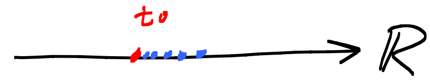
Lesson 11 The Maximum principle

HWK 5: Know $\int_{-\infty} e \, dt = \sqrt{\pi}$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta) \leftarrow$ true for $\theta \in \mathbb{C}$



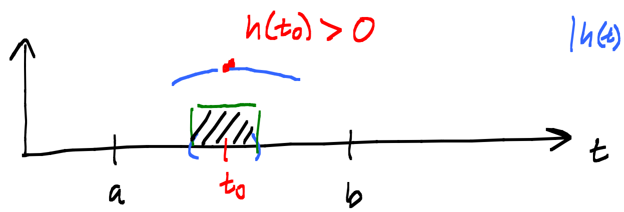
$\sin(\alpha+\beta) = \cos\alpha \sin\beta + \cos\beta \sin\alpha \leftarrow$ true for $\alpha, \beta \in \mathbb{C}$

Calculus lemma: $h: [a,b] \to \mathbb{R}$ continuous and $h(t) \geq 0$ on $[a,b]$.

Then $\int_a^b h(t)\,dt = 0 \implies h \equiv 0$.

$|h(t) - h(t_0)| < \varepsilon = \frac{h(t_0)}{2}$ if $|t - t_0| < \delta$

Pf:

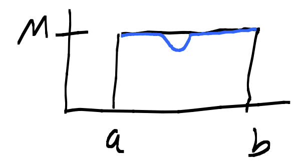


See $\int h > 2\delta \cdot \varepsilon > 0$ ↯.

Lemma: $h: [a,b] \to \mathbb{R}$, $h(t) \leq M$ on $[a,b]$, continuous.

$$\int_a^b h(t)\,dt = M(b-a) \implies h(t) \equiv M \text{ on } [a,b].$$

Pf: Use h to be $M-h$ in lemma above.



Cauchy integral formula $\implies$ Averaging property for analytic fcns.

$$f(z_0) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{it})}{(z_0 + re^{it}) - z_0} \, ire^{it}\,dt$$

$C_r(z_0)$: $z(t) = z_0 + re^{it}$

$z'(t) = ire^{it}$

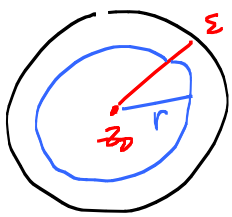
$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\,dt = \frac{1}{2\pi r} \int_0^{2\pi} f(z_0 + re^{it}) \underbrace{r\,dt}_{ds}$$

$= $ Average of f on $C_r(z_0)$.

Averaging prop $\Rightarrow$ Max princ.

<u>Maximum modulus principle #1</u>: Suppose f analytic on a <u>domain</u> Ω. If $|f|$ attains a <u>local</u> max at a point $z_0 \in \Omega$ (meaning $\exists$ $\varepsilon > 0$ such that $D_\varepsilon(z_0) \subset \Omega$ and $|f(z)| \le |f(z_0)|$ for all $z \in D_\varepsilon(z_0)$), then $f \equiv$ const on Ω.

<u>Pf</u>: Suppose $|f|$ has a local max at z_0. Get $D_\varepsilon(z_0) \subset \Omega$ as above.



$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt$$

$|f(z_0)| = M$

$$\underbrace{|f(z_0)|}_{=M} \le \frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(z_0 + re^{it})|}_{\le M}\,dt \le \frac{1}{2\pi} M \cdot 2\pi$$

Only way to get equality is if $|f(z_0 + re^{it})| \equiv M$, $t \in [0, 2\pi]$.

True for $0 < r < \varepsilon$. So $|f| \equiv M$ on $D_\varepsilon(z_0)$.

<u>Lemma</u>: $|f| \equiv M$ on $D_\varepsilon(z_0) \Rightarrow f \equiv$ const there.

<u>Pf of lemma</u>: $f(x+iy) = u(x,y) + iv(x,y)$. $\quad |f|^2 = M^2$

$$(*) \quad u^2 + v^2 = M^2$$

Case $M=0$. $f \equiv 0$. ✓

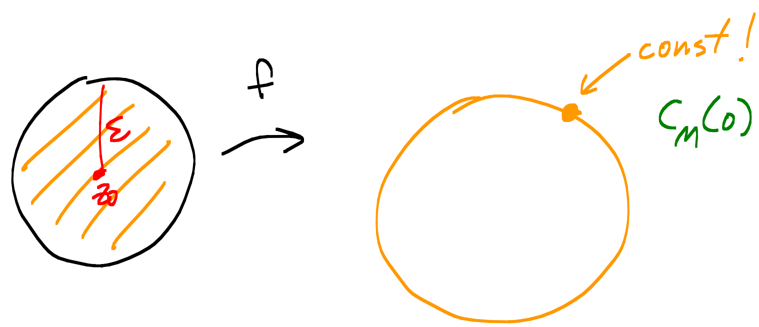
Case $M \neq 0$.
$$\begin{cases} \frac{\partial}{\partial x}(*) : & 2u\,u_x + 2v\,v_x = 0 \\ \frac{\partial}{\partial y}(*) : & 2u \underset{}{\overset{=-v_x}{u_y}} + 2v \overset{=u_x}{v_y} = 0 \end{cases}$$

$$\underbrace{\begin{bmatrix} u_x & v_x \\ -v_x & u_x \end{bmatrix}}_{\text{det must} = 0!} \begin{pmatrix} u \\ v \end{pmatrix} = \vec{0}$$

not the zero vector!

$$\det = (u_x)^2 + (v_x)^2 = |\underbrace{u_x + i v_x}_{\text{red box \#1}}|^2 = |f'|^2 \equiv 0$$

So $f \equiv$ const. HWK 3: $\rho = |z|^2 - M^2$



Pf: back to max princ pf: Have $f \equiv$ const on $D_\varepsilon(z_0) \subset \Omega$

Identity Thm $\Rightarrow$ $f \equiv$ const on Ω.

Max princ #2: Suppose Ω is a bounded domain ($\Omega \subset D_R(0)$ for some $R > 0$) and f is continuous on $\overline{\Omega}$ (the closure of Ω in $\mathbb{C}$) and f is analytic on Ω. Then $|f|$ attains its max modulus at a pt on the boundary. ($\exists\, z_0$ in bndry with $|f(z_0)| = \max_{\bar\Omega} |f|$.)

Pf: f continuous $\Rightarrow$ $|f|$ continuous.

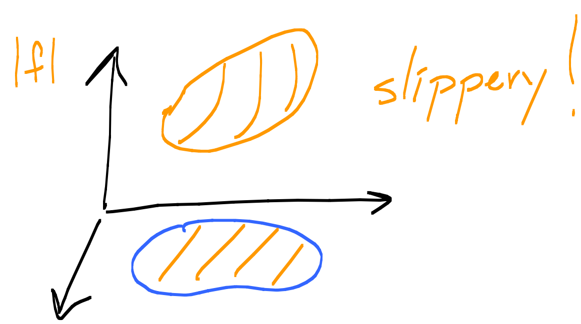
A continuous fcn on a compact set (closed and bounded set) attains its max at a pt $z_0 \in \bar\Omega$

Case 1: $z_0 \in b\Omega = \bar\Omega - \Omega$. ✓

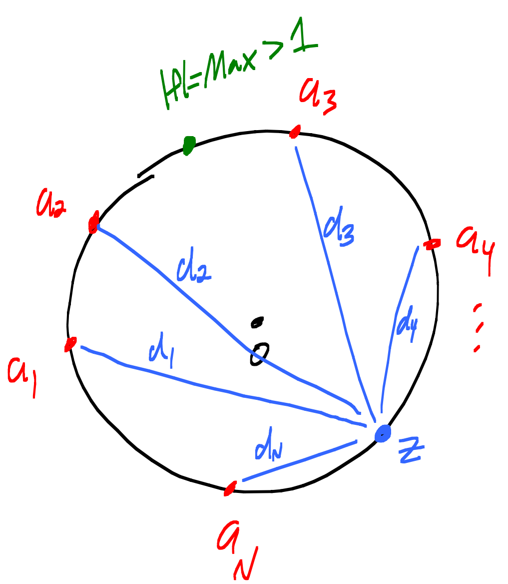
Case 2: $z_0 \in \Omega$. Max princ #1 $\Rightarrow$ f const.

So $|f| = M$ on Ω. Every pt in $\bar\Omega - \Omega$ is a limit pt of Ω. Continuity $\Rightarrow$ $|f| = M$ on $\bar\Omega$.

So max is attained on bndry.



Problem:



Unit disc

Show $\exists\, z$ with $|z| = 1$ such that $\prod_{n=1}^{N} d_n = 1$.

Let $f(z) = \prod_{n=1}^{N} (z - a_n)$. Notice $|f(z)| = \prod_{1}^{N} d_n$ when $|z| = 1$.

f is poly: $f(z) = z^N + \cdots$ <u>not</u> constant.

Aha! $|f(0)| = \left| \prod_1^N (-a_n) \right| = \prod_1^N |a_n| = 1.$

Aha! 1 can't be max because $f \not\equiv$ const.

So Max is > 1. M.P. #2 says M is <u>attained</u> on $|z| = 1$,

$|f| = 0$ at a_n's. Indermediate value thm $\Rightarrow \exists$ pt between a_1 and place where max modulus $M > 1$ is attained where $|f| = 1$. ✓

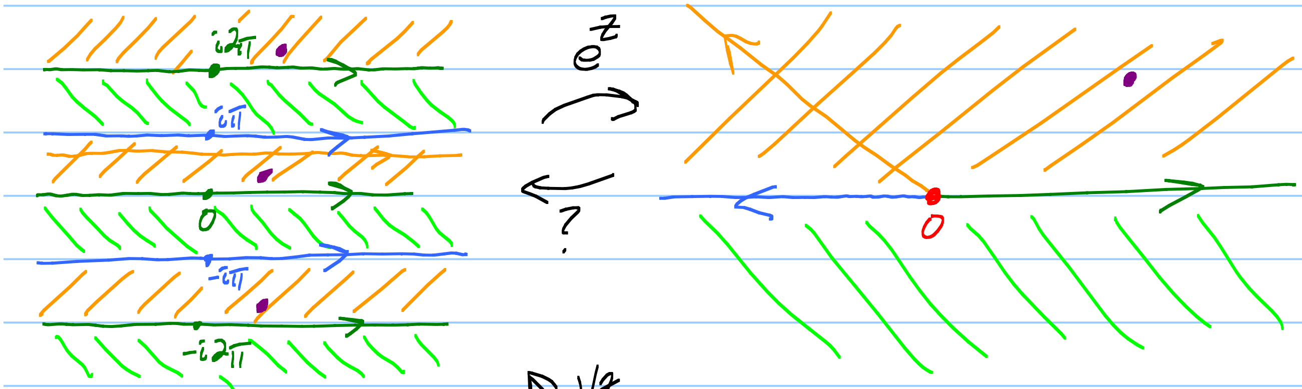
<u>Min princ?</u> z^n shows <u>no</u>, but if $f(z)$ analytic and <u>non-vanishing</u>, then max princ applied to $\frac{1}{f(z)}$ gives the min. princ.

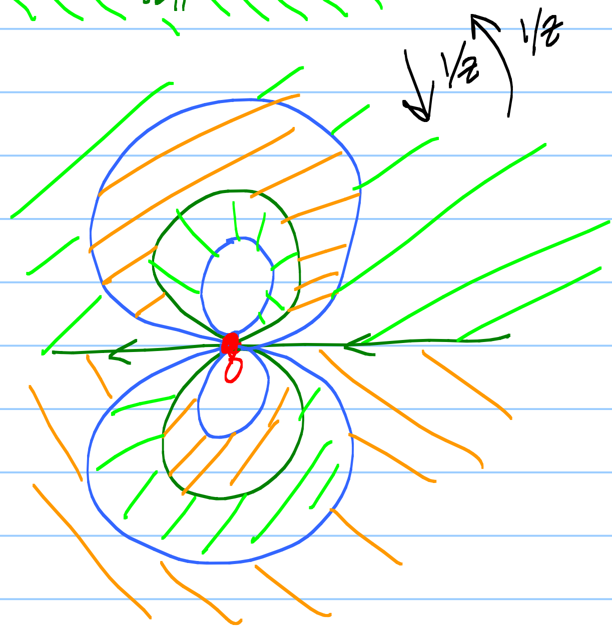
<u>EX</u>: f analytic on $D_1(0)$, continuous on $\overline{D_1(0)}$.

If $f : C_1(0) \to C_1(0)$ and f non-vanishing on $D_1(0)$, then $f \equiv$ const. of modulus 1.

ecture 13 Isolated singularities & the Riemann removable singularity thm

HWK 4 due <u>next</u> Thurs.

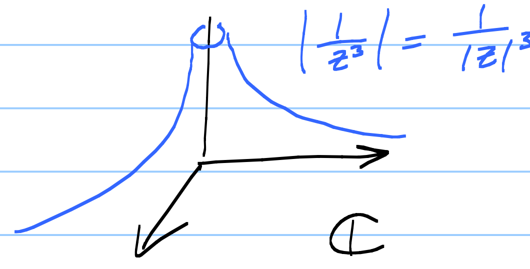




$e^{1/z}$ has a mind blowing singularity at $z=0$! No matter how small $\varepsilon>0$ is taken, $e^{1/z}$ maps $D_\varepsilon(0)-\{0\}$ <u>onto</u> $\mathbb{C}-\{0\}$ infinite-to-one!

0 is an "essential singularity" of $e^{1/z}$ at $z=0$.

<u>Other types of behavior:</u> 1) $\frac{1}{z^3}$



$$\lim_{z\to 0}\left|\frac{1}{z^3}\right| = +\infty$$

$z=0$ is a "pole" of $\frac{1}{z^3}$.

2) $$\frac{\sin z}{z} = \frac{z-\frac{z^3}{3!}+\frac{z^5}{5!}}{z} = \frac{z\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots\right)}{z}$$

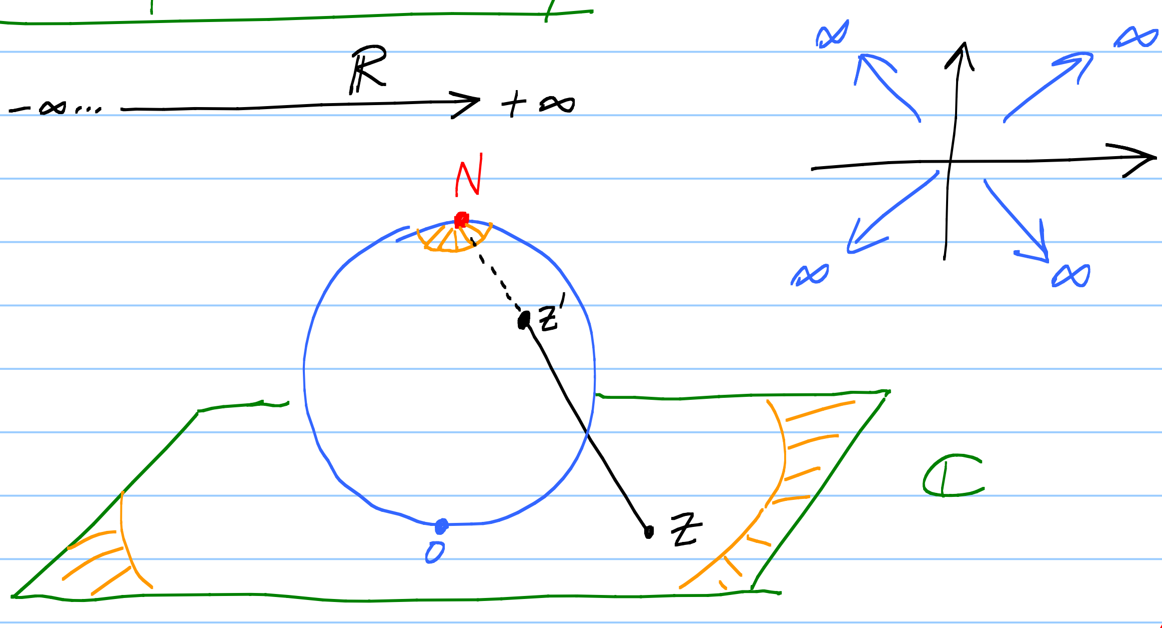
$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$ ← entire! $R=\infty$

$$f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$ is entire

0 is a "removable singularity" for $\frac{\sin z}{z}$.

Big fact: These are the only possible behaviors at an "isolated singularity" (meaning $\exists\ \varepsilon > 0$ such that fcn is analytic on $D_\varepsilon(z_0) - \{z_0\}$).

The "point at infinity": $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.



