MA 530 Complex analysis

Steve Bell
MATH 750
494-1497

Stein, Shakarchi : Complex analysis, Princeton Press

Office hours : T, Th 2:00–3:00 pm in MATH 750 plus zoom

HWK	100 pts
ME	100 pts
FE	200 pts

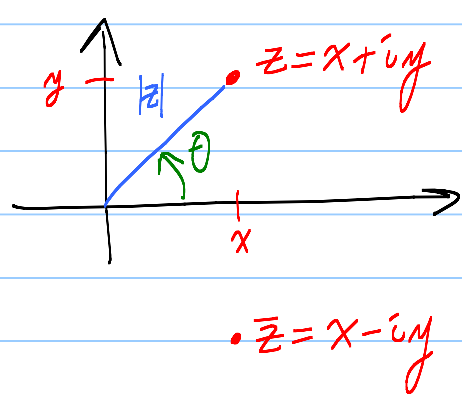
Quadratic formula : Complete square :

$$a(x-r)^2 = c \leftarrow c<0 !$$

Toss in i such that $i^2=-1$. Play games.

Algebra games, calculus games.

$\mathbb{C}$



$$|z|^2 = x^2+y^2 \qquad \begin{cases} x = \mathrm{Re}\, z \\ y = \mathrm{Im}\, z \end{cases}$$

$$-\pi < \theta \le \pi \qquad \theta = \mathrm{Arg}\, z$$

Principal arg of z

$$\arg z = \{ \mathrm{Arg}\, z + 2\pi n \ ; \ n \in \mathbb{Z} \}$$

Polar form: $z = re^{i\theta}$, $r=|z|$, $\theta \in \arg z$

where $e^{i\theta} = \cos\theta + i \sin\theta$

Calculus game : $e^{(a+b)} = e^a e^b$

$$e^{x+iy} = e^x e^{iy}$$

$$= e^x\left(1 + (iy) + \frac{(iy)^2}{2!} + \cdots\right)$$

$$= e^x\left[\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \cdots\right)\right]$$

$$e^{x+iy} \overset{\text{def}^n}{=} e^x\left[\cos y + i \sin y\right]$$

DeMoivre's formula: $(e^{i\theta})^N = e^{iN\theta}$ $\quad N \in \mathbb{Z}$

Super fun thing: $\displaystyle\sum_{n=-N}^{N} e^{in\theta} = \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}$

$$\begin{cases} \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} \\ \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} \end{cases}$$

Algebra games:

$$\begin{cases} (x_1, y_1) + (x_2, y_2) \overset{\text{def}^n}{=} (x_1 + x_2, y_1 + y_2) \\ (x_1, y_1)\cdot(x_2, y_2) = (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1) \\ 0 \sim (0,0),\quad 1 \sim (1,0),\quad i \sim (0,1) \end{cases}$$

Field! $\approx \mathbb{C}$ $\qquad$ A complete field!

Fund Thm Alg: A poly $P(z) = a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0$ of a deg $N \geq 1$ has a root r in $\mathbb{C}$. $[P(r) = 0]$

Consequently, $\displaystyle P(z) = a_N \prod_{n=1}^{N} (z - r_n)$ ← might repeat

$$= a_N \prod_{k=1}^{M} (z - r_k)^{m_k} \qquad \sum_{k=1}^{N} m_k = N$$

Algebra game:

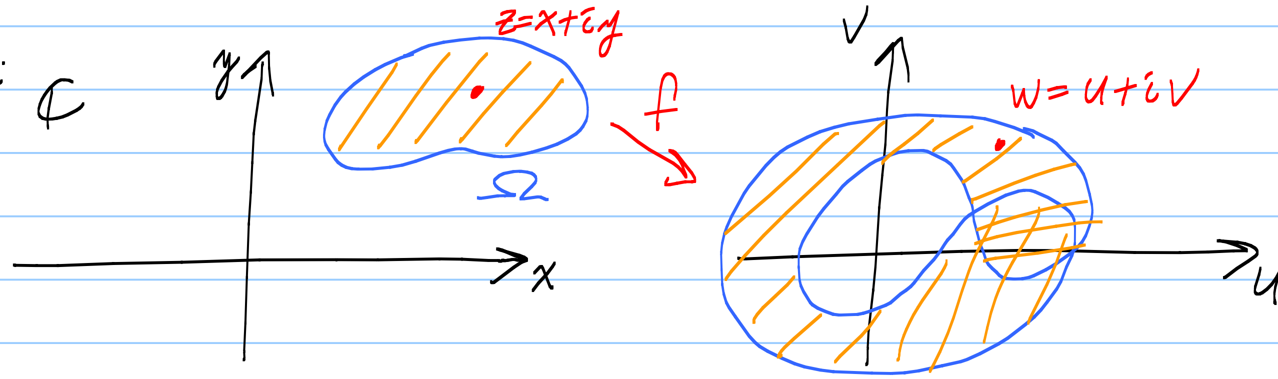
Frobenius: Every division ring gotten as a finite extension of $\mathbb{R}$ is either

1) $\approx \mathbb{R}$

2) $\approx \mathbb{C}$

3) $\approx \mathbb{H}$ ⟸ Quaternions

MA 530: $\mathbb{C}$



$f(z) = w$

$f(x+iy) = u + iv$

$u(x,y)$ $v(x,y)$

$\mathbb{C} \to \mathbb{C}$

$(x,y) \longmapsto (u(x,y), v(x,y))$

$\mathbb{R}^2 \to \mathbb{R}^2$

Ex: $w = z^2 = (x+iy)^2 = \underbrace{(x^2-y^2)}_{u(x,y)} + i\,\underbrace{(2xy)}_{v(x,y)}$

$\mathbb{C} \approx \mathbb{R}^2$ Topology

Ω open set: If $a \in \Omega$, then $\exists\, r > 0$ such that $D_r(a) \subset \Omega$.

$D_r(a) = \{z : \text{dist}(z,a) < r\} = \{z : |z-a| < r\}$

Biggy: f is <u>complex diff'ble</u>:

$$\frac{f(z)-f(a)}{z-a} \longrightarrow L \quad \text{as} \quad z \to a$$

$L = f'(a)$

MA 530: Complex diff'ble fcns on an open set.

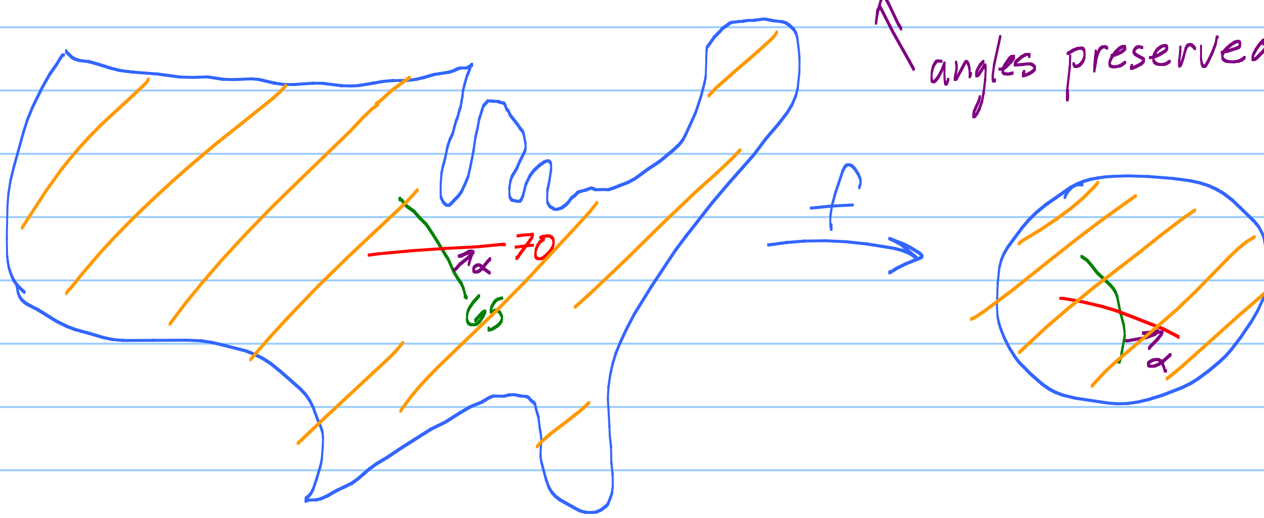
<u>Miracles:</u> 1) Fund Thm Alg

2) Riemann mapping theorem: Ω open connected set with no holes (simply connected), $\Omega \neq \mathbb{C}$. Then $\exists\, f : \Omega \to D_1(0)$ one-to-one, onto, complex diff'ble!

[f^{-1} $\mathbb{C}$-diff too, f' non-vanishing.

<u>And</u> f is <u>conformal</u>!]

angles preserved



<u>Def^n</u>: f $\mathbb{C}$ valued fcn on $D_r(z_0) - \{z_0\}$.

$\lim_{z \to z_0} f(z) = L$ means, given $\varepsilon > 0$, $\exists\, \delta > 0$ such that $|f(z) - L| < \varepsilon$ when $|z - z_0| < \delta$, $z \neq z_0$.

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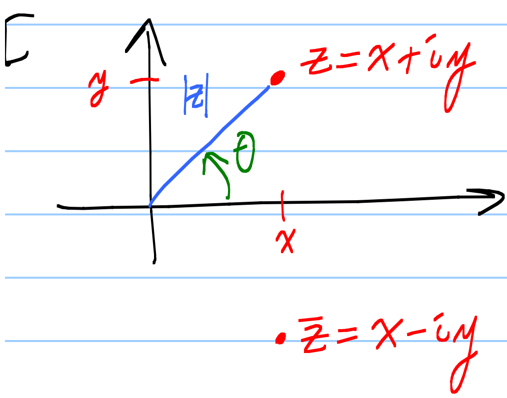

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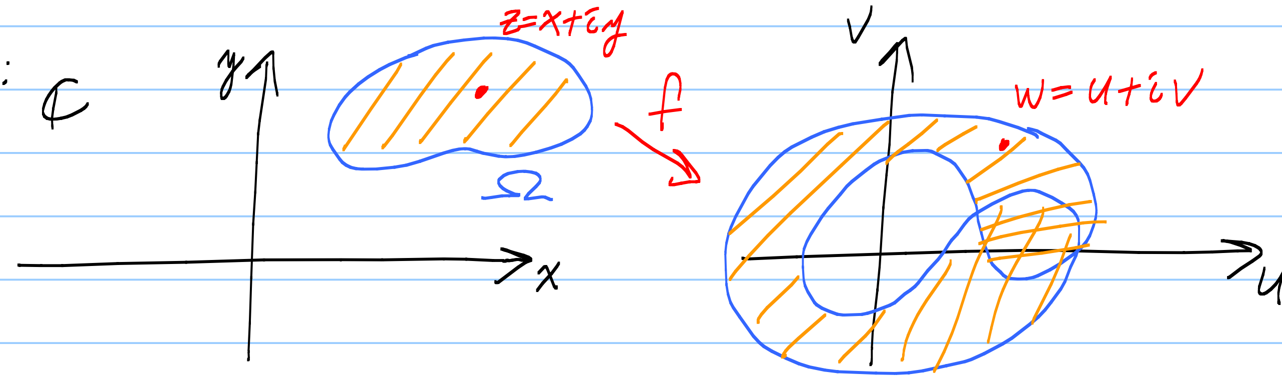

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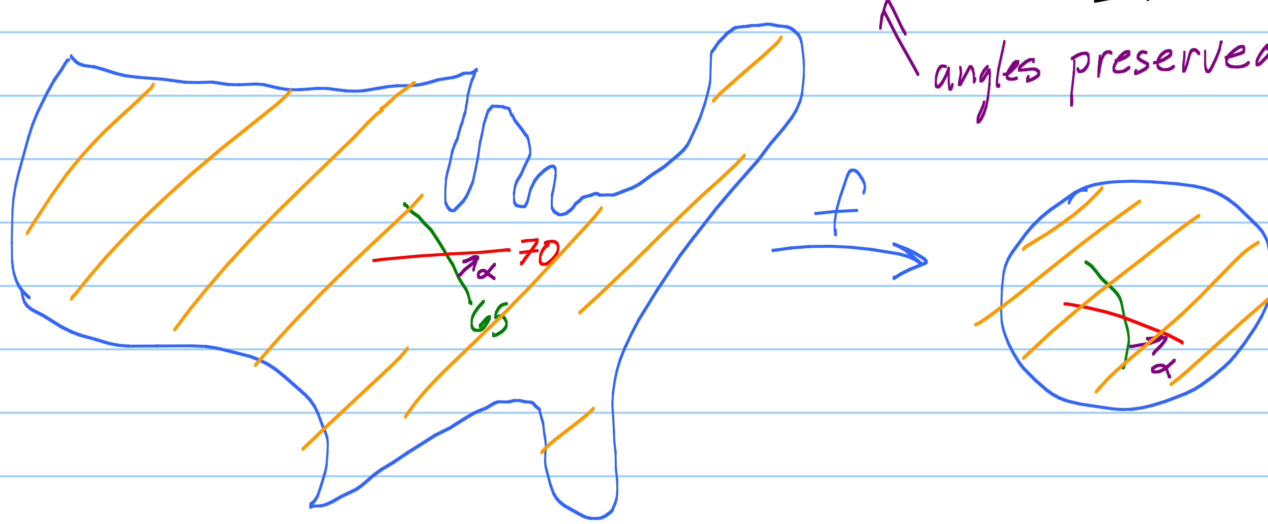

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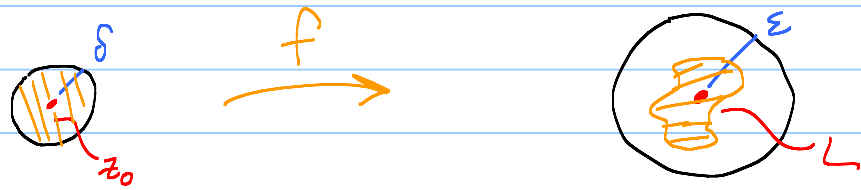
Lecture 2

HWK 1 due Thurs, Jan 20 at 11:59 pm in Gradescope

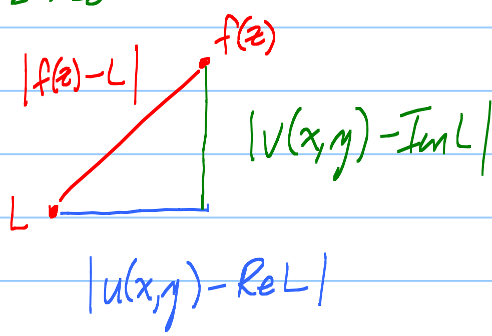
$\lim_{z \to z_0} f(z) = L$: Given $\varepsilon > 0$, $\exists\, \delta > 0$ such that $|f(z) - L| < \varepsilon$ when $|z - z_0| < \delta$, $z \neq z_0$.



