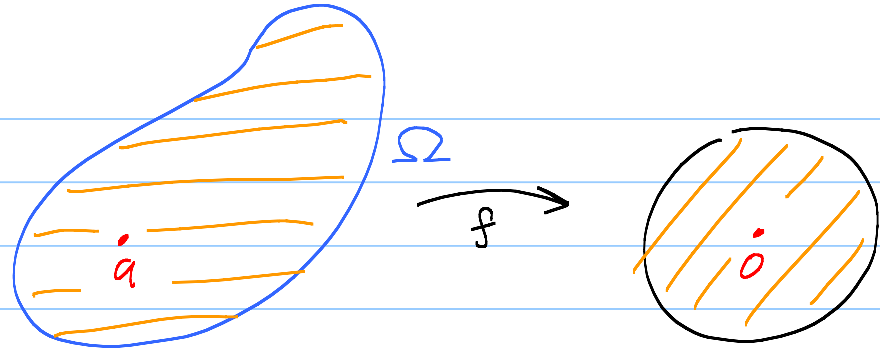
Lecture 32 Harmonic functions

HWK 7 due Tues in GS

Ave ≈ 91

Riemann map



The Riemann map $|f_a'(a)| = \text{Max}\left\{ |h'(a)| : h : \Omega \xrightarrow{\text{analytic}} D_1(0) \right\}$

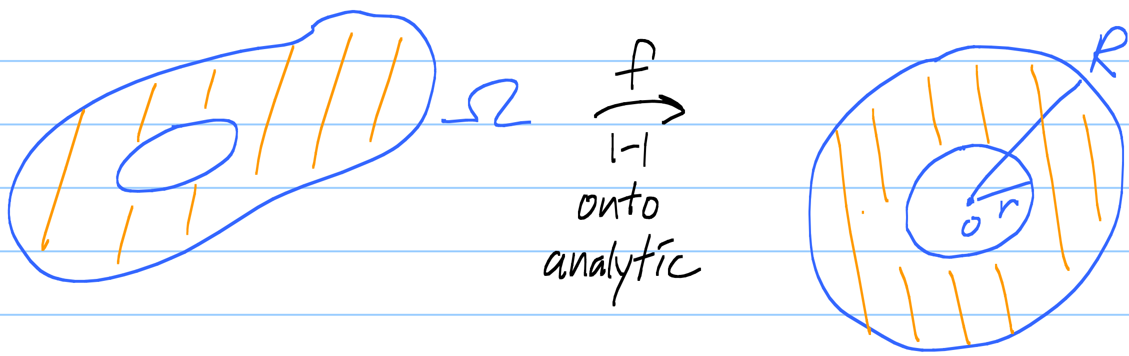
f_a solves this extremal problem

not assuming 1-1

Unique: $f_a'(a) \in \mathbb{R}^+$

Always: $f_a(a) = 0.$

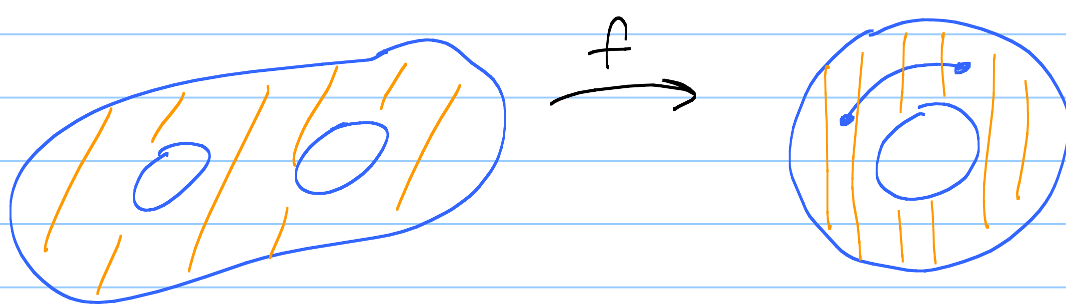
What about domains with holes?

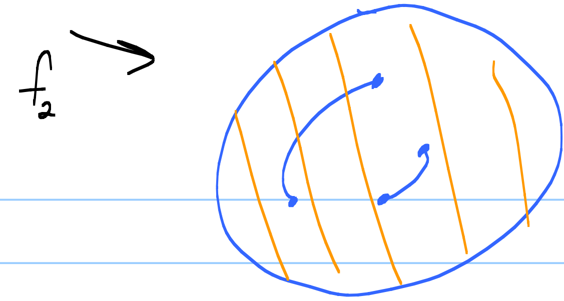


$\frac{r}{R}$ determines a "mapping class". $\frac{r}{R}$ = the "modulus" of Ω

$$\text{Aut}(\text{Annulus}) = \{ \lambda z : |\lambda| = 1 \} \cup \{ \text{inversion} \sim \tfrac{1}{z} \}$$

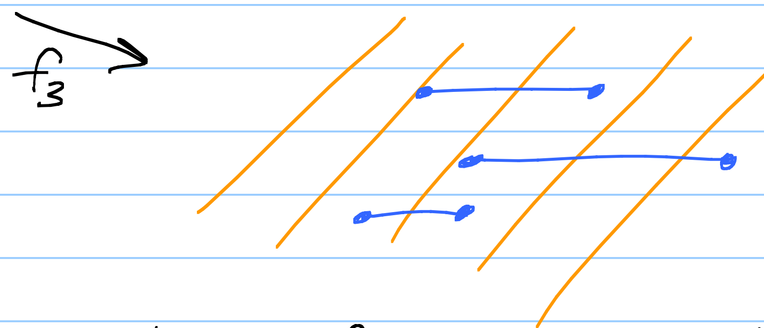
not transitive!





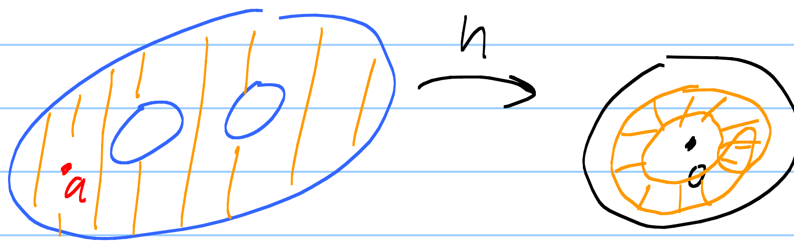
Disc minus circular slits

$\mathbb{C}$ - horiz slits.



3-connected domains and above: finitely many automorphisms, perhaps (most likely) none.

Ahlfors map:



Extremal map f_a : $|f_a'(a)| = \mathrm{Max}\{|h'(a)| : h : \Omega \to D_1(0)$ analytic $\}$

Unique: $f_a'(a) \in \mathbb{R}^+$

f_a is <u>the</u> Ahlfors map. Like a Riemann map for n-connected domains. $(n > 1)$.

Onto $D_1(0)$ ✓

Analytic ✓

Not 1-1. It is n-to-one, counting multiplicity.

Maps each boundary 1-1 onto unit circle.

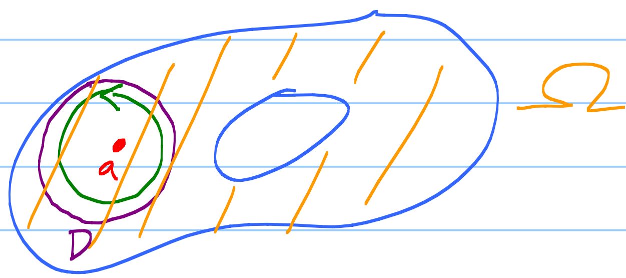
Harmonic fcn on a domain Ω is a C^∞-smooth real valued fcn such that $\Delta u \equiv 0$.

Fact: $\iff$ u is locally the real part of an analytic fcn.

Remark: Can get global harmonic conjugates when Ω is simply connected.

[$\operatorname{Ln}|z|$, no harm conj on $\mathbb{C}-\{0\}$.]

Important property: The averaging property



$$(*) \quad u(a) = \frac{1}{2\pi r}\int_0^{2\pi} u(a+re^{i\theta})\, \underbrace{r\,d\theta}_{ds}$$

Why: Get $f = u+iv$ on D, analytic.

Cauchy integral formula at a: $\underbrace{f(a)}_{u+iv} = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{f(a+re^{i\theta})}_{u+iv}\, d\theta$

Take real part to get $(*)$.

Thm: Ave prop $\implies$ Max prinicle.

Why: Suppose u has a local max at $z_0 \in \Omega$.

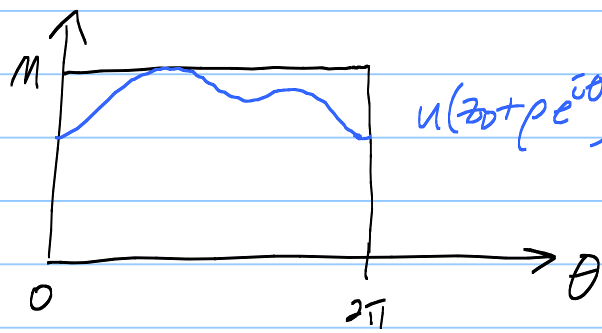
$\exists$ $D_\varepsilon(z_0) \subset \Omega$ such that $u(z) \le u(z_0) = M$ on $D_\varepsilon(z_0)$.

Ave prop $\quad \underbrace{u(z_0)}_{M} = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})\, d\theta$

$0<\rho<\varepsilon$

Thing inside $\int$ is $\leq M$ and continuous.

Calculus lemma. $\implies u(z_0+\rho e^{i\theta}) \equiv M, \quad \forall \theta$
$0<\rho<r$.



$u(z_0+\rho e^{i\theta})$ $\leftarrow$ picture can't happen!

$u \equiv M$.

Conclude that $u \equiv M$ on $D_\varepsilon(z_0)$.

Aha! $\quad F = u_x - i u_y$ is analytic on Ω

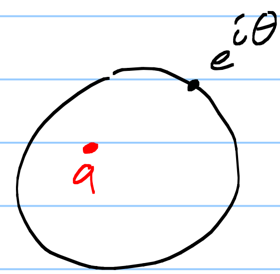
$u \equiv M$ on $D_\varepsilon \implies F \equiv 0$ on $D_\varepsilon \implies F \equiv 0$ on Ω

$\implies \nabla u \equiv 0$ on $\Omega \implies u \equiv \text{const}$ on Ω.
$= M$. ✓

Want a "Cauchy integral formula" for harmonic fcns.

See "Something about Poisson and Dirichlet" Find a link below AFTERMATH on home page.

<u>Poisson integral formula:</u>



$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|a|^2}{|a-e^{i\theta}|^2}\, u(e^{i\theta})\, d\theta$$

Simple Pf, easy to understand.

Formula true at $a=0$. Ave prop. ✓

Hmmm. $\varphi_{-a}(z) = \frac{z+a}{1+\bar{a}z}$: $D_1(0) \circlearrowleft$ $\overset{1\text{-}1}{\text{onto}}$ $\varphi_{-a}(0) = a$.

Fact: harmonic fcns composed with analytic fcns are harmonic.

Aha! Ave prop:

$$u(\underbrace{\varphi_{-a}(0)}_{a}) = \frac{1}{2\pi}\int_0^{2\pi} u(\underbrace{\varphi_{-a}(e^{i\theta})}_{e^{i\psi}})\, d\theta$$

$$e^{i\psi} = \frac{e^{i\theta}+a}{1+\bar{a}e^{i\theta}}$$

Calculus: $d\theta = \frac{1-|a|^2}{|a-e^{i\psi}|^2}\, d\psi$

Get $u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\psi})\, \frac{1-|a|^2}{|a-e^{i\psi}|^2}\, d\psi$ ✓

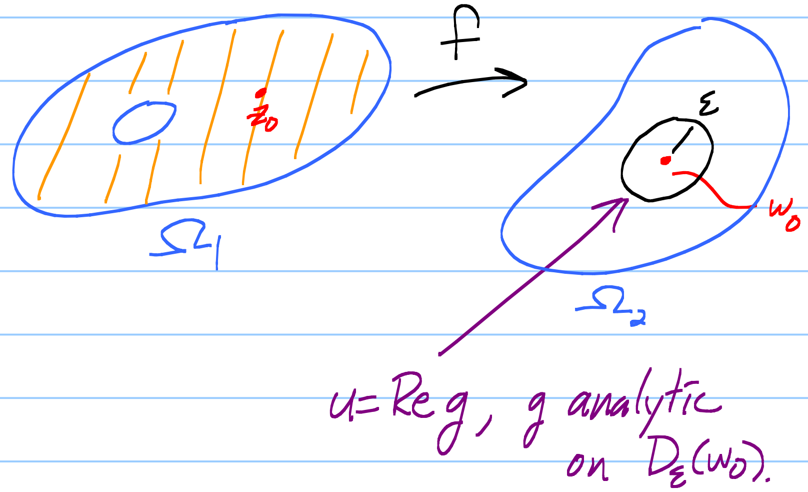
Stein: p. 66: prob 11.

Lecture 33 The Dirichlet problem, Schwarz's theorem

Fact $f: \Omega_1 \to \Omega_2$ analytic, u harmonic on Ω_2.
Then $u \circ f$ is harmonic on Ω_1.

$f(z_0) = w_0$

Why:



Ω_2 open: $\exists\, D_\varepsilon(w_0) \subset \Omega_2$.
f analytic $\Rightarrow$ f cont.
So $\exists\, \delta > 0$ so that
$D_\delta(z_0) \subset \Omega_1$ and
$f(D_\delta(z_0)) \subset D_\varepsilon(w_0)$.

Finally $u \circ f = \underbrace{\operatorname{Re}[g \circ f]}$ on $D_\delta(z_0)$.

Real part of an analytic fcn.
So $u \circ f$ is harmonic on $D_\delta(z_0)$.

Poisson kernel technical points. To get Poisson formula

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |a|^2}{|a - e^{i\theta}|^2}\, u(e^{i\theta})\, d\theta \qquad (*)$$

for harmonic fcns, needed u harmonic on $D_R(0)$, $R > 1$.
Get harmonic conj on $D_R(0)$, use ave prop., etc.

Easy extension: Suppose u is continuous on $\overline{D_1(0)}$ and harmonic on $D_1(0)$. Then u satisfies $(*)$ for $a \in D_1(0)$.

Why: $u_r(z) = u(rz)$, $0 < r < 1$, is harmonic on

$D_R(0)$ where $R = \frac{1}{r} > 1$. (*) holds for u_r.

Aha! Let $r \nearrow 1$. Use uniform continuity of u on $\overline{D_1(0)}$. Take limit. Get (*) for u.

<u>Remark:</u> Same trick works for analytic fcns continuous up to circle and Cauchy formula.

Hmmm. Famous problem: Given a continuous fcn $U(\theta)$ on $[0, 2\pi]$ with $U(0) = U(2\pi)$, is there a harmonic fcn u on $D_1(0)$ which extends continuously to $\overline{D_1(0)}$ such that $u(e^{i\theta}) = U(\theta)$?

Guess: Yes!

$$u(a) = \int_0^{2\pi} P(a, \theta)\, U(\theta)\, d\theta$$

where $P(a, \theta) = \frac{1}{2\pi} \frac{1 - |a|^2}{|a - e^{i\theta}|^2}$ is the Poisson kernel.

Dirichlet problem: Circular metal disc. Thermal equillibrium. Know temp on boundary. Want to know temp inside.

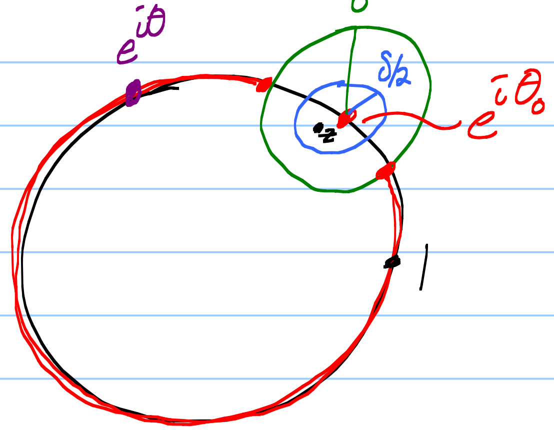
<u>Schwarz's thm:</u> Poisson formula solves D. prob.

