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1 Problem 1

Can someone do this one?

2 Problem 2

Suppose $u(z,s)$ is a continuous real values function on $\mathbb{C} \times \mathbb{R}$ such that $u(z,s)$ is harmonic in z for each fixed s. Define:

$$U(z) = \int_{-1}^{1} u(z,s)ds.$$

(a) Give an $\epsilon - \delta$ proof that U is continuous on $\mathbb{C}$.

Proof. Let u, U be as described above. Since u is continuous there exists δ such that for $0 < |z - z_0| < \delta$ we have $|u(z,s) - u(z_0,s)| < \frac{\epsilon}{2}$.

Then $|U(z) - U(z_0)| = |\int_{-1}^{1} u(z,s) - u(z_0,s)ds| \leq 2 \cdot |u(z,s) - u(z_0,s)| < 2 \cdot \frac{\epsilon}{2} = \epsilon$ by the basic integral estimate. □

(b) Prove that U is harmonic on $\mathbb{C}$ without taking derivatives.

Proof. We must show that U satisfies the averaging property. Note, that since $u(z,s)$ is harmonic in z for each fixed s we have: $\frac{1}{2\pi}\int_0^{2\pi} u(z+e^{i\theta},s)ds = u(z,s)$.

$$\frac{1}{2\pi}\int_0^{2\pi} U(z+e^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi}\int_{-1}^{1} u(z+e^{i\theta},s)dsd\theta =$$
$$\int_{-1}^{1}\frac{1}{2\pi}\int_0^{2\pi} u(z+e^{i\theta},s)d\theta ds = \int_{-1}^{1} u(z,s)ds = U(z)$$

Thus $U(z)$ satisfies the averaging property so that $U(z)$ is harmonic. □

3 Problem 3

Theorem 3.1. *Suppose that $f(z)$ is an entire function such that $f(z+\pi) = f(z)$ for all z and $f(z+i\pi) = f(z)$ for all z. Then f is the constant function.*

Proof. Consider the box B with vertices at $0, \pi, i\pi, \pi + i\pi$. By the periodicity of f, $f(B) = f(\mathbb{C})$. Since B is closed and bounded, it is compact. f is entire and hence continuous on B. Now, $f(B)$ is a closed and bounded set since continuous functions send compact sets to compact sets. Thus $|f(\mathbb{C})| = |f(B)| \leq M$ for some $M \in \mathbb{R}$. Thus f is a bounded, entire function so that by Louisville's theorem f must be constant. □

4 Problem 4

Theorem 4.1. *Suppose that $R(z) = \frac{P(z)}{Q(z)}$ for complex polynomials P, Q such that $deg(Q) \geq deg(P) + 2$. Then the sum of the residues of $R(z)$ in the complex plane must be 0.*

Proof. Let $C_r(0)$ be a circle of large enough radius such that all the zeros of P and Q are contained in its interior. We may apply the Baby Residue theorem. Let $deg(P) = n$ and $deg(Q) = m$.

$$2\pi i \sum_n Res_n(R(z)) = lim_{r\to\infty} \int_{C_r(0)} R(z)dz = lim_{r\to\infty} \int_{C_r(0)} \frac{P(z)}{Q(z)}dz$$
$$\leq lim_{r\to\infty} 2\pi r Max_{z\in C_r(0)} |\frac{P(z)}{Q(z)}| = lim_{r\to\infty} 2\pi r Max_{z\in C_r(0)} |\frac{a_n z^n + ... a_0}{b_m z^m + ... + b_0}|$$
$$\leq lim_{r\to\infty} 2\pi r Max_{z\in C_r(0)} |\frac{a_n r^n + ... a_0}{b_m r^m + ... + b_0}| = lim_{r\to\infty} 2\pi Max_{z\in C_r(0)} |\frac{a_n r^{n+1} + ... a_0 r}{b_m r^m + ... + b_0}| = 0$$

since $m \geq n + 2$ □

5 Problem 5

Theorem 5.1. *The family of 1-to-1 conformal mappings of the horizontal strip $\{z : 0 < Imz < 1\}$ onto itself is such that given any two points z_1 and z_2 in the strip, there is a mapping in the family that maps z_1 to z_2.*

Theorem 5.2. *It is enough to show that we can map z_1 to the origin of the unit disk.*

Consider the mappings:

$f : z \longmapsto \pi z$ *mapping* $\{z : 0 < Imz < 1\}$ *to* $\{0 < Imz < i\pi z\}$

$g : z \longmapsto e^z$ *mapping the strip* $\{0 < Imz < i\pi z\}$ *to the upper half plane.*

$h : z \longmapsto \frac{z - z_1}{z + z_1}$ *mapping the upper half plane to the unit disk such that the point z_1 is mapped to* 0.

Then $f \to g \to h$ maps $\{z : 0 < Imz < 1\}$ to the unit disk such that the point z_1 is mapped to 0.

Since we may map any point to the origin of the disk, and map the origin of the disk to any point we may follow these maps backwards replacing z_1 with z_2 so that z_1 is mapped to z_2. Since these are all conformal maps, their composition is conformal. Thus there is a mapping in the family that maps z_1 to z_2.

6 Problem 6

Explain why $\frac{sinz^2}{(z-1)(z+1)}$ has an analytic antiderivative on $\mathbb{C} - [-1,1]$.

Answer 6.1. *Let γ be a closed curve in $\mathbb{C} - [-1,1]$ and let $f(z) = \frac{sinz^2}{(z-1)(z+1)}$. Notice, there are no closed curves in $\mathbb{C} - \{\mathbb{C} - [-1,1]\} = [-1,1]$. Thus $Ind_\gamma(w) = 0$ for all $w \in [-1,1]$. Obviously, $f(z)$ is analytic on $\mathbb{C} - [-1,1]$ so that by the general Cauchy theorem: $\int_\gamma f(z)dz = 0$. Thus $f(z)$ has an analytic antiderivative on $\mathbb{C} - [-1,1]$.*

7 Problem 7

Compute $\int_\gamma \frac{sinz}{z^{10}}dz$ where γ denots an ellipse with one focus at the origin parameterized in the clockwise direction.

Answer 7.1. *By the baby residue theorem: $\int_\gamma \frac{sinz}{z^{10}}dz = -2\pi i Res_0(\frac{sinz}{z^{10}})$*

Using the taylor series expansion for sine about 0, we have: $\frac{1}{z^{10}}(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}) - \cdots \implies Res_0(\frac{sinz}{z^{10}}) = \frac{1}{9!}$.

Thus $\int_\gamma \frac{sinz}{z^{10}}dz = \frac{-2\pi i}{9!}$

8 Problem 8

Theorem 8.1. *Every harmonic function $u(z)$ on a simply connected domain Ω can be expressed as $u(z) = Ln|f(z)|$ were $f(z)$ is a nonvanishing analytic function on Ω.*

Proof. Consider $g(z) = u(z) + iv(z)$ where $v(z)$ is the harmonic conjugate of $u(z)$. In particular, $v(z)$ exists since $u(z)$ is harmonic on a simply connected domain. Then $g(z)$ is analytic and since $u(z)$ is the real part of $g(z)$:

$$|e^{g(z)}| = |e^{u(z)}e^{iv(z)}| = |e^{u(z)}| = e^{u(z)}$$

Let $|f(z)| = e^{u(z)}$. Then $ln|f(z)| = u(z)$.

$\square$

Question 8.2. *Is the function $f(z)$ unique?*

Answer 8.3. *No. We see in the proof above that $f(z)$ is not unique since the equality will hold for any $iv(z)$ since harmonic conjugates are unique up to adding constants.*