Little lemma : $f(x+iy) = u(x,y) + i\,v(x,y)$ $\qquad z_0 = x_0 + iy_0$

$$\lim_{z \to z_0} f(z) = L \iff \begin{cases} \lim_{(x,y)\to(x_0,y_0)} u(x,y) = \operatorname{Re} L \\ \lim_{(x,y)\to(x_0,y_0)} v(x,y) = \operatorname{Im} L \end{cases}$$

Pf:



$\Longrightarrow$ $|\text{Leg}| \le |\text{Hyp}|$

$\Longleftarrow$ Pyth

Def^n: f is $\mathbb{C}$-diff'ble at a :

$$\lim_{z \to a} \underbrace{\frac{f(z) - f(a)}{z - a}}_{DQ} \text{ exists. Call limit } f'(a).$$

EX: $f(z) = z^2$ $\qquad DQ = \dfrac{z^2 - a^2}{z - a} = z + a \longrightarrow 2a$ as $z \to a$.

$f'(a) = 2a$. $\qquad f'(z) = 2z$.

Fact: Basic rules of limits and differentiability $\mathbb{R} \to \mathbb{R}$ go over word for word $\mathbb{C} \to \mathbb{C}$. Proofs too!

EX: $[fg]' = f'g + fg'$ etc. Chain rule too.

EX: $f(z)=\bar{z}$ $f(x+iy)=x-iy$ is not $\mathbb{C}$-diff'ble!

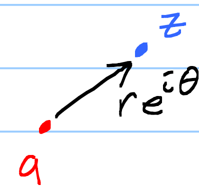
Facts: $|zw|=|z||w|$ $\overline{zw}=\bar{z}\cdot\bar{w}$

$\left|\frac{z}{w}\right|=\frac{|z|}{|w|}$, $w\neq 0$ $\overline{\left(\frac{z}{w}\right)}=\bar{z}/\bar{w}$, $w\neq 0$

$e^{i\theta}=\cos\theta+i\sin\theta$

$e^{-i\theta}=\cos\theta-i\sin\theta=\overline{e^{i\theta}}=\frac{1}{e^{i\theta}}$ ($N=-1$, De Moivre)

$DQ=\frac{\bar{z}-\bar{a}}{z-a}$



$z=a+re^{i\theta}$

$$=\frac{\overline{(re^{i\theta})}}{re^{i\theta}}=\frac{e^{-i\theta}}{e^{i\theta}}=e^{-i\theta}\cdot e^{-i\theta}=e^{-2i\theta}$$

Ouch! Limit as I slide z into a along a ray is diff for diff directions! So complex limit DNE.

Def^n: (Open set Ω) f is analytic on Ω if it is $\mathbb{C}$-diff at every pt in Ω. (holomorphic)

Def^n: Continuous: $\lim_{z\to z_0} f(z)=f(z_0)$

Thm: $\mathbb{C}$-diff'ble at a $\Rightarrow$ continuity at a.

Pf: $\frac{f(z)-f(a)}{z-a}=f'(a)+E(z)$ $\leftarrow$ defines $E(z)$

$\mathbb{C}$-diff'ble $\Rightarrow$ $E(z)\to 0$ as $z\to a$.

Solve for $f(z)$:

$$f(z) = f(a) + f'(a)(z-a) + E(z)\cdot(z-a)$$

$z \to a$

$$f(a) + f'(a)\cdot 0 + 0\cdot 0 = f(a) \checkmark$$

$$f(z) = (\mathbb{C}\text{-linear thing}) + (\text{Really small})$$

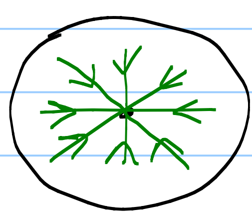
<u>Cauchy-Riemann equations</u> (CR-eqns)

If $f(x+iy) = u(x,y) + iv(x,y)$ is $\mathbb{C}$-diff'ble at $z_0 = x_0 + iy_0$, then the first partials of u and v <u>exist</u> and

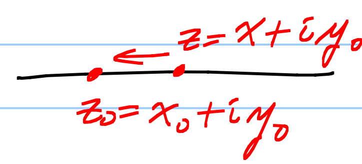
CR-eqns $\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$ at (x_0, y_0)

<u>Notation</u>: $u_x = \dfrac{\partial u}{\partial x}$ $\quad u^0 = u(x_0, y_0)$ $\quad u_x^0 = \dfrac{\partial u}{\partial x}(x_0, y_0)$, etc

<u>Pf</u>:



<u>Step 1</u>: Let $z \to z_0$ along horiz



$$DQ = \frac{[u(x,y_0) + iv(x,y_0)] - [u^0 + iv^0]}{(x+iy_0) - (x_0 + iy_0)}$$

$$= \left(\frac{u(x,y_0) - u^0}{x - x_0}\right) + i\left(\frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}\right)$$

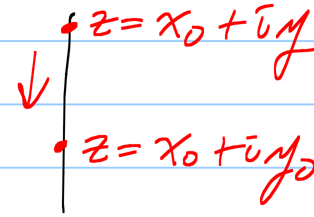
Little lem

$\operatorname{Re} f'(z_0) \qquad\qquad \operatorname{Im} f'(z_0)$

So first partials exist and

$$\boxed{f'(z_0) = u_x(x_0,y_0) + i\, v_x(x_0,y_0)}$$ Red box #1

Step 2 Vertical



$$DQ = \frac{\ell\ell\ell}{(x_0+iy)-(x_0+iy_0)} = \frac{1}{i}\cdot\frac{\ell\ell\ell}{(y-y_0)}$$

$$= \underbrace{\frac{1}{i}}_{-i}\ \underbrace{\frac{u(x_0,y)-u(x_0,y_0)}{y-y_0}} + \frac{v(x_0,y)-v(x_0,y_0)}{y-y_0}$$

$$\operatorname{Re} f'(z_0) \quad + \quad i\operatorname{Im} f'(z_0)$$

So u_y, v_y exist and

$$\boxed{f'(z_0) = v_y(x_0,y_0) - i\, u_y(x_0,y_0)}$$ Red box #2

Red box #1 = #2. C-R eqns ✓

EX: $\bar{z} = x - iy$. One of C-R eqns fail.

EX: $E(x+iy) = \underbrace{e^x \cos y}_{u} + i\,\underbrace{e^x \sin y}_{v}$ (e^z)

C-R eqns hold! Is $E(z)$ analytic? Yes, because

Partial converse to C-R eqns: Suppose u, v are C^1-smooth (meaning that u, v, and all first partials are cont.) on Ω, open and they satisfy the C-R eqns. Then $f = u + iv$ is analytic on Ω.

Looman-Menchoff thm:
1) u, v continuous on Ω
2) first partials exist and
3) satisfy the C-R eqns.

Then $u + iv$ is analytic on Ω.

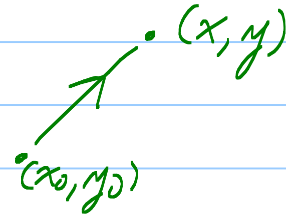
Narasimhan: Complex analysis in one variable, p. 43-50.

Calculus fact: Suppose u is C^1-smooth.

$$u(x,y) = u^0 + u_x^0 \cdot (x - x_0) + v_y^0 \cdot (y - y_0) + R_u(x,y)$$

where
$$\lim_{(x,y)\to(x_0,y_0)} \frac{R_u(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

Pf: (A): $u(x,y) - u^0 = \displaystyle\int_0^1 \frac{\partial}{\partial t}\Big[u\big(x_0+t(x-x_0),\ y_0+t(y-y_0)\big)\Big]\,dt$



$$= \int_0^1 u_x(\quad)\cdot(x-x_0) + u_y(\quad)\cdot(y-y_0)\ dt$$

(B): $u_x^0\cdot(x-x_0) + u_y^0\cdot(y-y_0)$

$$= \int_0^1 u_x^0\cdot(x-x_0) + u_y^0\cdot(y-y_0)\ dt$$

$$R(x,y) = (A)-(B) = \int_0^1 \underbrace{\big[u_x(\,)-u_x^0\big]}_{\varepsilon}\cdot(x-x_0) + \underbrace{\big[u_y(\,)-u_y^0\big]}_{\varepsilon}(y-y_0)\,dt$$

$$\frac{|R(x,y)|}{|z-z_0|} \le \int_0^1 \varepsilon\frac{|x-x_0|}{|z-z_0|} + \varepsilon\frac{|y-y_0|}{|z-z_0|}\ dt$$

$$\le 2\varepsilon$$

Lecture 3 C-R eqns and the complex exponential

Last time: u C^1-smooth:

$$u(x,y) = \underbrace{u(x_0,y_0)}_{u^0} + \underbrace{\frac{\partial u}{\partial x}(x_0,y_0)}_{u_x^0}(x-x_0) + \underbrace{\frac{\partial u}{\partial y}(x_0,y_0)}_{u_y^0}(y-y_0) + R_u(x,y)$$

where $\dfrac{R_u(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \longrightarrow 0$ as $(x,y) \to (x_0,y_0)$

$z = x+iy$ $\quad z_0 = x_0 + iy_0$

$\sqrt{} = |z-z_0|$

Theorem: u,v C^1-smooth and satisfy the C-R eqns.

Then $f(x+iy) = u(x,y) + iv(x,y)$ is analytic.

Pf: $$DQ = \frac{f(z)-f(z_0)}{z-z_0} = \frac{(u+iv)-(u^0+iv^0)}{z-z_0}$$

$$= \frac{\left[u_x^0\cdot(x-x_0) + \overset{-v_x^0}{\overset{\parallel}{u_y^0}}\cdot(y-y_0)\right] + i\left[v_x^0\cdot(x-x_0) + \overset{u_x^0}{\overset{\parallel}{v_y^0}}\cdot(y-y_0)\right]}{z-z_0} + \underbrace{\frac{R_u+iR_v}{z-z_0}}_{\mathcal{E}}$$

Aha!

$$= \frac{[u_x^0 + iv_x^0]\cdot\cancel{[(x-x_0)+i(y-y_0)]}}{\cancel{z-z_0}}$$

$$= (u_x^0 + iv_x^0) + \mathcal{E}$$

where $|\mathcal{E}| \le \dfrac{|R_u|}{|z-z_0|} + \dfrac{|R_v|}{|z-z_0|} \longrightarrow 0$ as $z \to z_0$.

Cor: $E(x+iy) = \underbrace{e^x\cos y}_{u} + i\underbrace{e^x \sin y}_{v}$ is analytic on $\mathbb{C}$

entire

Properties of $E(z)$

0) E is entire

1) $E'(z) = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} = u + iv = E(z)$

2) $E(0) = 1$

Fun thing: (1), (2) determine $E(z)$!

$$\begin{cases} u_x = u & v_x = v \\ v_y = u & -u_y = v \end{cases}$$

$u(\vec{0}) = 1$, $v(\vec{0}) = 0$ pins down u, v

e.g. $u_x = u \Rightarrow u(x,y) = Ce^x$ ← $C = C(y)$

3) $E(z+w) = E(z)E(w)$

Group homomorphism: (add grp) $\to$ (mult grp)

4) $E(-z) = 1/E(z)$ $(w = -z)$

5) $E(z)$ is never zero

6) $E(z-w) = E(z)/E(w)$

7) $E(x) = e^x$, $x \in \mathbb{R}$. $E(z)$ extends e^x, $\mathbb{R}$ to $\mathbb{C}$.