3 Key properties of $P(a,\theta)$

1) $P(a,\theta) > 0$ ✓ formula

2) $\displaystyle\int_0^{2\pi} P(a,\theta)\, d\theta = 1$ ✓ (*) using $u(z) \equiv 1$.

3)



$I_\delta = \{\theta : e^{i\theta} \notin D_\delta(e^{i\theta_0})\}$

$$P(z,\theta) \le \frac{4}{\pi\delta^2}(1-|z|)$$

$\longrightarrow 0$ uniformly on I_δ

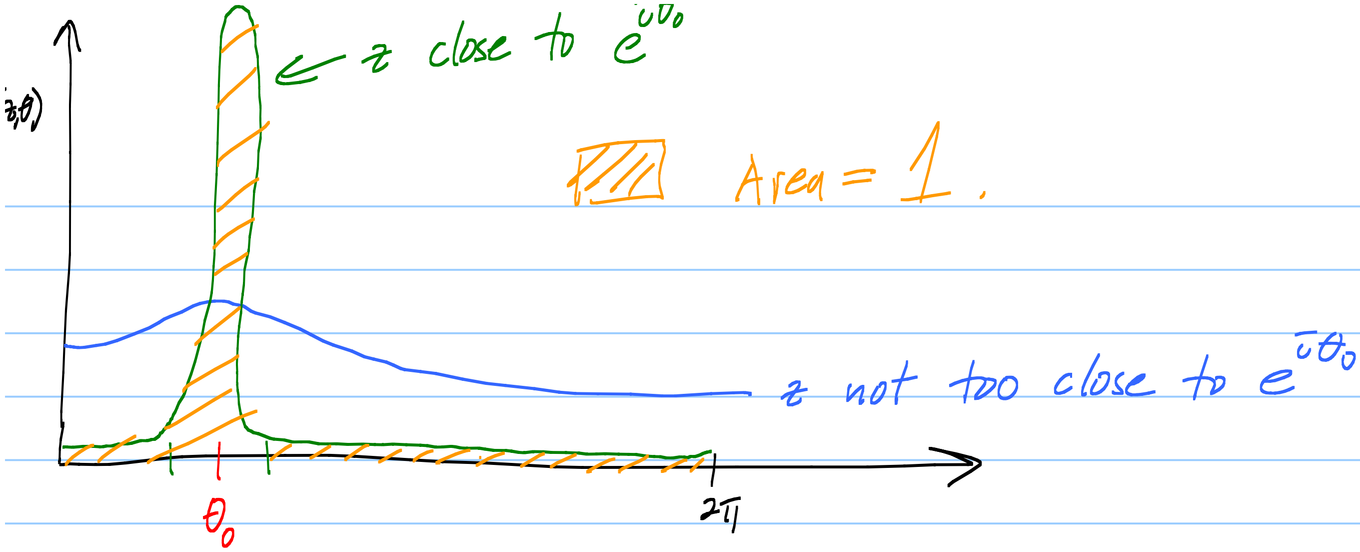
when $z \in D_{\delta/2}(e^{i\theta_0}) \cap D_1(0)$.

(3) easy: $\displaystyle \frac{1}{2\pi}\frac{1-|z|^2}{|z-e^{i\theta}|^2} \le \frac{1}{2\pi}\frac{1-|z|^2}{\left(\frac{\delta}{2}\right)^2} = \frac{1}{2\pi}\frac{(1-|z|)\overbrace{(1+|z|)}^{<2}}{\left(\frac{\delta}{2}\right)^2}$

$$< \frac{4}{\pi\delta^2}(1-|z|) \checkmark$$

4) $P(z,\theta)$ is harmonic in $z \in D_1(0)$.

because $\displaystyle P(z,\theta) = \frac{1}{2\pi}\operatorname{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right]$



Pf: Define $u(z) = \int_0^{2\pi} P(z,\theta)\, U(\theta)\, d\theta$

$$= \underbrace{\mathrm{Re}\, \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, U(\theta)\, d\theta}_{\text{analytic : HWK 2 : p. 1.}}$$

u is harmonic on $D_1(0)$. ✓

Step 2: Show $\lim_{z \to e^{i\theta_0}} u(z) = U(\theta_0)$.

$$u(z) - U(\theta_0) = \int_0^{2\pi} P(z,\theta)\left[U(\theta) - U(\theta_0)\right] d\theta = I$$

because $\int_0^{2\pi} P(z,\theta)\, d\theta = 1$.

Let $\varepsilon > 0$. $\exists\ \delta > 0$ such that $|U(\theta) - U(\theta_0)| < \frac{\varepsilon}{2}$ when $e^{i\theta} \in D_\delta(e^{i\theta_0})$.

Now $I = \int_{I_\delta} \underbrace{P(z,\theta)}_{\text{small}} \left(U(\theta) - U(\theta_0)\right) d\theta \qquad I_1$

$$+\int_{[0,2\pi]-I_\delta} P(z,\theta)\underbrace{\left(U(\theta)-U(\theta_0)\right)}_{\text{small}}d\theta \quad I_2$$

Suppose $z\in D_{\delta/2}(e^{i\theta_0})\cap D_1(0)$.

$$|I_1|\le \frac{4}{\delta^2\pi}(1-|z|)\,2M\,\underbrace{\text{Length}(I_\delta)}_{<2\pi} \to 0 \text{ as } z\to e^{i\theta_0}$$

inside set.

$M=\max_{[0,2\pi]}|U(\theta)|$

$$|I_2|\le \int_{[0,2\pi]-I_\delta} P(z,\theta)\underbrace{|U(\theta)-U(\theta_0)|}_{<\frac{\varepsilon}{2}}\,d\theta$$

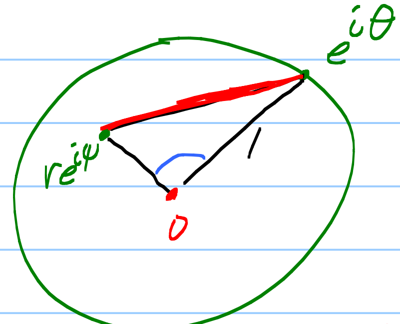
P possitive!

$$<\frac{\varepsilon}{2}\int_{[0,2\pi]-I_\delta} P(z,\theta)\,d\theta < \frac{\varepsilon}{2}\underbrace{\int_0^{2\pi} P(z,\theta)\,d\theta}_{=1\,!}$$

$P>0.$

$<\frac{\varepsilon}{2}$. Done!

Other formulas



blue angle $=\varphi-\theta$

red line $=$

$$\sqrt{1+r^2-2r\cos(\varphi-\theta)}$$

$$u(re^{i\psi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\psi-\theta)}\,\varphi(\theta)\,d\theta$$

$D_R(z_0)$ version:

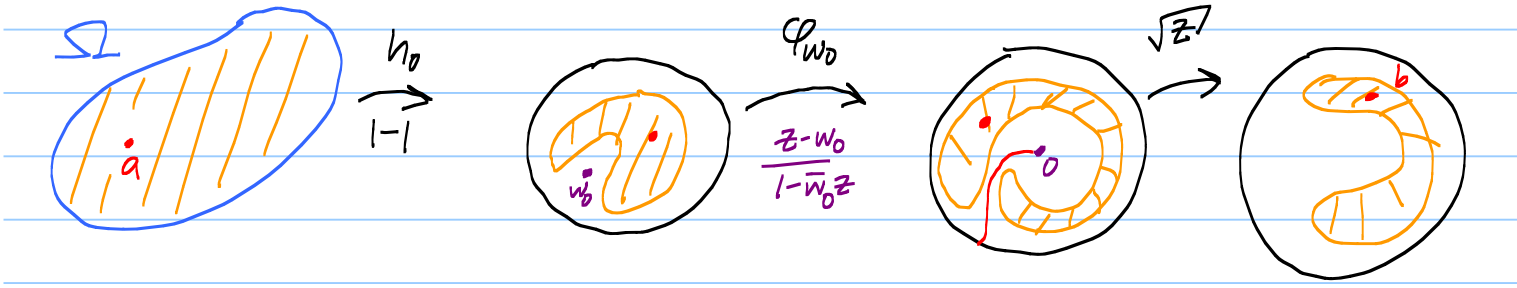
$$u(z_0+re^{i\psi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2-r^2)}{R^2-2rR\cos(\psi-\theta)+r^2}\,u(z_0+Re^{i\theta})\,d\theta$$

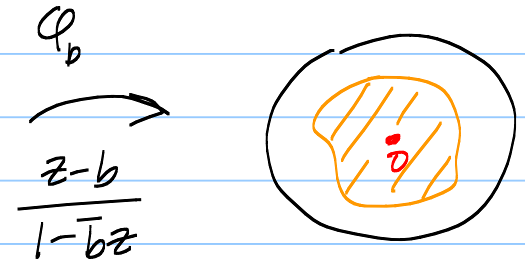
Note: $r=0$ gives ave property.

Explosion of theorems!

ecture 34 Applications of the Poisson formula

Friday odds and ends: Koebe's contructive pf of the RMT.





Construction: Composition

h_1 : $|h_1'(a)| > |h_0(a)|$

Repeat! Get seq h_j. Koebe: $h_j \twoheadrightarrow f_a$ ← Riemann map.

Much better ways to compute f_a are known now.

Remarks about maximum principles

Last time: Ave prop for harm fcns ⟹ Max princ for harm fcns.

Note: Max princ for harm fcns ⟹ Min princ for harm fcns

Pf: Max princ for $-u$ ⟹ min princ for $+u$.

Remark: Max modulus thm ⟹ Max princ for harm fcns.

Pf: Suppose harm fcn u has local max at z_0.

Shrink a disc $\overline{D_r(z_0)} \subset \Omega$ so that

$u(z) \leq u(z_0) = M$ on $\overline{D_r(z_0)}$.

Get $f' = u + iv$ analytic on $D_r(z_0)$.

Trick: $e^f = e^{u+iv} = e^u e^{iv}$

So $\boxed{|e^f| = e^u}$ ← has max at z_0!

So MMT $\Rightarrow$ $e^f \equiv$ const. on $D_r(z_0)$

Red box: $u = \ln|e^f| \equiv$ const too.

Know $u \equiv c$ on $D_r(z_0) \Rightarrow u \equiv c$ on domain Ω.

Remark: No Min princ for analytic fcns. e.g. z on $D_1(0)$.

However, there is a Min modulus thm for nonvanishing analytic fcns: Max princ for $1/f$ $\Rightarrow$ Min princ for f!

Big moment: Ave prop $\Longrightarrow$ harmonic.

Thm: Suppose u is a continuous real valued fcn on a domain Ω. If u satisfies the ave prop: $\overline{D_r(z_0)} \subset \Omega$: $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta$, then u is harmonic.

Pf: Being harmonic is a local property, so can restrict attention to a disc. $\frac{1}{R}(z - z_0)$ maps $D_R(z_0) \longleftrightarrow D_1(0)$. Can assume u satisfies ave prop on $D_r(0)$, $r > 1$.

Classic trick: Define

$$\tilde{u}(z) = \int_0^{2\pi} P(z,\theta)\, u(e^{i\theta})\, d\theta$$

Know $\tilde{u}$ harmonic on $D_1(0)$, continuous on $\overline{D_1(0)}$,

and $\tilde{u}(e^{i\theta}) \equiv u(e^{i\theta})$.

Know: harm fcns satisfy ave prop.

Aha! $u - \tilde{u}$ satisfies ave prop!

Ave prop $\Rightarrow$ max princ $\Rightarrow$ $u-\tilde{u}$ assumes max value on $C_1(0)$, where it is zero.

So $u - \tilde{u} \le 0$ on $\overline{D_1(0)}$.

Repeat using $\tilde{u} - u$ in place of $u - \tilde{u}$. Get $\tilde{u} - u \le 0$.

So $u \equiv \tilde{u}$ ← harmonic. ✓

Riemann removable singularity thm for harmonic fcns.

Pf: Can assume u harmonic on $D_R(0) - \{0\}$, $R > 1$.

We assume u is bounded there. Show 0 is removable

Let $\tilde{u}$ = Poisson integral of bndry values of u on $C_1(0)$.

Know $\tilde{u} \equiv u$ on $C_1(0)$, $\tilde{u}$ harm on $D_1(0)$, cont on $\overline{D_1(0)}$.

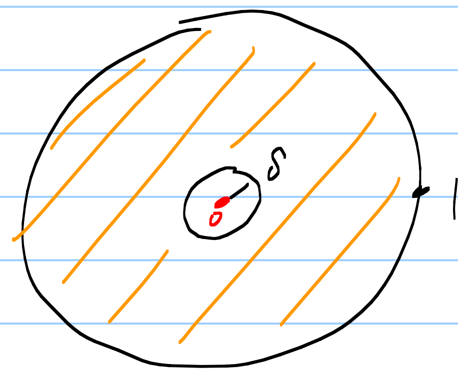
Sneaky trick: Define $v = u - \tilde{u} + \varepsilon \underbrace{\ln|z|}$ ← $\varepsilon > 0$

harmonic on $\mathbb{C} - \{0\}$,
$\equiv 0$ on $C_1(0)$,
$\to -\infty$ as $|z| \to 0$.

Have: v harmonic on $D_1(0) - \{0\}$,
v is continuous up to $C_1(0)$ and $v \equiv 0$ on $C_1(0)$.

Key: u bounded on $D_1(0) - \{0\} \implies v(z) \to -\infty$ as $z \to 0$.

Want to use max princ.



Choose $0 < \delta < 1$ such that $v(z) < -1$ when $|z| \le \delta$, $z \ne 0$.

Apply max princ to v on $\overline{D_1(0) - D_\delta(0)}$.

Max occurs on bndry: $v \le \text{Max}\{0, -1\}$ on domain.

Get $v \le 0$ on $\overline{D_1(0)} - D_\delta(0)$.

Can let $\delta \searrow 0$. Get $v \le 0$ on $\overline{D_1(0)} - \{0\}$.

$v = u - \tilde{u} + \varepsilon \ln|z| \le 0$

Aha! Can let $\varepsilon \searrow 0$!

Get $u - \tilde{u} \le 0$.

Repeat using $\tilde{u}-u$ in place of $u-\tilde{u}$.

So defining $u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\, d\theta$

makes u harm on $D_1(0)$.

<u>Fun fact</u>: Solution to the Dirichlet prob on $D_1(0)$ for polynomial data $p(x,y)$ on $C_1(0)$ is a polynomial harmonic fcn.

<u>Pf</u>: $p(x,y) = p\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = P(z,\bar{z})$

$$= \sum_{n,m=0}^{N} c_{nm} z^n \bar{z}^m$$

<u>Trick</u>:

$$z^n \bar{z}^m = \begin{cases} 1 & \text{on } C_1 & n=m \\ \underbrace{z^n \bar{z}^n}_{1} \bar{z}^{m-n} = \bar{z}^{m-n} & \text{on } C_1 & m>n \\ z^{n-m} \underbrace{z^m \bar{z}^m}_{1} = z^{n-m} & \text{on } C_1 & m<n \end{cases}$$

Lecture 35 More about the Dirichlet problem

HWK 7 due tomorrow Tues in GS.

Fun things to do: poly $p(x,y) = p\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = P(z,\bar{z})$

$$= \sum_{n,m=0}^{N} c_{nm} \underbrace{z^n \bar{z}^m}$$

$$\left.\begin{array}{ll} = 1 & n=m \\ = z^{n-m} & n>m \\ = \bar{z}^{m-n} & m>n \end{array}\right\} \text{ on } |z|=1$$

$$= c_{00} + p(z) + \overline{q(z)} \quad \text{on } C_1(0), \quad p, q \text{ polys!}$$

Aha! $p(z)$, $q(z)$ are analytic. So real and imag parts are harmonic: Write $c_{00} + p(z) + \overline{q(z)}$

$$= u(x,y) + i\,v(x,y)$$

real harmonic polys!