$N \sim \infty$

$$D_\varepsilon(\infty) = \{z : |z| > \tfrac{1}{\varepsilon}\} \cup \{\infty\}$$ ← "polar cap" on Riemann sphere

Def^n: $\lim_{z \to a} f(z) = \infty$ ← the point at infinity

means, given $M > 0$, there is a $\delta > 0$ such that

$|f(z)| > M$ when $|z-a| < \delta$, $z \neq a$. [Same as $\lim_{z \to a} |f(z)| = +\infty$.]

Think: Setting $f(a) = \infty$ makes f continuous at a as a map to $\hat{\mathbb{C}}$.

Goursat/Morera removable singularity theorem

Suppose f is continuous on a domain Ω and analytic on $\Omega - \{p\}$. Then p is removable. f is analytic on Ω.

Pf: Goursat $\Rightarrow \int_\Delta f\,dz = 0$ for Δ's $\subset \Omega$.

(f cont and $\int_{\Delta's} f\,dz = 0$). Morera's $\Rightarrow f$ analytic. ✓

Riemann removable singularity thm: Suppose f is analytic on $D_\rho(a) - \{a\}$ and bounded there (meaning $\exists\, M > 0$ such that $|f(z)| < M$ on $D_\rho(a) - \{a\}$.)

Then f can be given a value at a that makes it analytic on $D_\rho(a)$, i.e., a is removable.

Way false $\mathbb{R} \to \mathbb{R}$: $\sin(1/t)$

Pf: Define $$G(z) = \begin{cases} f(z)(z-a) & z \neq a \\ 0 & z = a \end{cases}$$

Note: $\lim_{z \to a} G(z) = 0$ because $|f| < M$ $\left[\begin{array}{l} \varepsilon > 0 \\ \delta = \frac{\varepsilon}{M} \checkmark \end{array}\right.$

G analytic on $D_\rho(a)-\{a\}$.

Goursat / Morera rem sing thm $\Longrightarrow$ G analytic on $D_\rho(a)$.

$$\text{So} \quad G(z) = \underbrace{G(a)}_{=0} + G'(a)(z-a) + \frac{G''(a)}{2!}(z-a)^2 + \cdots$$

$$= (z-a)\Big[\underbrace{G'(a) + \frac{G''(a)}{2!}(z-a)^2 + \cdots}_{g(z),\ \text{R. of C.} \geq \rho.}\Big]$$

$$f(z) = \begin{cases} \frac{G(z)}{z-a} = g(z), & z \neq a \\ g(a) = G'(a) \text{ should be value at } z=a. \checkmark \end{cases}$$

Alternate pf: Let $H(z) = \begin{cases} f(z)(z-a)^2 & z \neq a \\ 0 & z = a \end{cases}$

Show that $DQ = \frac{H(z)-0}{z-a} \longrightarrow 0$ as $z \to a$.

So H is analytic on $D_\rho(a)$. Have $H(a) = 0$
$H'(a) = 0$.

$$\text{So} \quad H(z) = \frac{H''(a)}{2!}(z-a)^2 + \cdots \quad \text{etc.}$$

Def^n: Suppose a is an isolated sing of f analytic on $D_\rho(a)-\{a\}$.

1) a is "removable" if f can be given a value at a that makes it analytic on $D_\rho(a)$.

2) a is a "pole" if $\lim_{z \to a} f(z) = \infty$

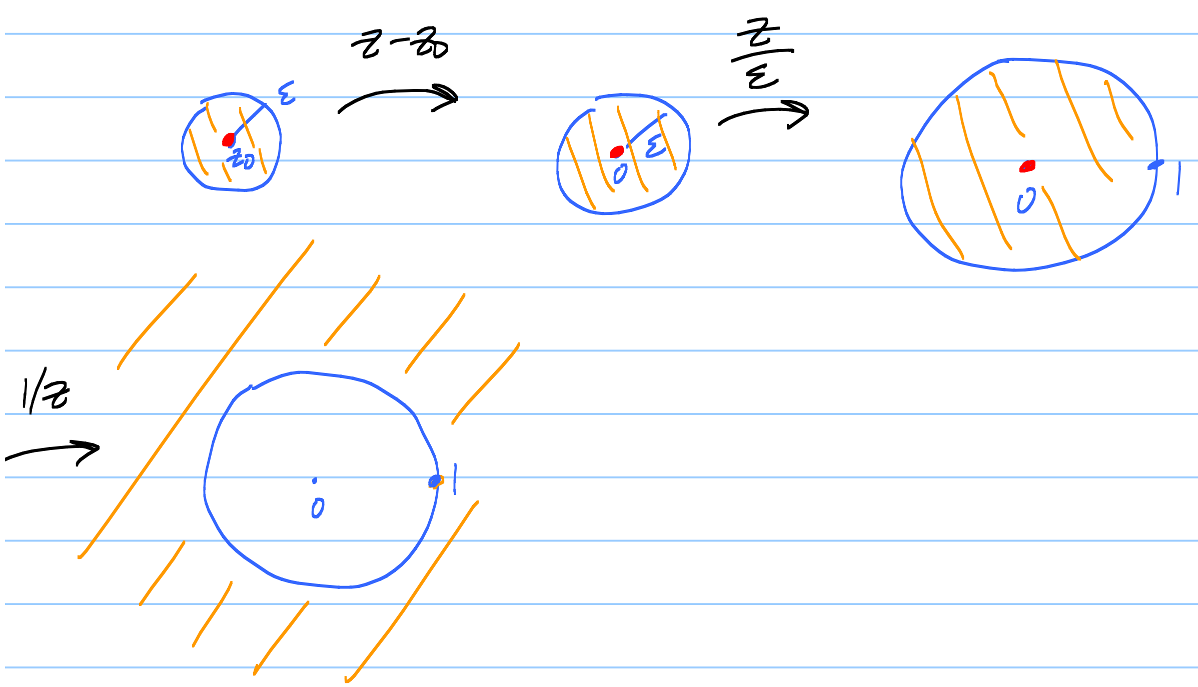
3) a is an "essential" if, for every ε with $0<\varepsilon<\rho$, $f(D_\varepsilon(a)-\{a\})$ is <u>dense</u> in $\mathbb{C}$,

meaning, every pt in $\mathbb{C}$ is a limit pt

$$\{w : w=f(z) \text{ for some } z\in D_\varepsilon(a)-\{a\}\}$$

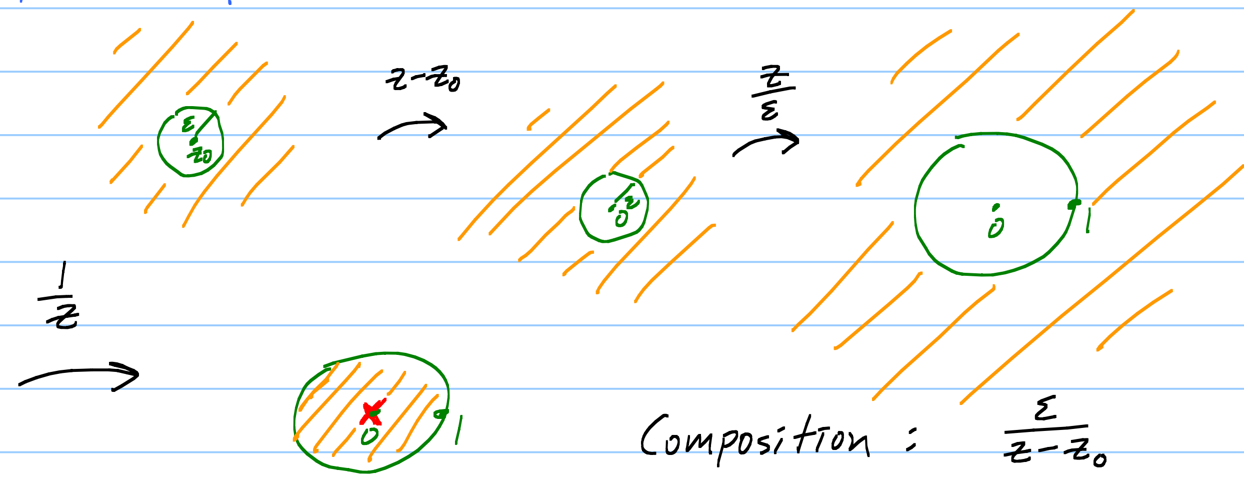
<u>Thm:</u> These are the only 3 possible behaviors.

<u>Important map</u>: Outside of a little disc to $D_1(0)$



Lecture 14 Isolated singularities

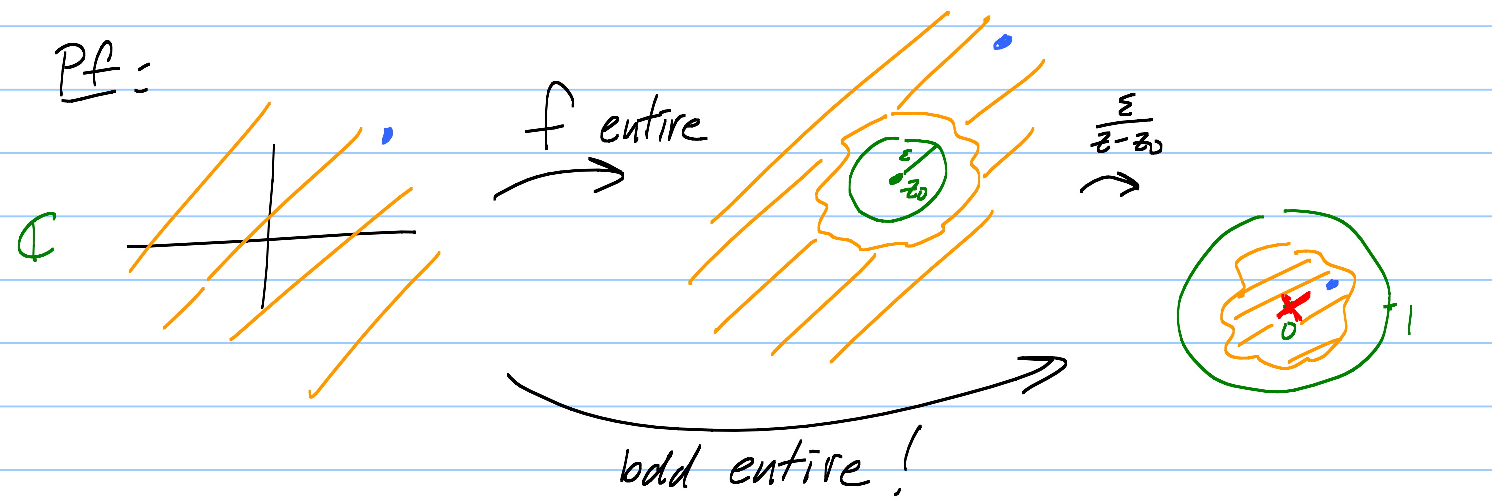
Important map: Outside of small disc to inside of unit disc.



Composition : $\frac{\varepsilon}{z - z_0}$

Casorati-Weierstraß thm If an entire fcn misses an open set, it must be constant.

Pf:



Liouville's $\Rightarrow$ const. $\frac{\varepsilon}{f(z) - z_0} = c \neq 0$

$\Rightarrow f$ const.

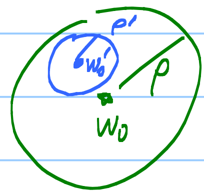
Thm Only 3 kinds of isolated singularities.

Pf: Suppose f is analytic on $D_\rho(z_0) - \{z_0\}$.

Suppose z_0 is not essential. Then $\exists$ $\varepsilon > 0$ with $0 < \varepsilon < \rho$ such that $\mathcal{A} = f(D_\varepsilon(z_0) - \{z_0\})$ is not dense in $\mathbb{C}$. Then $\exists\, w_0 \in \mathbb{C}$ that w_0 is <u>not</u> a limit pt of $\mathcal{A}$.

<u>Case $w_0 \notin \mathcal{A}$</u>: Then $\exists\, \rho > 0$ such that $D_\rho(w_0) \cap \mathcal{A} = \emptyset$.

<u>Case $w_0 \in \mathcal{A}$</u>: Then $\exists\, \rho > 0$ such that $D_\rho(w_0) \cap \mathcal{A} = \{w_0\}$



Aha! Replace w_0 by w_0', ρ by ρ'.

Get $\rho > 0$, $w_0 \in \mathbb{C}$ such that $D_\rho(w_0) \cap \mathcal{A} = \emptyset$.

bdd near z_0.

R.R.S.T. $\Longrightarrow$ z_0 is <u>removable</u> for $h(z) = \dfrac{\rho}{f(z) - w_0}$

Solve for f: $\quad f(z) = w_0 + \dfrac{\rho}{h(z)}$

<u>Case</u> $h(z_0) \neq 0$. See f has <u>removable</u> sing at z_0.

<u>Case</u> $h(z_0) = 0$. z_0 is an <u>isolated</u> zero.

See that $\lim_{z \to z_0} f(z) = \infty$. So z_0 is a <u>pole</u>.

<u>Zeroes & poles</u>:

Zeroes: $f(z) = a_N(z-a)^N + \cdots = (z-a)^N\left[a_N + a_{N+1}(z-a) + \cdots\right]$

$$= (z-a)^N F(z) \quad \text{where } F(a) = a_N = \frac{f^{(N)}(a)}{N!} \neq 0$$

f has a zero of order N at a.

Poles: $f(z) = (z-a)^{-N} F(z)$ where F analytic near a and $F(a) \neq 0$

f has a pole of order N at a.

__Why:__ If a is a pole of f, $\lim_{z\to a} f(z) = \infty$.

So $(M=1)$, $\exists\ \delta > 0$ such that $|f(z)| > 1$

if $z \in D_\delta(a) - \{a\}$. So f nonvanishing there.

$\frac{1}{f}$ has an isolated sing at a and $\left|\frac{1}{f}\right| < 1$

there. R.R.S.T. $\Rightarrow$ a is removable, and

$\lim_{z\to a} \frac{1}{f(z)} = 0 \Rightarrow a$ is an isolated __zero__

of some finite order N. $\frac{1}{f(z)} = (z-a)^N G(z)$, $G(a) \neq 0$,

So $f(z) = (z-a)^{-N} F(z)$ where $F = 1/G$. ✓

__Def^n:__ N = order of pole at a.

Power series routine:

$$f(z) = \frac{1}{(z-a)^N}\Big[\underbrace{A_0 + A_1(z-a) + A_2(z-a)^2 + \cdots}_{F(z) \qquad F(a) = A_0 \neq 0}\Big]$$

$$= \underbrace{\frac{A_0}{(z-a)^N} + \frac{A_1}{(z-a)^{N-1}} + \cdots + \frac{A_{N-1}}{z-a}}_{\substack{R_N(z) \\ \text{Principal part of } f \text{ at } a}} + \underbrace{\begin{pmatrix} \text{Convergent} \\ \text{power series} \end{pmatrix}}_{\substack{\text{analytic} \\ \text{near } a}}$$

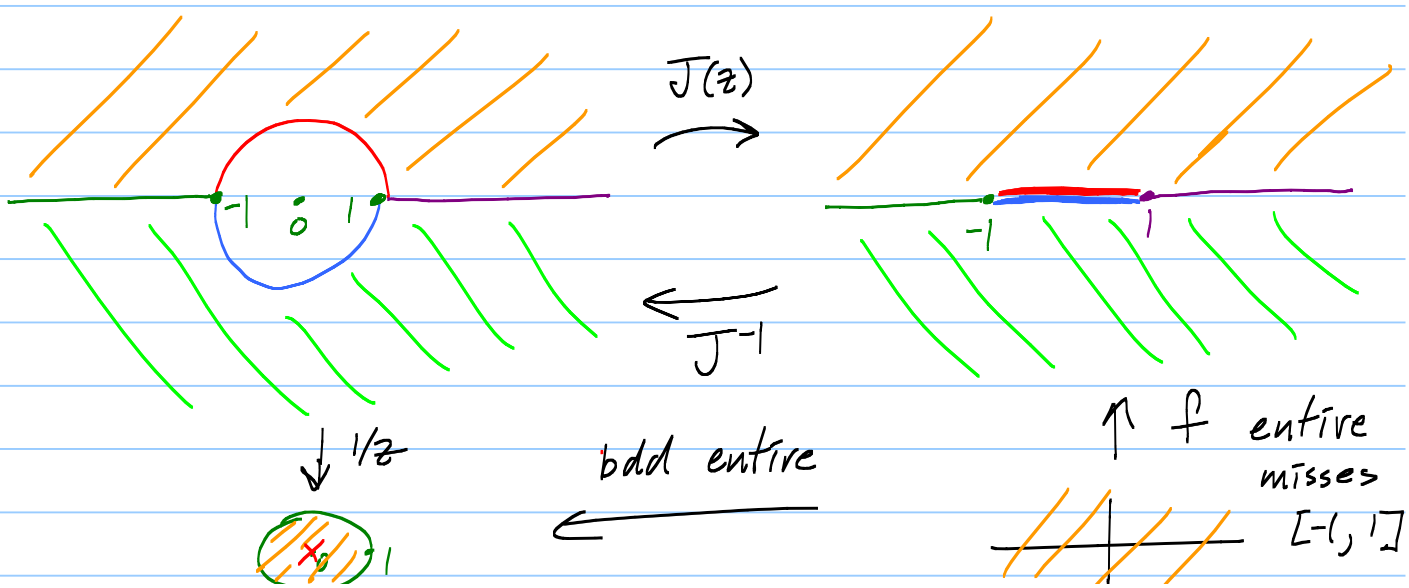
residue of f at a

$$f(z) = R_N(z) + h(z)$$ ← analytic

Fact: Principal part is uniquely determined by the condition that $f(z) - R_N(z)$ has a removable singularity at a.