8) $E(i\theta) = \cos\theta + i\sin\theta$

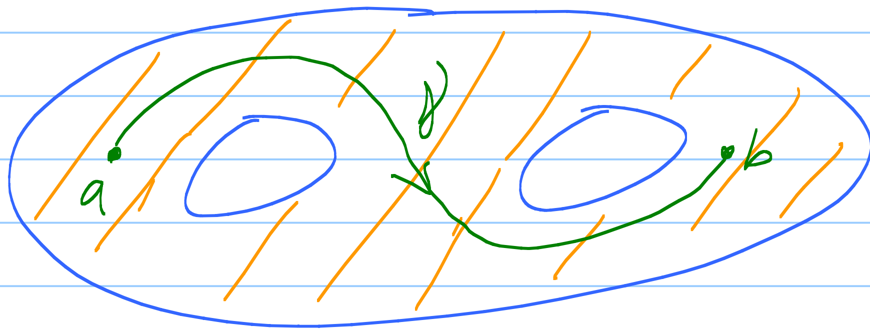
#3 $\Rightarrow$ DeMoivre's.

9) $E(z) = 1 \iff z = 2\pi n i$, $n \in \mathbb{Z}$

Just need to prove #3.

Lemma 1: If f analytic on $D_r(a)$ and $f' \equiv 0$, then f is const.

Wish pf: $f' = \begin{cases} u_x + i v_x \\ v_y - i u_y \end{cases} \equiv 0 \implies \begin{matrix} \nabla u \equiv 0 \\ \nabla v \equiv 0 \end{matrix}$

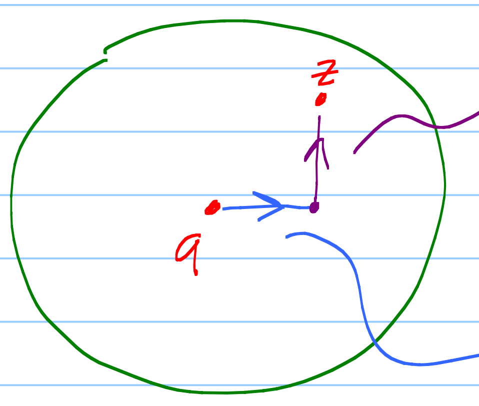


$$\int_\gamma \nabla u \cdot d\vec{s} = u(b) - u(a)$$ ← Ouch! Need to know u C^1-smooth.

Freshman calculus: $h(t)$ cont. on $[a,b]$ and diff'ble on (a,b) and $h' \equiv 0$ on (a,b). Then h is const. on $[a,b]$.

Pf: MVT

Pf of Lem 1:



$u_y, v_y \equiv 0$
u, v const on vert

$u_x, v_x \equiv 0$
$\implies u, v$ const on horiz

Lemma 2: If f is analytic on $D_R(0)$ and

$$f'(z) = k f(z), \quad k \in \mathbb{C} \text{ is a const.}$$

Then $f(z) = C E(kz)$ where $C = f(0)$.

Pf: $f' - kf = 0$. Int. factor: $E(-kz)$

$$E(-kz)f'(z) \underbrace{- kE(-kz)}\, f(z) \equiv 0$$

$$\frac{d}{dz}E(-kz) = \underbrace{E'(-kz)}_{E(-kz)} \underbrace{\frac{d}{dz}[-kz]}_{-k}$$

$$\frac{d}{dz}\Big[E(-kz)f(z)\Big] \equiv 0$$

So Lem 1 $\Longrightarrow$ $\boxed{E(-kz)f(z) \equiv C}$, const.

Plug in $z=0$ to see that $f(0) = C$.

Stop the proof! Start over using $f(z) = E(kz)$.

Check: $f'(z) = E'(kz)\frac{d}{dz}[kz] = kE(kz) = kf(z)$ ✓

Green box says: $E(-kz) \underbrace{f(z)}_{E(kz)} \equiv \underbrace{C}_{C=f(0)=1}$

$$E(-kz)E(kz) \equiv 1$$

So $E(kz) = 1/E(-kz)$

Back to the proof:

Green box $\quad E(-kz)f(z) = f(0) \quad$ [orig f]

so $\quad f(z) = \dfrac{f(0)}{E(-kz)} = f(0)E(kz)$ ✓

Pf of #3: Let $f(z) = E(z+w)$, w const.

$$f'(z) = E'(z+w)\frac{d}{dz}[z+w]$$

$$= E(z+w) = f(z) \;!$$

Lemma 2 $\Rightarrow$ $\underbrace{f(z)}_{E(z+w)} = \underbrace{f(0)}_{E(0+w)} E(1\cdot z)$

$$E(z+w) = E(w)E(z) \quad \checkmark$$

Big fact: f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f'' analytic $\Rightarrow \cdots$

Way false $\mathbb{R}\to\mathbb{R}$!

implying: f' exists $\Rightarrow$ f cont
f'' exists $\Rightarrow$ f' cont
$\vdots$

All red boxes $\Rightarrow$ u, v are C^∞-smooth.

Assuming big fact (or just that u and v are C^2-smooth), then $f = u + iv$ analytic $\Rightarrow$ u, v harmonic!

Why: C-R eqns $\begin{cases} u_x = v_y & (A) \\ u_y = -v_x & (B) \end{cases}$

$\frac{\partial}{\partial x}(A)$: $u_{xx} = v_{xy}$

$\frac{\partial}{\partial y}(B)$: $u_{yy} = -v_{yx}$

v C^2 smooth $\Rightarrow$ $v_{xy} = v_{yx}$

$\Delta u = u_{xx} + u_{yy} = 0$ $\qquad$ u harmonic!

Similarly, $\frac{\partial}{\partial y}(A) - \frac{\partial}{\partial x}(B)$ shows that v is harmonic.

Big fact: u C^2-smooth and harmonic $\Rightarrow$ u C^∞-smooth.

Problem: Given harmonic u, does there exist harmonic v such $u+iv$ is analytic? [v is called a harmonic conjugate for u.]

Ans: Yes, when Ω has no holes.

How to cook up v:

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \qquad v_x = -u_y, \quad v_y = u_x$$

v is a potential fcn for field $\vec{F} = (-u_y, u_x)$

Lecture 4   Complex power series   HWK 1 due Thurs 11:59 pm in GS
HWK 2 due the next Thurs.

How to cook up harmonic conjugate v for harmonic C^2 harmonic fcn u

Want v so that $u+iv$ is analytic: C-R equns $\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix}$ $\begin{matrix} v_x = -u_y \\ v_y = u_x \end{matrix}$

$\nabla v = \vec{F}$ where $\vec{F} = (-u_y, u_x)$

Prob: Find v with $\nabla v = \vec{F}$

potential fcn for $\vec{F}$

in $\mathbb{R}^3$: $\iff$ Curl $\vec{F} \equiv 0$ $\underbrace{\nabla v = \vec{F}}_{(v_x, v_y)}$

$\underbrace{\text{in } \mathbb{R}^2}_{\Omega \text{ with no holes}}$: $\iff$ $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$ $\leftarrow (v_{yx} = v_{xy})$

Shown in MA 261.   Check: $\dfrac{\partial}{\partial y}\big(\underbrace{-u_y}_{F_1}\big) \overset{?}{=} \dfrac{\partial}{\partial x}\big(\underbrace{u_x}_{F_2}\big)$

Yes! $u_{yy} + u_{xx} \equiv 0$ ✓

Fact: v is unique up to $+C$.

Power series:

Lem: $\sum_1^\infty a_n$ conv $\Longrightarrow$ $a_n \to 0$.

Pf: $S_N = \sum_1^N a_n$ $a_N = S_N - S_{N-1} \to L - L$ as $N \to \infty$ ✓

Def^n: $\sum_1^\infty a_n$ conv absolutely if $\sum_1^\infty |a_n| < \infty$.

Lemma: abs conv $\Longrightarrow$ conv.

Pf: Abs conv $\Longrightarrow$ S_N Cauchy seq $\Longrightarrow$ $\sum_1^\infty a_n$ conv.

Big Fact: Power series have a radius of conv. R, meaning

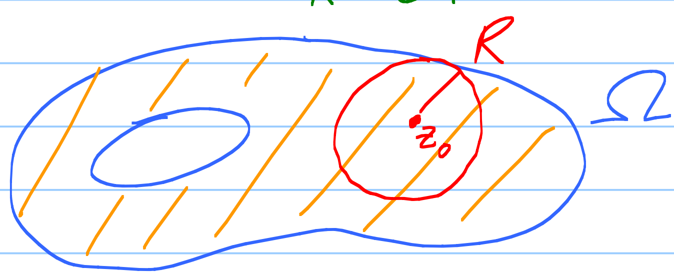
1) $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ conv. abs. in $D_R(z_0)$

2) conv uniformly on $D_r(z_0)$, $r < R$.

3) div if $|z-z_0| > R$.

$R=0$ case: only conv at z_0.
$R=\infty$ case: conv for all $z \in \mathbb{C}$.

Big Fact: Analytic fcns are given locally by conv power series with $R>0$.



Pf: later

Hadamard's Formula: $$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

Def^n: $$\limsup_{n\to\infty} \overset{\in\mathbb{R}}{r_n} = \lim_{N\to\infty} \underbrace{\sup\{r_n : n \geq N\}}_{\text{non-increasing in } N}$$

So limit exists as $N \to \infty$.
(Might be $-\infty$.)
(But, if $r_n = \sqrt[n]{|a_n|}$, $\limsup \geq 0$.)

Later: $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Hadamard's: $$\frac{1}{R} = \limsup_{n\to\infty} \frac{1}{\sqrt[n]{n!}}$$

Stirling's formula:

$$\underbrace{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n < n!}\ < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{4n}\right)$$

$$\frac{1}{\sqrt[n]{n!}} < \frac{1}{(2\pi n)^{1/2n}\left(\frac{n}{e}\right)} \longrightarrow 0 \quad \text{as } n\to\infty.$$

So $\frac{1}{R}$ "=" 0 and $R=\infty$.

Ratio test: much easier!

Geometric series: $\sum_{n=0}^{\infty} z^n$

$|z|\geq 1$ terms $\not\to 0$. So div.

$$S_N = 1+z+\cdots+z^N$$
$$zS_N = \quad z+\cdots+z^N+z^{N+1}$$
$$(1-z)S_N = 1-z^{N+1}$$

$$S_N = \frac{1}{1-z} + \underbrace{\frac{-z^{N+1}}{1-z}}_{E_N(z)}$$

Δ ineq: $|z+w|\leq|z|+|w|$ ⟵ Numerator estimate

$|z-w|\geq\big||z|-|w|\big|$ ⟵ Denom est

$|z+w|\geq\big||z|-|w|\big|$ (replace w by $-w$)

If $|z|<r<1$, then

$$|1-z|\geq\big|1-|z|\big| = 1-|z| > 1-r.$$

So $$|E_N(z)| = \frac{|z|^{N+1}}{|1-z|} \leq \underbrace{\frac{|z|^{N+1}}{1-|z|}}_{\text{see } \sum z^n \text{ conv}} < \underbrace{\frac{r^{N+1}}{1-r}}_{\text{see unif conv.}} \to 0 \text{ as } N\to\infty$$

Note: $E_N(x) = \frac{-x^{N+1}}{1-x} \longrightarrow -\infty$ as $x \nearrow 1$.

Do not get unif conv on $D_1(0)$.

Pf of Hadamard's. Case $0 < R < \infty$. Assume $z_0 = 0$.

Pick ρ, r with $|z| < r < \rho < R$ = Hadamard's

$$\frac{1}{R} < \frac{1}{\rho}$$

$\longleftarrow \sup_{n \geq N} \sqrt[n]{|a_n|}$

sliding down to $\frac{1}{R}$

So, $\exists\, N$ such that $\sqrt[n]{|a_n|} < \frac{1}{\rho}$ when $n \geq N$.

$$|a_n| < \frac{1}{\rho^n} \quad \text{when } n \geq N$$

$$|a_n z^n| < \left(\frac{|z|}{\rho}\right)^n \quad \text{when } n \geq N.$$

< 1

Aha! Comparison with geom series $\sum \left(\frac{|z|}{\rho}\right)^n$ shows $\sum_0^\infty a_n z^n$ is abs. conv., Can make r as close to R as desired.

So $\sum a_n z^n$ conv abs in $D_R(0)$.

Next: Get unif conv on $D_r(0)$. $f(z) = \sum_{n=0}^\infty a_n z^n$

$$\left| f(z) - \sum_{n=0}^{M} a_n z^n \right| = \left| \sum_{n=M+1}^{\infty} a_n z^n \right|$$

$$\le \sum_{n=M+1}^{\infty} |a_n z^n| \quad \text{and if } M > N:$$

$$\le \sum_{n=M+1}^{\infty} \left(\frac{|z|}{\rho}\right)^n = \frac{\left(\frac{|z|}{\rho}\right)^{M+1}}{1-\left(\frac{|z|}{\rho}\right)}$$

$$< \frac{\left(\frac{r}{\rho}\right)^{M+1}}{1-\frac{r}{\rho}}$$

$$\to 0 \text{ as } M \to \infty.$$

Get unif conv on $D_r(0)$.