Hmmm: on $C_1(0)$: $\underset{\text{real}}{p(x,y)} = u(x,y) + i \underbrace{v(x,y)}_{\equiv 0}$

Fact: On discs, poly boundary data yield poly sol^{n}s to D. Prob.

Something about Poisson and Dirichlet (see home page).

Fun: Algebra of $p(x,y)$ on $C_1(0)$ separates points.

Stone-Weirstraß $\Rightarrow$ polys dense among continuous fcns on $C_1(0)$.

Idea: Given cont. real valued fcn φ on $C_1(0)$,

take seq polys $p_n(x,y) \rightrightarrows \varphi$ on $C_1(0)$.

Solve D. Prob for $p_n(x,y)$. Get $\mathcal{P}_n$'s.

Show $\mathcal{P}_n \rightrightarrows$ sol^n to D. Prob for φ on $\overline{D_1(0)}$.

Important lemma: If a seq of harmonic fcns u_n converges unif on compact subsets of a domain Ω to u, then u is harmonic.

Pf: MA 504 implies that u is continuous.

Unif conv on circles $\Rightarrow$

$$\underbrace{\frac{1}{2\pi}\int_0^{2\pi} u_n(z_0+re^{i\theta})\,d\theta}_{u_n(z_0)} \longrightarrow \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})\,d\theta = u(z_0)$$

$$u_n(z_0) \longrightarrow u(z_0)$$

So u satisfies ave prop $\Rightarrow$ u harmonic.

Back to D. Prob. $\mathcal{P}_n \rightarrow$ harmonic fcn.

Hmmm. How to use Max princ?

Like a HWK prob. $p_n \twoheadrightarrow \varphi$ on $C_1(0)$.

So p_n unif Cauchy on $C_1(0)$.

Max princ $\Longrightarrow$ p_n are unif Cauchy on $\overline{D_1(0)}$.

So $p_n \twoheadrightarrow$ unif on $\overline{D_1(0)}$ to Φ.

Φ is cont on $\overline{D_1(0)}$, $\Phi = \varphi$ on $C_1(0)$,

and Φ harm on $D_1(0)$. Φ solves D. Prob.

Interesting fact Ellipses have poly $\rightarrow$ poly D. Prob propert

Pf: Ellipse: $\underbrace{\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1}_{\rho(x,y)} = 0$

$\rho(x,y)$ "defining fcn" for E.

$E = \{(x,y) : \rho(x,y) < 0\}$.

ρ is a degree two poly in x and y.

P_N = Vector space of polys of x and y of deg N or less.

Fisher map: poly $q(x,y) \longmapsto \Delta(\rho \cdot q)$

increases deg by 2

decreases deg by 2!

So $\widetilde{\Delta} : P_N \to P_N$ linear.

Claim: $\tilde{\nabla}$ is 1-1.

Why: If $\tilde{\nabla}(q) = 0$

$$\Delta(\rho q) \equiv 0$$

So ρq is harmonic. But $\rho \equiv 0$ on bndry!

Max princ $\Rightarrow \rho q \equiv 0. \Rightarrow q \equiv 0.$

Claim $\tilde{\nabla}$ is onto. Linear alg fact:

$$\tilde{\nabla}: P_N \xrightarrow{1\text{-}1} P_N \leftarrow \text{finite dim v. space}$$

$\Rightarrow \tilde{\nabla}$ onto.

Aha! Can use $\tilde{\nabla}$ to solve D. Prob on E with poly data.

Given poly data q. Let Δq.

$\tilde{\nabla}$ onto $\Rightarrow \exists\, p$ with $\tilde{\nabla} p = \Delta q.$

$$\Delta(\rho p) = \Delta q$$

Aha! $q - \rho p$ solves D. Prob with bndry data q.

$q - \rho p = q$ on bndry ✓

$$\Delta(\underbrace{q - \rho p}_{\text{harmonic ✓}}) = \Delta q - \Delta(\rho p) \equiv 0.$$

Khavinson–Shapiro conjecture: Only discs and ellipses have poly → poly D. Prob. property.

What about rational data?

On unit disc: $r(x,y) = R(z,\bar{z}) = R(z,\frac{1}{z})$ on $C_1(0)$

$S(z) = 1/z$ is the Schwarz fcn for unit disc.

$\frac{1}{z} = \bar{z}$ on $C_1(0)$.

$$\text{So} \quad R(z,\bar{z}) = R(z,\tfrac{1}{z}) \underset{\substack{\uparrow \\ \text{partial fractions}}}{=} \sum \underbrace{\frac{A_{kn}}{(z-a_n)^k}}_{a_n \in D_1(0)} + \sum \underbrace{\frac{B_{kn}}{(z-b_n)^k}}_{b_n \text{ outside } C_1(0)}$$

$$\underset{\substack{\uparrow \\ \text{on } C_1(0)}}{=} \sum \underbrace{\frac{A_{kn}}{(\frac{1}{\bar{z}}-a_n)}}_{\substack{\text{singularities} \\ \text{at } 1/\bar{a}_n \\ \text{outside } C_1(0)}} + \sum \frac{B_{kn}}{(z-b_n)^k}$$

$$= \underbrace{\overline{Q(z)} + r(z)}_{\text{solve D. Prob.}}$$

Fact: On discs, rational → rational for D. Prob.

B., Ebenfeld, Khavinson, Shapiro: Discs are only domains with this feature!

Important formula

$$P(a,\theta) = \frac{1}{2\pi}\frac{1-|a|^2}{|a-e^{i\theta}|} = \frac{1}{2\pi}\operatorname{Re}\left[\frac{a+e^{i\theta}}{a-e^{i\theta}}\right]$$

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z,\theta)\, u(e^{i\theta})\, d\theta$$

$$= \frac{1}{2\pi}\operatorname{Re} \underbrace{\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, u(e^{i\theta})\, d\theta}_{\text{analytic in } z}$$

Harmonic conjugate v for u = Im part of $\int$.

Hilbert transform: singular integral operator

Lecture 36 Weak averaging property and Schwarz reflection for harmonic fns

HWK 8 due Thurs, 4/21 in GS

$$P(z,\theta) = \frac{1}{2\pi} \operatorname{Re}\left[\frac{e^{i\theta}+z}{e^{i\theta}-z}\right] \qquad z(\theta) = e^{i\theta} \qquad z'(\theta) = ie^{i\theta}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, u(e^{i\theta}) \frac{z'(\theta)\, d\theta}{z'(\theta) = ie^{i\theta}} \quad dz$$

continuous on C

$$= \frac{1}{2\pi i}\int_C \frac{u(w)}{w-z}\, dw + \frac{z}{2\pi i}\int_C \frac{u(w)/w}{w-z}\, dw$$

HWK 2 : Prob 1 $\Rightarrow$ analytic in z.

Note: By differentiating under Poisson integral, get "Cauchy" estimates for harmonic fcns. Use them to prove:

Thm: Suppose u_n harmonic on a domain Ω and $u_n \rightrightarrows u$. Then u is harmonic and

$$\frac{\partial^{k+m} u_n}{\partial x^k \partial y^m} \rightrightarrows \frac{\partial^{k+m} u}{\partial x^k \partial y^m}$$

Theorem: Suppose u is

1) Continuous on Ω

2) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ exist on Ω, and

3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \equiv 0$ on Ω.

Then u is harmonic $\iff$ $\underbrace{\text{locally} = \operatorname{Re} f,\ f \text{ analytic}}_{C^\infty \text{ smooth!}}$

Pf: Thm local, so we can assume hyp valid on $D_R(0)$, $R > 1$.

Let $\tilde{u}$ = Poisson integral of $u(e^{i\theta})$.

Want to see that $u \equiv \tilde{u}$. If not, then

$\exists\, z_0 \in D_1(0)$ $\quad u(z_0) - \tilde{u}(z_0) \neq 0$. Can assume

positive by replacing $u - \tilde{u}$ by $\tilde{u} - u$.

Say $u(z_0) - \tilde{u}(z_0) = m > 0$.

Note $u - \tilde{u} \equiv 0$ on $C_1(0)$.

Sneaky trick: $\varepsilon|z|^2 = \varepsilon(x^2 + y^2)$ is small on $\overline{D_1(0)}$

and $\Delta(\varepsilon|z|^2) = 4\varepsilon > 0$ there.

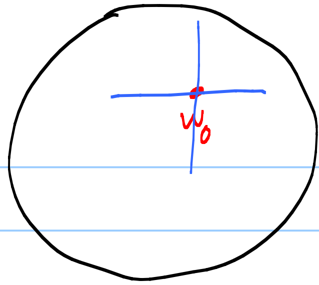
Let $v = u - \tilde{u} + \varepsilon|z|^2$ $\leftarrow$ want a positive max in $D_1(0)$.

Aha! Take $0 < \varepsilon < \frac{m}{2}$.

Then $v(z_0) = m + \varepsilon|z_0|^2 \geq m$

$v(e^{i\theta}) = 0 + \varepsilon < \frac{m}{2}$

max v on $\overline{D_1(0)}$ occurs at some pt $w_0 \in D_1(0)$.



2nd derivative test in horiz and vert directions at w_0

$\left[\tilde{u}''(t_0) > 0,\ \underline{\text{strict local min}}\right]$

Aha! Since $v(w_0)$ is a max,

$$\frac{\partial^2 v}{\partial x^2}(w_0) \le 0$$

$$+ \quad \frac{\partial^2 v}{\partial y^2}(w_0) \le 0$$

$$\Delta v\,(w_0) \le 0 \quad \lightning \quad \Delta v = 4\varepsilon > 0.$$

No such z_0. $u - \tilde{u} \equiv 0$. Done!

Elliptic regularity: Δu smooth $\Rightarrow$ u smooth.

__Weak averaging property__ A continuous fcn u on a domain Ω satisfies the W.A.P. if, given any point $z_0 \in \Omega$, there is $r > 0$ such tha $D_r(z_0) \subset \Omega$ and $u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})\,d\theta$ for $0 < \rho < r$.

__Remark__: r could be small. $\left(r < \mathrm{dist}(z_0, b\Omega)\right)$

__Thm__: W.A.P. $\Rightarrow$ u harmonic.

__Pf__: Local thm. Can assume u continuous on $D_R(0)$, $R>1$ and satisfies WAP.

Let $\tilde{u}$ = Poisson integral of $u(e^{i\theta})$.

Let $U = \tilde{u} - u$.

Note $U \equiv 0$ on $C_1(0)$.

U satisfies WAP too [$\tilde{u}$ is harmonic!]

Want $U \equiv 0$ on $D_1(0)$. Assume $U(z_0) \neq 0$.

Can assume $U(z_0) > 0$ by replacing $\tilde{u}-u$ with $u-\tilde{u}$.

Then U will a max M on $\overline{D_1(0)}$, $M>0$.

Since $U \equiv 0$ on $C_1(0)$, max occurs inside $D_1(0)$.

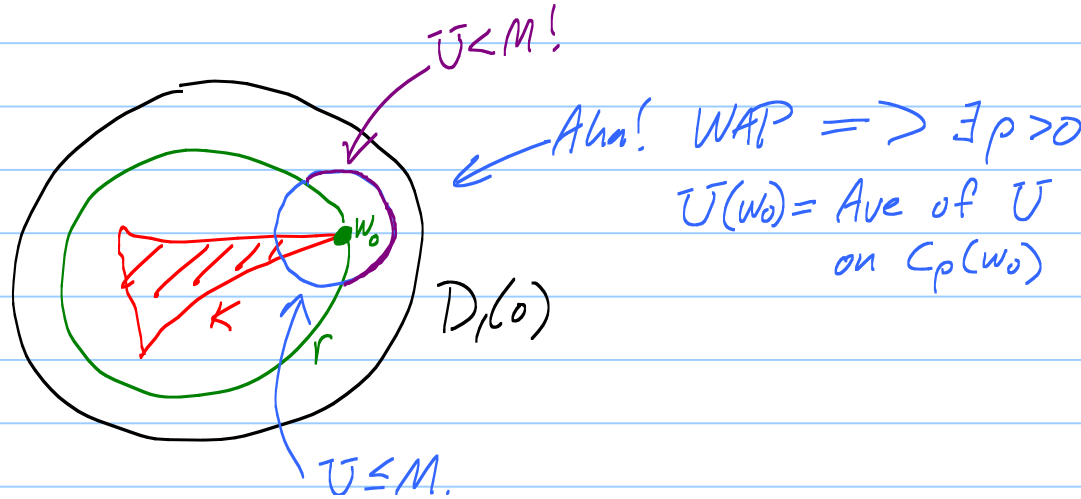
Hmmm. Let $K = \{z \in \overline{D_1(0)} : U(z) = M\}$.

K is closed (by continuity) and bounded

inside $D_1(0)$. K is compact.

$|z|$ is a continuous fcn on K, so it has a max r

on K.

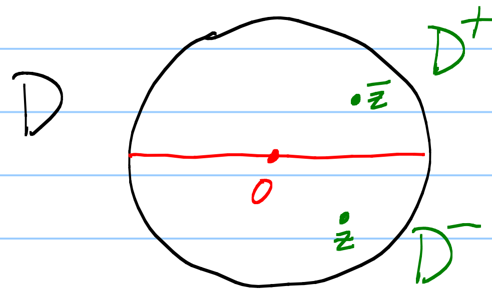


See Ave V of $C_\rho(w_0) < M$. ↯.

So no such z_0 where $\tilde{u}-u$ (or $u-\tilde{u}$) not zero.

Conclude $u \equiv \tilde{u}$ on $D_1(0)$. ✓

Schwarz reflection principle for harmonic fcns.



Suppose u is a real valued harmonic fcn on D^+ which is continuous up $D \cap \mathbb{R}$. If $u \equiv 0$ on $D \cap \mathbb{R}$,

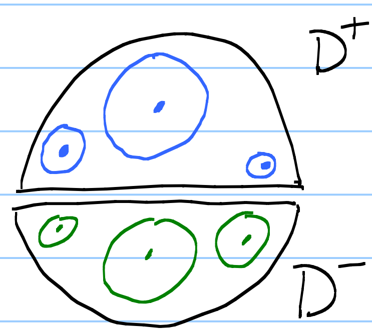
then

$$V(z) = \begin{cases} u(z) & z \in D^+ \\ 0 & z \in D \cap \mathbb{R} \\ -u(\bar{z}) & z \in D^- \end{cases}$$

is harmonic on D.

Note: $-u(\bar{z})$ is harmonic on D^-.

