Aug 2020, Q 2

a) No, since for an essential singularity at a point $a \in \Omega$, say,
$\exists\, \rho > 0$. s.t. $\forall \varepsilon > 0$, where $0 < \varepsilon < \rho$, $f(D_\varepsilon(a) \setminus \{a\})$ is dense in $\mathbb{C}$,
but $f(D_\varepsilon(a) \setminus \{a\})$ cannot contain $D_2(0)$

$\Rightarrow$ no pt. in $D_2(0)$ is contained $f(D_\varepsilon(a) \setminus \{a\})$ nor is a limit pt. in $f(D_\varepsilon(a) \setminus \{a\})$

This would contradict $f(D_\varepsilon(a) \setminus \{a\})$ being dense in $\mathbb{C}$.

(b) $\lim_{z \to \infty} f(z) = \infty$

$$\Rightarrow \lim_{z \to \infty} \frac{1}{f(z)} = 0.$$

$$\Rightarrow \lim_{z \to 0} \frac{1}{f(\frac{1}{z})} = 0$$

So, $\dfrac{1}{f(\frac{1}{z})}$ has a

removable singularity at $z=0$

(Handwavy?)

Let

$$F(z) = \begin{cases} \dfrac{1}{f(\frac{1}{z})}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

$F(z)$ is analytic on ???

PTO

(What domain does $\frac{1}{z}$ map to Ω?
It's the set $\{ z \in \mathbb{C} \mid |z| < \frac{1}{2} \}$)

So, $F(z)$ is analytic on $D_{\frac{1}{2}}(0)$.

$\Rightarrow F\left(\frac{z}{2}\right)$ is analytic on $D_1(0)$.

Note that $\max\limits_{z \in D_1(0)} \left| F\left(\frac{z}{2}\right) \right| \leq \frac{1}{2}$,

since f maps Ω to itself
and we've made it so

$$
\begin{tikzcd}
D_1(0) \arrow[r, "z/2"] \arrow[rrr, bend right, "F\left(\frac{z}{2}\right)"'] & D_{1/2}(0) \arrow[r, "1/z"] & \Omega \arrow[r, "\frac{1}{f(z)}"] & D_{\frac{1}{2}}(0)
\end{tikzcd}
$$

$\therefore\ G(z) = 2F\left(\frac{z}{2}\right)$ is analytic

on $D_1(0)$ and $G(D_1(0)) \subset D_1(0)$.

So, by Schwarz Lemma,

$$|G(z)| \le |z|, \quad z \in D_1(0)$$

$$\Rightarrow \left|2F\left(\frac{z}{2}\right)\right| \le |z| \quad \text{—''—}$$

$$\Rightarrow \left|F\left(\frac{z}{2}\right)\right| \le \left|\frac{z}{2}\right|, \quad \text{—''—}$$

$$\Rightarrow \left|\frac{1}{f\left(\frac{2}{z}\right)}\right| \le \left|\frac{z}{2}\right|, \quad \text{—''—}$$

$$\Rightarrow \left|\frac{1}{f(w)}\right| \le \left|\frac{1}{w}\right|, \quad w \in \Omega$$

(let $\frac{2}{z} = w$)

$$\Rightarrow |w| \le |f(w)|, \quad w \in \Omega$$