<u>Last step</u>: Div if $|z| > R$.

$$\frac{1}{|z|} < \frac{1}{R} \le \sup_{n \ge N} \sqrt[n]{|a_n|}$$

So, for each N, $\exists\, n$ with $n \ge N$ such that

$$\frac{1}{|z|} < \frac{1}{R} \le \sqrt[n]{|a_n|}$$

$$1 \le \sqrt[n]{|a_n|}\, |z| \quad \leftarrow \text{raise to } n\text{-th power}$$

$$1 \le |a_n z^n|$$

Aha! This shows that terms $a_n z^n \not\to 0$.
So series diverges.
$R = 0, \infty$ cases are similar and easier.

<u>EX</u>: $\sum_0^\infty n!\, z^n$. $R = \infty$

EX: $\sum_{0}^{\infty} \frac{z^n}{n^2}$, $R=1$ Converges uniformly on $\underbrace{\overline{D_1(0)}}_{\{z : |z| \leq 1\}}$

closed unit disc.

Lecture 5 Complex integration

HWK 2 due in GS
Thurs, Jan. 27, 11:59 pm

Ratio test Suppose $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right|$ exists $= L$

$L<1$: $\sum_1^\infty z_n$ converges. ← compare tail end of series with convergent geom series

$L>1$: $\sum_1^\infty z_n$ diverges ← terms $\not\to 0$

$L=1$: test fails : e.g. $\sum \frac{1}{n^2}$ conv, $\sum \frac{1}{n}$ div

Remark: Suppose $\sum_0^\infty a_n z^n$ has R. of C. R. Since $\lim_{n\to\infty} \sqrt[n]{n} = 1$,

Hadamard's $\Rightarrow$ $\sum_{n=1}^\infty n a_n z^{n-1}$ has same R. of C. R.

Big fact: Power series converge to analytic fcns inside $D_R(z_0)$,

$R =$ R. of C., and can diff'tiated term by term,

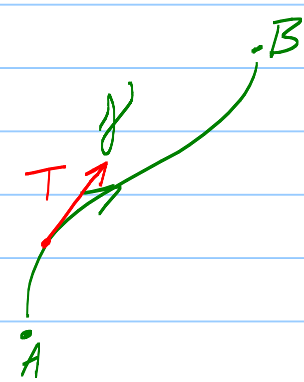
R. of C. of f' = R. of C. of f.

Stein p. 16–18.

Bigger fact: Analytic fcns are given locally by conv. power series.

Pf's: Later, sneaky.

Integration along paths



γ: $z(t) = x(t) + i y(t)$

$a \le t \le b$

$z : [a,b] \to \mathbb{C}$

$\underline{\text{Def}^n}$: $tr(\gamma)$ = "trace of γ" $= \{z(t) : t \in [a,b]\}$

$$z'(t_0) = \lim_{t \to t_0} \frac{z(t) - z(t_0)}{t - t_0} = \lim (DQ - x) + i\,(DQ - y)$$

$$= x'(t_0) + i\,y'(t_0)$$

Complex tangent vector $T = z'(t)$

Unit complex tang vect $\hat{T} = z'(t) / |z'(t)|$

$\underline{\text{Chain rule \#1}}$ $\quad g : \Omega_1 \to \Omega_2$ analytic

$f : \Omega_2 \to \mathbb{C}$ analytic

$h(z) = f(g(z))$ is analytic and $h'(z) = f'(g(z))\,g'(z)$

Pf: HWK 1 or Baby Rudin.

$\underline{\text{Chain rule \#2}}$: f analytic on an open set containing $tr(\gamma)$ and $z(t)$ is diff$^{\text{ble}}$, then

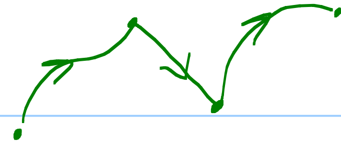
$$\frac{d}{dt} f(z(t)) = f'(z(t))\,z'(t)$$

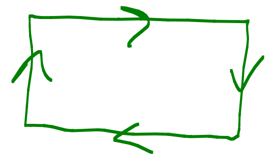
$\underline{\text{Pf}}$: $f(z(t)) = u(x(t), y(t)) + i\,v(x(t), y(t))$

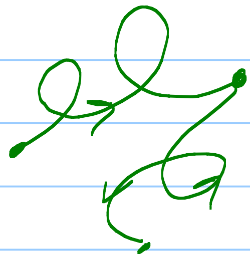
$$\frac{d}{dt} f(z(t)) = \left(u_x \dot{x} + \overset{-v_x}{\overset{\|}{u_y}}\, \dot{y}\right) + i\left(v_x \dot{x} + \overset{u_x}{\overset{\|}{v_y}}\, \dot{y}\right)$$

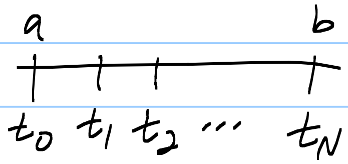
$$= \underbrace{[u_x + i\,v_x]}_{\text{red box \#1}} \cdot (\dot{x} + i\,\dot{y})$$

Piecewise $\mathcal{C}^1$ paths









$z(t)$ cont on $[a,b]$,
$z(t)$ continuously diff'ble on (t_n, t_{n+1}),
$z'(t)$ extends continuously to $[t_n, t_{n+1}]$

Def^n: $$\int_a^b z(t)\,dt = \lim_{\Delta t \to 0} \sum_{n=1}^{N} z(t_n)\,\Delta t_n$$

$$= \int_a^b x(t)\,dt + i\int_a^b y(t)\,dt$$

EX: $$\int_0^{2\pi} e^{it}\,dt = \int_0^{2\pi} \cos t\,dt + i\int_0^{2\pi} \sin t\,dt = 0$$

Fund Thm Calc #1: $$\int_a^b z'(t)\,dt = z(b) - z(a)$$

when $z(t)$ is continuously diff'ble.

True for piecewise $\mathcal{C}^1$ $z(t)$ too:

$$\int_a^b z'(t)\,dt = \sum_{n=0}^{N-1} \underbrace{\int_{t_n}^{t_{n+1}} z'(t)\,dt}_{z(t_{n+1}) - z(t_n)} = z(t_N) - z(t_0)$$

"telescopes"

Pf real and imag parts. Freshman calc.

Lemma: $\left| \int_a^b z(t)\,dt \right| \le \int_a^b |z(t)|\,dt$

Think: $\left| \sum z(t_n)\Delta t_n \right| \le \sum |z(t_n)|\,\Delta t_n$

Let $\Delta t \to 0$.

Pf: $\int_a^b z(t)\,dt = re^{i\theta}$ $\qquad \lambda = e^{-i\theta}$

$$\left| \int_a^b z(t)\,dt \right| = r = \lambda \int_a^b z(t)\,dt$$

$$\overset{\text{check}}{=} \int_a^b \lambda z(t)\,dt$$

$$= \int_a^b \underbrace{\operatorname{Re} \lambda z(t)}_{\le |\lambda z(t)| = |\lambda|\,|z(t)|} \,dt + \underbrace{i \int_a^b \operatorname{Im} \lambda z(t)\,dt}_{\text{must be zero}}$$

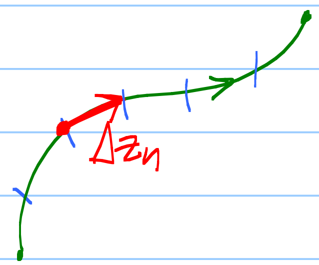
$$\le \int_a^b |z(t)|\,dt \quad \checkmark$$

Def^n: Suppose $f: tr(\gamma) \to \mathbb{C}$ is continuous.

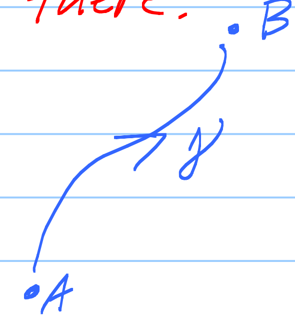
$$\int_\gamma f\,dz \overset{\text{def}^n}{=} \int_a^b f(z(t))\, \underbrace{z'(t)\,dt}_{dz}$$

Konrad Knopf

$$\int_\gamma f\,dz = \lim_{\Delta t \to 0} \sum f(z(t_n))\, \underbrace{\Delta z_n}_{\substack{z(t_{n+1}) - z(t_n) \\ \approx z'(t_n)\,(t_{n+1}-t_n)}}$$



<u>Fund Thm Calc #2</u>: f analytic on an open set containing $tr(\gamma)$ <u>and</u> f' is continuous there.



Then $\displaystyle\int_\gamma f'\,dz = f(B) - f(A)$

<u>Note</u>: Later, we will show that

$$f \text{ analytic} \Rightarrow f' \text{ analytic} \Rightarrow f'' \text{ analytic} \Rightarrow \cdots$$

$\Rightarrow f'$ continuous

Don't need red stuff.

<u>Pf</u>: $\displaystyle\int_\gamma f'\,dz = \int_a^b \underbrace{f'(z(t))\; z'(t)}_{= \frac{d}{dt} f(z(t))}\, dt$ Chain rule #2

$= f(z(b)) - f(z(a))$ Fund Thm Calc #1

✓

Lemma (Basic estimate)

$$\left|\int_\gamma f\,dz\right| \le \left(\max_{z\in tr(\gamma)} |f(z)|\right)\cdot \text{Length}(\gamma)$$

where $\text{Length}(\gamma) = \int_a^b \sqrt{\dot{x}^2+\dot{y}^2}\,dt = \int_a^b |z'(t)|\,dt.$

Pf: $\left|\int_\gamma f\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right|$

$$\le \int_a^b \underbrace{|f(z(t))|}_{\le M = \max_\gamma |f|}\,|z'(t)|\,dt$$

$$\le M \underbrace{\int_a^b |z'(t)|\,dt}_{=L} \quad \checkmark$$

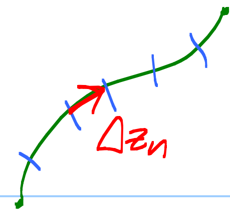
Fact: γ: $z(t)$ $\quad a\le t\le b$

$-\gamma$: $z(b+t(a-b))$ $\quad 0\le t\le 1$

$$\boxed{\int_{-\gamma} f\,dz = -\int_\gamma f\,dz}$$

Pf: Write out. Equate real and imag parts.
Use real variables change of var formula.

Or: Konrad Knopf

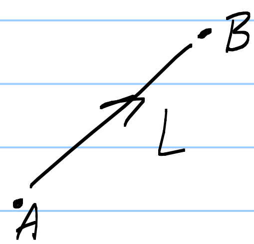


For $-\gamma$, $\Delta z_n = -\Delta z_n$.

Note: Assume $z'(t) \neq 0$ so we don't stop.

Fact: $\int$'s are independent of the parameterization.

EX:

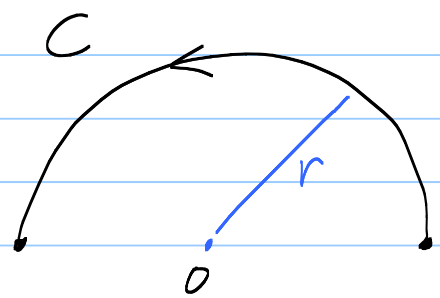


$L: \; z(t) = A + t(B-A)$

$0 \leq t \leq 1$

$z'(t) = B - A$

EX:



$C: \quad z(t) = re^{it}$

$0 \leq t \leq \pi$

$z'(t) = ire^{it}$

$$\frac{d}{dt} E(it) = \underbrace{E'(it)}_{E(it)} \underbrace{\frac{d}{dt}(it)}_{i} \quad \checkmark$$

EX: $\displaystyle\int_C z^n \, dz$, $n = -1$

$$\frac{d}{dz}\left[\frac{z^{n+1}}{n+1}\right]$$

Use Fund Thm Calc #2.

Lecture 6 Goursat's Lemma

HWK 2 due in GS Thurs 11:59 pm

$n \in \mathbb{Z}, \neq$

Important fact: γ closed piecewise C^1 path, then

$$\int_\gamma z^n\,dz = \int_\gamma \frac{d}{dz}\left[\frac{1}{n+1} z^{n+1}\right] dz = \left[\frac{1}{n+1} z^{n+1}\right]_{START}^{END} = \; = 0$$

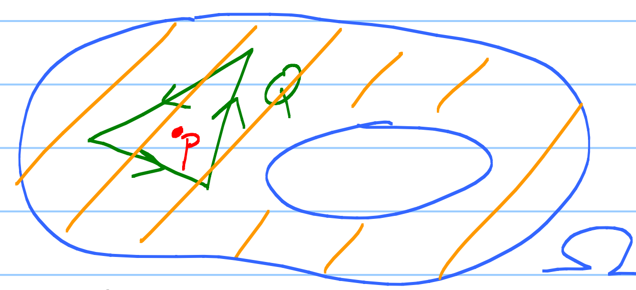
Fund Thm Calc #2

Consequently, $\int_\gamma P(z)\,dz = 0$, γ closed, P complex poly

In particular, $\int_\gamma Az + B\,dz = 0.$

Goursat's Lemma $\Omega \subset \mathbb{C}$ open, $p \in \Omega$, $f : \Omega \to \mathbb{C}$ continuous on Ω (including at p) and analytic on $\Omega - \{p\}$, Q is a Δ in Ω (and $\overline{\Delta} \subset \Omega$).