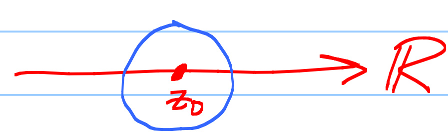
Pf:



WAP on D^+ ✓

WAP on D^- ✓

Last step



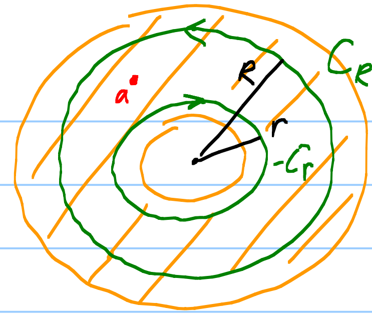
$$0 = u(z_0) = \underbrace{\int_{top} + \int_{Bot}}_{\text{Cancel!}}$$

WAP holds at $z_0 \in \mathbb{R}$ too. ✓

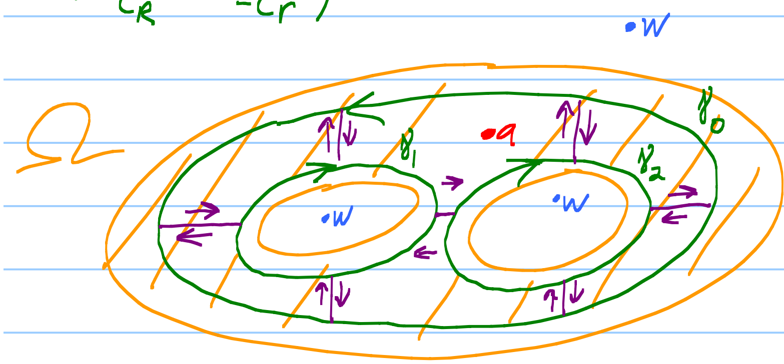
Lecture 37 The general Cauchy theorem

Cauchy theorem for an annulus

$$\left(\int_{C_R}+\int_{-C_r}\right) f\,dz = 0$$



Cauchy int formula too



$$\Gamma = \underbrace{\gamma_0 \cup \gamma_1 \cup \gamma_3}_{\gamma\text{'s closed curves in } \Omega} \leftarrow \text{cycle}$$

$$\int_\Gamma f\,dz = \sum \int_{\gamma_j} f\,dz$$

f analytic on Ω, then $\int_\Gamma f\,dz = 0$. And

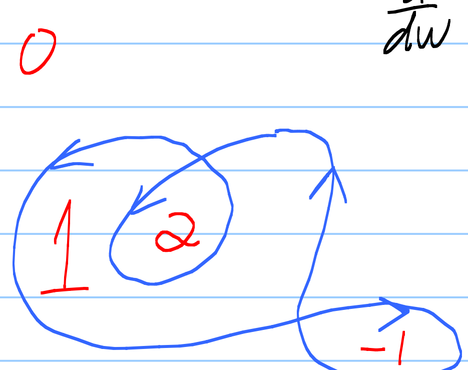
$$f(a)\,\mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\,dz$$

where $\mathrm{Ind}_\Gamma(a) = \sum \underbrace{\mathrm{Ind}_{\gamma_j}(a)}$

$$\frac{1}{2\pi i}\int_{\gamma_j}\frac{dz}{z-a} = \text{"winding number of } \gamma_j \text{ about } a\text{"}$$

Recall: Closed curve γ is such that $\mathbb{C} - tr(\gamma)$ is a union of open connected components. $\mathrm{Ind}_\gamma(w) \equiv$ const on each component because

$$\frac{d}{dw}\mathrm{Ind}_\gamma(w) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{(z-w)^2}$$



$$= \frac{1}{2\pi i}\int_\gamma \frac{d}{dz}\left[\frac{-1}{z-w}\right] dz = 0$$

$\mathrm{Ind}_\gamma(w)$

Important assumption to get a general Cauchy thm:

$$\mathrm{Ind}_\Gamma(w) = 0 \quad \text{for } \underline{\text{all}} \quad w \in \mathbb{C} - \Omega.$$

<u>Fancy word</u> Γ is "homologous to zero" in Ω.

Write $\Gamma \sim 0$ or $\Gamma \sim_\Omega 0$.

<u>General Cauchy Thm</u>: $\Gamma \sim 0$ and f analytic on Ω.

1) $\int_\Gamma f \, dz = 0$

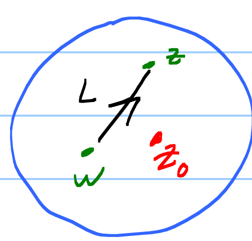
2) $f(a)\,\mathrm{Ind}_\Gamma(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\,dz \qquad a \in \Omega - \mathrm{tr}(\Gamma)$

<u>Pf</u>: (Ahlfors)

$$g(z,w) = \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \neq w \\[2ex] f'(z) & z = w \end{cases}$$

<u>Step 1</u> g is continuous on $\Omega \times \Omega$.

Clear at (z_0, w_0), $z_0 \neq w_0$. Only to check near pts (z_0, z_0).



$$\left| g(z,w) - g(z_0,z_0) \right| = \left| \underbrace{\frac{f(z)-f(w)}{z-w}}_{\frac{1}{z-w}\int_L f'(\zeta)\,d\zeta} - \underbrace{f'(z_0)}_{\frac{1}{z-w}\int_L f'(z_0)\,d\zeta} \right|$$

$$= \left| \frac{1}{z-w} \int_L \left[f'(\zeta) - f'(z_0) \right] d\zeta \right|$$

$$\le \frac{1}{|z-w|} \underbrace{\left(\max_{\zeta \in L} |f'(\zeta) - f'(z_0)| \right)}_{\substack{f' \text{ continuous.} \\ \text{Can easily make small} \checkmark}} \underbrace{|z-w|}_{\text{length}(L)}$$

Step 2 Define $h(z) = \int_\Gamma g(z,w)\, dw$

Plan: Extend h to $\mathbb{C}$ as an entire fcn that tends to zero at ∞. Liouville's $\Rightarrow$ Cauchy thm!

Step 3 h is analytic on Ω.

a) clear that h is continuous on Ω.

[Why: g cont on $\Omega \times \Omega$. $tr(\Gamma) = \bigcup tr(\gamma_j)$ is compact. g cont on compact $\overline{D_\delta(z_0)} \times tr(\Gamma)$ is unif continuous. Get h is cont.]

b) $\Delta \subset \Omega$ triangle.

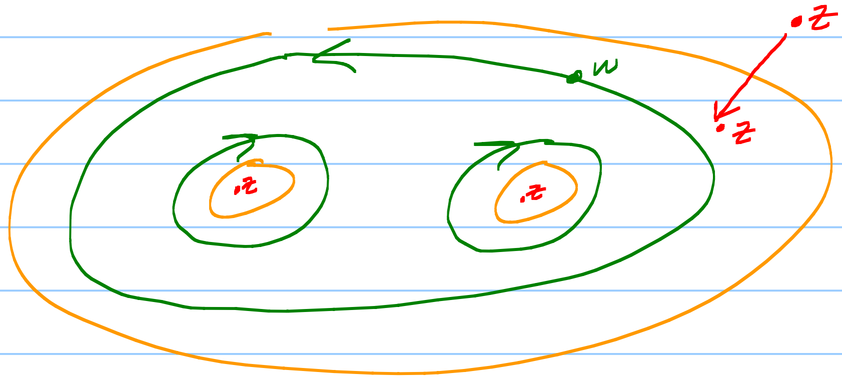
$$\int_\Delta h\, dz = \int_\Delta \left(\int_\Gamma g(z,w)\, dw \right) dz$$

$$\underset{\text{Fubinni}}{=} \int_\Gamma \left(\underbrace{\int_\Delta g(z,w)\, dz}_{=0} \right) dw = 0$$

Aha! When w fixed, $g(z,w)$ has a removable sing of z at $z=w$.

Morera $\Rightarrow$ h analytic!

Step 4 Extend h to $\mathbb{C}$



$$h(z) = \int_\Gamma \frac{f(z)-f(w)}{z-w}\,dw = f(z)\underbrace{\int_\Gamma \frac{1}{z-w}\,dw}_{=0} - \int_\Gamma \frac{f(w)}{z-w}\,dw$$

"outside" Γ

$$\tilde{h}(z) = -\int_\Gamma \frac{f(w)}{z-w}\,dw \quad \text{on} \quad \tilde{\Omega} = \{z \in \mathbb{C} - tr(\Gamma) \text{ where } \mathrm{Ind}_\Gamma(z)=0\}$$

Claim: $\tilde{h}$ defines an extension of h to $\mathbb{C}$ as an entire fcn.

Facts: 1) $\tilde{\Omega}$ is open.

2) $\Omega \subset \mathbb{C} - \tilde{\Omega}$

3) $\tilde{h}$ is analytic on $\tilde{\Omega}$ (HWK 2; prob 1).

4) $h = \tilde{h}$ on overlap $\Omega \cap \tilde{\Omega}$

So $\tilde{h}$ extends h to $\mathbb{C}$, analytic. Call ext h.

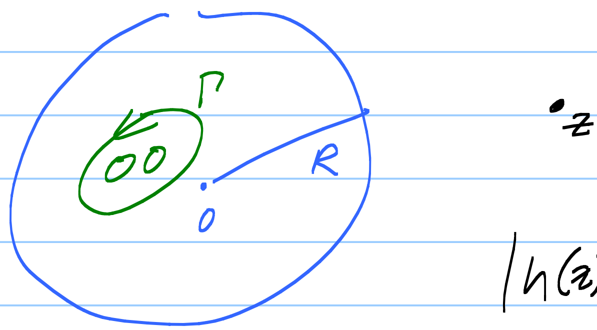
<u>Claim</u> $h(z) \to 0$ as $z \to \infty$.

<u>Why</u>: $tr(\Gamma) \subset D_R(0)$ for some $R > 0$.

So $Ind_\Gamma(w) = 0$ for $|w| > R$.

So, for $|z| > R$:

$$h(z) = \tilde{h}(z) = -\int_\Gamma \frac{f(w)}{z-w}\,dw$$



$$|h(z)| \le \max_{w \in \Gamma} \frac{|f|}{|z-w|}\,\text{length}(\Gamma)$$

$$\le \frac{M}{|z|-R}\,\text{length}(\Gamma)$$

$$\to 0 \text{ as } |z| \to \infty.$$

Liouville's $\Rightarrow h \equiv 0$.

So $$\int_\Gamma \frac{f(z)-f(w)}{z-w}\,dw = 0 \qquad z \in \Omega - tr(\Gamma),$$

$$f(z) \underbrace{\int_\Gamma \frac{1}{z-w}\,dw}_{2\pi i\, Ind_\Gamma(z)} = \int_\Gamma \frac{f(w)}{z-w}\,dw$$

Cauchy integral formula ✓ CIF $\Rightarrow$ Cauchy Thm

Lecture 38 The General residue theorem

General Cauchy thm Ω domain, Γ cycle, f analytic on Ω.

If $\Gamma \sim 0$ in Ω, then 1) $\int_\Gamma f \, dz = 0$

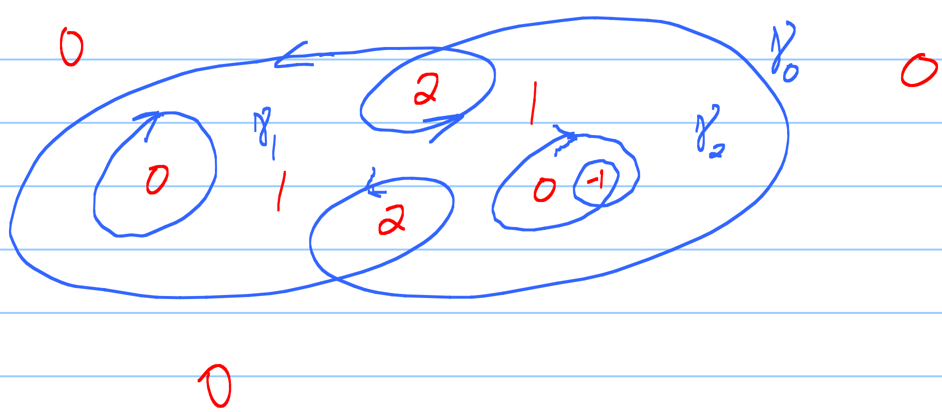
$a \notin tr(\Gamma)$, 2) $f(a) \, Ind_\Gamma(a) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-a} \, dz$

Def^{n}s: $\Gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$, γ_j closed curves in Ω.

$$Ind_\Gamma(a) = \frac{1}{2\pi i} \sum_1^n \int_{\gamma_j} \frac{1}{z-a} \, dz$$

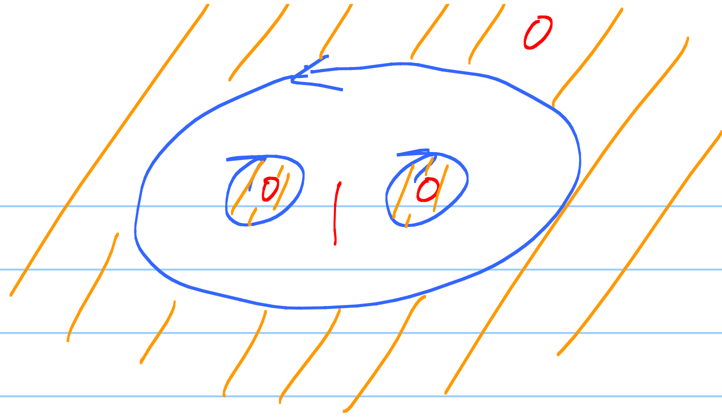
$$\int_\Gamma f \, dz = \sum_1^n \int_{\gamma_j} f \, dz$$

$\Gamma \sim_\Omega 0$: $Ind_\Gamma(a) = 0$ for $a \in \mathbb{C} - \Omega$.

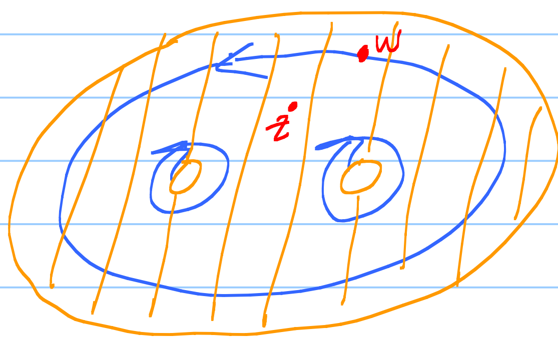


Fact: Set $K = tr(\Gamma) \cup \{z : Ind_\Gamma(z) \neq 0\}$ is a compact set in Ω. [K is closed ✓ bounded ✓

Pf: recap



$$\tilde{\Omega} = \{ z \in \mathbb{C} - tr(\Gamma) ; \ Ind_\Gamma(z) = 0 \}$$



Ω

Facts: $\tilde{\Omega}$ open $\subset \mathbb{C} - \Omega$. $\quad \mathbb{C} = \Omega \cup \tilde{\Omega}$.

$\tilde{\Omega}$ contains the unbounded component of $\mathbb{C} - tr(\Gamma)$

So $\exists \ R > 0$ such that $\{z : |z| > R\} \subset \tilde{\Omega}$.

DQ

$\int_\Gamma g(z,w)\,dw$

$$H(z) = \begin{cases} f(z) \int_\Gamma \frac{dw}{z-w} - \int_\Gamma \frac{f(w)}{z-w}\,dw & z \in \Omega \quad z \notin tr(\Gamma) \\ -\int_\Gamma \frac{f(w)}{z-w}\,dw & z \in \tilde{\Omega} \end{cases}$$

Last time: H is bnd and entire! $\to 0$ at ∞.

Liouvilles $\Rightarrow H \equiv 0$.

Top line of # def^n is the Cauchy Integral Formula (CIF)!

Fun fact: CIF $\Rightarrow$ Cauchy Thm.

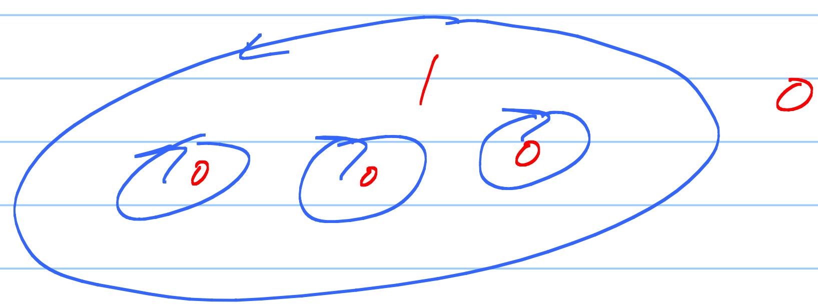
Why: Pick $z_0 \in \Omega - tr(\Gamma)$.

Define $F(z) = (z - z_0) f(z)$

CIF for F at z_0:

$$0 = F(z_0)\,\mathrm{Ind}_\Gamma(z_0) = \frac{1}{2\pi i}\int_\Gamma \underbrace{\frac{f(z)(z-z_0)}{z-z_0}}_{f(z)}\, dz \checkmark$$

Word A cycle is called *simple* if $\mathrm{Ind}_\Gamma(a)$ is only equal to zero or one on $\mathbb{C} - tr(\Gamma)$.



Think: $\{z : \mathrm{Ind}_\Gamma(z) = 1\}$ is the "inside" of Γ

$\{z : \mathrm{Ind}_\Gamma(z) = 0\}$ is the "outside"

"If f is analytic "inside" and on a simple cycle Γ, then $\int_\Gamma f\, dz = 0$."

Remark: Terminology: f "analytic on a compact K"

means $\exists$ open Ω with $K \subset \Omega$, analytic F on Ω such that $f = F$ on K.