Why: If R_1, R_2 both do this, then $R_1 - R_2$ has a removable sing at a. Show $N_1 = N_2$ and all coeff must be same.

Famous map Jukovsky map $J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$



Thm: An entire fcn that misses a line segment must be constant.

Picard's little thm Suppose f is entire. Then either, f is a polynomial, or f assumes every value in $\mathbb{C}$ infinitely many times with one possible exception (like e^z does).

Remark: Poly const. $\iff$ has a removable sing at ∞.

$P(z)$ poly deg $N>0$ $\iff$ pole at ∞.

Not poly $\iff$ ∞ is a essential sing.

Def^n: f analytic on $\{z: |z|>R\}$.

$$\Big(\text{Type of sing of } f \text{ at } \infty\Big) \underset{\text{Def}^n}{=} \Big(\text{Type of sing of } f\big(\tfrac{1}{z}\big) \text{ at } z=0\Big).$$

EX: e^z misses 0. $\cos z$ doesn't miss any value.

Picard's big thm: If f analytic on $D_\rho(z_0)-\{z_0\}$ has an ess sing at z_0, $0<\varepsilon<\rho$, then f maps $D_\varepsilon(z_0)-\{z_0\}$ to $\mathbb{C}$ ∞-to-one with one possible exception (like $e^{1/z}$ at $z=0$.)

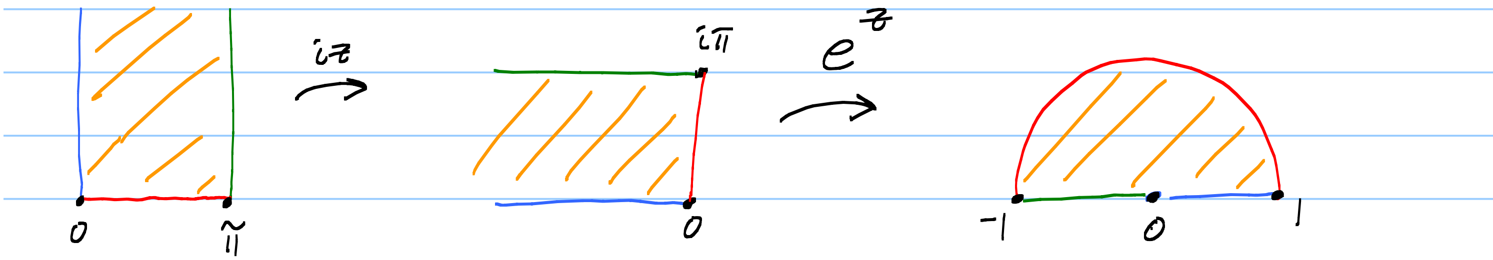
Cor of little Picard: An entire fcn that misses two complex values must be constant.

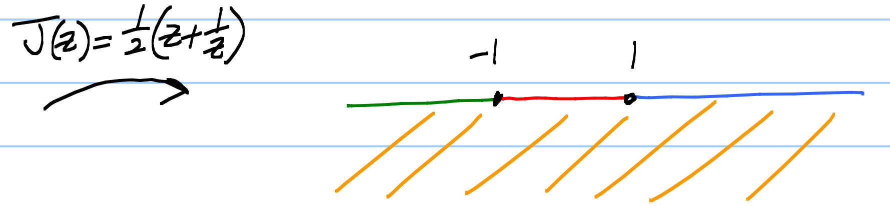
Pf: Polys don't miss anything. Picard ✓

Lecture 16 Partial fractions & Schwarz lemma

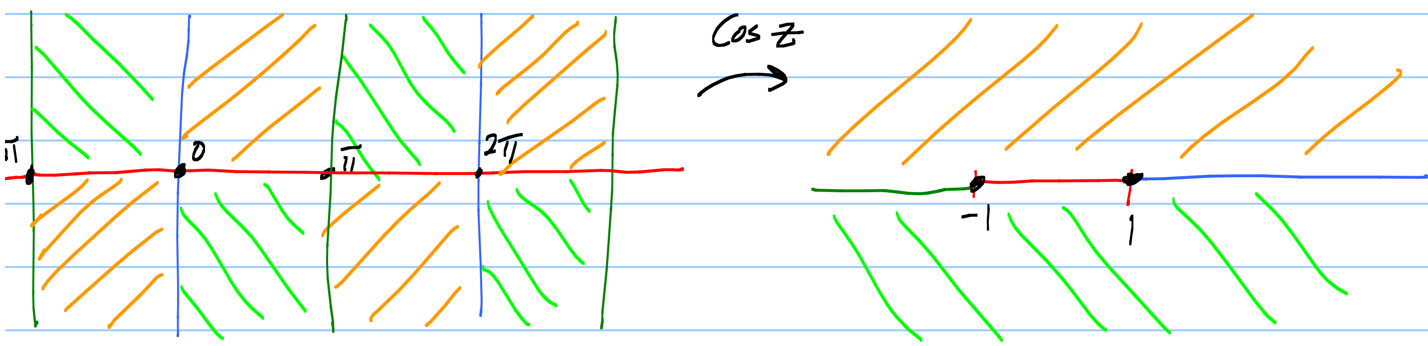
HWK 4 due Thurs, G>

Jukovsky map & $\cos z$





Composition $= \cos z$



$\cos(-z) = \cos z$

$\cos(z+\pi) = -\cos z$

Partial fractions: $\dfrac{p(z)}{q(z)}$ ← factor out common factors

If $\deg p \geq \deg q$, first do long division:

$$\frac{p}{q} = (\text{Poly}) + \frac{r}{q} \quad \text{where} \quad \deg r < \deg q$$

Partial fractions decomp thm: If P, Q polys with no common factors and $N_P = \deg P < \deg Q = N_Q$, then $\frac{P}{Q}$ has a part. frac. decomp as follows:

$$Q(z) = A\,(z-r_1)^{m_1}(z-r_2)^{m_2}\cdots(z-r_n)^{m_n}$$

where $\sum_{1}^{n} m_k = N_Q$.

Near r_k:
$$\frac{P(z)}{Q(z)} = \frac{1}{(z-r_k)^{m_k}}\Bigg[\underbrace{\frac{P(z)}{q_k(z)}}_{A_0 + A_1(z-r_k) + \cdots}\Bigg] \leftarrow q_k(r_k) \neq 0$$

analytic near r_k

$$= \underbrace{\frac{A_0}{(z-r_k)^{m_k}} + \cdots + \frac{A_{m_k-1}}{z-r_k}}_{} + \left(\text{analytic near } r_k\right)$$

R_k = Principal part at r_k

Claim: $\dfrac{P}{Q} = \sum_{k=1}^{n} R_k \leftarrow$ par. frac. decomp.

Let $f = \dfrac{P}{Q} - \sum_{1}^{n} R_k$

Claim: f is bndd entire.

Near r_j:
$$f = \underbrace{\left(\frac{P}{Q} - R_j\right)}_{\text{removable sing at } r_j} - \underbrace{\sum_{k\neq j} R_k}_{\text{analytic near } r_j}$$

Aha! Can give f values at r_j's to make it entire.

Claim: $f \to 0$ as $z \to \infty$.

Note: $\lim_{z\to\infty} R_j(z) = 0$. ✓

Basic poly est. : $a|z|^{N_P} \le \underline{|P(z)| \le A|z|^{N_P}}, \ |z| > R_P$

$\underline{b|z|^{N_Q} \le |Q(z)|} \le B|z|^{N_Q}, \ |z| > R_Q$

$$\left|\frac{P(z)}{Q(z)}\right| \le \frac{A|z|^{N_P}}{b|z|^{N_Q}} = \frac{A}{b}\frac{1}{|z|^{N_Q - N_P}} \text{ if } |z| > \text{Max}(R_P, R_Q)$$

$\longrightarrow 0$ as $z \to \infty$.

[$|f| < 1$ outside $D_R(0)$. $|f|$ cont on $\overline{D_R(0)}$.

See $|f|$ bdd on $\mathbb{C}$.]

Liouville's $\Rightarrow f \equiv c$, const. c must $= 0$. ✓

Freshman calc: $\dfrac{x^3 + 7x + 13}{(x-2)^2(x^2+x+1)} = \dfrac{A}{(x-2)^2} + \dfrac{B}{x-2} + \underbrace{\dfrac{Cx+D}{x^2+x+1}}_{\dfrac{E}{x-r} + \dfrac{\bar{E}}{x-\bar{r}}}$

__Schwarz lemma__ (Chap 8, sec 2)

Suppose $f: D_1(0) \to D_1(0)$ is analytic

($|f(z)| < 1$ when $|z| < 1$), __and__ $\boxed{f(0) = 0}$.

Then A) $|f(z)| \le |z|$

B) $|f'(0)| \le 1$

__and__ if equality holds in (A) for some $z \neq 0$, __or__

if equality holds in (B), then

$f(z) = \lambda z$ for some unimodular const $\lambda = e^{i\theta}$.

Pf: Let $F(z) = \begin{cases} \dfrac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$ $\quad DQ = \dfrac{f(z) - \overbrace{f(0)}^{0}}{z - 0}$

F analytic on $D_1(0) - \{0\}$ and cont. on $D_1(0)$.

So 0 is a removable sing.

Let $z_0 \in D_1(0)$. Pick r with $|z_0| < r < 1$.

Max Princ:

$$|F(z_0)| \leq \max_{|z|=r} |F| = \max_{|z|=r} \frac{|f(z)|}{r} < \frac{1}{r}$$

Aha! Can let $r \nearrow 1$. Then $\frac{1}{r} \searrow 1$.

Conclude that $|F(z_0)| \leq 1$. z_0 arbitrary!

$$|F(z)| = \begin{cases} \dfrac{|f(z)|}{|z|} & z \neq 0 \quad (A) \checkmark \qquad f(0) = 0 \checkmark \\ |f'(0)| & z = 0 \quad (B) \checkmark \end{cases}$$

Part 2 If a max of $|F|$ $(= 1)$ at a point in $D_1(0)$, then $F \equiv c$, const. $|c| = 1$.

$\dfrac{f(z)}{z} = c = e^{i\theta}$, $z \neq 0$. $\quad f(z) = e^{i\theta} z$, $z \neq 0$.

true at $z = 0$ too. $\checkmark$

Log on a convex open set

Given a nonvanishing analytic fcn F on a convex open Ω, $\exists$ analytic G on Ω such that $e^G = F$ on Ω. So $H = e^{G/N}$ is an analytic N-th root of F.

Idea: If $e^G = F$

$$\frac{d}{dz}\left(e^G\right) = F'$$

$$G' \underbrace{e^G}_{F} = F'$$

$$G' = \frac{F'}{F}$$

Idea: Define

$$G(z) = \int_{L_a^z} \frac{F'(w)}{F(w)}\, dw$$

Then $G' = \frac{F'}{F}$. Is $e^G = F$? Almost!

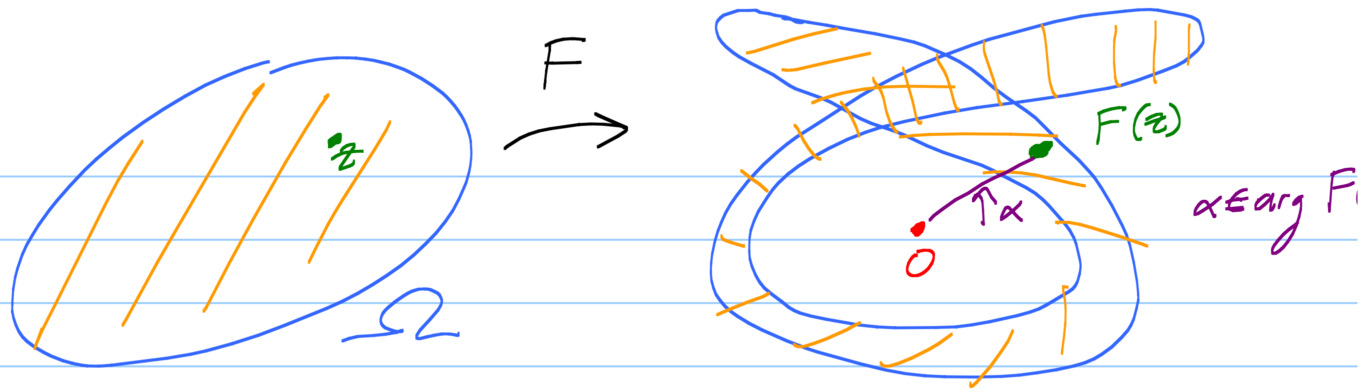
Trick: $$\frac{d}{dz}\left(\frac{F}{e^G}\right) = \frac{F' e^G - \overbrace{G'}^{F'/F} e^G F}{\left(e^G\right)^2} \equiv 0$$

So $\frac{F}{e^G} \equiv c$, const.

$$F = c\, e^G = e^{\operatorname{Log} c}\, e^G = e^{\underbrace{\operatorname{Log} c + G}}$$

that's my G!

Pic:



$$e^G = F \qquad \operatorname{Re} G = \operatorname{Ln}|F|$$

$$\operatorname{Im} G \in \arg F$$

$$G(z) = \operatorname{Ln}|F| + i\theta(z)$$ ← continuous choice of $\arg F$

Two G's on a domain:

$$G_1 \equiv G_2 + i2\pi n \quad n \in \mathbb{Z} \quad \text{on } \Omega,$$

Two N-th roots on a domain:

$$H_1 \equiv \lambda H_2 \qquad \lambda \ N\text{-th root of unity}$$

Lecture 16 The Open Mapping Theorem HWK 4 due tomorrow in Gi

OMT: Non-constant analytic fcns map open sets to open sets.

Way false $\mathbb{R}\to\mathbb{R}$: $h(t)=1-t^2$. $h(\underbrace{(-1,1)}_{\text{open}}) = (0,1]$ ← not open

Assume f analytic on a domain Ω.

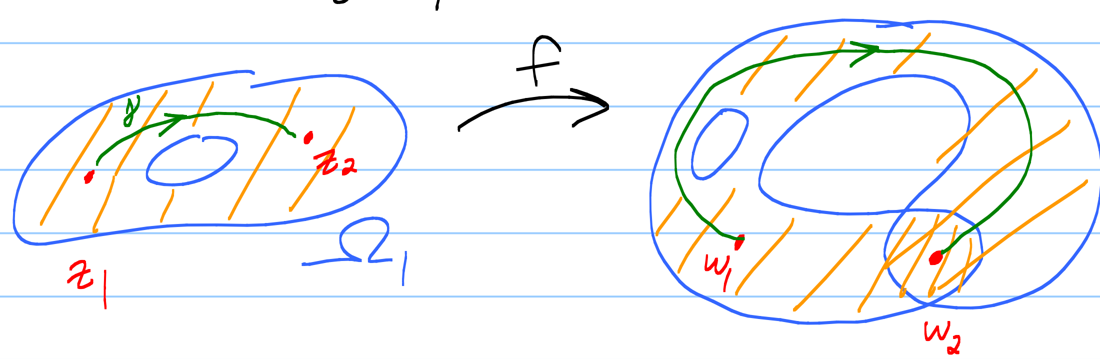
Notation: $f(U) = \{f(z) : z\in U\}$

$f^{-1}(U) = \{z : f(z)\in U\}$

Fact: f continuous $\Longleftrightarrow$ $f^{-1}(U)$ is open when U is open.

Cor of OMT: $f:\Omega_1 \to \Omega_2$ analytic on a domain Ω_1, $\Omega_2=f(\Omega_1)$. Then Ω_2 is also a domain.

Pf: OMT $\Rightarrow$ Ω_2 open. ✓



$f(\gamma)$ connects w_1 to w_2. Ω_2 connected ✓

Cor of OMT: $f:\Omega_1 \xrightarrow[\text{onto}]{1-1} \Omega_2$ analytic.

1) Then $F=f^{-1}$ is continuous on Ω_2.