Then $\int_Q f\,dz = 0.$



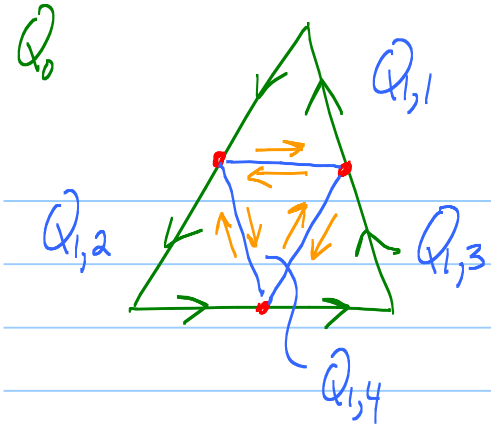
Abuse of notation: Q = open triangle Δ

$Q = \Delta$ piecewise C^1 boundary curve

$bQ = tr(Q)$ boundary of Q

$\overline{Q}$ = closed $\overline{\Delta}$

Pf: Case p outside $\overline{Q}$. Write $Q_0 = Q$.



• = midpoints of sides

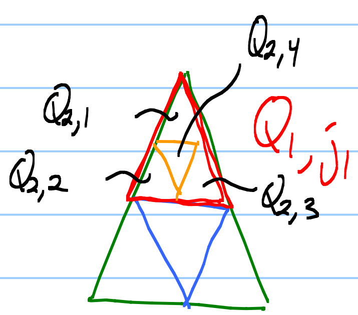
Aha! $I = \int_{Q_0} f \, dz = \sum_{j=1}^{4} \int_{Q_{1,j}} f \, dz$

Baby lemma: If $I = a_1 + a_2 + a_3 + a_4$, then $\exists j$ such that $|a_j| \geq \frac{|I|}{4}$.

Pf: $I = 0$, duh! $I \neq 0$: $|I| \leq \sum_1^4 |a_j|$.

If all $|a_j| < \frac{|I|}{4}$, ↯ ($|I| < |I|$. ↯)

So, $\exists j_1$ such $\left| \int_{Q_{1,j_1}} f \, dz \right| \geq \frac{|I|}{4}$.



Let $Q_1 = Q_{1,j_1}$.

Repeat!

Get $\int_{Q_1} f \, dz = \sum_1^4 \int_{Q_{2,j}} f \, dz$

$\exists j_2$ such that

$$\left| \int_{Q_{2,j_2}} f \, dz \right| \geq \frac{1}{4} \underbrace{\left| \int_{Q_1} f \, dz \right|}_{\geq \frac{|I|}{4}} \geq \frac{|I|}{4^2}$$

Continue! Get

$$\overline{Q_0} \supset \overline{Q_1} \supset \overline{Q_2} \supset \cdots$$

$$\left| \int_{Q_n} f \, dz \right| \geq \frac{|I|}{4^n}$$

Topology lemma $K_n \neq \emptyset$ compact in $\mathbb{R}^2$,

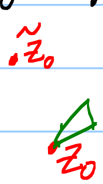
$$K_0 \supset K_1 \supset K_2 \supset \cdots$$

Then $\bigcap_{n=0}^{\infty} K_n \neq \emptyset$.

Furthermore, if $\operatorname{diam}(K_n) \to 0$ as $n \to \infty$, then

$\bigcap_{n=0}^{\infty} K_n = \{z_0\}$, a single pt in K_0.

Δ pf: Take leftmost (or lower leftmost corner if "leftmost" is a vert line) of each Δ. x-coord is nondecreasing, bdd above. They converge to x_0. (completeness of $\mathbb{R}$). Similarly y-coord converge to y_0. $z_0 = x_0 + iy_0$ is the pt. No other z_0 because:



Q_n : $\operatorname{diam}(Q_n) < \operatorname{dist}(\tilde{z}_0, z_0)$

Back to Goursat pf: $\bigcap_{n=0}^{\infty} \overline{Q_n} = \{z_0\}$, $z_0 \in \overline{Q_0}$.

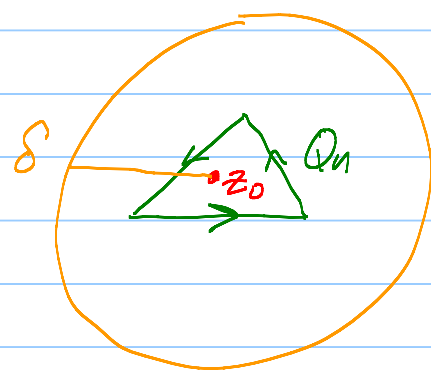
$$\frac{f(z)-f(z_0)}{z-z_0} = f'(z_0) + E(z)$$ ← defines $E(z)$, $z \neq z_0$. Define $E(z_0)=0$.

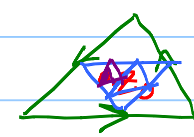
f complex diff'ble at $z_0 \neq p$. So $E(z) \to 0$ as $z \to z_0$.

$$f(z) = \underbrace{f(z_0) + f'(z_0)(z-z_0)}_{Az+B} + \underbrace{E(z)(z-z_0)}_{\text{really small near } z_0}$$ ← true at $z=z_0$ too.

Note: $E(z)$ is continuous on Ω.

Let $\varepsilon > 0$. $\exists\, \delta > 0$ such that $|E(z)| < \varepsilon$ when $|z-z_0| < \delta$. $\exists\, N$ such that $\overline{Q_n} \subset D_\delta(z_0)$ if $n > N$.





Now $$\int_{Q_n} f\,dz = \underbrace{\int_{Q_n} Az+B\,dz}_{=0} + \int_{Q_n} E(z)(z-z_0)\,dz$$

So $$\left|\int_{Q_n} f\,dz\right| \le \underbrace{\left(\max_{z\in bQ_n} |E(z)(z-z_0)|\right)}_{\le\, \varepsilon \max_{z \in bQ_n}|z-z_0| \;\le\, \varepsilon\, \text{Diam}(Q_n)} \text{Length}(bQ_n)$$

Aha! $\quad \text{Diam}(Q_n) = \frac{1}{2^n}\text{Diam}(Q_0)$

$\quad \text{Length}(Q_n) = \frac{1}{2^n}\text{Length}(bQ_0)$

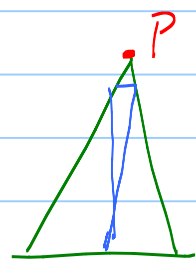
Hence, if $n > N$, then

$$\frac{|I|}{4^n} \leq \left|\int_{Q_n} f\,dz\right| \leq \varepsilon \frac{1}{4^n}\text{Diam}(Q_0)\text{Length}(bQ_0)$$

$$|I| \leq c\varepsilon \quad \text{where } c = \text{Diam}(Q_0)\text{Length}(bQ_0).$$

ε, arbitrary! Must be that $I = 0$. ✓

Case $p =$ vertex:



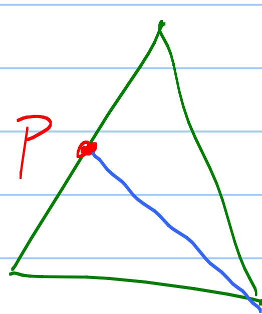
$$\int_Q = \text{Sum}$$

$$= \int_{\text{tiny top } \Delta} f \, dz$$

f continuous at p, so bdd on $D_\delta(p)$. $|f| \leq M$

$\left|\int_\Delta f\,dz\right| \leq M \,\text{Length}(b\Delta)$ can be made small.

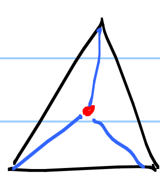
Case $p \in$ side:



$\int_Q f\,dz =$ Sum of 2

Reduce to $p =$ vertex case

Case p inside



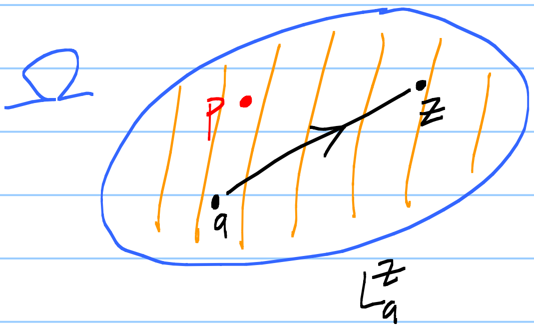
Reduce to $p =$ vertex case

Lecture 7 Cauchy Theorem on a convex open set

Hwk 2 due in G
Thurs, 11:59 pm

Thm: Suppose Ω is a convex open set in $\mathbb{C}$,
$f : \Omega \to \mathbb{C}$ is continuous on Ω,
f is analytic on $\Omega - \{p\}$. $(p \in \Omega)$

Then $\int_\gamma f\,dz = 0$ for any closed curve γ in Ω.



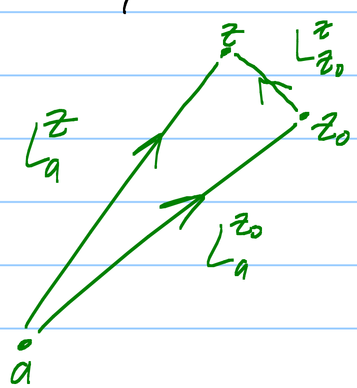
Pf: Plan: Given f, cook up F such that
$$F' = f$$
$\uparrow$ continuous!

If we had such an F, then $\int_{L_a^z} f\,dw = \int_{L_a^z} F'\,dw \underset{\text{F.Thm Calc. \#2}}{=} F(z) - \underbrace{F(a)}_{\text{a fixed const}}$

Aha! $$\frac{d}{dz}\Big(\underbrace{\int_{L_a^z} f\,dw}_{\text{Define to be } F(z)}\Big) = F'(z) = f(z)$$

Why we're doing this: $\int_\gamma f\,dz = \int_\gamma F'\,dz = F(\text{END}) - F(\text{START})$
$= 0$ when γ closed.

Show $F' = f$: Key: Goursat!



$$\Big(\int_{L_a^{z_0}} + \int_{L_{z_0}^z} + \underbrace{\int_{-L_a^z}}_{-\int_{L_a^z}}\Big) f\,dw = 0$$

Aha! $\quad DQ = \dfrac{F(z)-F(z_0)}{z-z_0} = \dfrac{1}{z-z_0}\displaystyle\int_{L_{z_0}^{z}} f \, dw$

Baby fact: $\displaystyle\int_{L_{z_0}^{z}} c \, dw = \int_{L} \frac{d}{dw}[cw]\, dw = [cw]_{z_0}^{z} = c(z-z_0)$

f continuous at z_0: $\quad f(z) = f(z_0) + E(z)$

where $E(z)$ is cont. on Ω and $E(z) \to 0$ as $z \to z_0$.

$$DQ = \frac{1}{z-z_0}\left[\int_{L_{z_0}^{z}} f(z_0) + E(w) \quad dw\right]$$

$$= \underbrace{\frac{1}{z-z_0}\cdot f(z_0)(z-z_0)}_{f(z_0)} + \underbrace{\frac{1}{z-z_0}\int_{L_{z_0}^{z}} E(w)\, dw}_{\mathcal{E}(z)}$$

where $\displaystyle |\mathcal{E}(z)| \le \frac{1}{|z-z_0|}\underbrace{\left(\max_{w\in L_{z_0}^{z}} |E(w)|\right)}_{\substack{<\varepsilon \\ \text{when } |z-z_0|<\delta, \text{ some } \delta.}} \underbrace{\text{Length}\left(L_{z_0}^{z}\right)}_{|z-z_0|}$

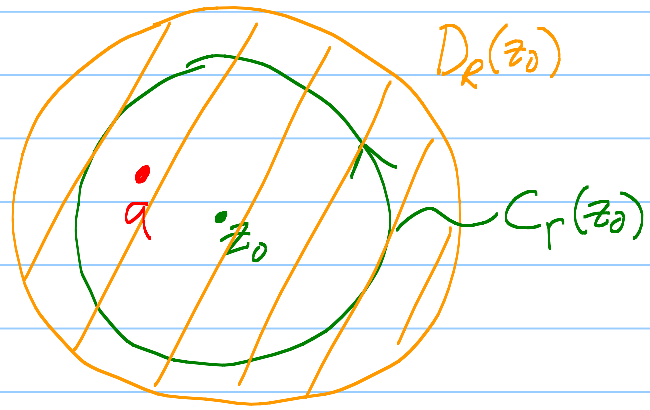
$\le \varepsilon$ when $|z-z_0|<\delta$. ✓

Cor: Analytic fcns on a convex open have an analytic antiderivative there given by $F(z) = \displaystyle\int_{L_a^z} f(w)\, dw$. $\leftarrow a$ fixed in Ω

Future: Improve "convex open" to

$\underbrace{\text{simply connected}}_{\text{no holes}}$ $\underbrace{\text{domain}}_{\text{open, connected}}$.

Cauchy integral Formula on a disc



f analytic on $D_R(z_0)$

$C = C_r(z_0)$ = circle of radius r about z_0 in counterclockwise sense

$a \in D_r(z_0)$

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

Lemma:
$$\int_{C_r(z_0)} \frac{1}{z-a}\, dz = \begin{cases} 2\pi i & a \in D_r(z_0) \\ 0 & |z-z_0| > r \end{cases}$$

Cauchy on convex.

Remark: $f(z) \equiv 1$ in C.I.F.