New improved Cauchy thm for simp conn. domains:

f analytic on s.c. domain Ω, γ closed curve in Ω

1) $\int_\gamma f \, dz = 0$ ← not new. $\gamma \sim \{z_0\}$.
homotopic

2) $f(z_0) \, \mathrm{Ind}_\gamma(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w - z_0} \, dw$

Pf: $\gamma \sim 0$ because $\int_\gamma \frac{1}{z-a} \, dz = 0$ by Cauchy thm when $a \in \mathbb{C} - \Omega$.
homologous to zero

General residue thm: Ω is a domain, Γ a cycle in Ω, $\Gamma \sim 0$ in Ω, f analytic on Ω except at finitely many isolated singular pts $A = \{a_1, a_2, \ldots, a_N\}$ ← distinct in $\Omega - \mathrm{tr}(\Gamma)$.

Then $$\int_\Gamma f \, dz = 2\pi i \sum_{n=1}^{N} \mathrm{Ind}_\Gamma(a_n) \, \mathrm{Res}_{a_n} f$$

[Simple cycle: $\int_\Gamma f \, dz = 2\pi i \left(\sum_{\text{“inside” } \Gamma} \text{Residues of } f \right)$]

Pf: Let $\tilde{\Omega} = \Omega - A$

Let $n_j = \mathrm{Ind}_\Gamma(a_j)$.

Def^n: $n_j C_\varepsilon(a_j)$: $z(t) = a_j + \varepsilon e^{i n_j t}$ $0 \le t \le 2\pi$

"Go around $C_\varepsilon(a_j)$ n_j times"

Easy facts $\mathrm{Ind}_{n_j C_\varepsilon(a_j)}(a_j) = n_j$.

$$\int_{n_j C_\varepsilon(a_j)} f \, dz = n_j \underbrace{\int_{C_\varepsilon(a_j)} f \, dz}_{= 2\pi i \, \mathrm{Res}_{a_j} f}$$

Can take $\varepsilon > 0$ so small that

1) Each $\overline{D_\varepsilon(a_j)} \subset \Omega$

2) $\overline{D_\varepsilon(a_j)} \cap tr(\Gamma) = \emptyset$

3) $\overline{D_\varepsilon(a_j)} \cap \overline{D_\varepsilon(a_k)} = \emptyset$, $j \neq k$.

Aha! $\tilde{\Omega} = \Omega - A$

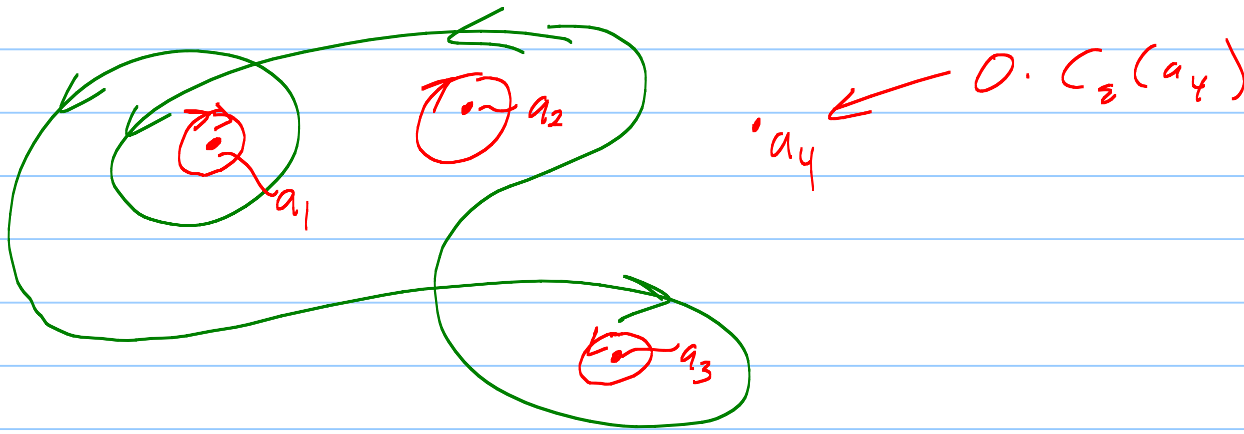
$$\tilde{\Gamma} = \Gamma \cup \Big[\bigcup_{j=1}^{N} -n_j C_\varepsilon(a_j) \Big]$$

$n_j = \mathrm{Ind}_\Gamma(a_j)$

Note: $\mathrm{Ind}_{\tilde{\Gamma}}(a_j) = \mathrm{Ind}_\Gamma(a_j) + (-n_j) + 0$'s from a_k $k \neq j$.

General Cauchy thm can be applied:

$$0 = \int_{\tilde{\Gamma}} f \, dz = \int_{\Gamma} f \, dz + 2\pi i \sum_{j=1}^{N} (-n_j) \operatorname{Res}_{a_j} f$$



Remark: Not necessary to let A be finite, just discrete.

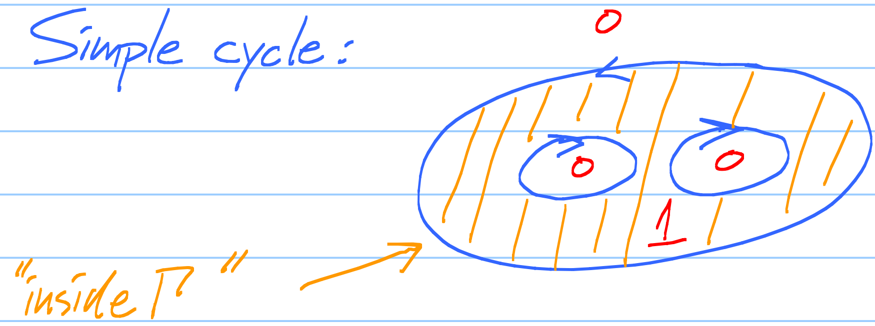
[because $K = tr(\Gamma) \cup \{z : Ind_\Gamma(z) \neq 0\}$ is compact. If $A \cap K$ is infinite, get limit pt. ↯.]

Even though Σ looks infinite, only finitely many $\neq 0$.

Lecture 39 Mittag-Leffler and Weierstraß theorems

HWK 8 due Thurs (last one)

Simple cycle:



$\mathrm{Ind}_\Gamma = 0$ or 1, need to add

$$\boxed{\{z : \mathrm{Ind}_\Gamma(z) = 1\} \text{ is connected.}}$$

Improved residue theorem Suppose Ω is a simply connected domain and A is a discrete subset of Ω and f is analytic on $\Omega - A$. If γ is a closed curve in $\Omega - A$, then

$$\int_\gamma f\,dz = 2\pi i \sum_{a \in A} \mathrm{Ind}_\gamma(a)\, \mathrm{Res}_f(a)$$

finitely many $\neq 0$, so finite sum

allow ess. sing.!

Remark Improved residue thms $\Rightarrow$ improved Arg princ, improved Rouché's, ...

e.g. Suppose f, g analytic inside and on a simple cycle Γ except at finitely many points inside where it has poles. No zeroes on Γ, too.

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'}{f}\,dz = \begin{pmatrix}\#\text{ zeroes}\\ \text{inside } \Gamma\end{pmatrix} - \begin{pmatrix}\#\text{ poles}\\ \text{inside } \Gamma\end{pmatrix}$$

with mult.

= # times $f(z)$ goes around origin in the counterclockwise <u>as</u> z goes around curves γ_j in Γ (in the direction of each γ_j).

Rouché's: $|f(z)-g(z)| < |f(z)|$ on $tr(\Gamma)$, then

f and g have the same number of zeroes

inside Γ (with mult) $\leftarrow$ no poles

(# zeroes) − (# poles) same for f, g $\leftarrow$ allow poles.

<u>Mittag-Leffler thm</u> Given a seq $\{a_n\}_{n=1}^{\infty}$ of distinct pts in $\mathbb{C}$ such that $a_n \to \infty$ as $n \to \infty$ (A discrete) and principal parts $R_n(z) = \frac{A_{N_n,n}}{(z-a_n)^{N_n}} + \cdots + \frac{A_{1,n}}{(z-a_n)}$,

there exists a meromorphic fcn f on $\mathbb{C}$ with poles exactly at the a_n with princ part R_n there.

[$f - R_n$ has a removable sing at a_n.]

<u>Def^n</u>: f is called meromorphic on a domain Ω if there is a discrete set of pts $A \subset \Omega$, f is analytic on $\Omega - A$, f only has poles at pts in A

Note: A could be empty.

Think: Give f the value ∞ at pts in A.

Get $f : \Omega \to \hat{\mathbb{C}}$ "analytic" from the point of view of $\hat{\mathbb{C}}$ as a complex manifold (Riemann surface).

[Coordinate chart at ∞:

$$\underbrace{\{z : |z| > R\} \cup \{\infty\}}_{\text{polar cap}} \xrightarrow{1/z} D_{1/R}(0)$$
]

Pf of M-L: First try: $f = \sum_{n=1}^{\infty} R_n(z)$

Ouch! Might not converge.

Second try: $f = \sum_{n=1}^{\infty} \left[R_n(z) - \underbrace{p_n(z)}_{\substack{\text{Taylor poly} \\ \text{for } R_n(z)}} \right]$

Taylor poly for $R_n(z)$ such that $|R_n(z) - p_n(z)| < \frac{1}{2^n}$ on $\overline{D_{r_n}(0)}$ where $r_n = \frac{|a_n|}{2}$

Claim: f solves M-L problem.

Pf: Take big disc $D_R(0)$. $\exists\, N$ such that $r_n = \frac{|a_n|}{2} > R$ if $n > N$.

$$f(z) = \underbrace{\sum_{n=1}^{N} \left(R_n(z) - p_n(z) \right)}_{\text{rational fcn with correct princ parts in } D_R(0)} + \sum_{n=N+1}^{\infty} \left(\underbrace{R_n(z) - p_n(z)}_{eeeeeee} \right)$$

$|eeee| < \frac{1}{2^n}$ on $D_R(0)$

Weierstraß M-test $\Rightarrow$ this Σ conv unif to analytic fcn on $D_R(0)$.

Remark M-L Thm is true on <u>any</u> domain Ω, discrete $A \subset \Omega$.

Lars Hörmander : Several complex variables : Chap 1.

Weierstraß thm Given seq $\{a_n\}_{n=1}^{\infty}$ as in M-L thm, and positive integers m_n, there is an <u>entire fcn</u> f that has zeroes of mult m_n at a_n's and <u>no other</u> zeroes

Classic proof $f(z) = \prod_{n=1}^{\infty} \underbrace{\left(1 - \frac{z}{a_n}\right)^{m_n} e^{P_n(z)}}_{\text{need terms to } \to 1 \text{ fast enough}}$ $\leftarrow$ Weier polys

My proof M-L $\Rightarrow$ Weier via general residue thm.

Hmmm. $\frac{f'}{f} = \frac{m_n}{z - a_n} + (\text{analytic})$ near a_n

Hmmm. $f(z) = e^{\log f} = e^{\int_\gamma \frac{f'}{f} dw}$

Aha! M-L says $\exists$ mero F with poles at a_n with p.p. $\frac{m_n}{z-a_n}$.

"Define" $f(z) = \exp\left(\int_{\gamma_a^z} F(w)\,dw \right)$

where γ_a^z is a curve in $\mathbb{C} - \{a_n\}_1^\infty$ from a fixed pt a to z.

<u>Step 1</u>: f is <u>well</u> defined.

$$\frac{f(z)}{\tilde{f}(z)} = \frac{\exp\left(\int_{\gamma_a^z} F\,dw\right)}{\exp\left(\int_{\tilde{\gamma}_a^z} F\,dw\right)} = \exp\left[\underbrace{\left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z}\right)}_{\int_\gamma} F\,dw\right]$$

$\gamma = \gamma_a^z \cup (-\tilde{\gamma}_a^z)$

$\uparrow$ closed

$$= \exp\left[2\pi i \underbrace{\sum_1^\infty \mathrm{Ind}_\gamma(a_n)\cdot m_n}_{\substack{\text{finite sum}\\ = \text{integer}}}\right] = \underline{1}$$

Next time: show f does it.

ecture 40 Mittag-Leffler, Weierstraß, infinite products

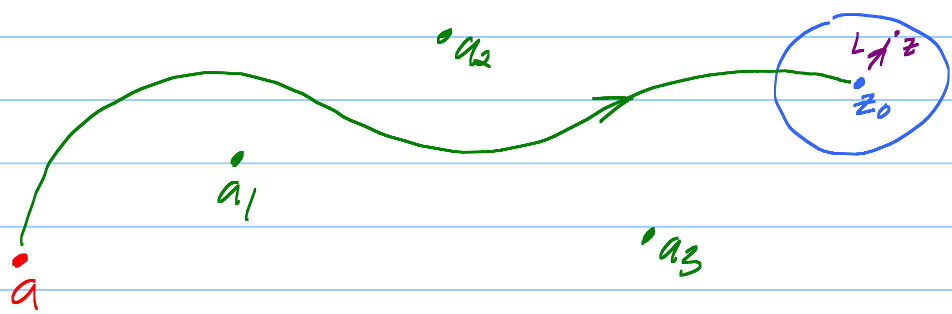
Practice problems for final on home page

Pf that M-L $\Rightarrow$ Weierstraß: $\exists$ entire fcn f with zeroes of multiplicity m_n at $a_n \longrightarrow \infty$.

M-L $\Rightarrow \exists$ F meromorphic on $\mathbb{C}$ with poles at a_n with principal part $\frac{m_n}{z-a_n}$ at a_n.

Last time showed $f(z)=\exp\left(\int_{\gamma_a^z} F(w)\,dw\right)$ well defined via residue thm.

Step 2 $\quad f$ is analytic on $\mathbb{C}-A$

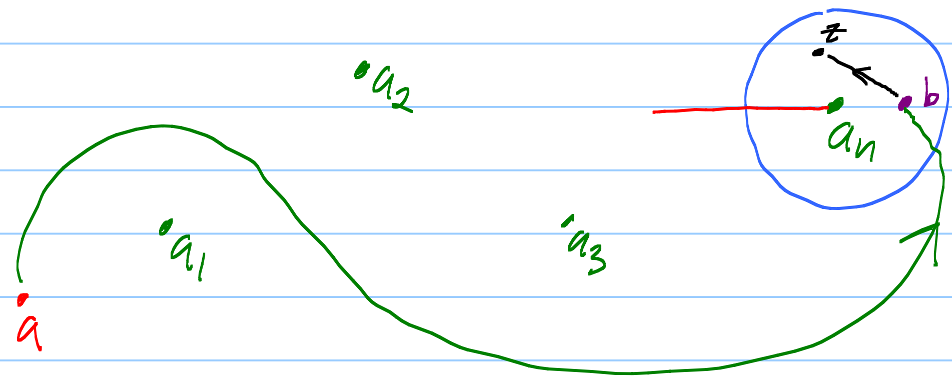


$$f(z) = \exp\left(\underbrace{\int_{\gamma_a^{z_0}} F\,dw}_{\text{const}} + \underbrace{\int_{L_{z_0}^{z}} F\,dw}_{\text{derivative } = F(z)}\right) \checkmark$$

see $\quad f'(z) = \underbrace{\exp\left(\quad\right)}_{f(z)} F(z)$

$$\frac{f'(z)}{f(z)} = F(z) \longleftarrow \text{hopefull}$$

Step 3: a_n removable sing for f. In fact, a zero of mult m_n.



$$f(z) = \exp\left(\underbrace{\int_{\gamma_a^b} F(w)\,dw}_{\text{const}} + \underbrace{\int_{L_b^z} \frac{m_n}{w-a_n}}_{m_n\left[\operatorname{Log}(z-a_n) - \underbrace{\operatorname{Log}(b-a_n)}_{\text{const}}\right]} + \underbrace{H(w)}_{\text{analytic}}\,dw \right)$$

$$f(z) = \exp\Big(\text{const} + m_n \operatorname{Log}(z-a_n) + \underbrace{h(z)}_{\text{antider of } H} \Big)$$

$$= \underbrace{\exp\big(m_n \operatorname{Log}(z-a_n)\big)}_{(z-a_n)^{m_n}} \left[\text{Analytic and } \neq 0\right]$$

Aha! Analytic above branch cut, and below and limits agree on cut. Analytic extension across cut! (Morera's).

