MA 53000 QUALIFIER, 1/3/2017

Each problem is worth 5 points. Make sure that you justify your answers.

Notes, books, crib sheets, and electronic devices are not allowed.

1. f is a function holomorphic in the half plane $\{\operatorname{Im} z > -3\}$, apart from a simple pole at $z = 2$. What can you say about the radius of convergence of its Taylor series about 0? Same question if the simple pole is, instead, at $z = 4$.

2. Compute the following integral (the path of integration is oriented counterclockwise):

$$\int_{|z|=2} \frac{e^z}{z - z^2}\, dz.$$

3. Suppose ϕ is holomorphic on some open set $\Omega \subset \mathbb{C}$, apart from isolated singularities. Suppose furthermore that for each $k \in \mathbb{N}$ we can write $\phi = \psi^k$ with a ψ that is also holomorphic on Ω, apart from isolated singularities. Prove that the singularities of ϕ are either removable or essential.

4. For positive numbers a, R let $\Gamma_{a,R} \subset \mathbb{C}$ stand for the path consisting of three segments as follows. It starts at $R - \pi i$, goes to $a - \pi i$, from there to $a + \pi i$ and then to $R + \pi i$. Prove that

$$\lim_{R\to\infty} \int_{\Gamma_{a,R}} \frac{e^{e^\zeta}}{\zeta - z}\, d\zeta = E_a(z)$$

exists and represents a holomorphic function E_a in the half plane $H_a = \{z \in \mathbb{C}\colon \operatorname{Re} z < a\}$. Prove also that if $a < b$ then $E_a = E_b$ in H_a.

5. Find a biholomorphic map between the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and the half disc $\{z \in D : \operatorname{Im} z > 0\}$. (If the map is found as the composition of simpler maps, it suffices to explain what the simpler maps are, there is no need to write down the composition.)

6. Suppose u is a harmonic function in $\mathbb{C}$, and $|u(z)| \le \sqrt{|z|}$ for $z \in \mathbb{C}$. Prove that u is constant.

7. Prove that if P is holomorphic on $\mathbb{C}$ and $\lim_{|z|\to\infty} |P(z)| = \infty$, then P is a polynomial.

1. **Note on Poles.** As was prompted by Jelena, we want to distinguish the concepts of an "infinite-order pole" and an essential singularity. An infinite-order pole at $z = \alpha$ would be akin to saying that on a neighborhood of α, $f \equiv \infty$. This comes from examining the definition of an order-k pole:

   Definition of pole: Suppose $f(z)$ is holomorphic on a punctured disk about $z = \alpha$, and let $g(z) := 1/f(z)$, so that $g(z)$ is holomorphic on the complete $D_r(\alpha)$. Then $f(z)$ has an order-k pole at α if $g(z)$ has an order-k zero at α. This is equivalent to $g(z) = (z-\alpha)^k h(z)$ where $h(z)$ is holomorphic *and nonvanishing* on $D_r(\alpha)$, which is also equivalent to saying: if $g(z) = a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots$, then $a_0 = a_1 = \cdots = a_{k-1} = 0$, and $a_k \neq 0$.

   Thus $f(z)$ having an infinite order pole would mean $g(z) = 1/f(z)$ had an infinite order zero at α, but this is equivalent to $a_0 = 0$, $a_1 = 0$, $a_2 = 0$, and in fact $a_j = 0$ for all $j \geq 0$, but then $g \equiv 0$ by the identity theorem, hence our descriptor that $f \equiv \infty$ on $D_r(\alpha)$.

   Said this way, it should be clear that this behavior is completely antithetical to f having an essential singularity at α—in such a case it would hold that the image under $f(z)$ of *any* $D_\epsilon(\alpha) \setminus \{\alpha\}$ should be *dense* in $\mathbb{C}$ by the C.W. theorem. Thus these two concepts are rather different!.

3. This is largely a polished version of the arguments Deion and Jack discussed with me/us. We are trying to exclude the possibility of ϕ having a pole, so we assume ϕ has an order-N pole, $N > 0$, at $z = \alpha$, so that near α,

$$\phi(z) = \frac{a_{-N}}{(z-\alpha)^N} + \cdots + a_0 + a_1(z-\alpha) + \cdots,$$

   with $a_{-N} \neq 0$. We now take an arbitrary positive integer k and consider the function ψ such that $\psi^k = \phi$. In particular we consider what kind of singularity ψ itself could have at α. So consider ψ having an order M-pole (we'll allow $M = 0$ possibly)

$$\psi(z) = \frac{b_{-M}}{(z-\alpha)^M} + \cdots + \frac{b_{-1}}{z-\alpha} + b_0 + b_1(z-\alpha) + \cdots.$$

   If $M = 0$, then $\psi(z) = b_0 + b_1(z-\alpha) + \cdots$, but then $\psi^k = b_0^M + (?)(z-\alpha)^{M+1} + \cdots$, and this would contradict $\psi^k = \phi$ having an order N pole at α, as clearly no negative powers of $(z-\alpha)$ will appear. So ψ cannot have a removable singularity if $\psi^k = \phi$ is going to have a pole.