Pf: $F^{-1}(U) = f(U)$ open when U open ✓

2) (1) $\Rightarrow$ F is analytic!

Pf: $w = f(z)$ $\quad F(w) = z$

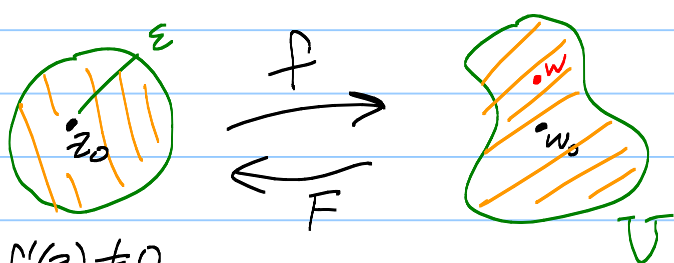
$w_0 = f(z_0)$ $\quad F(w_0) = z_0$

F cont. $\Longrightarrow$ $\underbrace{F(w)}_{z} \to \underbrace{F(w_0)}_{z_0}$ as $w \to w_0$

$$DQ = \frac{F(w) - F(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - f(z_0)} \longrightarrow \frac{1}{f'(z_0)} \quad \text{as } w \to w_0$$

provided that $f'(z_0) \neq 0$. $\quad$ [$F'(w_0)$ exists when $f'(z_0) \neq$

Hmmm. f not const. So $f' \not\equiv 0$. So zeroes of f' are isolated. Say $f'(z_0) = 0$.



$f'(z) \neq 0$ if $z \in \overline{D_\varepsilon(z_0)} - \{z_0\}$

F' exists and is nonvanishing on $U - \{w_0\}$. F is cont on U, w_0 is a removable sing of F!

Aha! $F(f(z)) = z$

$F'(f(z))f'(z) = 1$ $\leftarrow$ neither f nor F can have a vanishing derivative.

Fact: Locally 1-1 analytic fcns have nonvanishing derivative

Way false $\mathbb{R} \to \mathbb{R}$: $\quad h(t) = t^3$: $(-1,1) \xrightarrow[\text{onto}]{1\text{-}1} (-1,1)$

C^∞ smooth

but $h^{-1}(s) = s^{1/3}$ is not diff'ble at $s=0$!

Argument principle for a disc. Suppose f is analytic on $D_R(z_0)$ and $0 < r < R$ and suppose has no zeroes on $C_r(z_0) = \{z : |z-z_0| = r\}$. Then

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'(z)}{f(z)}\,dz = \left(\begin{array}{c}\#\text{ zeroes of } f \\ \text{inside } C_r(z_0) \\ \text{counted with multiplicity}\end{array}\right)$$

Lemma: Suppose f has a zero of order N at a. Then the principal part of $\frac{f'}{f}$ at a is $\frac{N}{z-a}$.

Pf: Know $f(z) = (z-a)^N F(z)$ where F analytic, $F(a) \neq 0$.

$$f'(z) = N(z-a)^{N-1}F(z) + (z-a)^N F'(z)$$

So
$$\frac{f'(z)}{f(z)} = \underbrace{\frac{N}{z-a}}_{\text{P.P. }\checkmark} + \underbrace{\frac{F'(z)}{F(z)}}_{\text{analytic near } a}$$

Pf of Arg princ: Let $a_1, \dots, a_N$ be the zeroes of f in $\overline{D_r(z_0)}$. [Only finitely many zeroes: If there were ∞ many zeroes in $\overline{D_r(z_0)}$ ← compact. They'd have a limit pt in $\overline{D_r(z_0)}$. f not zero on $C_r(z_0)$. So lim pt in $D_r(z_0)$. $\Longrightarrow f \equiv 0$. ↯.]

Let m_j = order of zero at a_j.

$$\frac{f'}{f} - \sum_{1}^{N} \frac{m_j}{z-a_j} = G(z) \text{ has removable sing at } a_j\text{'s}$$

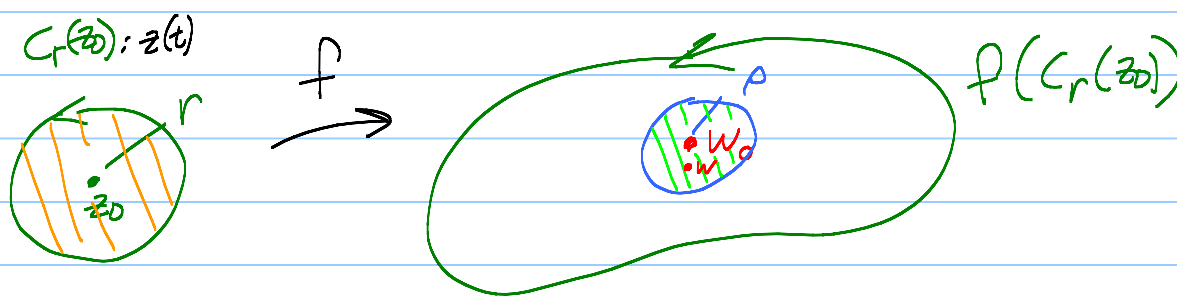
Give G the values at a_j to make it analytic.

$$\int_{C_r(z_0)} \frac{f'}{f}\,dz = \sum_1^N m_j \underbrace{\int_{C_r(z_0)} \frac{1}{z-a_j}\,dz}_{2\pi i} + \underbrace{\int_{C_r(z_0)} G\,dz}_{=0}$$

by Cauchy Thm on convex

$\sum_1^N m_j$ = total # zeroes, counted with mult. ✓

Pf of OMT: f analytic on a domain Ω and non-constant. Pick $w_0 \in f(\Omega)$. Then $\exists\, z_0 \in \Omega$ with $f(z_0) = w_0$. f not constant, so $f(z) - w_0$ has an isolated zero at z_0. So $\exists\, r > 0$ such that $\overline{D_r(z_0)} \subset \Omega$ and z_0 is the only zero of $f(z) - w_0$ in $\overline{D_r(z_0)}$.



Since $|f(z(t)) - w_0|$ is cont. and non-vanishing, it has a min $= m > 0$. Pick ρ with $0 < \rho < m$.

Claim: Every $w \in D_\rho(w_0)$ gets hit by f on $D_r(z_0)$.

$$\left(\begin{array}{c}\#\text{ zeroes of } f(z)-w \\ \text{inside } C_r(z_0)\end{array}\right) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f'(z)}{f(z)-w}\,dz = H(w)$$

Aha! Function of w on right is analytic!

Fact: A continuous integer valued fcn on a domain must be constant!

Pf that H is analytic:

$$H(w) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(z(t))}{f(z(t)) - w}\, z'(t)\, dt$$

$f(C_r(z_0))$: $\zeta(t) = f(z(t))$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\zeta'(t)}{\zeta(t) - w}\, dt$$

$$= \frac{1}{2\pi i}\int_{f(C_r(z_0))} \frac{1}{\zeta - w}\, d\zeta$$

HWK 2 prob 1.

$H(w_0) = \#$ zeroes of $f(z) - w_0$ in $C_r(z_0)$ ≥ 1.

So $H(w) \geq 1$ too for $w \in D_\rho(w_0)$ [= same #!]

So w gets hit on $D_r(z_0)$. Pf of OMT is done.

Lecture 17 Inverse fcn thm, Local mapping thm

HWK 5 due Thurs, March 3

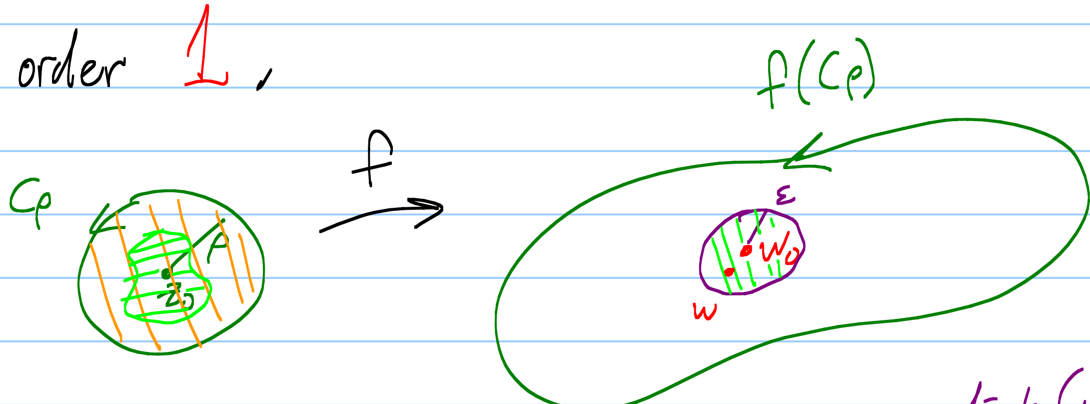
Super inverse fcn thm. Suppose f analytic on a domain Ω_1 and one-to-one on Ω_1. Then $\Omega_2 = f(\Omega_1)$ is a domain and $f^{-1}: \Omega_2 \to \Omega_1$ is analytic. Furthermore, f' is nonvanishing on Ω_1 (and $(f^{-1})'$ is nonvanishing too). $(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}$

"Locally 1-1 analytic fcns have nonvanishing derivatives."

Inverse fcn thm: If f is analytic on $D_\delta(z_0)$ and $f'(z_0) \neq 0$, $\exists\ \varepsilon > 0$ and a domain $U = f^{-1}(D_\varepsilon(w_0))$ $[w_0 = f(z_0)]$ such that $f: U \xrightarrow[\text{onto}]{1-1} D_\varepsilon(w_0)$ and f^{-1} is analytic on $D_\varepsilon(w_0)$. Chain rule: $f'(f^{-1}(w))(f^{-1})'(w) =$

Pf: $f(z_0) = w_0$, $f'(z_0) \neq 0$. So $f(z) - w_0$ is not constant. So zero of $f(z) - w_0$ at z_0 is isolated.

Pick ρ with $0 < \rho < \delta$ such that z_0 is only zero of $f(z) - w_0$ in $\overline{D_\rho(z_0)}$. $f'(z_0) \neq 0 \Rightarrow z_0$ is a zero of order 1.



$\varepsilon < \text{dist}(w_0, f(C_\rho))$

$$\begin{pmatrix}\#\text{ zeroes of}\\ f(z)-w\\ \text{inside } C_\rho\end{pmatrix} = \frac{1}{2\pi i}\int_{C_\rho}\frac{f'(z)}{f(z)-w}\,dz = H(w)$$

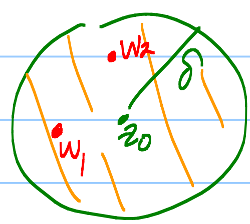
H is analytic on $D_\varepsilon(w_0)$ and integer valued there.

So $H(w)\equiv H(w_0) = 1$. Every $w\in D_\varepsilon(w_0)$ gets hit exactly once. Let $U=f^{-1}(D_\varepsilon(w_0))$.

U is open because f cont. OMT $\Rightarrow$ f^{-1} cont. $\Rightarrow$ U is a domain. Use Super IFT to see f^{-1} analytic.

Another way to see f is locally 1-1 if $f'(z_0)\neq 0$.

$f'(z)=f'(z_0)+E(z)$ where $E(z)\to 0$ as $z\to z_0$,
so $\exists\ \delta>0$ such that $|E(z)|<\varepsilon<\frac{|f'(z_0)|}{2}$ on $\overline{D_\delta(z_0)}$.



$$f(w_2)-f(w_1) = \int_{L_{w_1}^{w_2}} f'(z)\,dz$$

$$= \int_{L_{w_1}^{w_2}} f'(z_0)+E(z)\,dz$$

$$= f'(z_0)(w_2-w_1) + \underbrace{\int_{L_{w_1}^{w_2}} E(z)\,dz}_{\varepsilon}$$

$$|f(w_2)-f(w_1)| \geq \Big|\, \underbrace{|f'(z_0)(w_2-w_1)|} - |\varepsilon| \,\Big|$$

$|f'(z_0)|\,|w_2-w_1|$ $\qquad |\mathcal{E}| \le \left(\underset{L_{w_1}^{w_2}}{\mathrm{Max}}\,|E|\right)|w_2-w_1|$

$< \varepsilon \cdot |w_2-w_1|$

$$\ge |f'(z_0)|\,|w_2-w_1| - \frac{|f'(z_0)|}{2}|w_2-w_1| \qquad < \frac{|f'(z_0)|}{2}|w_2-w_1|$$

$$\ge \frac{|f'(z_0)|}{2}|w_2-w_1| \leftarrow \text{not zero if } w_2 \neq w_1.$$

So f is 1-1. Use Super IFT to see f^{-1} is analytic.

Local mapping theorem. $f(z_0)=w_0$, f analytic, not const.

Then zero of $f(z)-w_0$ at z_0 is isolated. Get $D_\rho(z_0)$

is only zero on $\overline{D_\rho(z_0)}$.

Def^n: Say $f(z_0)=w_0$ with mult N if order of zero at $f(z)-w_0$ is N.

$$f(z)-w_0 = (z-z_0)^N F(z), \quad F \text{ analytic}, \ F(z_0)\neq 0.$$

F nonvanishing on convex $D_\rho(z_0)$. So $\exists$ analytic

N-th root of F: $\quad g(z)^N = F(z)$ on $D_\rho(z_0)$.

$$f(z)-w_0 = \Big[\underbrace{(z-z_0)\,g(z)}_{G(z)}\Big]^N$$

Notice that $G(z_0)=0$.

Claim: $G'(z_0)\neq 0$

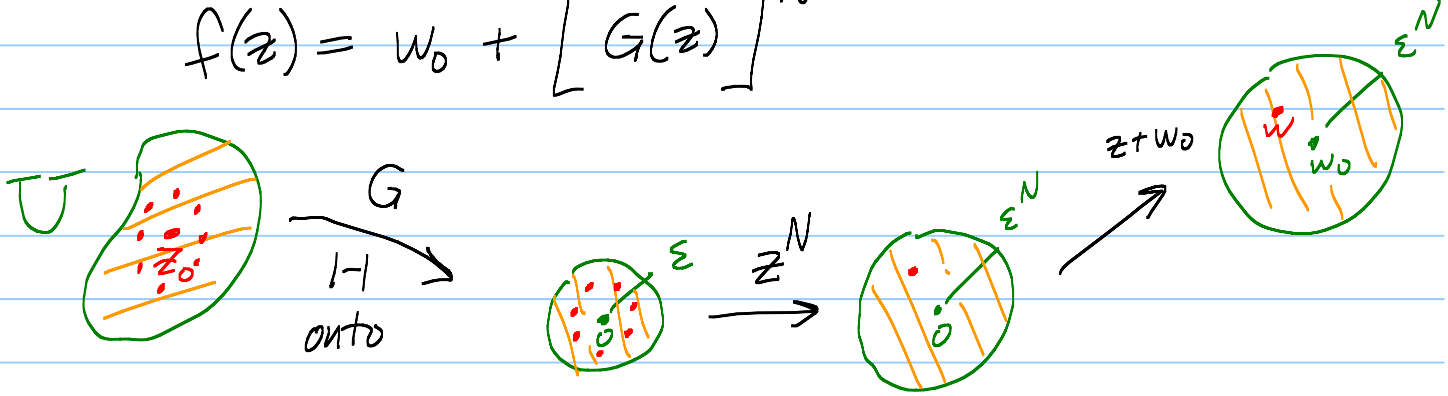
$$G'(z) = 1\cdot g(z) + (z-z_0)\,g'(z)$$

$G'(z_0) = g(z_0) + 0 \neq 0$ (because $F(z_0) \neq 0$)

Get $D_\varepsilon(0)$ and U from IFT such

$G : U \xrightarrow[\text{onto}]{1\text{-}1} D_\varepsilon(0)$, G' nonvanishing, G^{-1} analytic.

$$f(z) = w_0 + \left[G(z) \right]^N$$



$f : U - \{z_0\} \longrightarrow D_{\varepsilon^N}(w_0) - \{w_0\}$ is N-to-one

$f^{-1}(w) = \{N$ distinct pts in U of mult one$\}$

if $w \neq w_0$. $\quad f^{-1}(w_0) = \{z_0\}$.

f is an "N-sheeted covering map" of $U - \{z_0\}$ onto

$D_{\varepsilon^N}(w_0)$. $\quad f : U \to D_{\varepsilon^N}(w_0)$ is a "branched

covering map"

Notice: f is locally 1-1 $\iff N = 1$.

(Local mapping thm $\Rightarrow$ Super IFT and OMT.)