Cor: Combine M-L and Weier to be able to prescribe finitely terms in power series at seq $a_n \to \infty$.

Cor Meromorphic fcns on $\mathbb{C}$ $= \{\frac{f}{g} :$ f, g entire, $g \not\equiv 0\}$.

Pf: Suppose F mero on $\mathbb{C}$ with poles of order m_n at discrete $a_n \to \infty$. Weir $\Rightarrow \exists$ entire g with zeroes of mult m_n at a_n. $[g \not\equiv 0]$.

Aha! $gF := f$ has removable sing at a_n's. ✓

Infinit products $\prod_{n=1}^{\infty} p_n = \lim_{N\to\infty} \prod_{n=1}^{N} p_n$ where $p_n \neq 0$, $\forall n$.

Lemma If limit exists $\neq 0$, then $p_n \to 1$ as $n \to \infty$.

Pf: $$p_N = \frac{\prod_1^N p_n}{\prod_1^{N-1} p_n} \to \frac{L}{L} = 1$$

Traditional to write $\prod_{n=1}^{\infty}(1+a_n)$ instead.

Lemma $\prod_{n=1}^{\infty}(1+a_n)$ converges to $L \neq 0$

$$\Longleftrightarrow \sum_{n=1}^{\infty} \operatorname{Log}(1+a_n) \text{ converges}$$

Pf: $(\Leftarrow)$ easy. $\prod_{n=1}^{N}(1+a_n) = \exp\left(\sum_1^N \operatorname{Log}(1+a_n)\right)$

$(\Rightarrow)$ Read Stein p. 141.

Fact $\operatorname{Log}(1+z)$ analytic on $D_1(0)$ and has a simple zero. And $\left.\frac{d}{dz}\operatorname{Log}(1+z)\right|_{z=0} = 1$

So $(1-\varepsilon)|a_n| \le \left|\operatorname{Log}(1+a_n)\right| \le (1+\varepsilon)|a_n|$. $(*)$

Given ε with $0<\varepsilon<1$, $\exists\ \delta>0$ such that $(*)$ holds when $|a_n|<\delta$.

Def^n: $\prod_{n=1}^{\infty}(1+a_n)$ converges absolutely when

$$\sum_{n=1}^{\infty}|a_n| < \infty.$$

So, absolute conv $\Longrightarrow$ conv.

Fact If $\sum_{n=1}^{\infty} f_n(z)$ converges absolutely and uniformly on compact sets of a domain Ω, f_n analytic, then $\prod_{n=1}^{\infty}(1+f_n(z))$ converges absolutely and uniformly on compacts on Ω to analytic limit.

EX: $f(z) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$ $\leftarrow$ entire. Simple zeroes at $\mathbb{Z}$.

$$\sum_{n=1}^{\infty}\left|\frac{z^2}{n^2}\right| = |z|^2\left(\frac{\pi^2}{6}\right) < \infty$$

Remark: $\prod (z-a_n)$

$\prod \left(1-\frac{z}{a_n}\right) \leftarrow$ better

If $a_n \to \infty$ fast enough, can see conv.

EX: $\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)$ Ouch! $\sum_{1}^{\infty}\left|\frac{z}{n}\right| = |z|\sum_{1}^{\infty}\frac{1}{n} \to \infty.$

Weier: multiply by something here.

Inspiration:

$$1 = (1-z)\, e^{-\mathrm{Log}(1-z)}$$

$$-\mathrm{Log}(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$$

$$P_N(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \frac{z^N}{N}$$

Can make $1 \approx (1-z)\, e^{P_N(z)}$ as close as desired on $D_{1/2}(0)$ by taking N big.

Weier: $$f(z) = \prod_{n=1}^{\infty}\left[\left(1-\frac{z}{a_n}\right) e^{P_{N_n}\left(\frac{z}{a_n}\right)}\right]^{m_n}$$

Famous formulas

$$\sin \pi z = \pi z \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(1-\frac{z}{n}\right) e^{z/n}$$

$$= \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2}\right)$$

$$\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

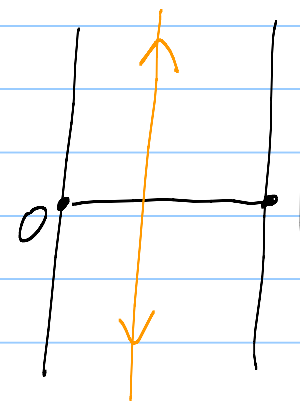
both periodic with period 1.

Same principal parts at $\mathbb{Z}$

$H(z) = (\text{left}) - (\text{right})$ entire

both tend zero along vert lines $\to \pm\infty$.

H bndd entire. $\to 0$. So $\equiv 0$.



ecture 41 The inhomogeneous Cauchy-Riemann eqns

Review problems on home page

The $\bar{\partial}$-operator

$$\begin{cases} dz = dx + i\,dy \\ d\bar{z} = dx - i\,dy \end{cases} \qquad dx = \frac{1}{2}(dz + d\bar{z}), \quad dy = \frac{1}{2i}(dz - d\bar{z})$$

If $f = u + iv$, then $\frac{\partial f}{\partial x} = u_x + i v_x$ and $\frac{\partial f}{\partial y} = u_y + i v_y$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = \underbrace{\left(\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)\right)}_{\frac{\partial f}{\partial z}} dz + \underbrace{\left(\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)\right)}_{\frac{\partial f}{\partial \bar{z}}} d\bar{z}$$

$\frac{\partial f}{\partial z} \leftarrow$ def^{n}s $\rightarrow \frac{\partial f}{\partial \bar{z}}$

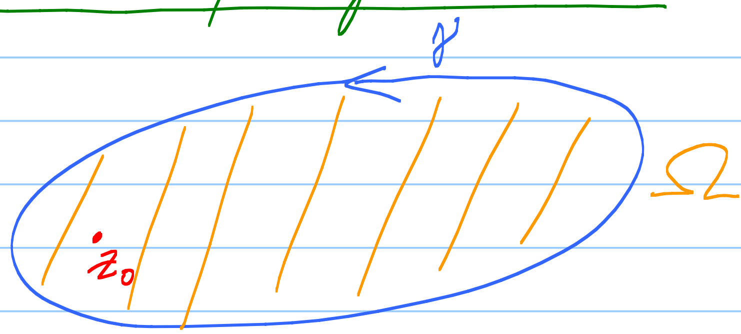
Facts: $f = u + iv$. u & v satisfy the C-R eqns $\iff \frac{\partial f}{\partial \bar{z}} \equiv 0$.

and, if f is analytic, then $\frac{\partial f}{\partial z} = f'$.

Complex chain rule $\quad w = f(\zeta)$ where $\zeta = g(z)$.

$$\begin{cases} \dfrac{\partial w}{\partial z} = \dfrac{\partial w}{\partial \zeta}\dfrac{\partial \zeta}{\partial z} + \dfrac{\partial w}{\partial \bar{\zeta}}\dfrac{\partial \bar{\zeta}}{\partial z} \\[2ex] \dfrac{\partial w}{\partial \bar{z}} = \dfrac{\partial w}{\partial \zeta}\dfrac{\partial \zeta}{\partial \bar{z}} + \dfrac{\partial w}{\partial \bar{\zeta}}\dfrac{\partial \bar{\zeta}}{\partial \bar{z}} \end{cases}$$

Improved Cauchy integral formula (Pompieu)



$$u(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{u(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\iint_\Omega \frac{\frac{\partial u}{\partial \bar z}(z)}{z-z_0}\,dz\wedge d\bar z$$

for any $u \in C^\infty(\bar\Omega)$. [$u$ complex valued]

<u>Note</u>: u analytic, then $\frac{\partial u}{\partial \bar z} \equiv 0$ and get CIF.

$$dz\wedge d\bar z = (dx + i\,dy)\wedge(dx - i\,dy)$$

$$= \underbrace{dx\wedge dx}_{0} - i\,dx\wedge dy + i\,\underbrace{dy\wedge dx}_{-dx\wedge dy} + \underbrace{dy\wedge dy}_{0}$$

$$= -2i\,dx\wedge dy \leftarrow dx\wedge dy \sim dx\,dy \text{ area measure}$$

<u>Theorem</u>: (Lars Hörmander, Several complex variables, Chap 1)

Given a complex valued C^∞ fcn v on a domain Ω, there is a $u \in C^\infty(\Omega)$ with $\frac{\partial u}{\partial \bar z} = v$.

<u>Note</u>: If u_1 and u_2 both sol^{n}s, then

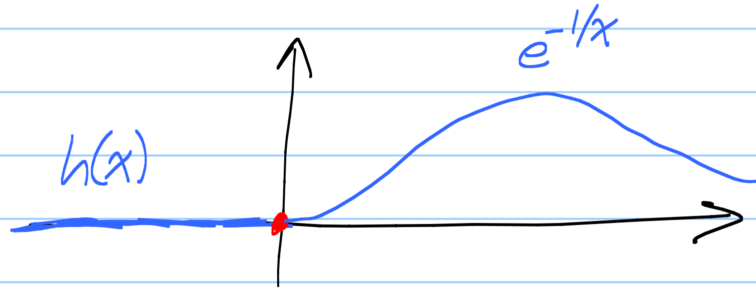
$\frac{\partial}{\partial \bar z}(u_1 - u_2) = v - v \equiv 0$ and $u_1 - u_2 = h$, an analytic fcn. Sol^n unique up to + (analytic fcn).

<u>Cor</u>: Mittag-Leffler thm holds on any domain Ω.

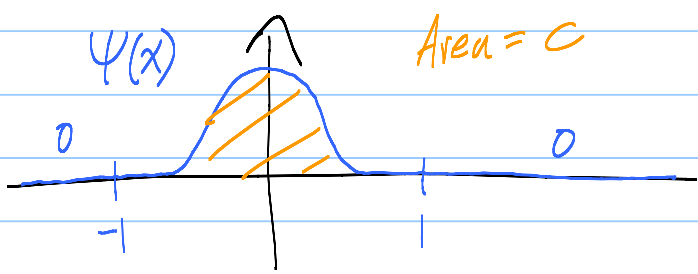
Given discrete $A = \{a_n\}_{n=1}^\infty \subset \Omega$ and princ parts $K_n(z)$ at each a_n, $\exists$ mero f fcn on Ω with poles only at pts in A with prescribed princ part.

<u>Pf</u>: Plan: Solve M-L prob in C^∞.
Correct it by solving a $\bar\partial$-problem.

<u>Step 0</u> Cook C^∞ "bump fcns" and cut-off fcns.



h is $C^\infty(\mathbb{R})$



$h(x+\frac{1}{2})\,h(-x+\frac{1}{2})$

Support of ψ = Supp ψ = closure of $\{x : \psi(x) \neq 0\}$.

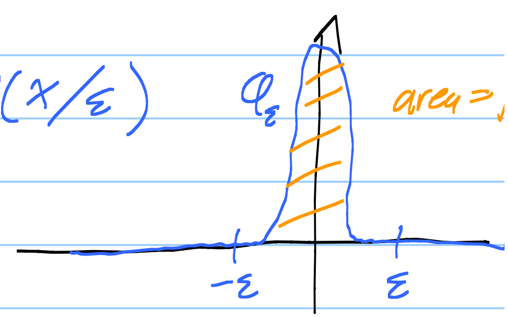
Supp ψ is <u>compact</u> in $(-1,1)$.

$$\varphi(x) = \frac{1}{c}\psi(x)$$

even, $\in C_0^\infty(-1,1)$

$$\int_{-\infty}^{\infty} \varphi \, dx = 1.$$

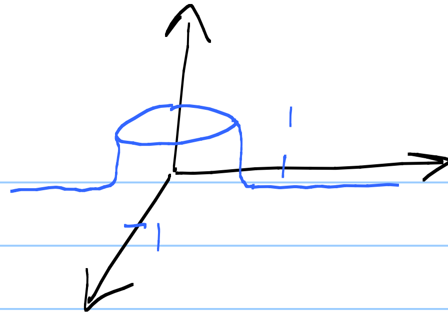
$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(x/\varepsilon)$$

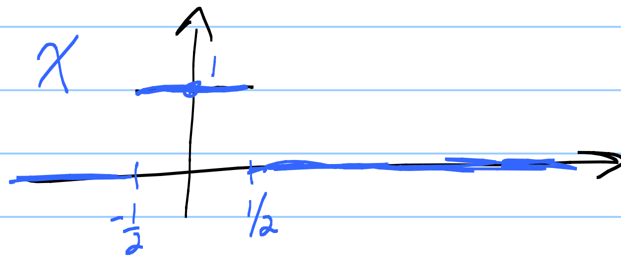


Classic fact: $\varphi_\varepsilon * f$ ← smooths out f

"close to f in L^2, etc"

C^∞ "birthday cake fcn"





$$\psi = (\varphi_\varepsilon * \chi)(x)$$

$$= \int_{-\infty}^{\infty} \varphi_\varepsilon(x-y)\,\chi(y)\,dy$$

If $\varepsilon < \frac{1}{2}$, $\psi \in C_0^\infty(-1,1)$ and $\psi \equiv 1$ near 0.

Birthday cake fcn : $\chi_r^{z_0}(z) = \psi\left(\frac{|z-z_0|^2}{r^2}\right)$

centered at z_0, compact supported in $D_r(z_0)$, C^∞ smooth, $\equiv 1$ on a nbhd of z_0.

<u>Pf of M-L</u> : A discrete. So $\exists$ discs $\overline{D_{r_n}(a_n)} \subset \Omega$ and $\overline{D_{r_n}(a_n)} \cap \overline{D_{r_m}(a_m)} = \emptyset$ when $n \neq m$. Let $U = \sum_{n=1}^{\infty} \chi_n R_n$ ⟸ finite sum at any spot.

where χ_n is a C^∞ cut-off fcn suppored in $D_{r_n}(a_n)$ and $\equiv 1$ on a nbhd of a_n.

U is a "C^∞ solution to M-L prob"

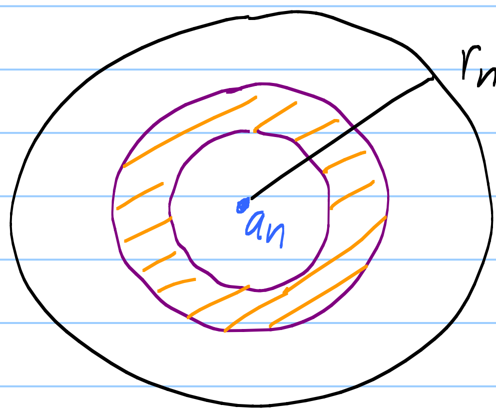
Near a_n: $\quad U - R_n = (\chi_n R_n) - R_n$

$\equiv 0$ near a_n.

Hmmm. $$\frac{\partial U}{\partial \bar z} = \sum_1^\infty \left(\underbrace{\frac{\partial \chi_n}{\partial \bar z}}_{\text{zero near } a_n} \underbrace{R_n}_{\text{bad at } a_n} + \chi_n \underbrace{\frac{\partial R_n}{\partial \bar z}}_{\equiv \text{ zero (not defined at } a_n)} \right)$$

Aha! Define $$v = \begin{cases} 0 & \text{at } a_n\text{'s} \\ \frac{\partial U}{\partial \bar z} & \text{on } \Omega - A \end{cases}$$

Support of $\frac{\partial U}{\partial \bar z}$



v is C^∞-smooth on Ω.

Get $u \in C^\infty(\Omega)$ with $\frac{\partial u}{\partial \bar z} = v$.

Claim: $f = U - u$ solves M-L prob!

Why: $\frac{\partial f}{\partial \bar z} = \frac{\partial U}{\partial \bar z} - \frac{\partial u}{\partial \bar z} = v - v \equiv 0 \quad z \neq a_n$.

f analytic on $\Omega - A$. ✓

Near a_n: $f - R_n = (\underbrace{\tilde{U}}_{\chi_n R_n} - u) - R_n$

$$= \underbrace{(1-\chi_n)R_n}_{\equiv 0 \text{ near } a_n} - \underset{\substack{\uparrow \\ \text{bndd}}}{u}$$

a_n is a removable singularity for $f - R_n$.