   We can argue similarly for ψ not having an essential singularity at α, as then the Laurent expansion for ψ having infinitely-many negative powers would mean ψ^k will almost certainly also have infinitely-many negative powers, but because making that precise might be troublesome we can just appeal to the Casorati-Weierstrass theorem: If the range of $\psi(z)$ near α is dense in $\mathbb{C}$, as it would be if α is an essential singularity, then certainly the range of ψ^k will be dense in $\mathbb{C}$ as well.

   So now we assume ψ has a pole at α, i.e. $M > 0$ now. Applying the k-th power, we see

$$\psi^k(z) = \frac{b_{-M}^k}{(z-\alpha)^{Mk}} + \frac{\sim}{(z-\alpha)^{Mk-1}} + \cdots b_0^k + \cdots.$$

Because this must equal ϕ, we need $Mk = N$. This is not a problem, *yet.* Specifically, because k is arbitrary, take $k = N+1$. Then a (*potentially different*) ψ will need to have $\psi^{N+1} = \phi$. However, if we examine the previous discussion, we see that if this new ψ even has an order $M = 1$ pole at α, then $\phi = \psi^{N+1}$ will have an order $N+1$ pole at α, contradicting our assumption on ϕ. Thus ϕ can only have either a removable or essential singularity at $z = \alpha$.

4. Draw pictures to help with large amount of notation. The plan is to approximate $E_a(z)$ as the uniform limit of holomorphic functions, specifically the ones using the finite-length curves $\Gamma_{a,R}$. Given a, R, define

$$\gamma_{a,R}(s) : [0, 2(R-a) + 2\pi] \to \Gamma_{a,R}, \tag{1}$$

a unit-speed (to avoid chain rule tedium in $\gamma'(s)$) parameterization of $\Gamma_{a,R}$, and

$$F(z,\zeta) := \frac{e^{e^{\zeta}}}{\zeta - z}. \tag{2}$$

Then defining

$$E_{a,R}(z) := \int_0^{2(R-a+\pi)} F(z, \gamma_{a,R}(s))\, ds, \tag{3}$$

we know this is holomorphic by the theorem on "integral of holomorphic function over compact interval is holormophic," e.g. [Stein, Shakarchi, Complex Analysis, thm. 5.4, p. 56], wherein it should make sense that "$[0,1]$" there can be replaced by any finite-length interval.

Now let $r : a < r < R$ and consider $E_{a,R}(z) - E_{a,r}(z)$. This integral will (after cancelling the other segments) only have segments of integration $\{t \pm i\pi : r < t \le R\}$. On these segements we can make two bounds: if $z := x + iy$ with $x < a$, then $|\zeta - z| > a - x$ on all of the $\Gamma_{a,R}$, so $1/|\zeta - z| < 1/(a-x)$. Second, if $\zeta = t \pm i\pi$, then

$$\left| e^{e^{\zeta}} \right| = e^{Re\left(e^t e^{\pm i\pi}\right)} = e^{e^t \cos(\pm\pi)} = e^{-e^t}.$$

Because $t > 0 \implies t < e^t$, we can simplify this and just say $e^{-e^t} < e^{-t}$. Putting everything together now, we see

$$|E_{a,R}(z) - E_{a,r}(z)| \le 2 \cdot \int_r^R \frac{e^{-t}}{(a-x)}\, dt,$$

the 2 coming from just combining the interals over the two segments. Then

$$2 \cdot \int_r^R \frac{e^{-t}}{(a-x)}\, dt \le \frac{2}{a-x} \int_r^\infty e^{-t}\, dt = \frac{2e^{-r}}{a-x}.$$

From here we see that by increasing r, the differences $E_{a,R}(z) - E_{a,r}(z)$ can be made arbitrarily small *independent of* R, so that the $E_{a,R}(z)$ must converge uniformly to the limit function which is $E_a(z)$. As such, $E_a(z)$ must be holormophic (on $\{Re(z) < a\}$ still).

For the final part, with $a < b$ we can similarly define $E_{b,R}(z)$, and we see by Cauchy's theorem (since the integrand is holomorphic away from z) that once $R > b$, $E_{a,R}(z) = E_{b,R}(z)$, and so the functions $E_{b,R}(z)$ would converge to $E_a(z)$, as well as $E_b(z)$ by their definition.