Cor of LMT: Zeroes of nonconstant harmonic

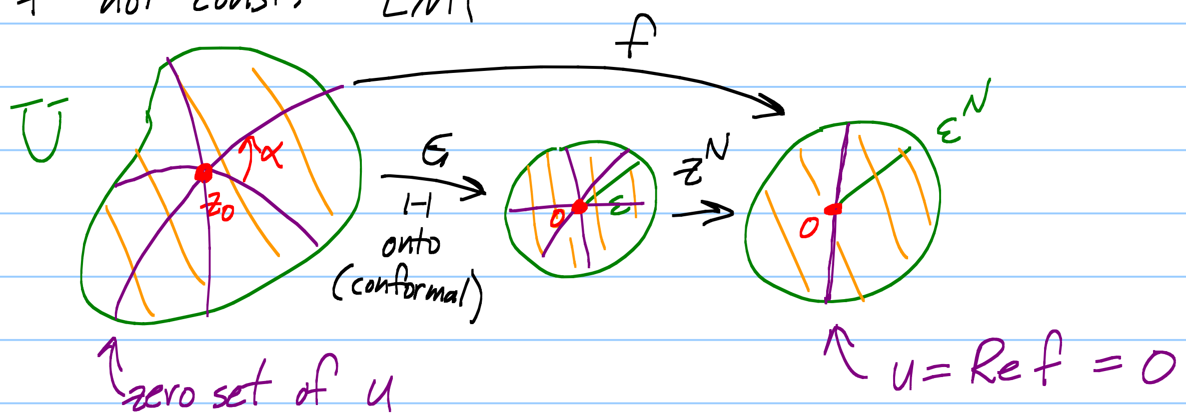
cannot be isolated. Given locally as curves that

cross at angles $2\pi/N$.

Pf: $w_0 = 0$ in LMT. $f = u + iv$

harm harm conj

f not const. LMT

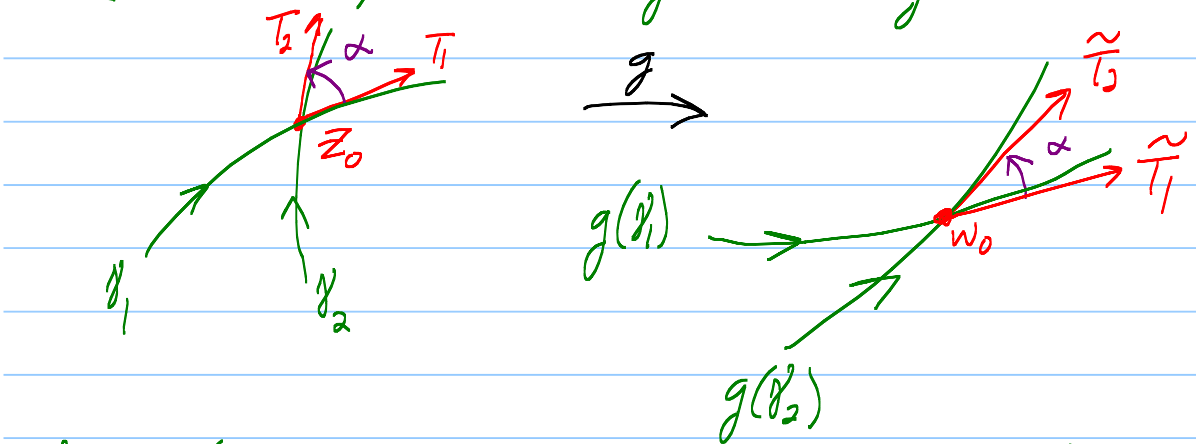


Conformality $\Rightarrow$ angles $= \frac{2\pi}{N}$.

Lecture 18 Conformal mapping, Baby Residue theorem on toy domains

HWK 5 due Thurs March 3 in GS

Big fact: Analytic g that is locally one-to-one preserves angles (is conformal). Note: g 1-1 $\Rightarrow$ g' nonvanishing.



$\gamma_j : z_j(t) \qquad z_j(0) = z_0 \qquad j = 1, 2. \qquad T_j = z_j'(0)$

$g(\gamma_j) : g(z_j(t)) \qquad w_0 = g(z_0) \qquad \tilde{T}_j = g'(z_j(t))\, z_j'(t) \Big|_{t=0}$

$$\boxed{\tilde{T}_j = \underbrace{g'(z_0)}_{re^{i\theta}} T_j}$$

stretch by factor of $r > 0$, rotate counterclockwise by θ. ✓

Converse is "almost" true:

$f(x+iy) = u(x,y) + i v(x,y)$. Suppose u, v C^1-smooth and $\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \neq 0$.

If f is conformal, then f is analytic.

Pf: $J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \underset{\text{analytic}}{=} \begin{bmatrix} R\cos\theta & -R\sin\theta \\ R\sin\theta & R\cos\theta \end{bmatrix}$

causes rotation by θ see C-R equs!

MA 525 $f(x+iy) = u + iv$

Def^n: Analytic : u, v $\mathcal{C}^1$-smooth satisfy C-R equs.

I.F.T.

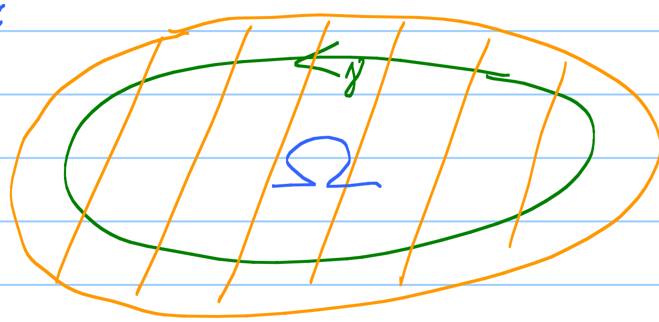
$$J = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

$$\det J = u_x^2 + v_x^2 = |f'|^2$$

$f'(z_0) \neq 0 \implies \det J \neq 0$, $\quad \mathbb{R}^2 \to \mathbb{R}^2$ I.F.T.

$$J_{f^{-1}} = \left(J_f\right)^{-1} = \underbrace{\frac{1}{A^2+B^2}\begin{bmatrix} A & B \\ -B & A \end{bmatrix}}_{\text{see C-R equs!}}$$

Cauchy Theorem:



f analytic

Ω = "inside of γ"

$$\int_\gamma f\,dz = \int_a^b \left[u(x(t),y(t)) + i\,v(x(t),y(t))\right]\cdot\left[\dot{x}(t) + i\,\dot{y}(t)\right] dt$$

$$= \int_a^b (u\dot{x} - v\dot{y})\,dt + i\int_a^b (u\dot{y} + v\dot{x})\,dt$$

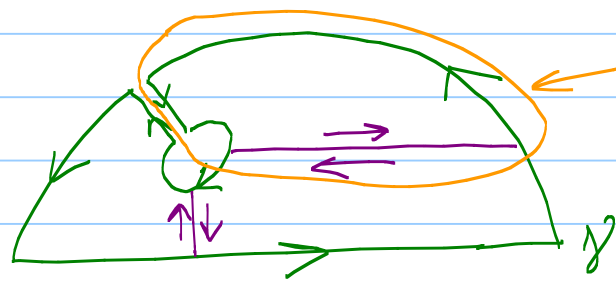
$$= \int_\gamma u\,dx - v\,dy + i\int_\gamma u\,dy + v\,dx$$

$$\underset{\uparrow}{=} \iint_\Omega \underbrace{-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}}_{\substack{=0 \\ \text{CR-eqn \#2}}} dA + i \iint_\Omega \underbrace{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}}_{\substack{=0 \\ \text{CR-eqn \#1}}} dA$$

Green's

$$\int_\gamma P\,dx + Q\,dy = \iint_\Omega \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

Back to MA 530 Stein: Toy regions



Cauchy on convex

$$\int f\,dz = 0$$

Add up. See Cauchy thm.

Thm by definition: Cauchy thm holds on Toy regions.

Def^n: If f has a pole at a, we know that

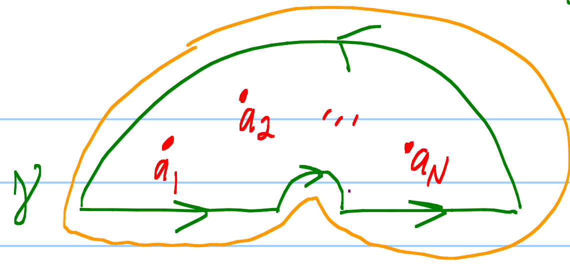
near a,

$$f(z) = \underbrace{\frac{A_N}{(z-a)^N} + \dots + \frac{A_1}{z-a}}_{\text{Princ part}} + (\text{analytic})$$

$\leftarrow \operatorname{Res}_a f$

A_1 = Residue of f at a.

Baby Residue thm on Toy regions



f is analytic "inside and on" γ except at finitely many poles inside γ.

$$\int_\gamma f\,dz = 2\pi i \sum_{j=1}^{N} \mathrm{Res}_{a_j} f = 2\pi i \left(\text{Sum of residues of } f \text{ inside } \gamma\right)$$

Pf: $f - \sum_{j=1}^{N} R_j$ has removable singularities at each a_j.

Princ part of f at a_j

Give it values at a_j's to make it analytic.

Cauchy thm
$$0 = \int_\gamma f - \sum_1^N R_j \, dz$$

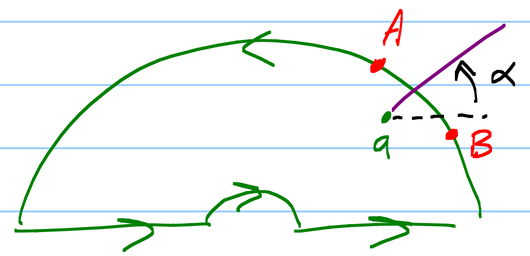
$$\int_\gamma f\,dz = \sum_1^N \int_\gamma R_j \, dz$$

Aha! $$\int_\gamma \frac{1}{(z-a)^n}\,dz = 0 \quad \text{if} \quad n \neq 1$$

$$\frac{d}{dz}\left[\frac{1}{-n+1}(z-a)^{-n+1}\right] = (z-a)^{-n}.$$

Key: $\displaystyle\int_\gamma \frac{1}{z-a}\,dz = \begin{cases} 2\pi i & a \text{ inside } \gamma \\ 0 & a \text{ outside} \end{cases}$

$= H(a)$ Cauchy thm



Pick a close to γ so ray cuts γ at an angle in exactly one place.

$$\int_\gamma \frac{1}{z-a}\,dz = \lim_{\substack{A \text{ slides down} \\ B \text{ slides up}}} \int_{\gamma_A^B} \underbrace{\frac{1}{z-a}}_{\frac{d}{dz}\log_\alpha(z-a)}\,dz$$

$$= \lim \log_\alpha(B-a) - \log_\alpha(A-a)$$

$$= \lim \Big[\big(\ln|B-a| + i(\alpha+2\pi-\varepsilon)\big) - \big(\ln|A-a| + i(\alpha+\varepsilon)\big)\Big]$$

$$= 2\pi i \quad \text{as } A\downarrow,\ B\uparrow$$

$H(w) = \displaystyle\int_\gamma \frac{1}{z-w}\,dz$ is analytic in w.

$$H'(w) = \int_\gamma \frac{1}{(z-w)^2}\,dz = \int_\gamma \frac{d}{dz}\left[\frac{-1}{z-w}\right]dz \equiv 0$$

$H(a) = 2\pi i$. So $H \equiv 2\pi i$ inside γ.

Only $\dfrac{A_1}{z-a_i}$ terms contribute: $A_1 2\pi i$ ✓

Fact: $\frac{d}{dz} \log_\alpha z = \frac{1}{z}$

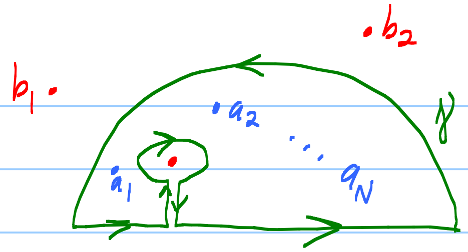
because $e^{\log_\alpha z} = z$

Chain rule: $\underbrace{e^{\log_\alpha z}}_{z} \frac{d}{dz}(\log_\alpha z) = 1$ ✓

Lecture 19 Fun with the Baby Residue Theorem

HWK 5 due next Thurs, March 3 in GS

Midterm Exam: Fri after Spring break



$$\int_\gamma f\,dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}_{a_j} f$$

f has finitely many poles in a Toy region enclosed by γ

Thm: Suppose P, Q polynomials and

1) $\deg Q \geq \deg P + 2$

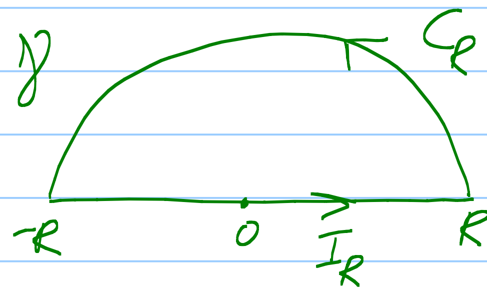
2) Q has no zeroes on $\mathbb{R}$

Then $$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{a \in Z_Q^+} \operatorname{Res}_a \frac{P}{Q}$$

where $Z_Q^+ = \{$ zeroes of Q in the UHP $\}$

upper half plane

Pf:



$I_R: z(t) = t, \quad -R \leq t \leq R$

$z'(t) = 1$

$$\int_{I_R} \frac{P(z)}{Q(z)}\,dz = \int_{-R}^{R} \frac{P(t)}{Q(t)} \cdot 1\,dt$$ want

Take R big enough that γ contains Z_Q^+.

Claim: $\int_{C_R} \frac{P}{Q}\,dz \to 0$ as $R \to \infty$.

Key: Basic poly estimate

$$a|z|^{\deg P} \le \underline{|P(z)| \le A|z|^{\deg P}}, \quad |z| > R_P$$

$$\underline{b|z|^{\deg Q} \le |Q(z)|} \le B|z|^{\deg Q}, \quad |z| > R_Q$$

So $\left|\frac{P(z)}{Q(z)}\right| \le \frac{A}{b}\frac{1}{|z|^{\underbrace{\deg Q - \deg P}_{\ge 2}}} \le \frac{A}{b}\frac{1}{|z|^2}$ $\quad |z| > \mathrm{Max}(R_P, R_Q, 1)$

$\left(\text{Note } \frac{1}{|z|^{2+n}} < \frac{1}{|z|^2} \text{ when } |z| > 1. \right)$

Now $\left|\int_{C_R} \frac{P}{Q}\,dz\right| \le \underbrace{\left(\max_{C_R} \left|\frac{P}{Q}\right|\right)}_{\frac{A}{b}\frac{1}{R^2}} \cdot \underbrace{\mathrm{Length}(C_R)}_{\pi R}$

$$\longrightarrow 0 \quad \text{as} \quad R \to \infty.$$

Res Thm $\quad \int_{I_R} + \int_{C_R} = 2\pi i \sum \mathrm{Res}$

$$\downarrow \qquad \downarrow$$

$$\int_{-\infty}^{\infty} + 0 = \checkmark$$

EX: $I = \int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$ $\qquad z^4+1 = 0$

Roots $\underbrace{e^{i\pi/4}, e^{i3\pi/4}}_{\text{inside}}, \underbrace{e^{i5\pi/4}, e^{i7\pi/4}}_{\text{outside}}$

$$z^4+1 = (z-r_1)(z-r_2)(z-r_3)(z-r_4)$$

Near r_1: $\displaystyle \frac{1}{z^4+1} = \frac{1}{z-r_1}\left[\underbrace{\frac{1}{(z-r_2)(z-r_3)(z-r_4)}}_{A_0 + A_1(z-r_1) + A_2(z-r_1)^2 + \cdots}\right]$

$$= \frac{A_0}{z-r_1} + \left(\begin{matrix}\text{conv}\\\text{power}\\\text{series}\end{matrix} \text{ near } r_1\right)$$

where $\displaystyle \mathrm{Res}_{r_1} = A_0 = \frac{1}{(r_1-r_2)(r_1-r_3)(r_1-r_4)} \leftarrow$ Ugh!

Fact: Suppose f, g analytic near a and g has a simple zero at a [meaning $g(a)=0$, $g'(a) \neq 0$].

Then $\displaystyle \boxed{\mathrm{Res}_a \frac{f}{g} = \frac{f(a)}{g'(a)}}$

Why: $g(z) = \underset{\substack{\uparrow\\ g(a)=0}}{a_0} + \underset{\substack{\uparrow\\ \frac{g'(a)}{1!} \neq 0}}{a_1}(z-a) + a_2(z-a)^2 + \cdots$

$$= (z-a)\Big[\underbrace{a_1 + a_2(z-a) + \cdots}_{G(z)}\Big] \qquad G(a) = a_1 = \frac{g'(a)}{1!}$$

$$\frac{f(z)}{g(z)} = \frac{1}{z-a}\left[\underbrace{\frac{f(z)}{G(z)}}_{A_0 + A_1(z-a) + \cdots}\right]$$

$$= \frac{A_0}{z-a} + \left(\begin{matrix}\text{conv pow}\\\text{series}\end{matrix}\right)$$

So $\quad \mathrm{Res}_a \dfrac{f}{g} = A_0 = \dfrac{f(a)}{G(a)} = \dfrac{f(a)}{g'(a)}$ ✓

Think about case: g has a double zero at a.