Pf: $C = C_r(z_0)$: $z(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$
$z'(t) = ire^{it}$

Easy case: $a = z_0$: $\int_C \frac{1}{z-z_0}\, dz$

$$= \int_0^{2\pi} \frac{1}{\underbrace{(z_0 + re^{it})}_{z(t)} - z_0} \underbrace{ire^{it}}_{z'(t)}\, dt$$

$$= \int_0^{2\pi} i\, dt = 2\pi i \quad \checkmark$$

Define

$$H(w) = \int_C \frac{1}{z-w}\, dz$$

Claim:

$$H'(w) = \int_C \frac{d}{dw}\left[\frac{1}{z-w}\right] dz = \int_C \frac{1}{(z-w)^2}\, dz$$

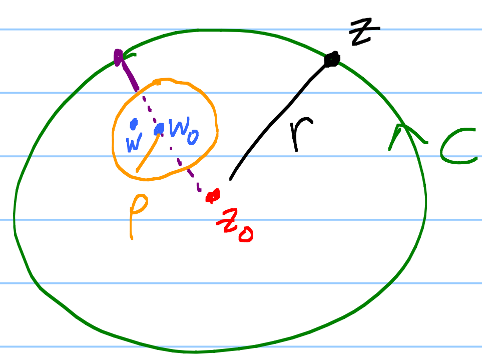
(HWK 2: prob 1)

Pf: $$DQ = \frac{H(w)-H(w_0)}{w-w_0} = \frac{1}{w-w_0}\int_C \underbrace{\frac{1}{z-w} - \frac{1}{z-w_0}}_{\frac{w-w_0}{(z-w)(z-w_0)}}\, dz$$

$$= \int_C \frac{1}{(z-w)(z-w_0)}\, dz \xrightarrow[\text{as } w\to w_0]{\text{guess}} \int_C \frac{1}{(z-w_0)^2}\, dw$$

$$DQ - \text{guess} = \int_C \underbrace{\frac{1}{(z-w)(z-w_0)} - \frac{1}{(z-w_0)^2}}_{\frac{w-w_0}{(z-w)(z-w_0)^2}}\, dz$$

$$|DQ - \text{guess}| = |w-w_0| \cdot \underbrace{\left|\int_C \frac{1}{(z-w)(z-w_0)^2}\, dz\right|}_{\text{just need to bound}}$$



$$\rho = \operatorname{dist}(w_0, C)/2$$
$$= \left(r - |w_0 - z_0|\right)/2$$

$|z-w| \geq \rho \qquad |z-w_0| \geq 2\rho$

$$\left| \int_C \frac{1}{(z-w)(z-w_0)^2}\, dz \right| \leq \frac{1}{\rho(2\rho)^2} \cdot \underbrace{\text{Length}(C)}_{2\pi r}$$

Let $w \to w_0$ inside $D_\rho(w_0)$. ✓

Claim: $H'(w) \equiv 0$

$$H'(w) = \int_C \frac{d}{dw}\left[\frac{1}{z-w}\right] dz = \int_C \frac{1}{(z-w)^2}\, dz$$

$$\underset{\text{sneaky!}}{=} \int_C \frac{d}{dz}\left[\frac{-1}{z-w}\right] dz$$

$$= 0 \quad \text{by F. Thm Calc \#2!}$$

So $H \equiv$ const. $H(z_0) = 2\pi i$. $c = 2\pi i$ ✓

Pf of Cauchy Int. Formula:

$$G(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases} \qquad a = P!$$

G cont. on $D_R(z_0)$, analytic on $D_R(z_0) - \{a\}$.

Cauchy on convex $\Rightarrow \displaystyle\int_C G\, dz = 0$

$$\int_C \frac{f(z)-f(a)}{z-a}\,dz = 0$$

$$\int_C \frac{f(z)}{z-a} - f(a)\underbrace{\int_C \frac{1}{z-a}\,dz}_{2\pi i} = 0 \checkmark$$

Higher order Cauchy Integral Formulas

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz \qquad \text{HWK 2 : Prob 1}$$

$$f''(a) = \frac{2}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz$$

$$\vdots$$

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Aha! This is how you see that

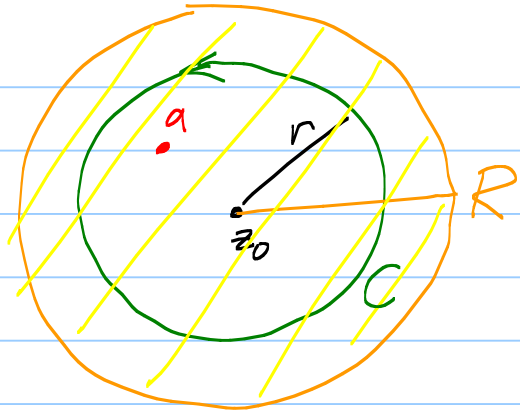
f analytic $\Rightarrow$ f' analytic $\Rightarrow$ f'' analytic $\Rightarrow \cdots$

$\Rightarrow f'$ cont $\qquad$ f'' cont

Soon: Cauchy Int Formula $\Rightarrow$ f = power series !

Lecture 8 Liouville's Theorem & the Fundamental Theorem of Algebra

HWK 3 due Thurs, GS



$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz \leftarrow \text{Cauchy Integral Formula}$$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz \leftarrow \text{HWK 2: \#}$$

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz \leftarrow \text{same pf!}$$

Thm: f analytic on $\Omega \Rightarrow f'$ analytic on Ω

$=u+iv \Rightarrow u,v$ C^∞-smooth harmonic!

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Cauchy estimates: Let $a = z_0$. Get

$$|f^{(n)}(z_0)| \le \frac{n!\,M}{r^n} \quad \text{where } M = \max_C |f|$$

Pf: $$|f^{(n)}(z_0)| \le \frac{n!}{2\pi}\left(\max_{z\in C}\frac{|f(z)|}{\underbrace{|z-z_0|}_{r}{}^{n+1}}\right)\underbrace{\text{Length}(C)}_{2\pi r} \checkmark$$

Case $n=1$: $$|f'(z_0)| \le \frac{M}{r}$$

Liouville's Thm: A $\underbrace{\text{bounded}}_{|f|\le M \text{ on } \mathbb{C}}$ $\underbrace{\text{entire}}_{\text{analytic on } \mathbb{C}}$ function must be constan

Pf: Pick z_0. Cauchy est $(n=1)$: $|f'(z_0)| \leq \frac{M}{r}$

Let $r \to \infty$. See $f'(z_0)=0$. z_0 arbitrary.

So $f' \equiv 0$. $\Longrightarrow$ $f \equiv$ const. ✓

Way false $\mathbb{R} \to \mathbb{R}$: $\sin x$ [Hmmm. $\sin z$ is not bounded on $\mathbb{C}$.]

Basic polynomial estimate: $P(z) = a_N z^N + \cdots + a_1 z + a_0$

complex poly of deg $N \geq 1$ $(a_N \neq 0)$. There exists

a radius $R>0$ and real constants $0 < A < B$

such that

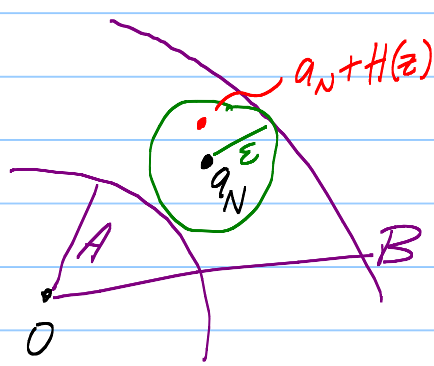
$$A|z|^N \leq |P(z)| \leq B|z|^N \text{ when } |z|>R.$$

Remark: $A < |a_N| < B$ ← A, B can be taken as close to $|a_N|$ as desired.

Pf:

$$\frac{P(z)}{z^N} = a_N + \underbrace{\left(\frac{a_{N-1}}{z} + \cdots + \frac{a_1}{z^{N-1}} + \frac{a_0}{z^N}\right)}_{H(z)}$$

$H(z)$, $H(z) \to 0$ as $|z| \to \infty$.



Take $0 < \varepsilon < |a_N|$.

$|H(z)| < \varepsilon$ if $|z| > R$.

$$A < \left|\frac{P(z)}{z^N}\right| < B \quad \text{if} \quad |z| > R. \checkmark$$

Books: $\varepsilon = \frac{|a_N|}{2}$.

<u>Fundamental Theorem of Algebra</u>: A complex poly of deg $N \geq 1$ has a <u>root</u> in $\mathbb{C}$. Consequently, can factor polys over $\mathbb{C}$.

<u>Pf</u>: Suppose $P(z)$ does <u>not</u> have a root.

Basic poly est: $A|z|^N \leq |P(z)| \leq B|z|^N$, $|z| > R$.

$\frac{1}{P(z)}$ is an entire fcn!

$$\left|\frac{1}{P(z)}\right| \leq \frac{1}{A|z|^N} \quad \text{when} \quad |z| > R$$

$$< \frac{1}{AR^N} \quad \text{when} \quad |z| > R$$

<u>And</u>: $\left|\frac{1}{P(z)}\right|$ continuous on $\overline{D_R(0)}$ ← closed, bdd, so compact

has max M on $\overline{D_R(0)}$.

$$\left|\frac{1}{P(z)}\right| \leq \operatorname{Max}\left(M, \frac{1}{AR^N}\right) \quad \text{on} \quad \mathbb{C} \;!$$

Liouville's $\Rightarrow$ $\frac{1}{P(z)} \equiv \text{const}$ $\Rightarrow$ $P(z) \equiv \text{const}$. $\lightning$

$$\left[\; P^{N}(0) = N!\, a_N \neq 0. \quad ↯ \;\right]$$

<u>Morera's theorem</u>: Suppose $f : \Omega \to \mathbb{C}$ is a continuous fcn on open set Ω.

If $\displaystyle\int_\gamma f\, dz = 0$ for every closed γ in Ω,

then f is <u>analytic</u> on Ω.

<u>Note</u>: Enough to know only $\displaystyle\int_Q f\, dz$ for $\bar{\Delta} \subset \Omega$.

<u>Pf</u>: Restrict attention to $D_r(z_0) \subset \Omega$.

Define $F(z) = \displaystyle\int_{[z_0, z]} f\, dw$ on $D_r(z_0)$.

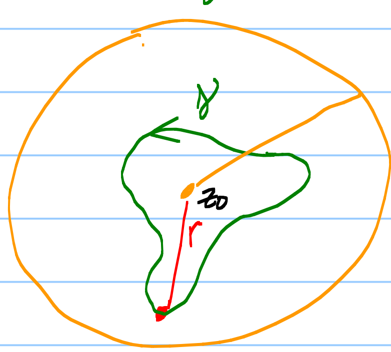
Aha! Our hyp $\Rightarrow$ conclusion of Goursat's!

$\Rightarrow$ $F' = f$. Today: know F analytic

$\Rightarrow F' = f$ analytic.

<u>Cor</u>: Power series converge to analytic fcns inside their circle of convergence.

<u>Pf</u>:



$R = R.$ of $C.$

$\gamma : z(t)$, $a \le t \le b$

$|z(t)|$ continuous on $[a,b]$. It has a max $= r < R$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n = \lim_{N\to\infty} \underbrace{P_N(z)}_{\text{polys}}$$

Note $\displaystyle\int_\gamma P_N(z)\,dz = \sum_0^N A_n \int_\gamma \underset{\frac{d}{dz}\left[\frac{z^{n+1}}{n+1}\right]}{z^n}\,dz = 0$

Know power series $\longrightarrow f$ <u>uniformly</u> on $D_r(z_0)$.

$$\left|\int_\gamma f\,dz\right| = \left|\int_\gamma f\,dz - \underbrace{\int_\gamma P_N(z)\,dz}_{=0}\right|$$

$$= \left|\int_\gamma f - P_N\,dz\right| \le \underbrace{\left(\max_\gamma |f-P_N|\right)}_{\substack{\to 0 \text{ as } N\to\infty \\ \text{by unif conv on } \overline{D_r(z_0)}.}} \text{Length}(\gamma)$$

So $\int_\gamma f\,dz = 0$. γ arbitrary. Morera's $\Rightarrow$ f analytic!

<u>Def^n</u>: Given a seq of continuous fcns f_n on an open set $\Omega \subset \mathbb{C}$, we say that <u>$f_n$ converges uniformly on compact subsets of Ω to f</u> if, for each compact $K \subset\subset \Omega$, given $\varepsilon > 0$, there

exists an N such that $|f(z) - f_n(z)| < \varepsilon$ for all $z \in K$ when $n > N$. Write $f_n \rightrightarrows f$.

Fact: Uniform limits of continuous are continuous.

So f has to be cont. too.

Thm: f_n analytic and $f_n \rightrightarrows f$, then f is analytic.

Pf: $\Omega =$ Disc. $\left| \int_\gamma f \, dz \right| = \left| \int_\gamma f \, dz - \underbrace{\int_\gamma f_n \, dz}_{=0 \text{ by Cauchy's}} \right|$

$$= \left| \int_\gamma f - f_n \, dz \right| \leq \underset{\underset{\text{compact}}{tr(\gamma)}}{\operatorname{Max}} |f - f_n| \cdot \operatorname{Length}(\gamma)$$

$$\to 0.$$

Morera's $\Rightarrow f$ analytic.