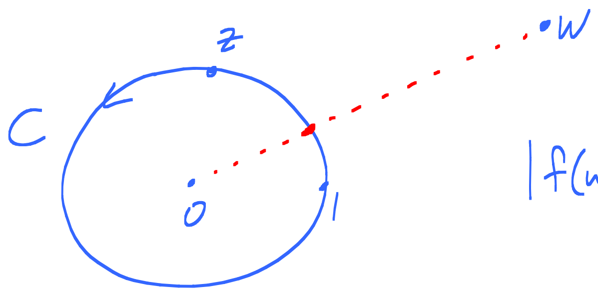
Review lecture 1

Final exam a take home exam.
Tuesday, Due 11:59 pm Wed. Open book, notes only.
3 hours max.

1. Suppose that ϕ is a continuous function on the unit circle in the complex plane, and let C denote the unit circle parametrized via $z(t) = e^{it}$, $0 \le t \le 2\pi$. Let $\Omega = \{w \in \mathbb{C} : |w| > 1\}$. For $w \in \Omega$, define

$$f(w) = \int_C \frac{\phi(z)}{z-w}\, dz.$$

What kind of singularity does f have at infinity? Use careful estimates and explain.



$$|f(w)| \le \frac{\max_C |\varphi|}{|w|-1}\, 2\pi \cdot 1$$

$$\longrightarrow 0 \text{ as } w \to \infty.$$

Type of sing of f at ∞ $\underset{\text{def}^n}{=}$ Type of sing of $f(\frac{1}{z})$ at $z=0$.

Removable sing at ∞. In fact, f has a "zero at ∞".

f analytic on $\{z : |z| > R\}$. ∞ is an "isolated sing" on $\hat{\mathbb{C}}$. 3 types: removable, pole, essential.

2. Suppose that A is a finite set and that $f(z)$ is analytic on $\mathbb{C} - A$ with poles at each point in A. Prove that if f has a removable singularity at infinity, then f must be a rational function.

Aha! $\underbrace{f}_{\substack{\text{value} \\ \text{as } \to \infty.}} - \underbrace{\left(\sum_{1}^{N} \text{princ parts } A \right)}_{\to 0 \text{ at } \infty} = F$

F entire, bnd. Liouville's $\Rightarrow F \equiv c.$ ✓

Remark: Same thing true if f has a pole at ∞.

(If A infinite discrete set. $A = \{a_n\}_{n=1}^{\infty}$ and $a_n \to \infty$ as $n \to \infty$. ∞ is <u>not</u> an isolated sing.)

3. Let $a_0 = 0$ and $a_1 = 1$. The Fibonacci numbers are defined inductively via

$$a_{n+2} = a_{n+1} + a_n \qquad \text{for } n = 0, 1, 2, \ldots$$

Find the radius of convergence of the power series $\sum a_n z^n$. Hint: Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and compare the power series for $z^2 f(z)$, $zf(z)$, and $f(z)$. Find a closed form formula for f. When you get the picture, make sure everything you say is true. (For example, don't say that $f(z)$ is defined someplace until you know it is.)

$$z^2 f(z) = \sum_{n=0}^{\infty} a_n z^{n+2} = \sum_{n=2}^{\infty} a_{n-2} z^n$$

$$z f(z) = \;.\;.\;.\;.\;.\; = \sum_{1}^{\infty} a_{n-1} z^n$$

$$f(z) = z^2 f(z) + z f(z) \quad \text{almost}$$

$$-0-1\cdot z$$

Suspect: $f(z)\left[1 - z - z^2\right] = z$

Radius of conv = dist (0, nearest pole)

To nail, have to go backwards from.

Define $f(z) = \dfrac{z}{1-z-z^2}$, get R. of C.

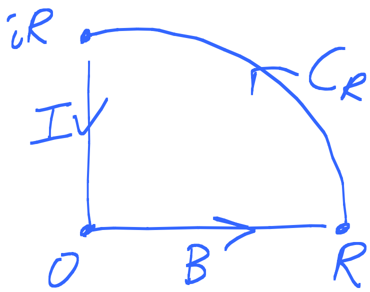
Show $(1-z-z^2)f(z) = z$ yields a_n = F. #s.

4. How many zeroes does the polynomial

$$z^{1998} + z + 2001$$

have in the first quadrant? Explain your answer.

Arg princ: $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \left(\begin{array}{c}\#\text{ zeroes } f \\ \text{inside } \gamma \text{ with mult}\end{array}\right)$



$=$ # times $f(z)$ spins around 0 as z goes around γ in C.C. sense.

Get $\int_{C_R} \frac{f'}{f}\,dz$ by letting $R \to \infty$.

Along B: $t^{1998} + t + 2001$ sticks to $\mathbb{R}$!

$\Delta_B \arg = 0.$

Along I: $(it)^{1998} + (it) + 2001$

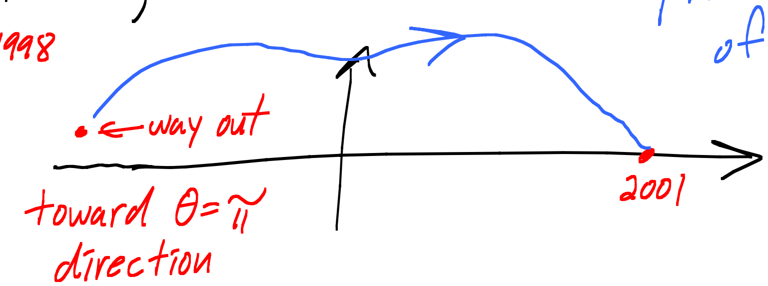
$-t^{1998} + it \quad 2001$

$(2001 - t^{1998}) + it \leftarrow$ in UHP!

iR: $\underset{2001 - R^{1998}}{(\text{Big negative }\#)} + iR$

Can use princ branch of arg.

$i \cdot 0$: $\quad 2001$



$\Delta_I \arg \to -\pi$

5. Suppose that f is a non-vanishing analytic function on the complex plane minus the origin. Let γ denote the curve given by $z(t) = e^{it}$ where $0 \le t \le 2\pi$. Suppose that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$$

is divisible by 3. Prove that f has an analytic cube root on $\mathbb{C} - \{0\}$.

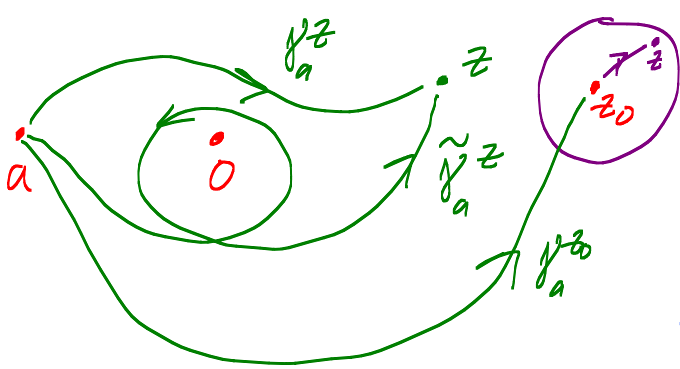
Hmmm $\displaystyle\int_{\gamma_a^z} \frac{f'(w)}{f(w)}\,dw \leftarrow ?\ \log f$

Not well defined!

Aha! "Define" $\displaystyle F(z) = \exp\left(\frac{1}{3}\int_{\gamma_a^z} \frac{f'(w)}{f(w)}\,dw\right)$

Step 1 F well defined.

$$\frac{F(z)}{\tilde{F}(z)} = \exp\left(\frac{1}{3}\underbrace{\left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z}\right)}_{\int_\gamma\ \gamma \text{ closed}} \frac{f'}{f}\,dw\right)$$



$$\frac{1}{3}\left(2\pi i \cdot N \cdot 3\right)$$

$$= e^{2\pi i N} = 1$$

Step 2 Is $F^3 = f$? Almost!

Key F analytic: $\displaystyle F' = \exp\left(\tfrac{1}{3}\int\right)\left[0 + \frac{1}{3}\frac{f'(z)}{f(z)}\right]$

$$F' = F \cdot \frac{1}{3}\frac{f'}{f}$$

Step 3 Trick: $\displaystyle\frac{d}{dz}\left[\frac{F^3}{f}\right] \equiv 0.$

6. Suppose that $\{a_k\}_{k=1}^N$ is a finite sequence of distinct complex numbers and that f is analytic on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$. Prove that there exist constants c_j, $j = 1, 2, \ldots, N$, such that

$$f(z) - \sum_{k=1}^{N} \frac{c_k}{z - a_k}$$

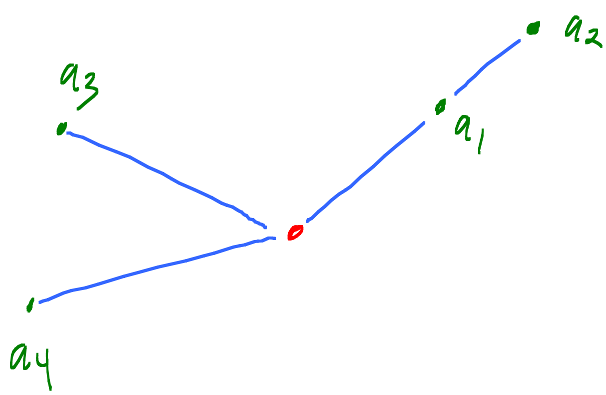
has an analytic antiderivative on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$.

__Key__: analytic f has an antiderivative on a domain $\Longleftrightarrow$ $\int_\gamma f\,dz = 0$ for __all__ closed γ in Ω.

Hmmm. $\displaystyle\int_\gamma f\,dz \underset{\substack{\uparrow \\ \text{Gen Res thm.}}}{=} 2\pi i \sum \mathrm{Ind}_\gamma(a_n)\,\mathrm{Res}_{a_n} f$

7. Suppose $a_1, a_2, \ldots, a_N$ are distinct nonzero complex numbers and let Ω denote the domain obtained from $\mathbb{C}$ by removing each of the closed line segments joining a_k to the origin, $k = 1, \ldots, N$. Prove that there is an analytic function f on Ω such that

$$f(z)^N = \prod_{k=1}^{N} (z - a_k).$$

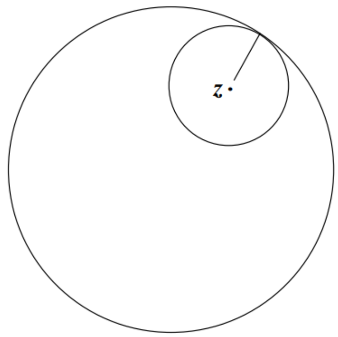


8. Find a one-to-one conformal mapping of the "piece of pie" $\{re^{i\theta} : 0 < r < 1,\ 0 < \theta < \pi/4\}$ onto the horizontal strip $\{z : 0 < \operatorname{Im} z < 1\}$.

9. Suppose that u is a continuous real valued function on $\overline{D_1(0)}$ and that

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + (1 - |z|)e^{i\theta})\, d\theta \tag{$*$}$$

for each $z \in D_1(0)$. (This equality means that u is only known to satisfy the averaging property on circles like the one pictured below.) Prove that u is harmonic in $D_1(0)$.



Review lecture 2

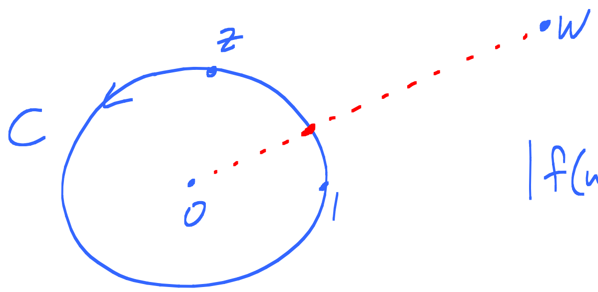
Final exam a take home exam.

Tuesday, Due 11:59 pm Wed. Open book, notes only.
3 hours max. Scan, upload in Gradescope.

1. Suppose that ϕ is a continuous function on the unit circle in the complex plane, and let C denote the unit circle parametrized via $z(t) = e^{it}$, $0 \le t \le 2\pi$. Let $\Omega = \{w \in \mathbb{C} : |w| > 1\}$. For $w \in \Omega$, define

$$f(w) = \int_C \frac{\phi(z)}{z - w}\, dz.$$

What kind of singularity does f have at infinity? Use careful estimates and explain.



$$|f(w)| \le \frac{\max_C |\varphi|}{|w| - 1}\, 2\pi \cdot 1$$

$$\longrightarrow 0 \text{ as } w \to \infty.$$

Type of sing of f at ∞ $\underset{\text{def}^n}{=}$ Type of sing of $f(\frac{1}{z})$ at $z = 0$.

Removable sing at ∞. In fact, f has a "zero at ∞".

f analytic on $\{z : |z| > R\}$. ∞ is an "isolated sing" on $\hat{\mathbb{C}}$. 3 types: removable, pole, essential.

2. Suppose that A is a finite set and that $f(z)$ is analytic on $\mathbb{C} - A$ with poles at each point in A. Prove that if f has a removable singularity at infinity, then f must be a rational function.

Aha! $\underbrace{f}_{\substack{\text{value}\\ \text{as } \to \infty.}} - \underbrace{\left(\sum_1^N \text{princ parts } A \right)}_{\to 0 \text{ at } \infty} = F$

F entire, bnd. Liouville's $\Rightarrow F \equiv c.$ ✓

Remark: Same thing true if f has a pole at ∞.

(If A infinite discrete set. $A = \{a_n\}_{n=1}^{\infty}$ and $a_n \to \infty$ as $n \to \infty$. ∞ is <u>not</u> an isolated sing.)

3. Let $a_0 = 0$ and $a_1 = 1$. The Fibonacci numbers are defined inductively via

$$a_{n+2} = a_{n+1} + a_n \qquad \text{for } n = 0, 1, 2, \ldots$$

Find the radius of convergence of the power series $\sum a_n z^n$. Hint: Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and compare the power series for $z^2 f(z)$, $zf(z)$, and $f(z)$. Find a closed form formula for f. When you get the picture, make sure everything you say is true. (For example, don't say that $f(z)$ is defined someplace until you know it is.)

$$z^2 f(z) = \sum_{n=0}^{\infty} a_n z^{n+2} = \sum_{n=2}^{\infty} a_{n-2} z^n$$

$$z f(z) = \;\cdots\cdots\; = \sum_{1}^{\infty} a_{n-1} z^n$$

$$f(z) = z^2 f(z) + z f(z) \qquad \text{almost}$$

$$-0 - 1 \cdot z$$

Suspect: $f(z)\left[1 - z - z^2\right] = z$

Radius of conv = dist (0, nearest pole)

To nail, have to go backwards from.

Define $f(z) = \dfrac{z}{1 - z - z^2}$, get R. of C.

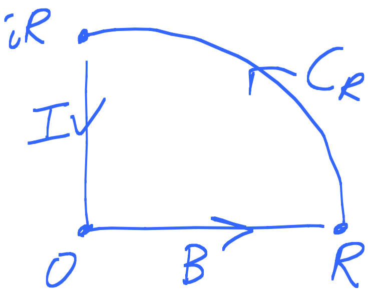
Show $(1 - z - z^2) f(z) = z$ yields a_n = F. #s.

4. How many zeroes does the polynomial

$$z^{1998} + z + 2001$$

have in the first quadrant? Explain your answer.

Arg princ: $\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = \left(\begin{matrix}\#\text{ zeroes } f \\ \text{inside } \gamma \text{ with mult}\end{matrix}\right)$



$=$ # times $f(z)$ spins around 0 as z goes around γ in C.C. sense.

Get $\int_{C_R} \frac{f'}{f}\,dz$ by letting $R \to \infty$.

Along B: $t^{1998} + t + 2001$ sticks to $\mathbb{R}$!