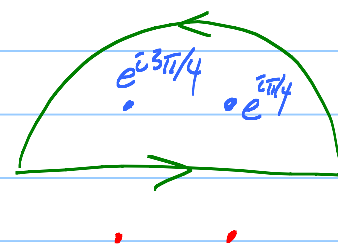
Now, $\quad I = 2\pi i \left(\mathrm{Res}_{e^{i\pi/4}} + \mathrm{Res}_{e^{i3\pi/4}} \right) \qquad \dfrac{1}{z^4+1}$ $\begin{array}{l} \leftarrow f(z) \\ \leftarrow g(z) \end{array}$

$g(e^{i\pi/4}) = 0$ ✓

$g'(e^{i\pi/4}) = 4(e^{i\pi/4})^3 \neq 0$ ✓

$$= 2\pi i \left[\frac{1}{4e^{i3\pi/4}} + \frac{1}{4e^{i9\pi/4}} \right]$$

$$= \frac{1}{2}\pi i \left[\underbrace{e^{-i3\pi/4}}_{-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}} + \underbrace{e^{-i9\pi/4}}_{\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}} \right] = \frac{\pi}{\sqrt{2}}$$



Fourier transform: $\qquad \hat{f}(s) = \displaystyle\int_{-\infty}^{\infty} e^{isx} f(x)\,dx$

EX: $\qquad I(s) = \displaystyle\int_{-\infty}^{\infty} \frac{e^{isx}}{x^2+1}\,dx$

Case $s>0$:

$$= 2\pi i \, \mathrm{Res}_i \frac{e^{isz}}{z^2+1}$$

$$= 2\pi i \, \frac{e^{is(i)}}{2(i)} = \pi e^{-s}$$

because

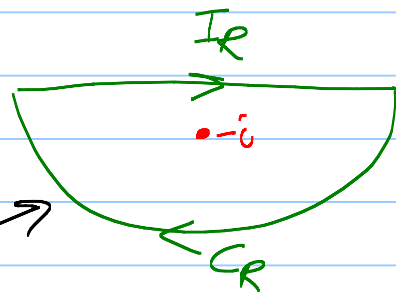
$$e^{isz} = e^{is(x+iy)} = e^{isx}\, e^{-sy}$$

So $\left|e^{isz}\right| = e^{-sy} \le 1$ when $\begin{matrix} s>0 \\ y \ge 0 \end{matrix} \leftarrow$ UHP

and $|z^2+1| \ge \big||z^2|-1\big| \ge R^2-1$ if $|z|=R>1$.

So $\left|\int_{C_R} \frac{e^{isz}}{z^2+1}\,dz\right| \le \frac{1}{R^2-1}\cdot \pi R \to 0$ as $R\to\infty$.

<u>Case $s<0$:</u>



clockwise!

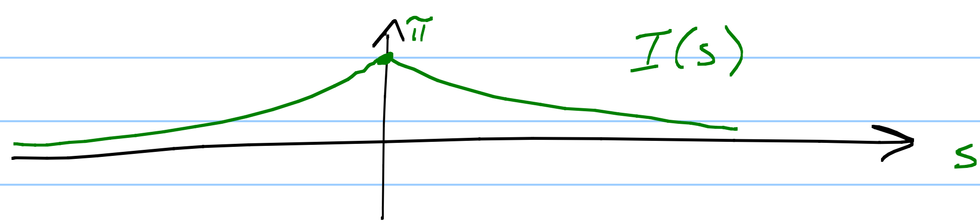
$\left|e^{isz}\right| = e^{-sy} \le 1$

$\begin{matrix} s<0 \\ y \le 0 \end{matrix} \leftarrow$ LHP

$$I(s) = -2\pi i \operatorname{Res}_{-i} \frac{e^{isz}}{z^2+1}$$

clockwise

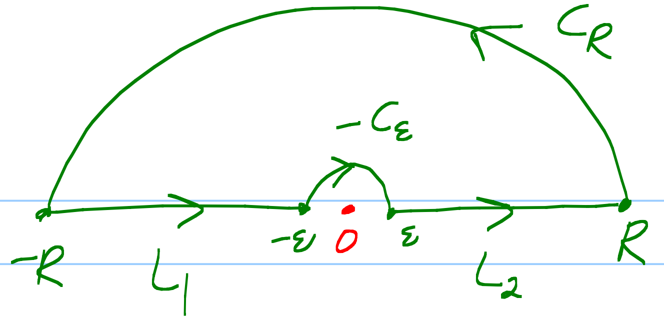
$$= -2\pi i \frac{e^{is(-i)}}{2(-i)} = \pi e^{s}$$



<u>Next:</u> $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$

$f(z) \overset{?}{=} \frac{\sin z}{z}$ No!

Trick: $\frac{e^{ix}}{x} = \frac{\cos x}{x} + i\,\frac{\sin x}{x}$



$f(z) = \dfrac{e^{iz}}{z}$

$$\int_L = \left(\int_{L_1} + \int_{L_2} \right) \frac{e^{iz}}{z}\, dz$$

$$= \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) \underbrace{\frac{\cos t}{t}}_{\text{odd}} + i \underbrace{\frac{\sin t}{t}}_{\text{even}} \; dt$$

$\int$'s cancel

$$= 2i \int_{\varepsilon}^{R} \frac{\sin t}{t}\, dt \leftarrow \text{want}$$

Cauchy's thm: $\displaystyle\int_\gamma \frac{e^{iz}}{z}\, dz = 0$

Claim: $\displaystyle\int_{C_R} \frac{e^{iz}}{z}\, dz \to 0$ as $R \to \infty$. (Jordan's lemma)

$\displaystyle\int_{-C_\varepsilon} \frac{e^{iz}}{z}\, dz \longrightarrow ?$ as $\varepsilon \to 0$.

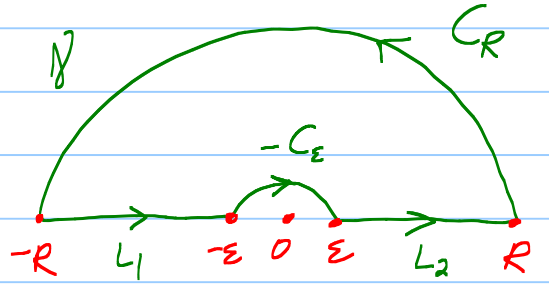
$$\frac{e^{iz}}{z} = \frac{1 + (iz) + \frac{(iz)^2}{2!} + \cdots}{z} = \frac{1}{z} + \underbrace{\left(i - \frac{1}{2!} z + \cdots \right)}_{\text{entire!}}$$

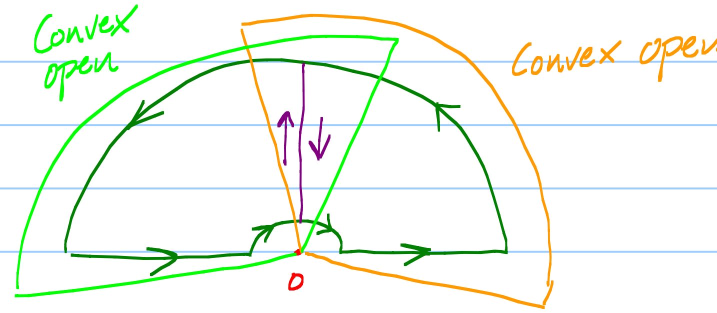
$$= \frac{1}{z} + H(z) \leftarrow |H| < M \text{ on } \overline{D_1(0)}$$

HWK 5 due Thurs

Lecture 20

$$\int_{-\infty}^{\infty} \frac{\sin t}{t}\,dt = \pi$$





$$0 = \int_\gamma \frac{e^{iz}}{z}\,dz = \underbrace{\left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right) \underset{\text{odd}}{\frac{\cos t}{t}} + i\,\underset{\text{even}}{\frac{\sin t}{t}}\,dt}_{2i\int_\varepsilon^R \frac{\sin t}{t}\,dt} + \underbrace{\int_{-C_\varepsilon} \frac{e^{iz}}{z}\,dz}_{-\int_{C_\varepsilon}\frac{e^{iz}}{z}\,dz} + \underbrace{\int_{C_R} \frac{e^{iz}}{z}\,dz}_{}$$

$-\int_{C_\varepsilon}\frac{e^{iz}}{z}\,dz \to -i\pi$ as $\varepsilon \to 0$

$\int_{C_R} \to 0$ as $R \to \infty$ via Jordans lemma

First, let $\varepsilon \to 0$. Then let $R \to \infty$.

Near 0, $$\frac{e^{iz}}{z} = \frac{1 + (iz) + \frac{(iz)^2}{2!} + \cdots}{z}$$

$$= \frac{1}{z} + \underbrace{\left(i - \frac{1}{2!}z - \frac{i}{3!}z^2 + \cdots\right)}_{H(z)\text{ entire!}}$$

$|H|$ bndd by M on $\overline{D_1(0)}$

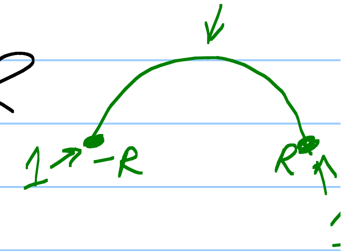
$$\int_{C_\varepsilon} \frac{e^{iz}}{z}\,dz = \underbrace{\int_{C_\varepsilon} \frac{1}{z}\,dz}_{} + \underbrace{\int_{C_\varepsilon} H\,dz}_{\to 0 \text{ as } \varepsilon \to 0}$$

$$\int_0^\pi \frac{1}{\varepsilon e^{it}}\, i\varepsilon e^{it}\,dt \qquad \left|\int_{C_\varepsilon} H\,dz\right| \le \underbrace{\left(\max_{C_\varepsilon} |H|\right)}_{<M,\ \varepsilon<1} \cdot \pi\varepsilon$$

$$= \int_0^\pi i\,dt = i\pi \qquad < M\pi\varepsilon \to 0$$

Jordan's lemma: $\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| \to 0$ as $R\to\infty$

$|e^{iz}|$ small



Basic est fails us: $\left| \quad \right| \le \left(\max_{C_R} \frac{|e^{iz}|}{R}\right)\pi R$

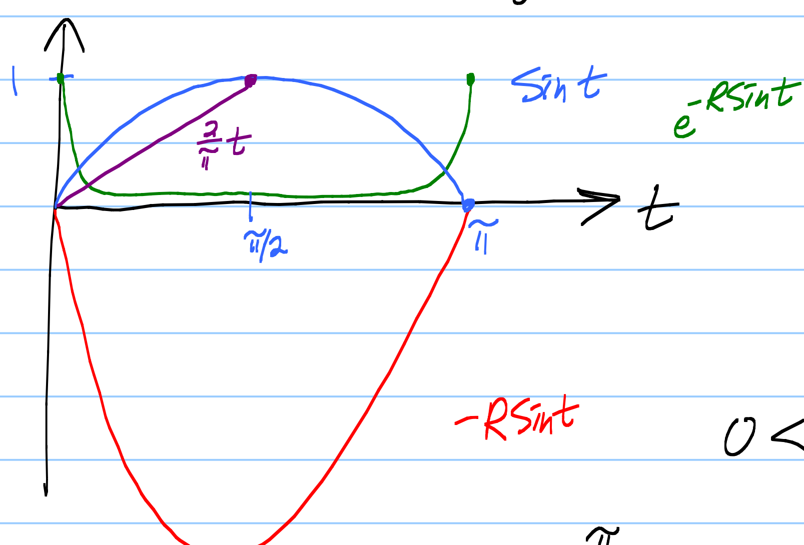
$$\left|e^{i(x+iy)}\right| = \left|e^{ix-y}\right| = e^{-y}$$

$$\le \frac{1}{R}\cdot \pi R = \pi \quad \text{Ouch!}$$

$$\left|\int_{C_R}\frac{e^{iz}}{z}\,dz\right| = \left|\int_0^{\pi} \frac{e^{iRe^{it}}}{Re^{it}}\, iRe^{it}\,dt\right|$$

$$= \left| i\int_0^{\pi} \underbrace{e^{iR\cos t - R\sin t}}_{e^{iR\cos t}\, e^{-R\sin t}}\,dt\right|$$

$$\le \int_0^{\pi} \left| \text{eee} \right| dt = \int_0^{\pi} e^{-R\sin t}\,dt$$



$0 \le \frac{2}{\pi}t \le \sin t \le 1 \quad [0, \frac{\pi}{2}]$

$-R\sin t \le -\frac{2R}{\pi}t \le 0$

$0 < e^{-R\sin t} \le e^{-\frac{2R}{\pi}t} \le 1$

So $\int_0^{\pi} e^{-R\sin t}\,dt = 2\int_0^{\pi/2} e^{-R\sin t}\,dt$

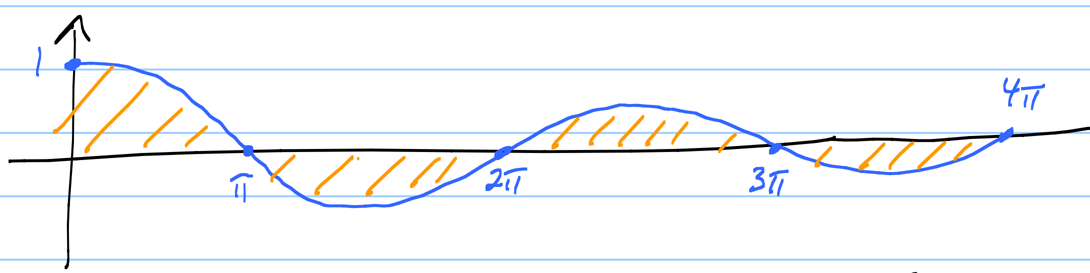
$$\leq 2\int_0^{\pi/2} e^{-\frac{2R}{\pi}t}\,dt$$

$$2\left[\frac{1}{\left(-\frac{2R}{\pi}\right)} e^{-\frac{2R}{\pi}t}\right]_0^{\pi/2}$$

$$= \frac{\pi}{R}\left(1-e^{-R}\right) < \frac{\pi}{R}$$

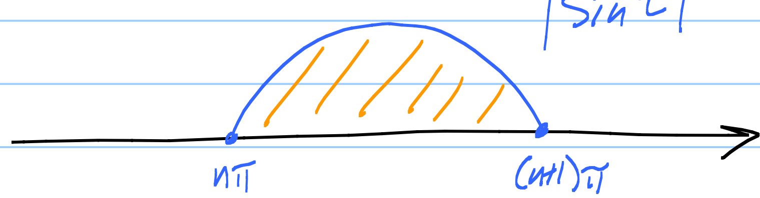
$$\to 0 \quad \text{as } R\to\infty.$$

$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$ is famous



Alternating series test shows that $\int_0^R$ converges as $R\to\infty$

but $\int_0^R \left|\frac{\sin t}{t}\right| dt \to \infty$. Conditionally convergent.



$\int_{n\pi}^{(n+1)\pi}$: $\frac{|\sin t|}{(n+1)\pi} \leq \frac{|\sin t|}{t} \leq \frac{|\sin t|}{n\pi}$ $[n\pi, (n+1)\pi]$

$$\frac{2}{(n+1)\pi} \leq \int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{t}\,dt \leq \frac{2}{n\pi}$$

Shows that $\int_0^{(N+1)\pi}$ blows up like

harmonic series $\sum_1^N \frac{1}{n} \sim \ln N$

<u>Thm</u>: P, Q polynomials

1) $\deg Q \geq \deg P + 1$ ← instead of $+2$

2) Q has no zeroes on $\mathbb{R}$

Then $$\lim_{R\to\infty} \int_{-R}^{R} \frac{P(x)}{Q(x)} e^{ix}\, dx = 2\pi i \sum_{a \in Z_Q^+} \operatorname{Res}_a \frac{P(z)}{Q(z)} e^{iz}$$

where Z_Q^+ = zeroes of Q in UHP.

<u>Pf</u>: Basic est fails us, but Jordan's lemma saves the day.