Lecture 9 Analytic functions are given locally by convergent power series

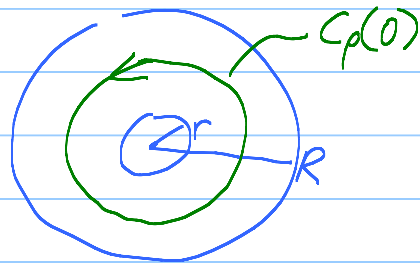
Simple fact: On $D_r(z_0)$, $f_n \twoheadrightarrow f$ $\Longleftrightarrow$

$f_n \to f$ uniformly on each $\overline{D_\rho(z_0)}$, $0<\rho<r$. [or $D_\rho(z_0)$]

Why: $|z-z_0|$ continuous on compact $K \subset\subset D_r(z_0)$. $\rho = \max_{z\in K} |z-z_0|$

EX: $f(z)=\frac{1}{z}$ on $\{z: r<|z|<R\}$, $0<r<R$,

has a hole. Pick ρ with $r<\rho<R$. $H(z)\equiv 1$



$$\int_{C_\rho} \frac{1}{z}\,dz = \int_{C_\rho} \frac{H(z)}{z-0}\,dz$$

$$= 2\pi i\, H(0) = 2\pi i \quad \text{not zero}$$

Cauchy theorem for Ω with holes will be different.

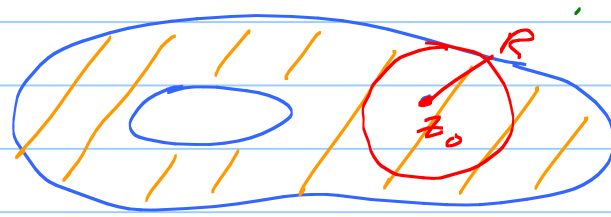
Notation: $S_N(z) = 1 + z + \cdots + z^N$

$$\frac{1}{1-z} = S_N(z) + \underbrace{\frac{z^{N+1}}{1-z}}_{E_N(z)}$$

Showed that if $|z|<\rho<1$, then $|E_N(z)| \le \frac{\rho^{N+1}}{1-\rho}$

Thm: Suppose f analytic on $D_R(z_0)$. Then f is given by a convergent $\sum_0^\infty a_n(z-z_0)^n$ where $a_n = \frac{f^{(n)}(z_0)}{n!}$

where R. of C. $\geq R$.

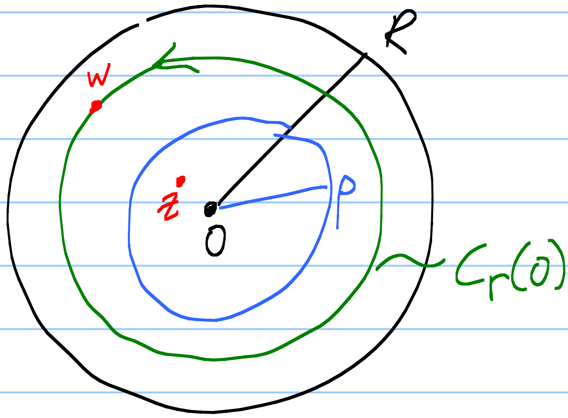


f analytic on Ω

Note: $f(z)=e^z$ or poly, R of $C=\infty$. Maybe $>R$.

Pf: Can assume $z_0=0$ (otherwise $w=z-z_0$, etc.)

Let $0<\rho<r<R$. $\quad |z|<\rho \qquad \left|\frac{z}{w}\right|<\frac{\rho}{r}<1$



$$f(z)=\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w-z}\,dw$$

$$=\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w}\underbrace{\frac{1}{1-\frac{z}{w}}}_{S_N\left(\frac{z}{w}\right)+E_N\left(\frac{z}{w}\right)}dw$$

$$=\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w}\left[1+\left(\frac{z}{w}\right)+\cdots+\left(\frac{z}{w}\right)^N\right]dw+\underbrace{\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w}E_N\left(\frac{z}{w}\right)dw}_{\mathcal{E}_N(z)}$$

$$=\sum_{n=0}^{N}\underbrace{\left(\frac{1}{2\pi i}\int_{C_r}\frac{f(w)}{w^{n+1}}\,dw\right)}_{\frac{f^{(n)}(0)}{n!}}z^n+\mathcal{E}_N(z)$$

and $$|\mathcal{E}_N(z)|\le\frac{1}{2\pi}\left(\max_{C_r}\frac{|f|}{r}\right)\frac{\left(\frac{\rho}{r}\right)^{N+1}}{1-\frac{\rho}{r}}\underbrace{\text{Length}(C_r)}_{2\pi r}$$

$$\le\frac{M\left(\frac{\rho}{r}\right)^{N+1}}{1-\frac{\rho}{r}}\quad\text{where } M=\max_{C_r}|f|$$

$\longrightarrow 0$ as $N\to\infty$ because $\frac{\rho}{r}<1$

So series converges uniformly on $D_\rho(0)$ to f.

Can squeeze ρ, r as close to R as desired.

Get convergence on $D_R(0)$. ✓ So R.of.C $\geq R$

Today: $E(z) = e^z = \sum_0^\infty \frac{z^n}{n!}$, $R = \infty$.

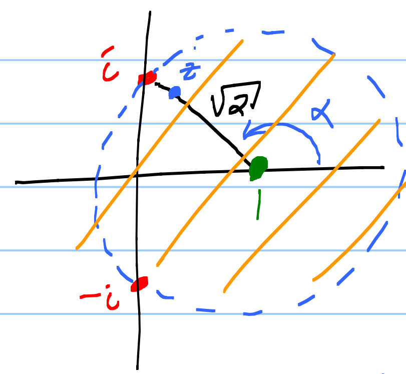
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \quad R = \infty$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots, \quad R = \infty$$

$$\tan z = \frac{\sin z}{\cos z} \qquad R = ?$$

EX: $f(z) = \frac{1}{z^2+1}$, $z_0 = 1$

$= \sum_0^\infty a_n (z-1)^n$



Thm $\Rightarrow$ $R_f \geq \sqrt{2}$

$z = 1 + re^{i\alpha}$, $\alpha = \frac{3\pi}{4}$

Claim: $R_f \leq \sqrt{2}$ too.

Why: If power series for f converges on $D_\rho(1)$, $\rho > \sqrt{2}$,

it converges to F analytic on $D_\rho(1)$. ← F continuous on $D_\rho(1)$.

$F \equiv f$ on $D_{\sqrt{2}}(1)$. F is cont at i, f is not! ↯.

Thm: Can differentiate power series term by term:

$$f(z) = \sum_0^\infty a_n (z - z_0)^n, \quad R_f > 0.$$

$$f'(z) = \sum_1 n a_n (z-z_0)^{\cdots}; \quad R_{f'} = R_f$$

$$\left(= \sum_0^\infty b_n (z-z_0)^n \quad \text{where } b_n = (n+1)a_{n+1} \right)$$

Pf: f' analytic on $D_{R_f}(z_0)$ because f is.

So $R_{f'} \geq R_f$ by Thm. And

$$f'(z) = \sum_0^\infty b_n (z-z_0)^n$$

where
$$b_n = \frac{(f')^{(n)}(z_0)}{n!} = \frac{f^{(n+1)}(z_0)}{n!} \cdot \frac{n+1}{n+1}$$

$$= (n+1)\frac{f^{(n+1)}(z_0)}{(n+1)!} = (n+1)a_{n+1}$$

Claim: Know $R_{f'} \geq R_f$. $\leq$ too.

Hmmm. If $R_{f'} > R_f$, the power series for f'

would converg to analytic $\tilde{f}$ on $D_{R_{f'}}(0)$. Let F be an antiderivat

Note: $\frac{d}{dz}[F - f] \equiv 0$ on $D_{R_f}(0)$.

So $F - f \equiv c$, a const on

$f = F - c \leftarrow$ R. of C. $> R_f$ ↯.

Alternatively: Use Hadamard's and $\lim \sqrt[n]{n} = 1$ and LimSup fac

Next time: Zeroes of analytic functions are "polynomialesque". They are <u>isolated</u> and we can "factor them out."

Lecture 10 Zeroes of analytic functions   HWK 5 due GS Thurs 11:59 pm

Missing piece from Lecture 9 : $f(z) = \sum a_n z^n$ ← unif convergence on $D_r(0) \subset D_R(0)$

$$\underbrace{\frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z^{N+1}}\,dz}_{\frac{1}{N!}f^{(N)}(0) = a_N} = \sum a_n \Big(\underbrace{\frac{1}{2\pi i}\int_{C_\rho} \frac{1}{z^{1+N-n}}\,dz}_{\substack{=0 \text{ if } N\neq n \\ =1 \text{ if } N=n}}\Big)$$

← $0<\rho<r$

$\int \sum = \sum \int$ because of unif conv

Thm: Power series coeff are unique. Taylor's formula.

Lemma If f is analytic on $D_r(z_0)$, $f(z_0)=0$.

Then, either A) $f \equiv 0$ or

B) zero at z_0 is isolated.

Pf: $f(z) = a_N (z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots$

↑ first $a_n \neq 0$. $(a_0 = f(z_0) = 0)$

$$= (z-z_0)^N \Big[\underbrace{a_N + a_{N+1}(z-z_0) + \cdots}_{F(z)}\Big]$$

Aha! Power series for F converges for exactly same z as p.s. for f
They have same R.of C. $F(z_0) = a_N \neq 0$.
F analytic on $D_r(z_0) \Rightarrow F$ cont. there. $\exists\ \delta > 0$ such that $F(z) \neq 0$ for all $z \in D_\delta(z_0)$.
z_0 is the only zero of f in $D_\delta(z_0)$ ← "isolated"

Def^n: N is the order (or multiplicity) of the zero of f at z_0.

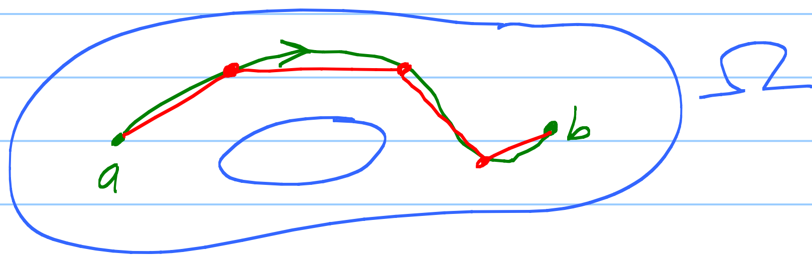
$$f(z) = (z-z_0)^N F(z)$$

where F is analytic where f is and $F(z_0) \neq 0$.

Connected open sets in $\mathbb{C}$ (Domain)

Def^n #1 An open set $\Omega \subset \mathbb{C}$ is path-connected if any two points $a, b \in \Omega$ can be joined by a piecewise C^1 curved γ_a^b in Ω. $(tr(\gamma_a^b) \subset \Omega)$

Topology: γ_a^b is assumed to be merely continuous



Def^n #2 If U_1 and U_2 are open subsets of Ω (open) and

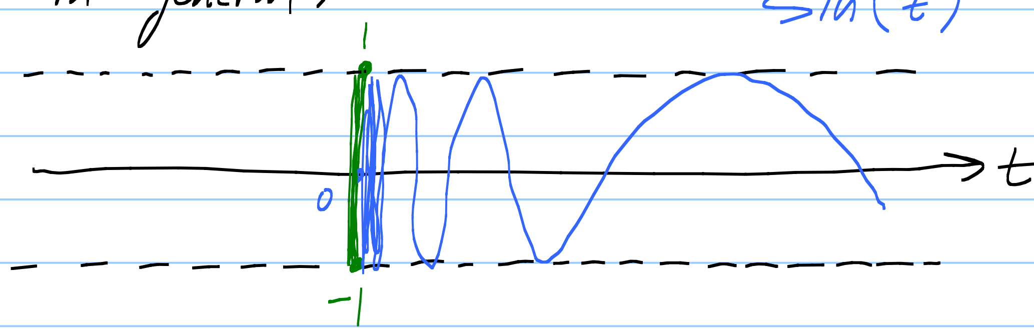
1) $U_1 \cup U_2 = \Omega$

2) $U_1 \cap U_2 = \emptyset$

then either $U_1 = \Omega,\ U_2 = \emptyset$ or $U_1 = \emptyset,\ U_2 = \Omega$.

Fact Def #1 $\Longleftrightarrow$ Def #2 for open sets Ω.

But in general:



union graph of $\sin\left(\frac{1}{t}\right)$, $t>0$.

is connected, but not path-connected.

MA 530: Analytic fcns on domains

Thm f analytic on a domain and $f' \equiv 0$ there, then $f \equiv c$, a const.

Pf: $$\int_{\gamma_a^b} \underbrace{f'}_{0}\, dz = f(b) - f(a) = 0$$

Thm (Identity theorem) Suppose f is analytic on a domain Ω. Let $Z_f = \{z \in \Omega : f(z) = 0\}$.

Then either A) $f \equiv 0$ $\quad (Z_f = \Omega)$

B) Z_f has no limit points in Ω

Pf: Suppose $z_0 \in \Omega$ is a limit pt of Z_f, meaning $\exists$ seq $\{z_n\}_{n=1}^{\infty}$ in $Z_f \subset \Omega$, $z_n \neq z_0$, and $z_n \to z_0$ as $n \to \infty$.

