.e. if $A = \{a_k\}_{k=1}^{N}$,

$\exists\, M \in \mathbb{N}$ s.t.

$\mathrm{Ind}_\gamma(a_k) = M \quad \forall\, k \in \{1, 2, \ldots, N\}$.

So,

$$\int_\gamma f\,dz = 2\pi i \sum_{k=1}^{N} \mathrm{Ind}_\gamma(a_k)\,\mathrm{Res}_{a_k} f$$

(Improved res. thm from L38 applied to $\mathbb{C} \setminus A$.
Note: we're not applying it to all possible curves; just those contained in $\mathbb{C} \setminus \overline{D_R(0)}$.)

$$= 2\pi i \sum_{k=1}^{N} M\,\mathrm{Res}_{a_k} f$$

$$= 2\pi i\, M \underbrace{\sum_{k=1}^{N} \mathrm{Res}_{a_k} f}_{=0}$$

(given)

$\therefore \int_\gamma f\,dz = 0$ for any closed curve γ where $tr(\gamma) \subset \mathbb{C} \setminus \overline{D_R(0)}$

$\iff$ f has an analytic antiderivative on $\mathbb{C} \setminus \overline{D_R(0)}$

(result from the review lecture before the midterm).