$\Delta_B \arg = 0.$

Along I: $(it)^{1998} + (it) + 2001$

$-t^{1998} + it \quad 2001$

$(2001 - t^{1998}) + it$ ← in $\underline{\underline{\text{UHP}}}$!

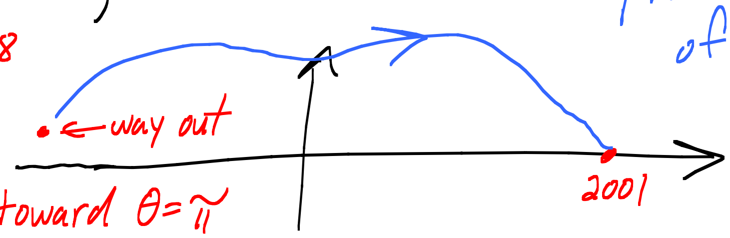
iR: (Big negative #) $+ iR$

$2001 - R^{1998}$

ArcTan $\frac{R}{2001 - R^{1998}}$

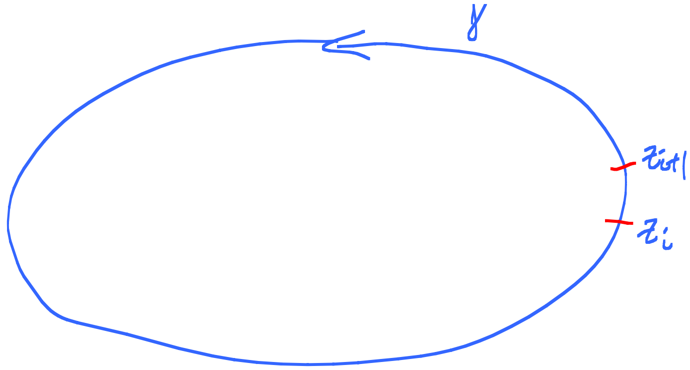
$i \cdot 0$: $\quad 2001$

$\Delta_I \arg \to -\pi$



toward $\theta = \pi$ direction

Can use princ branch of arg.



$$\int_\gamma \frac{f'}{f}\,dz = \sum \log f(z)\Big|_{z_i}^{z_{i+1}}$$

$$= \underbrace{\sum \ln|f(z)|\Big|_{z_i}^{z_{i+1}}}_{=0} + i\Delta_\gamma \operatorname{Arg} f(z)$$

Important point $\quad \operatorname{Im}\left(\int_{\gamma_a^b} \frac{f'}{f}\,dz\right) = \Delta_{\gamma_a^b} \operatorname{Arg} f$

$$\int_{C_R} \underbrace{\frac{1998 z^{1997} - 1}{z^{1998} - z + 2001}}_{\sim \frac{1998}{z}}\,dz$$

$$\int_{C_R} \frac{1998}{z}\,dz = \int_0^{\pi/2} \frac{1998}{Re^{it}}\, iRe^{it}\,dt = i\,\underbrace{\frac{\pi}{2}\,1998}_{\Delta_{C_R}\arg}$$

Show $\quad \int_{C_R}\left(\frac{1998 z^{1997} - 1}{z^{1998} - z + 2001} - \frac{1998}{z}\right) dz \longrightarrow 0$

$R \to \infty$

5. Suppose that f is a non-vanishing analytic function on the complex plane minus the origin. Let γ denote the curve given by $z(t) = e^{it}$ where $0 \le t \le 2\pi$. Suppose that

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz$$

is divisible by 3. Prove that f has an analytic cube root on $\mathbb{C} - \{0\}$.

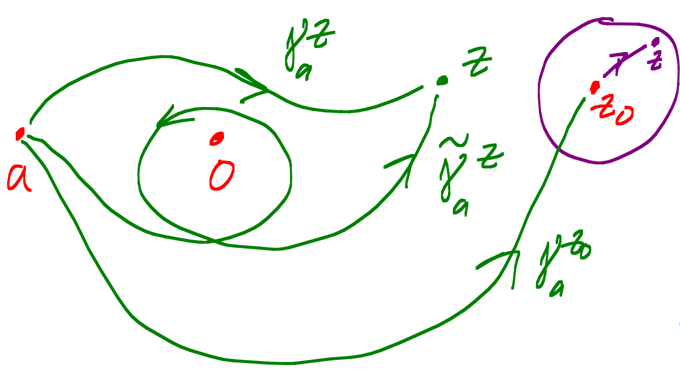
Hmmm $\displaystyle\int_{\gamma_a^z} \frac{f'(w)}{f(w)}\, dw \leftarrow ?\ \log f$

Not well defined!

Aha! "Define" $\displaystyle F(z) = \exp\left(\frac{1}{3}\int_{\gamma_a^z} \frac{f'(w)}{f(w)}\, dw\right)$

Step 1 F well defined.

$$\frac{F(z)}{\tilde{F}(z)} = \exp\left(\frac{1}{3}\underbrace{\left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z}\right)}_{\int_\gamma\ \gamma \text{ closed}} \frac{f'}{f}\, dw\right)$$



$$\frac{1}{3}\left(2\pi i \cdot N \cdot 3\right)$$

$$= e^{2\pi i N} = 1$$

Step 2 Is $F^3 = f$? Almost!

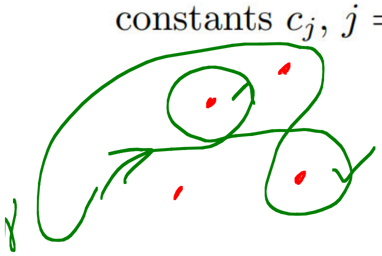
Key F analytic: $\displaystyle F' = \exp\left(\tfrac{1}{3}\int\right)\left[0 + \frac{1}{3}\frac{f'(z)}{f(z)}\right]$

$$F' = F \cdot \frac{1}{3}\frac{f'}{f}$$

Step 3 Trick: $\displaystyle\frac{d}{dz}\left[\frac{F^3}{f}\right] \equiv 0.$ $cF^3 = f$

Ans: $c^{1/3} F$

6. Suppose that $\{a_k\}_{k=1}^N$ is a finite sequence of distinct complex numbers and that f is analytic on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$. Prove that there exist constants c_j, $j = 1, 2, \ldots, N$, such that



$$f(z) - \sum_{k=1}^{N} \frac{c_k}{z - a_k}$$

$\Omega = \mathbb{C} - \{a_n\}_{n=1}^N$

has an analytic antiderivative on $\mathbb{C} - \{a_k : k = 1, 2, \ldots, N\}$.

Key: analytic f has an antiderivative on a domain $\iff \int_\gamma f\,dz = 0$ for all closed γ in Ω.

Hmmm. $\int_\gamma f\,dz = 2\pi i \sum \text{Ind}_\gamma(a_n)\,\text{Res}_{a_n} f$

Gen Res thm.

$\underbrace{F}_{=0}$

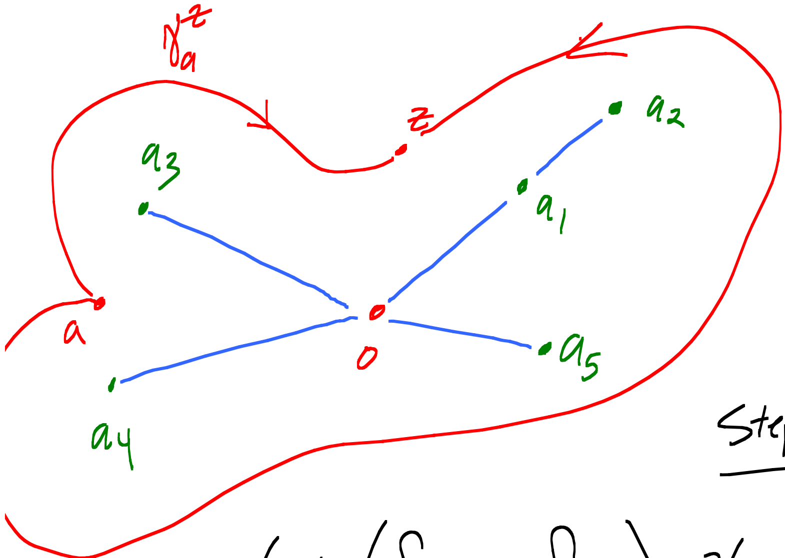
$F = f - \sum \frac{c_k}{z - a_k}$

$c_k = \text{Res}_{a_k} f$

So $\text{Res}_{a_k} F = 0$

7. Suppose $a_1, a_2, \ldots, a_N$ are distinct nonzero complex numbers and let Ω denote the domain obtained from $\mathbb{C}$ by removing each of the closed line segments joining a_k to the origin, $k = 1, \ldots, N$. Prove that there is an analytic function f on Ω such that

$$f(z)^N = \prod_{k=1}^{N} (z - a_k). \quad = p(z)$$



Try

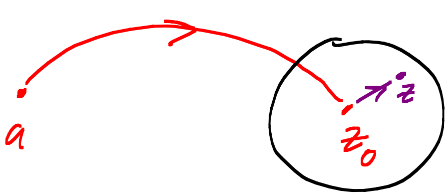
$$F(z) = \exp\left(\frac{1}{N}\int_{\gamma_a^z} \frac{p'(w)}{p(w)}\,dw\right)$$

Step 1 $\quad \dfrac{F(z)}{\tilde{F}(z)} =$

$$\exp\left(\frac{1}{N}\underbrace{\left(\int_{\gamma_a^z} - \int_{\tilde{\gamma}_a^z}\right)}_{\gamma \text{ closed}} \frac{p'}{p}\,dw\right)$$

$$= \exp\left(\frac{1}{N}\, 2\pi i \sum_{1}^{N} \underbrace{\mathrm{Ind}_\gamma(a_k)}_{\substack{\text{all} \\ \text{same!} \\ n}}\ \underbrace{\mathrm{Res}_{a_k} \frac{p'}{p}}_{\substack{= \text{order of simple zero} \\ = 1}}\right)$$

$$= \exp\left(\frac{1}{N}\, 2\pi i \cdot n \cdot N\right) = e^{2\pi n i} = 1 \ \checkmark$$

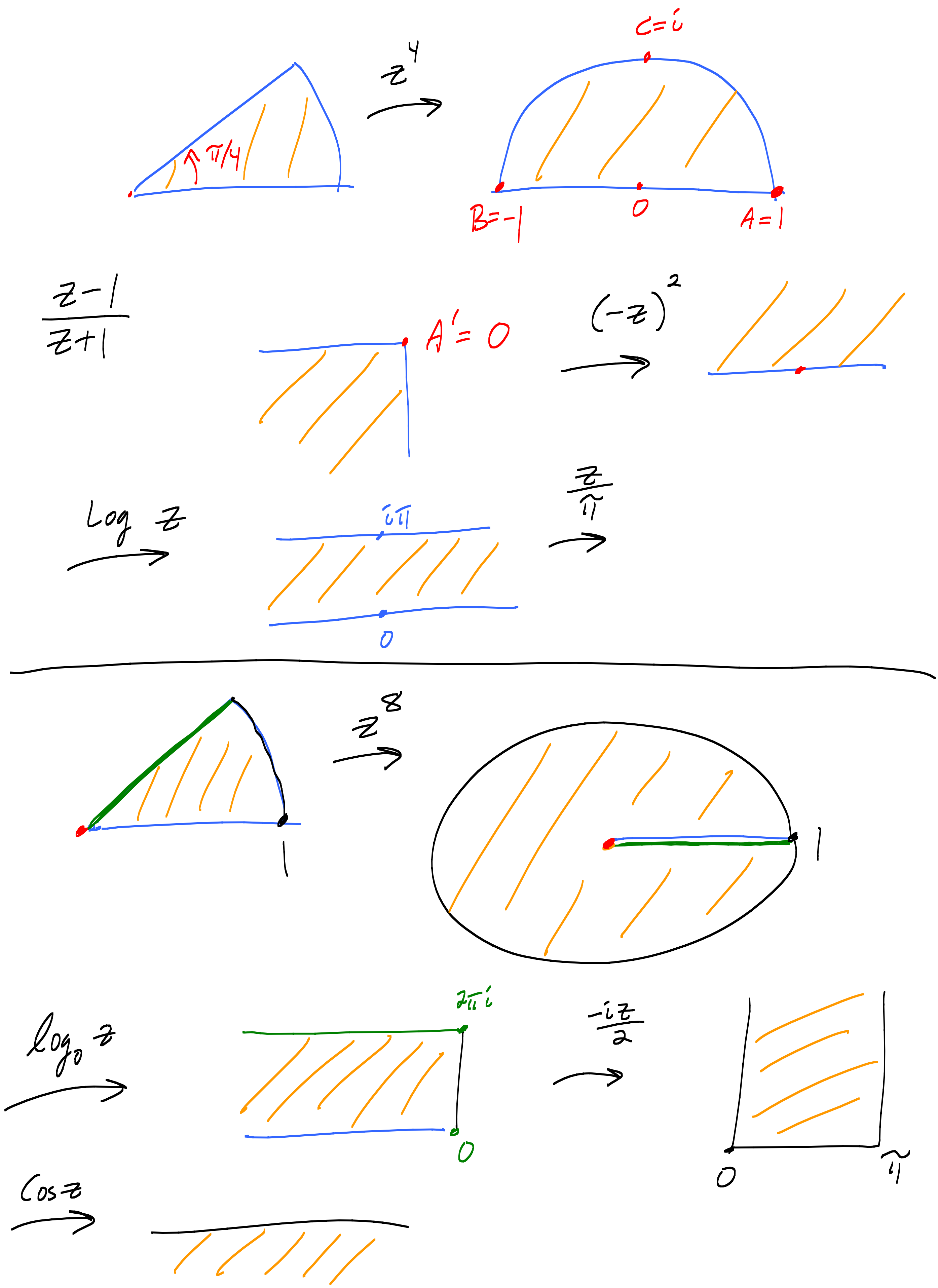


$$F' = \exp\left(\tfrac{1}{N}\int\right)\left[0 + \frac{1}{N}\frac{p'(z)}{p(z)}\right]$$

$$F' = F\,\frac{1}{N}\,\frac{p'}{p}$$

$$\frac{d}{dz}\left(\frac{F^N}{p}\right) \equiv 0, \text{ etc.}$$

8. Find a one-to-one conformal mapping of the "piece of pie" $\{re^{i\theta} : 0 < r < 1,\ 0 < \theta < \pi/4\}$ onto the horizontal strip $\{z : 0 < \operatorname{Im} z < 1\}$.

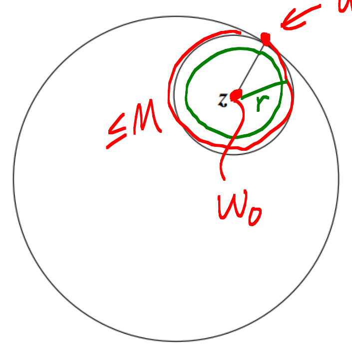


9. Suppose that u is a continuous real valued function on $\overline{D_1(0)}$ and that

$$(*) \qquad u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + (1-|z|)e^{i\theta})\, d\theta$$

for each $z \in D_1(0)$. (This equality means that u is only known to satisfy the averaging property on circles like the one pictured below.) Prove that u is harmonic in $D_1(0)$.

$\tilde{u} - u = 0$ here!



$r \nearrow (1-|z|)$
unif cont. $\Rightarrow$
$\tilde{u}$ satisfies $(*)$ too.

Let $\tilde{u}$ = Poisson integral of $u(e^{i\theta})$.

$\tilde{u} - u \not\equiv 0$, then $\tilde{u}(z_0) - u(z_0) = c \neq 0$. Suppose $c > 0$.

$\tilde{u} - u$ has max M on $\overline{D_1(0)}$ at some pt. $w_0 \in D_1(0)$.

$\left(\text{Ave of } \underbrace{\tilde{u} - u}_{\substack{< M \\ \text{near edge}}} \text{ on circle}\right) = M$ $\lightning$

Case $c < 0$: Apply to $u - \tilde{u}$ instead. Same $\lightning$.

So $\tilde{u} - u \equiv 0$. $\quad u = \tilde{u}$ is harmonic.