Note: $f(z_0) = \lim_{n\to\infty} \underbrace{f(z_n)}_{0} = 0$ by continuity.

so $z_0 \in Z_f$. Z_f is closed.

Ω open. So $\exists\ D_r(z_0) \subset \Omega$, $r > 0$.

Lemma $\Rightarrow$ $f \equiv 0$ on $D_r(z_0)$.

Pick $w_0 \in \Omega$ and let $\gamma : z(t)$, $a \le t \le b$ be curve in Ω connecting z_0 to w_0.

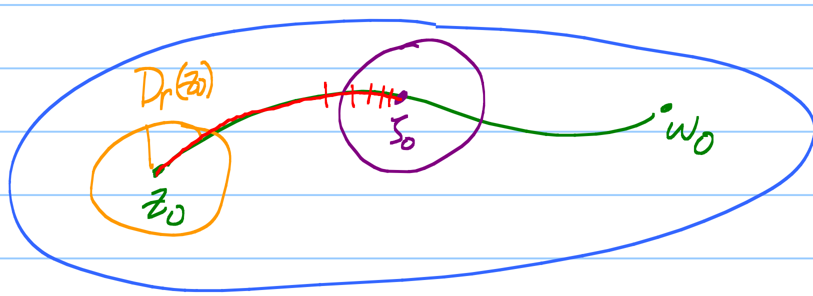
Let $t_{max} = \operatorname{Sup}\{T : f(z(t)) \equiv 0,\ t \in [a, T)\}$

$f \equiv 0$ on $D_r(z_0)$ $\Rightarrow$ $t_{max} > a$.

<u>Claim</u> $t_{max} = b$. So $f(w_0) = 0$ too. $f \equiv 0$ on Ω.

<u>Why</u>: Pick seq t_n, $a < t_1 < t_2 < \cdots < t_{max}$ and $t_n \to t_{max}$ as $n \to \infty$. Let $z_n = z(t_n)$.

z_n are zeroes of f that pile up at $\zeta_0 = z(t_{max})$.



Ω open.
$\exists\ D_\rho(\zeta_0) \subset \Omega$
$\rho > 0$.

Lemma $\Rightarrow$ $f \equiv 0$ on $D_\rho(\zeta_0)$. Can get $T > t_{max}$. $\lightning$.

So $t_{max} = b$. $\checkmark$

<u>Books</u>: $U_1 = \{z : z \text{ is a limit pt of } Z_f\}$

$U_2 = \Omega - U_1$. Use Def^n #2 if $U_1 \neq \emptyset$.

<u>Consequence</u> e^z is the only analytic extension of e^x from $\mathbb{R}$ to $\mathbb{C}$.

<u>Why</u>: If $f(z)$ and $g(z)$ both extend e^x, then $f-g\equiv 0$ on $\mathbb{R}$. Every pt in $\mathbb{R}$ is a limit pt of $\mathbb{R}$. $\Longrightarrow$ $f-g\equiv 0$ on $\mathbb{C}$.

<u>Cor</u>: Analytic fcns that agree on a subset with a limit pt in a domain agree on the domain.

<u>Consequence</u>: Trig identities extend for comple angles.

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

First let $\alpha=z\in\mathbb{C}$. Fix z. Let $\beta=w\in\mathbb{C}$.

Identity Thm way false $\mathbb{R}\to\mathbb{R}$

$$h(t)=\begin{cases} e^{-1/t^2}\sin\left(\frac{1}{t}\right) & t\neq 0 \\ 0 & t=0 \end{cases}$$

is a C^∞ smooth fcn on $\mathbb{R}$. Zeroes pile up at 0.

All derivatives $=0$ at $t=0$.

$h\neq$ its power series $(\equiv 0)$.

$f(z)=(z-z_0)^N F(z) = (z-z_0)^N(z-w_0)^M H(z)$ $\quad H(z_0)\neq 0$

$H(w_0)\neq 0$

Lesson 10 Zeroes of analytic functions, the Identity theorem

Lemma: Suppose f is analytic on $D_R(z_0)$ and $f(z_0)=0$.

Then, either A) $f \equiv 0$ on $D_R(z_0)$, or

B) z_0 is an isolated zero of f, meaning $\exists\, \delta > 0$

such that z_0 is the only zero of f in $D_\delta(z_0)$.

Pf: $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ Case A: $\forall\, a_n = 0$.

Case B: a_N first $a_n \neq 0$. $f(z_0)=0, \ldots, f^{(N-1)}(z_0) = 0$

$$f(z) = a_N (z-z_0)^N + \cdots = (z-z_0)^N \underbrace{\left(a_N + a_{N-1}(z-z_0) + \cdots\right)}_{H(z),\ \text{same R. of C. as series for } f!}$$

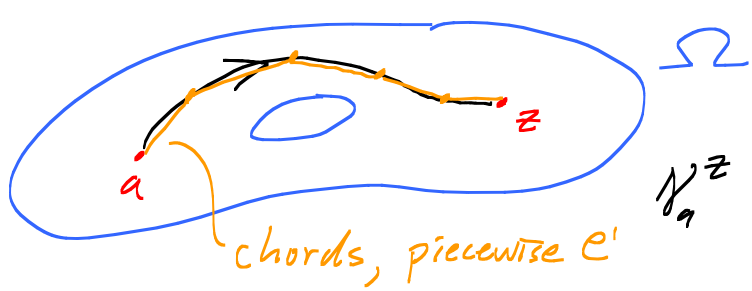
$$= (z-z_0)^N \underbrace{H(z)}_{H(z_0) = a_N \neq 0}$$

Def^n: An open connected set in $\mathbb{C}$ is called a *domain*.

Def^n #1: If Ω is open, we say it is *path connected* if any two pts in Ω can be connected by a piecewise $\mathcal{C}^1$-smooth curve in Ω.

Remark: Topology MA 571 uses continuous curves in def^n.

Gives same concept:



Def^n #2: Ω open is connected if, whenever

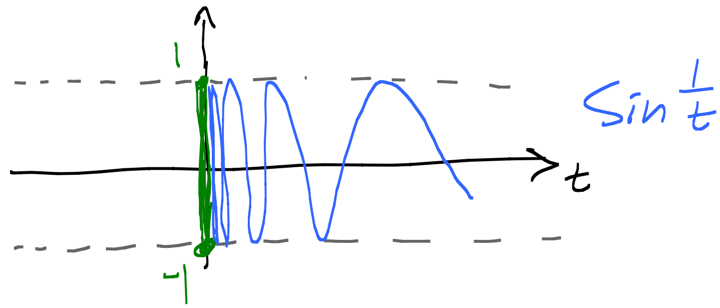
$$\Omega = U_1 \cup U_2 \quad \text{where } U_1, U_2 \text{ open}$$

$$\text{and } U_1 \cap U_2 = \emptyset,$$

one U_j must be empty and the other $= \Omega$.

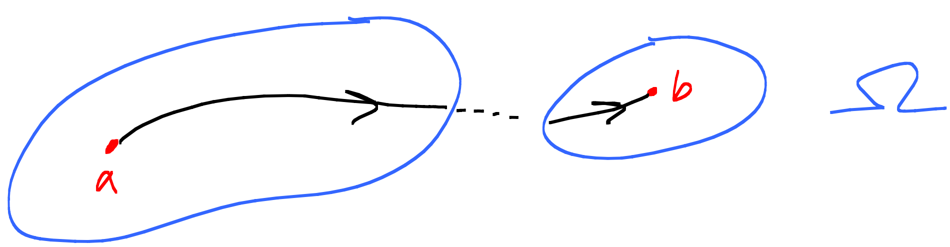
Fact: Def #1 and #2 are equiv for open sets in $\mathbb{C}$.

EX:



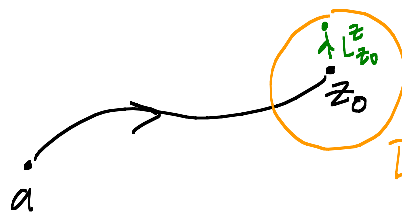
$S' =$ segment $\cup$ graph
Connected,
but not
path connected

Pf of Fact: 1) Ω open connected #2.



Let $a \in \Omega$. Let $U_1 = \{ z : \exists\ \gamma_a^z \subset \Omega \}$

U_1 is open:



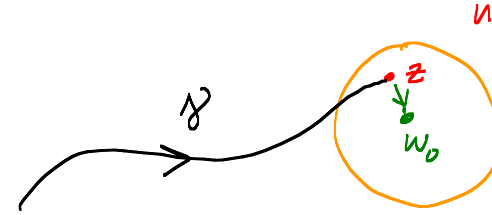
$\sim \Omega$ open

$D_r(z_0) \subset U_1$. ✓

$U_2 = \Omega - U_1$.

U_2 is open. $w_0 \in U_2$:

no such z!

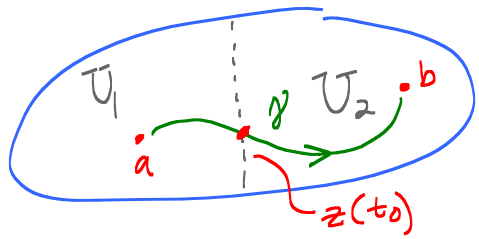


$\sim \Omega$ open

$D_r(w_0) \subset U_2$

Conclude $U_2 = \emptyset$ (because $a \in U_1$).

2) Suppose Ω is path-connected, and $\Omega = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$.



Suppose $a \in U_1$, $b \in U_2$.

Let t_0 be

$\sup\{t : z(\tau) \in U_1,\ \tau \in [\alpha, t)\}$

$\gamma : z(t) : \alpha \le t \le \beta$

Where is $z(t_0)$? In Ω, but not U_1 or U_2! $\lightning$.

MA 530: Analytic fcns on domains.

Thm: If f is analytic on a domain Ω and $f' \equiv 0$, then f is const.

Pf: Fix $z_0 \in \Omega$. For $z \in \Omega$, let $\gamma_{z_0}^{z}$ be a C^1-path in Ω connecting z_0 to z.

$$0 = \int_{\gamma_{z_0}^{z}} f' \, dw = f(z) - f(z_0). \quad \checkmark$$

Theorem: (Identity Thm) Suppose f is analytic on a domain Ω. Let $Z_f = \{z \in \Omega : f(z) = 0\}$.

Then either A) $f \equiv 0$ on Ω, or

B) Z_f has no limit points in Ω.

Def^n: z_0 is a limit pt of Z_f means $\exists$ seq $z_n \in Z_f$ with $z_n \neq z_0$ for all n and $\lim_{n\to\infty} z_n = z_0$.

Note: If $z_0 \in \Omega$ and z_0 is a limit pt of Z_f, then

$$f(z_0) = \lim_{n\to\infty} \underbrace{f(z_n)}_{0} = 0$$ by continuity.

Pf of Thm: Suppose Z_f has a limit pt in Ω.

Let U_1 = set of all limit pts of Z_f in Ω.

Claim: U_1 is open: If $z_0 \in U_1$, then Ω open $\Rightarrow$ $\exists$ $D_r(z_0) \subset \Omega$. z_0 is not an isolated zero of f, Lemma $\Rightarrow$ $f \equiv 0$ on $D_r(z_0)$. So $D_r(z_0) \subset U_1$.

Next, show $U_2 = \Omega - U_1$ is open. Suppose $w_0 \in U_2$.

Case 1: $f(w_0) \neq 0$. Then $\exists$ $\delta > 0$ such f is nonvanishing on $D_\delta(w_0) \subset \Omega$. See $D_r(w_0) \subset U_2$.

Case 2: $f(w_0) = 0$. w_0 not a limit pt of zeroes $\Rightarrow$ w_0 is an isolated zero. f non-vanishing on $D_\delta(w_0) - \{w_0\}$. See $D_\delta(w_0) \subset U_2$.

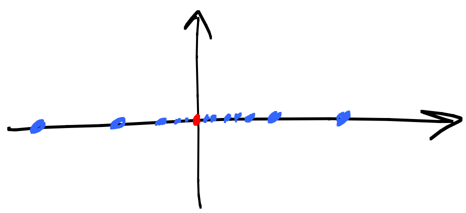
Done: $U_1 \neq \emptyset$. So $U_2 = \emptyset$ and $U_1 = \Omega$.

$U_1 \subset Z_f$. So $Z_f = \Omega$.

Cor: If two analytic fcns f, g a domain Ω agree on a set with a limit pt in Ω, the $f \equiv g$ on Ω.

b) If f and g agree on $D_\varepsilon(z_0) \subset \Omega$, then $f \equiv g$ on Ω.

EX: $\sin \frac{1}{z}$ has zeroes at $z = \frac{1}{n\pi}$, $n \in \mathbb{Z}$.



zeroes pile up at origin.

OK. $\Omega = \mathbb{C} - \{0\}$. OK for zeroes to pile up at a boundary pt.

Identity Theorem way false $\mathbb{R} \to \mathbb{R}$:

$$h(t) = \begin{cases} e^{-1/t^2} \sin \frac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

h is C^∞ smooth on $\mathbb{R}$, zeroes pile up at $t = 0$.

Taylor series at origin $\equiv 0 \neq h$ near zero.

Identity Thm $\Rightarrow$ e^z is only way to extend e^x from $\mathbb{R}$ to $\mathbb{C}$ as a complex diff'ble fcn!