<u>EX</u>: $$I = \int_0^{2\pi} \sin^4\theta \, d\theta$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{e^{i\theta} - \frac{1}{e^{i\theta}}}{2i}$$

Aha! $$= \frac{z(\theta) - \frac{1}{z(\theta)}}{2i}$$ where $z(\theta) = e^{i\theta}$ $\quad C$

$z'(\theta) = i e^{i\theta}$

$$I = \int_0^{2\pi} \left(\frac{z(\theta) - \frac{1}{z(\theta)}}{2i} \right)^4 \frac{ie^{i\theta}}{\underbrace{ie^{i\theta}}_{iz(\theta)}} dt \quad dz$$

$$= \int_C \left(\frac{z - \frac{1}{z}}{2i}\right)^4 \cdot \frac{1}{iz} \, dz$$

residue term!

$$= \frac{1}{16}\int_C \left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right) \cdot \frac{1}{iz} \, dz$$

$$= \frac{1}{16} \cdot 2\pi i \, \mathrm{Res}_0 \left(\ell\ell\ell \right) = \frac{1}{16} \cdot 2\pi i \cdot \frac{6}{i} = \frac{3\pi}{4}$$

<u>Method</u>: $R(z,w)$ rational

no blowing up

$$\int_0^{2\pi} R(\cos\theta, \sin\theta) \, d\theta$$

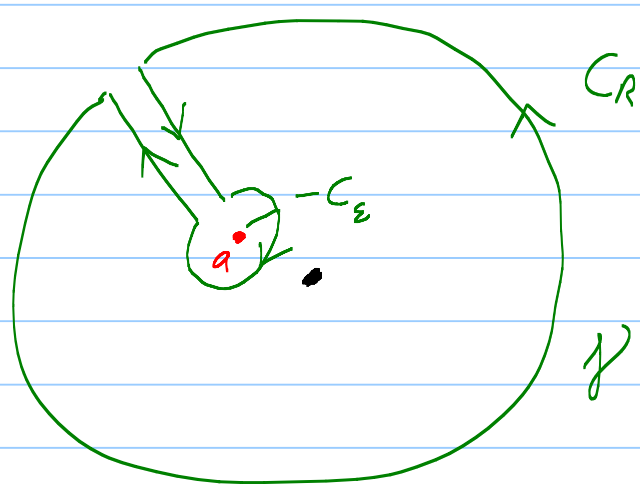
no poles on C

$$= \int_C \underbrace{R\left(\frac{z + \frac{1}{z}}{2}, \frac{z - \frac{1}{z}}{2i}\right) \frac{1}{iz}}_{H(z)} \, dz$$

$$= 2\pi i \sum_{\substack{a \in \text{poles of} \\ H \text{ inside } C}} \mathrm{Res}_a H$$

<u>Stein</u> Keyhole trick

$$) = \int_\gamma \frac{f(z)}{z-a} \, dz$$



Push parallel lines together.
They cancel.

Get $\left(\int_{-C_\varepsilon} + \int_{C_R} \right) \frac{f(z)}{z-a} \, dz = 0$

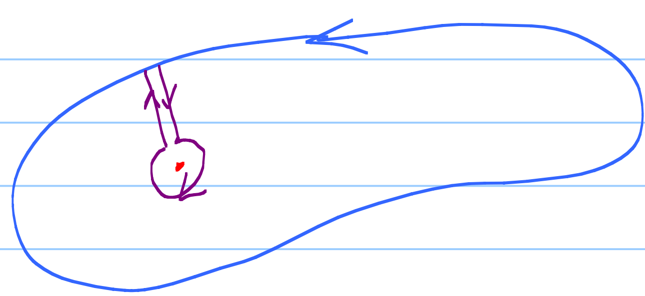
$$\int_{-C_\varepsilon} \frac{f(z)}{z-a} \, dz = - \int_{C_\varepsilon} \frac{f(z)}{z-a} \, dz = - \int_0^{2\pi} \frac{f(a+\varepsilon e^{it})}{(a+\varepsilon e^{it}) - a} \, i\varepsilon e^{it} \, dt$$

$$= -i \int_0^{2\pi} f(a+\varepsilon e^{it}) \, dt$$

$\longrightarrow -i \, 2\pi \, f(a)$ because

analytic f is continuous at a.

Cauchy thm $\Longrightarrow$ Cauchy integral formula



Lecture 21 Why the "argument" principle

HWK 5 due Thurs, Midterm exam Friday after spring break

f analytic on $D_R(z_0)$, nonvanishing on $C_r(z_0)$, $0 < r < R$.

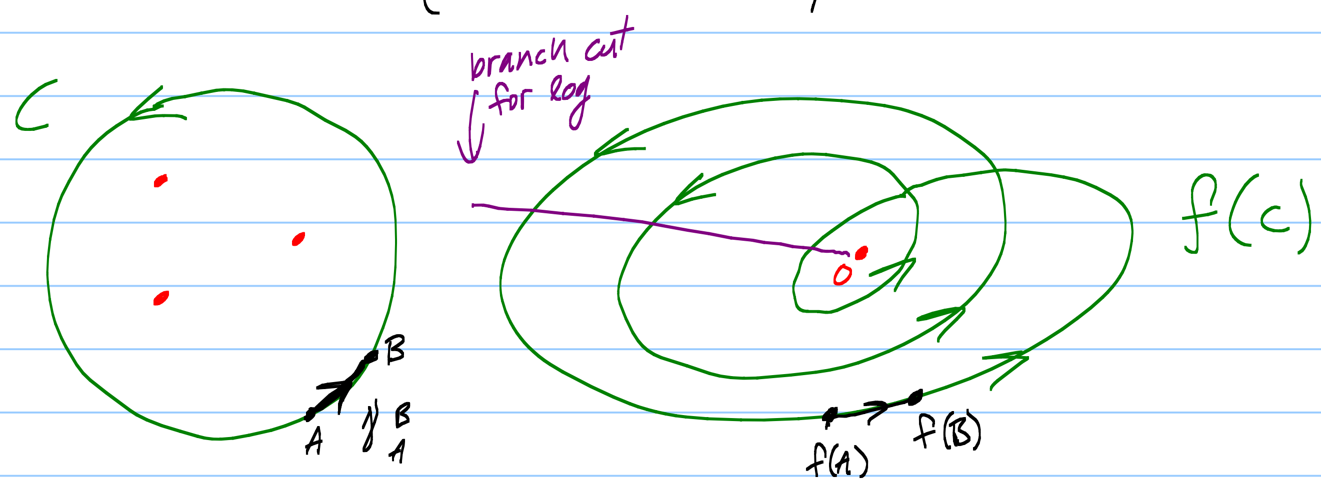
$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'}{f}\,dz = \left(\begin{array}{c}\#\text{ zeroes of } f \\ \text{inside } C_r(z_0) \\ \text{counted with multiplicity}\end{array}\right)$$

If f allowed to have finitely many poles in $C_r(z_0)$ too, then

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'}{f}\,dz = \left(\begin{array}{c}\#\text{ zeroes } f \\ \text{in } C_r, \\ \text{w/ mult.}\end{array}\right) - \left(\begin{array}{c}\#\text{ poles } f \\ \text{in } C_r, \\ \text{w/ order}\end{array}\right)$$

Pf:

$$\operatorname{Res}_a \frac{f'}{f} = \begin{cases} N & a \text{ is zero of order } N \\ -N & a \text{ is pole of order } N \end{cases}$$



$$\int_{\gamma_A^B} \frac{f'}{f}\,dz = \int_{\gamma_A^B} \frac{d}{dz}\left[\log f(z)\right] dz$$

$$= \log f(B) - \log f(A)$$

$$= \left(\operatorname{Ln}|f(B)| + i \arg f(B)\right) - \left(\operatorname{Ln}|f(A)| + i \arg f(A)\right)$$

$$= \left(\operatorname{Ln}|f(B)| - \operatorname{Ln}|f(A)| \right) + i \underbrace{\Delta_{\gamma_A^B} \arg f}_{}$$

does not depend on choice of branch!

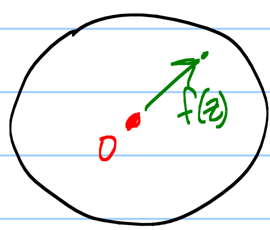
Add up $\int_{\gamma_{A_i}^{A_{i+1}}}$ going around C:

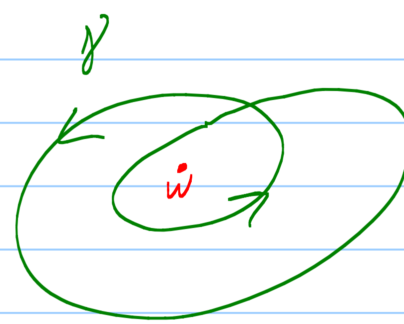
$\operatorname{Ln}|f|$ terms pairwise cancel and first one cancels last one!

Aha! $$\int_C \frac{f'}{f}\,dz = i \sum \Delta_{\gamma_{A_i}^{A_{i+1}}} \arg f \overset{\text{def}^{\text{n}}}{=} i \Delta_C \arg f$$

$$\frac{1}{2\pi i}\int_C \frac{f'}{f}\,dz = \frac{1}{2\pi}\Delta_C \arg f = \left(\begin{array}{c} \#\text{ times } f(z) \\ \text{goes around} \\ \text{the origin} \end{array} \right)$$

Arg watch:





<u>Def$^{\text{n}}$</u>: $$\frac{1}{2\pi i}\int_\gamma \frac{1}{z-w}\,dz$$

$$= \left(\begin{array}{c} \text{winding \#} \\ \text{of } \gamma \text{ about } w \end{array} \right) = \operatorname{Ind}_\gamma(w)$$ ← Index of w with respect to γ

$$= \left(\begin{array}{l} \#\text{ times } \gamma \\ \text{wraps around} \\ w\text{, counterclockwise} \end{array} \right)$$

Note: Argument integral idea with $f(z) = z - w$

Fact: $C : z(t), \quad a \le t \le b$

$$f(C) : f(z(t)), \quad a \le t \le b$$

$$\underbrace{\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)-w}\,dz}_{H(w)} = \frac{1}{2\pi i}\int_a^b \frac{f'(z(t))\,z'(t)\,dt}{\underbrace{f(z(t))}_{\zeta}-w}$$

$$\frac{d}{dt} f(z(t))\,dt$$

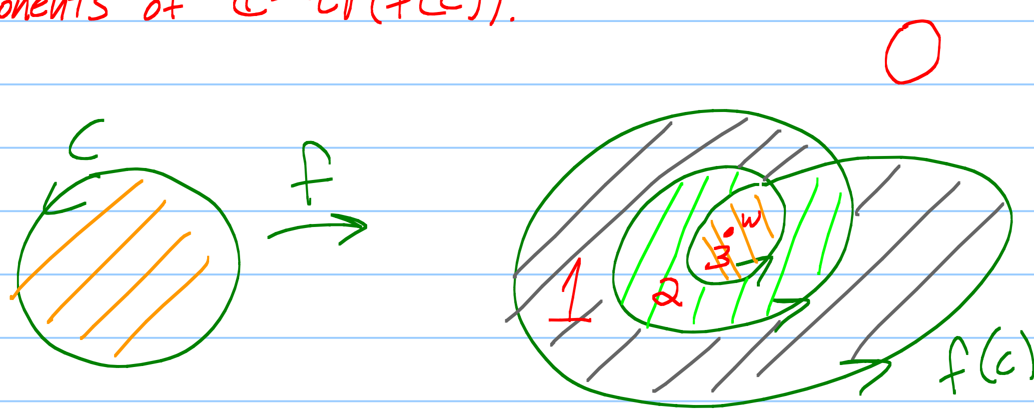
$$= \frac{1}{2\pi i}\int_{f(C)} \frac{1}{\zeta - w}\,d\zeta$$

$$= \mathrm{Ind}_{f(C)}(w)$$

Last time $H'(w) \equiv 0$,

So H is constant on connected components of $\mathbb{C} - tr(f(C))$.

Picture:

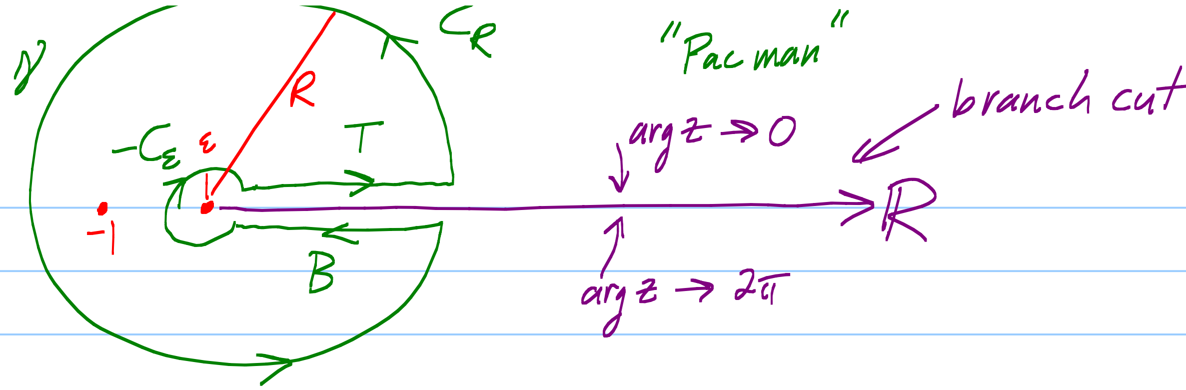


Think about z^3.

EX: $$I = \int_0^\infty \frac{x^{-\alpha}}{x+1}\,dx, \quad 0 < \alpha < 1$$

$$x^{-\alpha} = e^{-\alpha \operatorname{Ln} x}, \quad (x > 0)$$

$$z^{-\alpha} = e^{-\alpha \log z} \leftarrow \text{branch ?}$$



Res Thm
$$\int_\gamma \underbrace{\frac{e^{-\alpha \log z}}{z+1}}_{f(z)}\, dz = 2\pi i \, \operatorname{Res}_{-1} \frac{e^{-\alpha \log z}}{z+1}$$
$$= 2\pi i \, \frac{e^{-\alpha \log(-1)}}{(1)}$$
$$= 2\pi i \, e^{-\alpha(\operatorname{Ln}|-1| + i\pi)}$$
$$= 2\pi i \, e^{-i\alpha\pi}$$

Limits as "mouth" closes:
$$\int_T f(z)\, dz = \int_\varepsilon^R \frac{e^{-\alpha(\operatorname{Ln} t + i\cdot 0)}}{t+1}\, dt$$ ← want!
$$\int_B f(z)\, dz = -\int_\varepsilon^R \frac{e^{-\alpha(\operatorname{Ln} t + i\cdot 2\pi)}}{t+1}\, dt$$
$$= \underbrace{-e^{-i\alpha 2\pi}}_{\text{not} = -1.} \underbrace{\int_\varepsilon^R \frac{e^{-\alpha \operatorname{Ln} t}}{t+1}\, dt}_{\text{want}}$$

On C_R: $\left|z^{-\alpha}\right| = |z|^{-\alpha}$ (claim)

$$\left|e^{-\alpha \log z}\right| = \left|e^{-\alpha \log Re^{it}}\right| = \left|e^{-\alpha(\ln R + it)}\right|$$

$$= \underbrace{\left|e^{-\alpha \ln R}\right|}_{R^{-\alpha}} \underbrace{\left|e^{-\alpha i t}\right|}_{1} = R^{-\alpha} \checkmark$$

$$|z+1| \geq \big||z| - |1|\big| = R-1 \quad \text{if} \quad R > 1$$

$$\left|\int_{C_R} f\, dz\right| \leq \left(\frac{R^{-\alpha}}{R-1}\right) \cdot (2\pi R) \quad \text{if} \quad R > 1$$

$$\longrightarrow 0 \quad \text{as} \quad R \to \infty \quad \text{because} \quad \underline{\alpha > 0}.$$

<u>On C_ε:</u> $|z+1| \geq \big||z| - |1|\big| \geq 1 - \varepsilon$ if $|z| = \varepsilon < 1$

$$\left|\int_{C_\varepsilon} f\, dz\right| \leq \frac{\varepsilon^{-\alpha}}{1-\varepsilon} \cdot 2\pi\varepsilon \quad \text{if} \quad 0 < \varepsilon < 1$$

$$\longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{because} \quad \underline{\alpha < 1}.$$

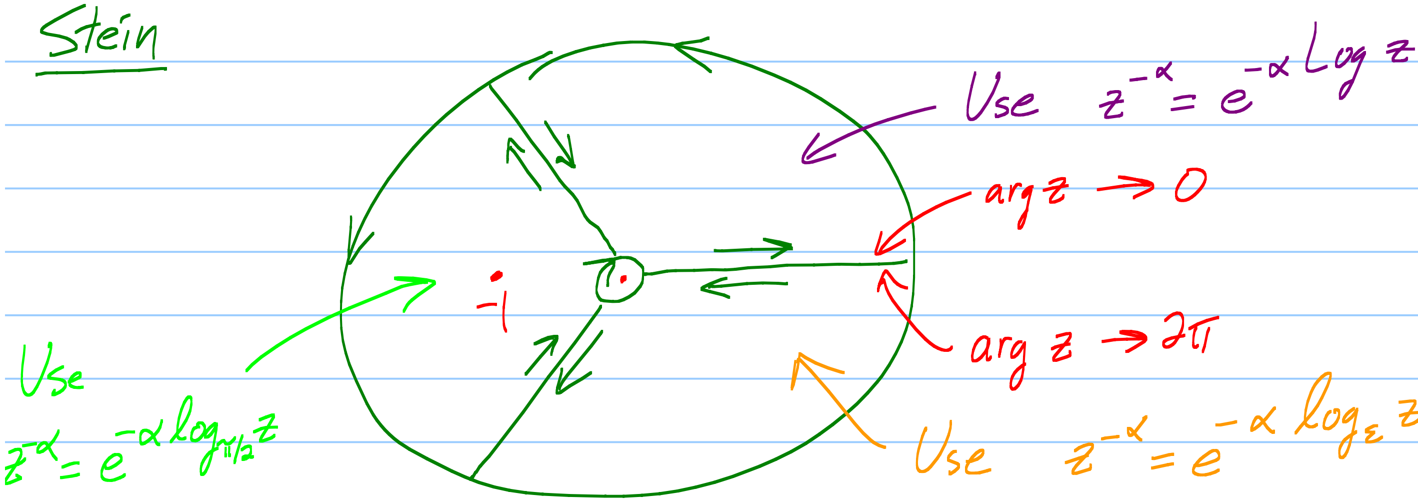
Close mouth, let $\varepsilon \to 0$, let $R \to \infty$. Get

$$\Big(\underbrace{\int_T + \int_B}_{(1-e^{-i\alpha 2\pi})\, I} + \underbrace{\int_{-C_\varepsilon}}_{\downarrow\, 0} + \underbrace{\int_{C_R}}_{\downarrow\, 0}\Big) f\, dz = 2\pi i\, e^{-i\alpha\pi}$$

So $I = \dfrac{2\pi i\, e^{-i\alpha\pi}}{1-e^{-i\alpha 2\pi}} \cdot \dfrac{e^{i\alpha\pi}}{e^{i\alpha\pi}}$

$$= \pi\left(\frac{2i}{e^{i\alpha\pi}-e^{-i\alpha\pi}}\right) = \frac{\pi}{\sin \alpha\pi}$$

<u>Stein</u>

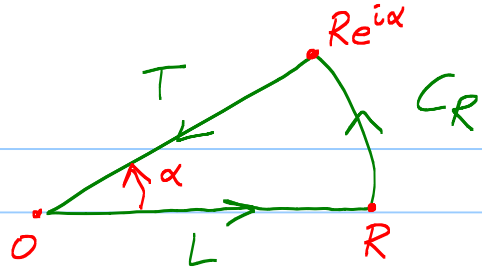


Add up 3 $\int$'s using different branches of log to cancel out first two cuts. Third cuts don't cancel!

Lecture 22 Important consequences of the Schwarz lemma

HWK 5 due THURS (tomorrow)

EX $I = \int_0^\infty \underbrace{\frac{1}{x^3+1}}_{f(z)=\frac{1}{z^3+1}}\, dx$

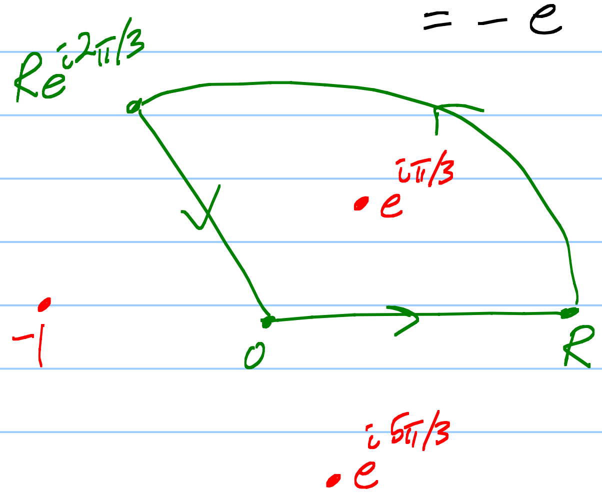


$\int_L f\, dz = \int_0^R \frac{1}{t^3+1}\, dt \leftarrow$ want , $\left| \int_{C_R} \frac{1}{z^3+1}\, dz \right| \le \frac{1}{R^3-1} \cdot \alpha\pi \to 0$ as $R \to \infty$

T : $-T$: $z(t) = te^{i\alpha}$, $0 \le t \le R$, $z'(t) = e^{i\alpha}$.

$\int_T f\, dz = -\int_{-T} f\, dz = -\int_0^R \frac{1}{\underbrace{(te^{i\alpha})^3}_{t^3 e^{i3\alpha}}+1} \cdot e^{i\alpha}\, dt$ $\leftarrow$ Aha! Want $3\alpha = 2\pi$, $\alpha = \frac{2\pi}{3}$

$= -e^{i2\pi/3} \int_0^R \frac{1}{t^3+1}\, dt$



$z^3+1=0$

$z = e^{i\pi/3}, -1, e^{i5\pi/3}$

$$\left(\int_L + \int_T \right) + \int_{C_R} f\, dz = 2\pi i \operatorname{Res}_{e^{i\pi/3}} \frac{1}{z^3+1}$$

$$\left(1 - e^{i2\pi/3}\right) I + 0 = 2\pi i \frac{1}{3\left(e^{i\pi/3}\right)^2}$$

$$I = \frac{2\pi i}{3e^{i2\pi/3}\left(1 - e^{i2\pi/3}\right)}$$

$$= \frac{\pi}{3} \cdot \frac{2i}{1-e^{i2\pi/3}} \, e^{-i\pi/3} \cdot \left(\frac{-e^{-i\pi/3}}{-e^{-i\pi/3}} \right)$$

$$= \frac{\pi}{3} \quad \frac{2i}{e^{i\pi/3} - e^{-i\pi/3}} \underbrace{\left(-e^{-i3\pi/3} \right)}_{-(-1)}$$

$$= \frac{\pi/3}{\sin \pi/3}$$

Applications of the Schwarz lemma

HWK 1: prob. 1 : $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$ $\quad a \in D_1(0)$.

Möbius transformation

Facts: φ_a analytic, $\varphi_a : D_1(0) \circlearrowleft$ 1-1 onto

$$\varphi_a^{-1} = \varphi_{-a}(z) = \frac{z+a}{1+\bar{a}z}$$

Def^n: $Aut(\Omega) = \{ f : \Omega \circlearrowleft \text{ 1-1 onto, analytic} \}$

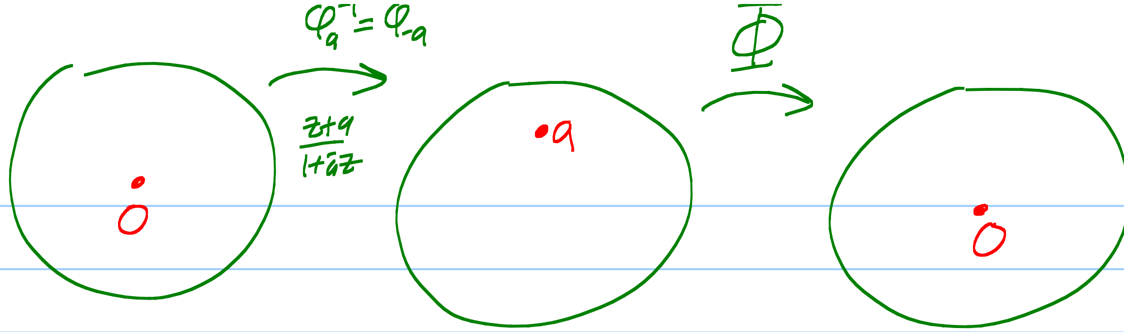
domain biholomorphic self map of Ω

Fact: $Aut(\Omega)$ is a group under composition

Pf: Super inverse function theorem: $(f^{-1}) \in Aut(\Omega)$ when $f \in Aut(\Omega)$.

Thm: $Aut(D_1(0)) = \{ \lambda \varphi_a : a \in D_1(0), |\lambda| = 1 \}$

Pf: Suppose $\Phi \in Aut(D_1(0))$



$\Psi = \Phi(\varphi_{-a})$

Lemma: If $\Psi \in \mathrm{Aut}(D_1(0))$ and $\Psi(0) = 0$, then

$\Psi(z) = \lambda z$ for some unimodular const λ.

Pf: Schwarz $\Rightarrow$ $|\Psi'(0)| \le 1$.

But Ψ^{-1} satisfies same hyp! (Super IFT too)

So $\underbrace{|(\Psi^{-1})'(0)|}_{\frac{1}{|\Psi'(0)|}} \le 1$

$\Rightarrow |\Psi'(0)| \ge 1$

So $|\Psi'(0)| = 1$ and Schwarz $\Rightarrow$ $\Psi(z) = \lambda z$ ✓

Back to proof: Now have

$$\Phi(\underbrace{\varphi_{-a}(z)}_{w}) = \lambda z$$

$\varphi_a(w)$

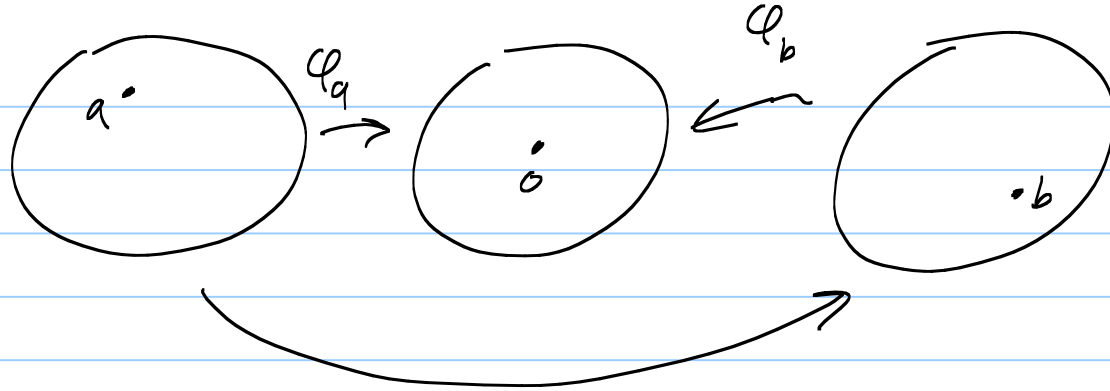
$w = \varphi_{-a}(z)$

$z = (\varphi_{-a})^{-1}(w)$

$= \varphi_a(w)$

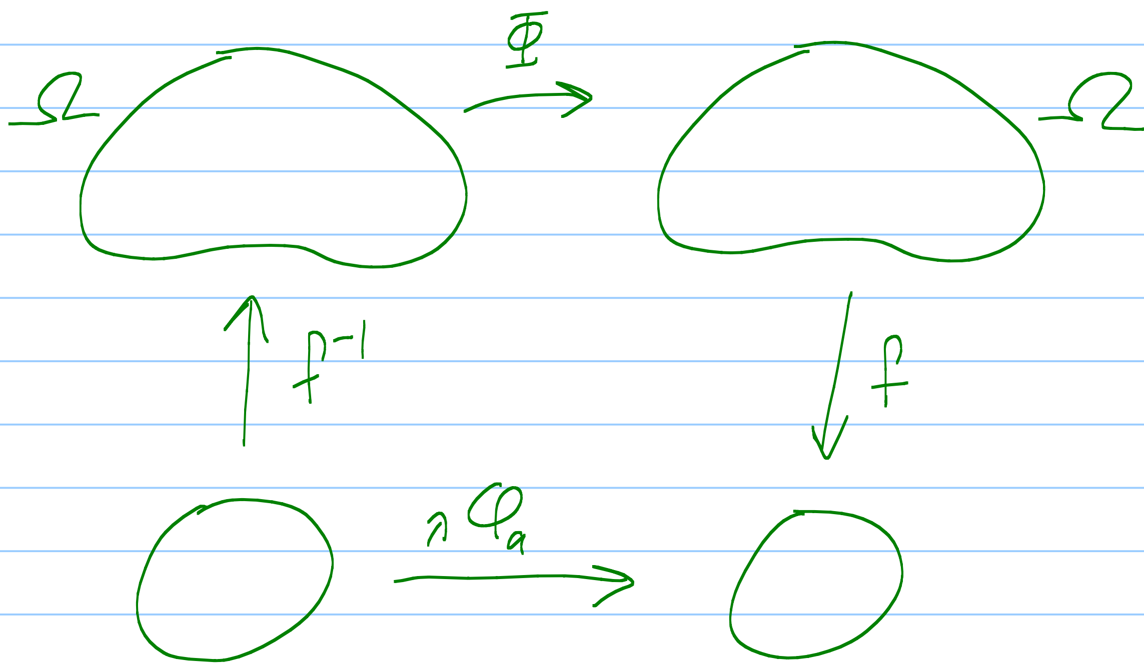
So $\Phi(w) = \lambda \varphi_a(w)$ ✓

Fact: $\mathrm{Aut}(D_1(0))$ is transitive

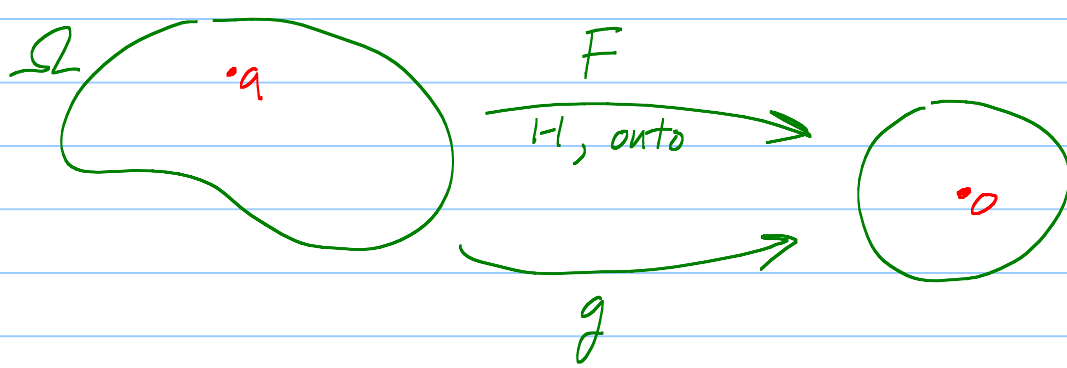


$$\varphi_b^{-1} \circ \varphi_a = \varphi_{-b} \circ \varphi_a$$

Riemann mapping thm: $\Omega \subsetneq \mathbb{C}$, simply connected (no holes), $\exists\ f : \Omega \to D_1(0)$ 1-1, onto, analytic.



Important HWK 5 problem:



F, g analy

$$\begin{cases} F(a) = 0 \\ g(a) = 0 \end{cases}$$

Schwarz: $|g'(a)| \leq |F'(a)|$

$M = \sup\{|g'(a)| : g: \Omega \to D_1(0) \text{ analytic}, g(a)=0\}$

Riemann map F: $|F'(a)| = M$.

Strategy to find F:

Step 1: $M < \infty$. (Cauchy estimates)

Step 2: Get seq $g_n: \Omega \to D_1(0)$, $g_n(a) = 0$

analytic, $|g_n'(a)| \to M$.

Step 3: Show $\exists$ subseq of $\{g_n\}$ that converges

uniformly on compact sets of Ω to f. (Montel's Thm)

Know f is analytic.

Step 4: Show $f: \Omega \to D_1(0)$ is 1-1. (Hurwicz)

Step 5: Show f onto. (Schwarz lemma)

f is Riemann map!

Linear fractional transformations (LFTs)

$$L(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

Some books: LFTs = Möbius transf, not here.

Facts: $L : \hat{\mathbb{C}} \xrightarrow[\text{onto}]{1\text{-}1} \hat{\mathbb{C}}$

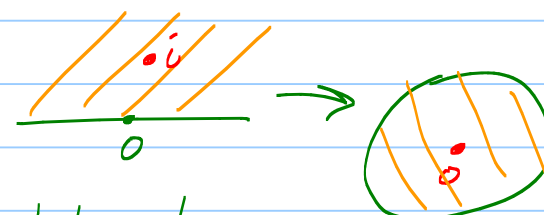
$$\lim_{z\to\infty} \frac{az+b}{cz+d} = \frac{a}{c} \qquad L(\infty) = \frac{a}{c}$$

(If $c=0$, then $d \neq 0$. $L(z) = Az+B$ $\quad L(\infty)=\infty$.)

$z = \frac{-d}{c}$ is a simple pole, $c \neq 0$.

Think $L\left(\frac{-d}{c}\right) = \infty$

$\mathrm{Aut}(UHP) = ?$ $\qquad \frac{z-i}{z+i}$:



1-1, onto

Big fact: LFTs map $\left\{\begin{matrix}\text{lines,}\\ \text{circles}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{lines,}\\ \text{circles}\end{matrix}\right\}$